

DOMINATION OF SEMIGROUPS GENERATED BY REGULAR FORMS

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ABSTRACT. We give a representation for regular forms associated with dominated C_0 -semigroups which, in turn, characterises the domination of C_0 -semigroups associated with regular forms. In addition, we prove a relationship between the positivity of (dominated) C_0 -semigroups and the locality of the associated forms.

1. INTRODUCTION

In the theory of semigroups on the Hilbert space L^2 generated by bilinear forms, there is a well-known characterisation due to Ouhabaz [22, Theorem 2.21], which characterizes domination of semigroups in terms of the generating forms. This result is a consequence of another characterisation, namely of the situation when a semigroup (orbit) on an abstract Hilbert space leaves a closed and convex subset invariant [22, Theorem 2.2]. In fact, the Beurling-Deny criteria – which characterise positivity and L^∞ -contractivity of a semigroup – are also consequences of this abstract result.

In this short note, we revisit the domination of semigroups generated by forms. The well-known characterization of domination [22, Theorem 2.21] states that if T and \widehat{T} are two semigroups on $L^2_\mu(\Omega)$ generated by quadratic forms \mathfrak{a} and $\widehat{\mathfrak{a}}$ respectively such that \widehat{T} is positive, then the semigroup T is dominated by \widehat{T} if and only if $D(\mathfrak{a})$ is an ideal of $D(\widehat{\mathfrak{a}})$ and

$$\widehat{\mathfrak{a}}(|u|, |v|) \leq \operatorname{Re} \mathfrak{a}(u, v)$$

for every $u, v \in D(\mathfrak{a})$ with $u\bar{v} \geq 0$. In our first main result (Theorem 1.1), we give an integral representation theorem for the difference $\mathfrak{a} - \widehat{\mathfrak{a}}$ under the assumption that the measure space Ω is a topological measure space and that the form \mathfrak{a} is in some sense regular; thereby showing explicitly which type of form perturbations lead to domination. In particular, we see that some non-local perturbations may also lead to domination (compare with the proof of [4, Theorem 4.1]). Secondly, we show that (Theorem 3.2) if $\widehat{\mathfrak{a}}$ is local and if T is dominated by \widehat{T} , then the positivity of T is equivalent to the locality of \mathfrak{a} . In particular, we give a simpler proof of [1, Theorem 4.3].

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A characterisation of domination. Throughout, we let $(\Omega, \mathfrak{A}, \mu)$ be a topological measure space. By this we mean that Ω is a topological space, \mathfrak{A} is the Borel σ -algebra, and μ is a (positive) Borel measure on (Ω, \mathfrak{A}) . Without loss of generality, we assume that μ has full support in the sense that there exists no nonempty open subset $U \subseteq \Omega$ which has zero measure. All vector spaces in this note are *complex* vector spaces.

We now state our main result:

Theorem 1.1. *Let $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \rightarrow \mathbb{C}$ and $\widehat{\mathfrak{a}} : D(\widehat{\mathfrak{a}}) \times D(\widehat{\mathfrak{a}}) \rightarrow \mathbb{C}$ be two sesquilinear, Hermitian, closed, accretive, and densely defined forms on $L^2_\mu(\Omega)$ such that the associated self-adjoint C_0 -semigroups (denoted by T and \widehat{T} respectively) are real. Assume that the semigroup \widehat{T} is positive, and that the form \mathfrak{a} is \mathcal{A} -regular for some closed, Hermitian, and unital subalgebra \mathcal{A} of $C^b(\Omega)$. Further, let $\widetilde{\Omega}$ denote the compactification of Ω associated with \mathcal{A} .*

Then T is dominated by \widehat{T} , in the sense that

$$|T(t)u| \leq \widehat{T}(t)|u|, \quad \text{for every } u \in L^2_\mu(\Omega),$$

if and only if $D(\mathfrak{a})$ is an ideal of $D(\widehat{\mathfrak{a}})$ and there exists a (positive) Hermitian Borel measure ν on $\widetilde{\Omega} \times \widetilde{\Omega}$ such that ν is absolutely continuous with respect to the \mathfrak{a} -capacity, the equality

$$\mathfrak{a}(u, v) = \widehat{\mathfrak{a}}(u, v) + \int_{\widetilde{\Omega} \times \widetilde{\Omega}} u(x) \overline{v(y)} d\nu(x, y) \quad (1.1)$$

holds for every $u, v \in D(\mathfrak{a})$, and

$$\operatorname{Re} \int_{\widetilde{\Omega} \times \widetilde{\Omega}} u(x) \overline{v(y)} d\nu(x, y) \geq \widehat{\mathfrak{a}}(|u|, |v|) - \operatorname{Re} \widehat{\mathfrak{a}}(u, v) \quad (1.2)$$

for every $u, v \in D(\mathfrak{a})$ which satisfy $u\bar{v} \geq 0$ on Ω .

The rest of this section will be dedicated to explaining the terminology used in the statement and that we need to prove Theorem 1.1. Then, in Section 2 we provide the proof of Theorem 1.1. In Section 3, we turn our attention to local forms. Finally, we leave the reader with some additional remarks in Section 4.

Notation and Preliminaries. If u is a vector in a Banach lattice E , then we use the usual notation u^+ and u^- to denote the positive and negative parts of u respectively. Recall that $u = u^+ - u^-$ and the modulus of u satisfies $|u| = u^+ + u^-$.

Let $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \rightarrow \mathbb{C}$ be a sesquilinear form, where $D(\mathfrak{a})$ is a dense subspace of $L^2_\mu(\Omega)$ known as the *form domain*. We say that the form \mathfrak{a} is *Hermitian* if for every $u, v \in D(\mathfrak{a})$,

$$\mathfrak{a}(u, v) = \overline{\mathfrak{a}(v, u)}.$$

The form \mathfrak{a} is said to be *accretive* if for every $u \in D(\mathfrak{a})$,

$$\operatorname{Re} \mathfrak{a}(u) := \operatorname{Re} \mathfrak{a}(u, u) \geq 0,$$

and an accretive form is called *closed* if the form domain $D(\mathfrak{a})$ is complete with respect to the norm

$$\|u\|_{D(\mathfrak{a})} := \sqrt{\operatorname{Re} \mathfrak{a}(u) + \|u\|_{L^2_\mu}^2} \quad (u \in D(\mathfrak{a})).$$

Clearly, $D(\mathfrak{a})$ embeds continuously into $L^2_\mu(\Omega)$ when equipped with this norm. Given two Hermitian forms \mathfrak{a} and $\widehat{\mathfrak{a}}$, we say that $D(\mathfrak{a})$ is an *ideal* of $D(\widehat{\mathfrak{a}})$ if

- (a) the implication $u \in D(\mathfrak{a}) \Rightarrow |u| \in D(\widehat{\mathfrak{a}})$ holds and
- (b) whenever $u \in D(\mathfrak{a})$ and $v \in D(\widehat{\mathfrak{a}})$, the inequality $|v| \leq |u|$ implies $v \operatorname{sgn} u \in D(\mathfrak{a})$.

In particular, the above ideal property is stronger than the notion of lattice ideal in Banach lattices. For a situation when the two definitions are equivalent, see [22, Proposition 2.23].

For every Hermitian, accretive, and closed sesquilinear form, the operator given by

$$D(A) := \{u \in D(\mathfrak{a}) \mid \exists f \in L^2_\mu(\Omega) \forall v \in D(\mathfrak{a}) : \mathfrak{a}(u, v) = \langle f, v \rangle_{L^2_\mu}\},$$

$$Au := f,$$

is self-adjoint and positive semi-definite. This operator is the negative generator of a self-adjoint contraction semigroup $T := (T(t))_{t \geq 0}$. Actually, there is a one-to-one correspondence between Hermitian, closed, accretive, and densely defined forms, the self-adjoint and positive semi-definite operators, and the self-adjoint contraction semigroups. For all these facts, we refer the reader to standard monographs, for instance, [16, Chapter XVII] or [22, 23]. Note that, by a semigroup, we always mean a C_0 -semigroup.

The semigroup T is said to be *real* if each operator $T(t)$ leaves the real part of $L^2_\mu(\Omega)$ invariant and it is called *positive* if each $T(t)$ leaves the positive cone of $L^2_\mu(\Omega)$ invariant. Clearly, a positive semigroup is always real. By [22, Proposition 2.5], the semigroup T is real if and only if

$$\forall u \in D(\mathfrak{a}) : \operatorname{Re} u \in D(\mathfrak{a}) \text{ and } \mathfrak{a}(\operatorname{Re} u, \operatorname{Im} u) \in \mathbb{R},$$

Moreover, by the well-known *first Beurling-Deny criterion* (see, for instance, [17, Theorem 1.3.2], [23, Theorem XIII.50], or [22, Theorem 2.7]), positivity of a real semigroup T is characterised by the property

$$\forall u \in D(\mathfrak{a}) : |u| \in D(\mathfrak{a}) \text{ and } \mathfrak{a}(|u|) \leq \mathfrak{a}(u).$$

Let \mathcal{A} be a closed, unital, and Hermitian subalgebra of $C^b(\Omega)$, the space of bounded continuous functions. Here, as usual, an algebra is called *Hermitian* if the spectrum of each of its self-adjoint elements is real. We say that a closed, accretive, and sesquilinear form \mathfrak{a} is *\mathcal{A} -regular* if

- $D(\mathfrak{a}) \cap \mathcal{A}$ is dense in the Hilbert space $D(\mathfrak{a})$ and
- $\operatorname{span}((D(\mathfrak{a}) \cap \mathcal{A}) \cup \{1\})$ is dense in \mathcal{A} with the supremum norm.

In this note, we use an abstract compactification $\widetilde{\Omega}$ of Ω , an abstract boundary of Ω , and a capacity on $\widetilde{\Omega} \times \widetilde{\Omega}$, which were recently considered in the case of nonlinear Dirichlet forms by Claus [8]. Let us explain this in more detail. First, by the Gelfand representation theorem, the Banach algebra \mathcal{A} is isometrically isomorphic to the Banach algebra $C(\widetilde{\Omega})$ for some compact space $\widetilde{\Omega}$. This compact space $\widetilde{\Omega}$ is a compactification of Ω . The construction of the above isometric isomorphism – via the so-called Gelfand space or Gelfand spectrum – shows that there is a natural, continuous

mapping

$$\begin{aligned} \iota : \Omega &\rightarrow \tilde{\Omega}, \\ x &\mapsto \iota(x), \end{aligned}$$

such that the inverse of the above-mentioned isometric isomorphism is given by

$$\begin{aligned} J : C(\tilde{\Omega}) &\rightarrow \mathcal{A}, \\ f &\mapsto f \circ \iota. \end{aligned}$$

The mapping ι allows us to define a push-forward Borel measure $\tilde{\mu}$ on $\tilde{\Omega}$ by setting

$$\tilde{\mu}(B) := \mu(\iota^{-1}(B))$$

for every Borel set $B \subseteq \tilde{\Omega}$. Now, the mapping J above extends to an isometric isomorphism

$$\begin{aligned} J : L^2_{\tilde{\mu}}(\tilde{\Omega}) &\rightarrow L^2_{\mu}(\Omega), \\ f &\mapsto f \circ \iota. \end{aligned}$$

This means that in the general setting above, if the form \mathfrak{a} is \mathcal{A} -regular, then we may assume without loss of generality that Ω (actually, $\tilde{\Omega}$) is a compact, topological space, $D(\mathfrak{a})$ (actually, $J^{-1}D(\mathfrak{a})$) is a dense subspace of $L^2(\tilde{\Omega})$, and $D(\mathfrak{a}) \cap C(\tilde{\Omega})$ is dense in $D(\mathfrak{a})$ and a fortiori in $L^2_{\mu}(\tilde{\Omega})$.

In order to prove Theorem 1.1, we freely use the theory of tensor norms. For this, we refer to the monograph [18]. For example, we use the projective tensor product $D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$, but later also the injective tensor product $C(\tilde{\Omega}) \otimes_{\varepsilon} C(\tilde{\Omega})$. Note that the projective tensor product $D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$ is a subspace of the Hilbert space tensor product $D(\mathfrak{a}) \otimes_H D(\mathfrak{a})$. The latter is a subspace of the Hilbert space tensor product $L^2_{\mu}(\tilde{\Omega}) \otimes_H L^2_{\mu}(\tilde{\Omega})$ which in turn is isometrically isomorphic to $L^2_{\mu \otimes \mu}(\tilde{\Omega} \times \tilde{\Omega})$, where $\mu \otimes \mu$ is the product measure. Hence, elements of $D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$ can be identified with (equivalence classes of) functions on the product space $\tilde{\Omega} \times \tilde{\Omega}$.

If \mathfrak{a} is a closed and accretive form, then we define the \mathfrak{a} -capacity on the product space $\tilde{\Omega} \times \tilde{\Omega}$ by setting, for every subset $B \subseteq \tilde{\Omega} \times \tilde{\Omega}$,

$$\begin{aligned} \mathcal{L}_B &:= \{w \in L^2_{\mu \otimes \mu}(\tilde{\Omega} \times \tilde{\Omega}) : w \geq 1 \text{ } \mu \otimes \mu\text{-a.e. on an open set } U \supseteq B\}, \\ \text{cap}(B) &:= \inf\{\|w\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})} : w \in \mathcal{L}_B \cap (D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a}))\}. \end{aligned}$$

Here, $\|\cdot\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})}$ is the usual projective tensor norm, which on the algebraic tensor product is given by

$$\|w\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})} := \inf \left\{ \sum_{i=1}^n \|u_i\|_{D(\mathfrak{a})} \|v_i\|_{D(\mathfrak{a})} \mid n \in \mathbb{N} \text{ and } \sum_{i=1}^n u_i \otimes v_i = w \right\}.$$

The projective tensor product is the completion of the algebraic tensor product with respect to this norm. Our definition of the capacity is perhaps unusual in two aspects: first, our assumption on the form is minimal (accretive, closed) in order to define some capacity, and second, we do not define

the capacity on $\tilde{\Omega}$ but on the product space $\tilde{\Omega} \times \tilde{\Omega}$. However, when dealing with non-local forms, it is natural to define the capacity on the product space.

From the above definition of capacity and properties of infimum, we immediately have that $\text{cap}(A) \leq \text{cap}(B)$ for all $A, B \subseteq \tilde{\Omega} \times \tilde{\Omega}$ satisfying $A \subseteq B$. More importantly, we have the following properties:

Lemma 1.2. *Let $A, B \subseteq \tilde{\Omega} \times \tilde{\Omega}$.*

(a) *There exists $C > 0$ such that $(\mu \otimes \mu)(B) \leq C \text{cap}(B)$.*

(b) *Subadditivity: $\text{cap}(A \cup B) \leq \text{cap}(A) + \text{cap}(B)$.*

Proof. First of all, due to the continuous embedding of $D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$ in the space $L^2_{\mu \otimes \mu}(\tilde{\Omega} \times \tilde{\Omega})$, there exists $C > 0$ such that

$$(\mu \otimes \mu)(B) \leq \|w\|_{L^2_{\mu \otimes \mu}(\tilde{\Omega} \times \tilde{\Omega})} \leq C \|w\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})}.$$

for all $w \in \mathcal{L}_B \cap (D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a}))$. Taking infimum over all such w yields (a).

(b) Let $w_A \in \mathcal{L}_A \cap (D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a}))$ and $w_B \in \mathcal{L}_B \cap (D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a}))$. Then $w_A + w_B \in \mathcal{L}_{A \cup B} \cap (D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a}))$ and

$$\text{cap}(A \cup B) \leq \|w_A + w_B\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})} \leq \|w_A\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})} + \|w_B\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})},$$

from which the assertion readily follows. \square

A Borel measure ν on the product space $\tilde{\Omega} \times \tilde{\Omega}$ is called *symmetric* if $\nu(B) = \nu(\tilde{B})$ for every Borel set B , where $\tilde{B} := \{(x, y) \in \tilde{\Omega} \times \tilde{\Omega} \mid (y, x) \in B\}$ is the reflection of B . We say that a subset $B \subseteq \tilde{\Omega} \times \tilde{\Omega}$ is *\mathfrak{a} -polar* if $\text{cap}(B) = 0$ and the measure ν is said to be *absolutely continuous with respect to the \mathfrak{a} -capacity* if $\nu(B) = 0$ for every \mathfrak{a} -polar Borel set $B \subseteq \tilde{\Omega} \times \tilde{\Omega}$. Let us mention that if the capacity and the measure ν were only defined on $\tilde{\Omega}$, then one could also define absolute continuity of the measure with respect to the capacity; this property was called *admissibility* in [4].

We also need the concept of quasi-continuity. Let Y be a topological space. A function $f : \tilde{\Omega} \times \tilde{\Omega} \rightarrow Y$ is said to be *quasi-continuous* if for each $\epsilon > 0$, there exists an open set $U \subseteq \tilde{\Omega} \times \tilde{\Omega}$ such that $\text{cap}(U) < \epsilon$ and $f|_{U^c}$ is continuous. In particular, $f \in L^2_{\mu \otimes \mu}(\tilde{\Omega} \times \tilde{\Omega})$ is said to be *quasi-continuous* if it has a representative (which we again denote by f) which is quasi-continuous. Finally, a property is said to hold *quasi-everywhere (q.e.)* if it holds everywhere except possibly on an \mathfrak{a} -polar set.

2. PROOF OF THE MAIN RESULT

In this section, we provide a proof of Theorem 1.1. First, we make a brief remark on the boundedness condition (1.2).

Remark 2.1. *In some typical examples – for instance, forms associated with the Laplace operator with local boundary conditions – the sesquilinear form $\hat{\mathfrak{a}}$ satisfies $\hat{\mathfrak{a}}(u, v) = \hat{\mathfrak{a}}(|u|, |v|)$ for every real $u, v \in D(\mathfrak{a})$ such that $uv \geq 0$. In such*

examples, the boundedness condition (1.2) implies the condition

$$\int_{\tilde{\Omega} \times \tilde{\Omega}} u(x)v(y) \, dv(x, y) \geq 0 \text{ for every real } u, v \in D(\mathfrak{a}) \text{ such that } uv \geq 0. \quad (2.1)$$

Note that the product uv at the end of (2.1) is a function on $\tilde{\Omega}$, while the tensor product $u \otimes v$ appearing under the integral is a function on $\tilde{\Omega} \times \tilde{\Omega}$. The positivity of the product uv only means that the tensor product $u \otimes v$ is positive on the diagonal $\Delta := \{(x, y) \in \tilde{\Omega} \times \tilde{\Omega} \mid x = y\}$. The condition (2.1) can be rewritten in the form

$$\int_{\Delta} u(x)v(x) \, dv(x, x) \geq - \int_{\tilde{\Omega} \times \tilde{\Omega} \setminus \Delta} u(x)v(y) \, dv(x, y) \quad (2.2)$$

for every real $u, v \in D(\mathfrak{a})$ such that $uv \geq 0$.

We say that the measure ν is diagonally dominant if it satisfies (2.2). Diagonal dominance and also the condition (1.2) is a certain boundedness condition on ν .

Note that, there are positive measures ν which are not diagonally dominant. The sesquilinear form $\mathfrak{b} : H^1(0, 1) \times H^1(0, 1) \rightarrow \mathbb{C}$ given by

$$\mathfrak{b}(u, v) = \int_0^{\frac{1}{2}} u \cdot \int_{\frac{1}{2}}^1 \bar{v} + \int_0^{\frac{1}{2}} \bar{v} \cdot \int_{\frac{1}{2}}^1 u$$

is well defined, continuous, and positive on $H^1(0, 1)$, it is of the form of the integral term in (2.1) for the Lebesgue measure ν restricted to the union of the squares $([0, \frac{1}{2}] \times [\frac{1}{2}, 1]) \cup ([\frac{1}{2}, 1] \times [0, \frac{1}{2}])$, but this measure ν is not diagonally dominant in the sense of condition (2.2); indeed this can be seen by taking $u = v$ where $u(x) = 1$ for $x \in [0, \frac{1}{2}]$ and $u(x) = -1$ for $x \in (\frac{1}{2}, 1]$.

The importance of the following technical result will become clear in Remark 2.4. The proof mimics the arguments of [20, Theorem 2.1.3].

Proposition 2.2. *The functions in $D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$ are quasi-continuous, that is, their equivalence classes contain quasi-continuous representatives.*

Proof. First of all, we assert that the inequality

$$\text{cap}\{|w| > \lambda\} \leq \frac{\|w\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})}}{2\lambda} \quad (2.3)$$

holds for each $\lambda > 0$ and all continuous $w \in D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$. Indeed, let $\lambda > 0$ and $w \in D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$ be continuous. Then $A = \{w > \lambda\}$ is open and $\frac{w}{\lambda} \in \mathcal{L}_A \cap (D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a}))$. Therefore, $\text{cap}(A) \leq \frac{\|w\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})}}{\lambda}$. Analogously, taking $B = \{w < -\lambda\}$ satisfies $\text{cap}(B) \leq \frac{\|w\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})}}{\lambda}$. The inequality (2.3) now follows using subadditivity (Lemma 1.2(b)).

Next, let $w \in D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$ be arbitrary. By the regularity of \mathfrak{a} , there exists a sequence of continuous functions (w_n) in $D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$ such that $w_n \rightarrow w$ in $D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})$. Replacing by a subsequence, we may assume without loss of generality that $\|w_{k+1} - w_k\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})} \leq 2^{-2k}$ holds for each $k \in \mathbb{N}$. Thus, $G_k := \{|w_{k+1} - w_k| > 2^{-k}\}$ satisfies $\text{cap}(G_k) \leq 2^{-(k+1)}$ by (2.3). Set $F_k := \bigcap_{j=k}^{\infty} G_j^c$. Then (F_k) is a sequence of increasing closed sets

such that $\lim_{k \rightarrow \infty} \text{cap}(F_k^c) = 0$. Moreover, on F_k , we have the inequality

$$|w_n - w_m| \leq \sum_{j=k}^{\infty} |w_{j+1} - w_j| \leq 2^{-(k-1)}$$

for each $n, m \geq k$. Therefore, for each k , the sequence $(w_n|_{F_k})$ is uniformly convergent on F_k . Setting $\tilde{w}(x) = \lim w_n(x)$ for $x \in \bigcup_{k=1}^{\infty} F_k$, we get that \tilde{w} is continuous on each F_k and $w = \tilde{w}$ quasi-everywhere and hence also a.e. (Lemma 1.2(a)). As $\lim_{k \rightarrow \infty} \text{cap}(F_k^c) = 0$, it follows that \tilde{w} is quasi-continuous. \square

Next, we outsource the proof of the necessity of the measure ν to be absolutely continuous with respect to the \mathfrak{a} -capacity to the following lemma.

Lemma 2.3. *Let the sesquilinear form \mathfrak{a} , the algebra \mathcal{A} , and the compactification $\tilde{\Omega}$ be as in Theorem 1.1. Further, let ν be a positive Borel measure on $\tilde{\Omega} \times \tilde{\Omega}$ and let the form $\mathfrak{b} : D(\mathfrak{a}) \times D(\mathfrak{a}) \rightarrow \mathbb{C}$ be given by*

$$\mathfrak{b}(u, v) = \int_{\tilde{\Omega} \times \tilde{\Omega}} u(x) \overline{v(y)} d\nu(x, y)$$

for all continuous $u, v \in D(\mathfrak{a})$.

Assume that \mathfrak{b} is continuous on $D(\mathfrak{a})$, i.e., there exists $C > 0$ such that the inequality $|\mathfrak{b}(u, v)| \leq C \|u\|_{D(\mathfrak{a})} \|v\|_{D(\mathfrak{a})}$ is true for every $u, v \in D(\mathfrak{a})$. Then $\nu(B) \leq C \text{cap}(B)$ for every $B \subseteq \tilde{\Omega} \times \tilde{\Omega}$. In particular, ν is absolutely continuous with respect to the \mathfrak{a} -capacity.

Proof. Let $B \subseteq \tilde{\Omega} \times \tilde{\Omega}$. Then, for every non-negative $w \in D(\mathfrak{a}) \otimes_{\text{alg}} D(\mathfrak{a})$ with $w = \sum_{i=1}^n u_i \otimes v_i$ for continuous $u_i, v_i \in D(\mathfrak{a})$ and $w \geq 1$ on a neighbourhood of B , we have

$$\begin{aligned} \nu(B) &\leq \int_{\tilde{\Omega} \times \tilde{\Omega}} w(x, y) d\nu(x, y) \\ &= \sum_{i=1}^n \int_{\tilde{\Omega} \times \tilde{\Omega}} u_i(x) v_i(y) d\nu(x, y) \end{aligned}$$

Since \mathfrak{b} is continuous on $D(\mathfrak{a})$, the above inequality yields

$$\nu(B) \leq \sum_{i=1}^n C \|u_i\|_{D(\mathfrak{a})} \|v_i\|_{D(\mathfrak{a})}.$$

The regularity of the form and [18, Proposition 3.9 on page 36] implies now that $\nu(B) \leq C \|w\|_{D(\mathfrak{a}) \otimes_{\pi} D(\mathfrak{a})}$ for all $w \in D(\mathfrak{a}) \otimes_{\text{alg}} D(\mathfrak{a})$ with $w \geq 1$ on a neighbourhood of B . Taking the infimum over all such w , the assertion $\nu(B) \leq C \text{cap}(B)$, follows. \square

Remark 2.4. *We point out that when speaking of an integral of the elementary tensors $u \otimes v \in D(\mathfrak{a}) \otimes D(\mathfrak{a})$ with respect to a measure ν that is absolutely continuous with respect to the \mathfrak{a} -capacity, we implicitly take the quasi-continuous representatives of u and v (this is possible due to Proposition 2.2).*

Proof of Theorem 1.1. Recall that, by assumption, both the semigroups T and \hat{T} are real. Also, by [22, Theorem 2.21], the semigroup T is dominated by \hat{T} if and only if $D(\mathfrak{a})$ is an ideal of $D(\hat{\mathfrak{a}})$ and

$$\hat{\mathfrak{a}}(|u|, |v|) \leq \text{Re } \mathfrak{a}(u, v) \tag{2.4}$$

for every $u, v \in D(\mathfrak{a})$ with $u\bar{v} \geq 0$.

Now, if $D(\mathfrak{a})$ is an ideal of $D(\widehat{\mathfrak{a}})$ and if \mathfrak{a} is given as in equation (1.1) for some appropriate measure ν , which is absolutely continuous with respect to the \mathfrak{a} -capacity, and which satisfies the lower bound (1.2), then

$$\begin{aligned} \widehat{\mathfrak{a}}(|u|, |v|) &\leq \operatorname{Re} \widehat{\mathfrak{a}}(u, v) + \operatorname{Re} \int_{\widehat{\Omega} \times \widehat{\Omega}} u(x) \overline{v(y)} \, d\nu(x, y) \\ &= \operatorname{Re} \mathfrak{a}(u, v). \end{aligned}$$

for all $u, v \in D(\mathfrak{a})$ with $u\bar{v} \geq 0$. Hence, the inequality (2.4) is fulfilled for every $u, v \in D(\mathfrak{a})$ such that $u\bar{v} \geq 0$. Therefore, T is dominated by \widehat{T} .

Conversely, suppose that T is dominated by \widehat{T} . In particular, the inequality (2.4) is fulfilled for every $u, v \in D(\mathfrak{a})$ such that $u\bar{v} \geq 0$. Define the sesquilinear form

$$\begin{aligned} \Psi : D(\mathfrak{a}) \times D(\mathfrak{a}) &\rightarrow \mathbb{C}, \\ (u, v) &\mapsto \mathfrak{a}(u, v) - \widehat{\mathfrak{a}}(u, v). \end{aligned}$$

Let u, v be positive elements in $D(\mathfrak{a})$. In particular, u, v are real. Since the semigroups T and \widehat{T} are real, therefore by a characterisation for real semigroups [22, Proposition 2.5], we obtain that $\mathfrak{a}(u, v)$ and $\widehat{\mathfrak{a}}(u, v)$ are also real. Thus, the inequality (2.4) implies that

$$\Psi(u, v) \geq 0,$$

that is, Ψ is positive on $D(\mathfrak{a}) \times D(\mathfrak{a})$ in the order sense. By the universal property of the algebraic tensor product $D(\mathfrak{a}) \otimes_{\text{alg}} D(\mathfrak{a})$, associated with the sesquilinear form Ψ , a unique linear mapping $\psi : D(\mathfrak{a}) \otimes_{\text{alg}} D(\mathfrak{a}) \rightarrow \mathbb{C}$ exists such that the diagram

$$\begin{array}{ccc} D(\mathfrak{a}) \times D(\mathfrak{a}) & \xrightarrow{\mathfrak{b}} & D(\mathfrak{a}) \otimes_{\text{alg}} D(\mathfrak{a}) \\ & \searrow \Psi & \downarrow \psi \\ & & \mathbb{C} \end{array}$$

commutes. Here, $\mathfrak{b} : D(\mathfrak{a}) \times D(\mathfrak{a}) \rightarrow D(\mathfrak{a}) \otimes_{\text{alg}} D(\mathfrak{a})$, $(u, v) \mapsto u \otimes v$ is the canonical sesquilinear form. By the construction of the algebraic tensor product and by the positivity of Ψ , we obtain that $\psi(u \otimes v)$ is positive whenever u and v are positive. This readily implies that ψ is a positive functional on $D(\mathfrak{a}) \otimes_{\text{alg}} D(\mathfrak{a})$ (see, for instance, [19, Section 7]). In particular, ψ also positive on the smaller tensor product $(D(\mathfrak{a}) \cap \mathcal{A}) \otimes_{\text{alg}} (D(\mathfrak{a}) \cap \mathcal{A})$.

In what follows, we write $C(\widetilde{\Omega})$ instead of \mathcal{A} ; where $\widetilde{\Omega}$ denotes the compactification of Ω with respect to \mathcal{A} . We consider two cases. Assume first, that the span of $D(\mathfrak{a}) \cap C(\widetilde{\Omega})$ is dense in $C(\widetilde{\Omega})$ with respect to the supremum norm. Then there exists a positive $u_0 \in D(\mathfrak{a}) \cap C(\widetilde{\Omega})$ such that $\|u_0 - 1\|_{\infty} \leq \frac{1}{2}$. This function u_0 is an order unit in $C(\widetilde{\Omega})$ and similarly, $u_0 \otimes u_0$ is an order unit in the injective tensor product $C(\widetilde{\Omega}) \otimes_{\varepsilon} C(\widetilde{\Omega})$; the injective tensor product actually is isometrically isomorphic to the space $C(\widetilde{\Omega} \times \widetilde{\Omega})$ (see [18, Example I.4.2(3)]). The tensor product $(D(\mathfrak{a}) \cap C(\widetilde{\Omega})) \otimes_{\text{alg}} (D(\mathfrak{a}) \cap C(\widetilde{\Omega}))$ therefore is majorizing in $C(\widetilde{\Omega} \times \widetilde{\Omega})$ in the sense that for every $w \in C(\widetilde{\Omega} \times \widetilde{\Omega})$ there exists $z \in (D(\mathfrak{a}) \cap C(\widetilde{\Omega})) \otimes_{\text{alg}} (D(\mathfrak{a}) \cap C(\widetilde{\Omega}))$

such that $w \leq z$. The positivity of the mapping ψ and Kantorovich's theorem [21, Corollary 1.5.9] now imply that ψ uniquely extends to a positive and bounded linear form on the space $C(\tilde{\Omega} \times \tilde{\Omega})$; actually, Kantorovich's theorem in this special situation is an exercise. By the Riesz-Markov representation theorem, there exists a unique, positive, and finite measure ν on $\tilde{\Omega} \times \tilde{\Omega}$ such that

$$\psi(u \otimes \bar{v}) = \Psi(u, v) = \int_{\tilde{\Omega} \times \tilde{\Omega}} u(x) \overline{v(y)} d\nu(x, y), \quad (2.5)$$

and the implication is proved.

Secondly, assume that the span $D(\mathfrak{a}) \cap C(\tilde{\Omega})$ is not dense in $C(\tilde{\Omega})$. By assumption of \mathcal{A} -regularity, however, the span of $(D(\mathfrak{a}) \cap C(\tilde{\Omega}) \cup \{1\})$ is dense in $C(\tilde{\Omega})$. Hence, the closure \mathcal{A}_0 of the space $D(\mathfrak{a}) \cap C(\tilde{\Omega})$ has codimension one in the space of continuous functions. The fact that $D(\mathfrak{a})$ is a lattice implies that \mathcal{A}_0 is actually a maximal ideal in $C(\tilde{\Omega})$. Hence, there exists a vector $x_\infty \in \tilde{\Omega}$ such that

$$\mathcal{A}_0 = \{u \in C(\tilde{\Omega}) \mid u(x_\infty) = 0\} = C_0(\tilde{\Omega}_\infty);$$

where $\tilde{\Omega}_\infty := \tilde{\Omega} \setminus \{x_\infty\}$. Similarly as above, one shows that the tensor product $(D(\mathfrak{a}) \cap C(\tilde{\Omega})) \otimes_{\text{alg}} (D(\mathfrak{a}) \cap C(\tilde{\Omega}))$ is majorizing in $C_c(\tilde{\Omega}_\infty \times \tilde{\Omega}_\infty)$, and therefore, ψ uniquely extends to a positive and bounded linear form on $C_c(\tilde{\Omega}_\infty \times \tilde{\Omega}_\infty)$. By the Riesz-Markov representation theorem, there exists a unique, positive Radon measure ν on $\tilde{\Omega}_\infty$ such that (2.5) holds. Note that in this case, the measure ν need not be finite, but for all functions $u, v \in D(\mathfrak{a})$ the product $u \otimes \bar{v}$ is ν -integrable.

Now, the measure ν is positive and the forms \mathfrak{a} , $\hat{\mathfrak{a}}$, and in turn, Ψ are Hermitian. This implies that the measure ν can be chosen to be symmetric. In fact, if necessary, it suffices to replace the measure ν by the measure $\tilde{\nu}$ given by $\tilde{\nu}(B) = \frac{1}{2}(\nu(B) + \nu(\tilde{B}))$ for every Borel set $B \subseteq \tilde{\Omega} \times \tilde{\Omega}$ (where again, \tilde{B} is the reflection of B). Finally, the continuity of ψ on the space $D(\mathfrak{a}) \otimes_{\text{alg}} D(\mathfrak{a})$ together with Lemma 2.3 implies that ν is absolutely continuous with respect to the \mathfrak{a} -capacity. \square

3. POSITIVITY AND LOCALITY

In a recent article, Akhlil found a surprising connection between the positivity of a dominated semigroup and the locality of the generating form, at least when the dominating semigroup is generated by a local operator or a local form [1, Theorem 4.3]. Let us recall, that an operator A on $L^2_\mu(\Omega)$ is called *local*, if for every $u \in D(A)$, $v \in L^2_\mu(\Omega)$ satisfying $uv = 0$ one has $\langle Au, v \rangle_{L^2_\mu} = 0$. Note that the condition $uv = 0$ for two L^2 -functions is equivalent to the condition $|u| \wedge |v| = 0$. Similarly, we say that a sesquilinear form \mathfrak{a} is *local*, if for every $u, v \in D(\mathfrak{a})$ satisfying $uv = 0$ one has $\mathfrak{a}(u, v) = 0$. This is equivalent to the condition that for every $u, v \in D(\mathfrak{a})$ satisfying $uv = 0$ one has

$$\mathfrak{a}(u + v) = \mathfrak{a}(u) + \mathfrak{a}(v). \quad (3.1)$$

The proof of the following is straightforward.

Proposition 3.1. *Let \mathfrak{a} be a closed, accretive, and \mathcal{A} -regular sesquilinear form on $L^2_\mu(\Omega)$; where \mathcal{A} is a closed, Hermitian, and unital subalgebra of $C^b(\Omega)$. Let $\tilde{\Omega}$ be the compactification of Ω associated with \mathcal{A} and let ν be a positive Borel measure on $\tilde{\Omega} \times \tilde{\Omega}$ such that the form*

$$\mathfrak{b}(u, v) = \int_{\tilde{\Omega} \times \tilde{\Omega}} u(x) \overline{v(y)} d\nu(x, y)$$

is well defined and continuous on $D(\mathfrak{a})$. Then \mathfrak{b} is local if and only if $\text{supp } \nu$ is contained in the diagonal $\Delta := \{(x, y) \in \tilde{\Omega} \times \tilde{\Omega} | x = y\}$.

The first statement in the following theorem gives a condition under which the locality of a form implies the positivity of the generated semigroup. The second statement in the theorem is basically [1, Theorem 4.3], although with weaker assumptions. The proof in [1] uses the Beurling-Deny and Lejan representation of regular Dirichlet forms [2, 3]; in particular, the dominated semigroup is a submarkovian semigroup, that is, it is positive and L^∞ -contractive. We give here a different proof that only uses the positivity of the dominated semigroup.

Theorem 3.2. *Let $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \rightarrow \mathbb{C}$ and $\hat{\mathfrak{a}} : D(\hat{\mathfrak{a}}) \times D(\hat{\mathfrak{a}}) \rightarrow \mathbb{C}$ be two sesquilinear Hermitian forms on $L^2_\mu(\Omega)$. Denote the associate semigroups by T and \hat{T} respectively and assume they are real.*

- (a) *If the form \mathfrak{a} is local and $D(\mathfrak{a})$ is a sublattice of $L^2_\mu(\Omega)$, then T is a positive semigroup.*
- (b) *Assume that the semigroup T is positive and T is dominated by \hat{T} . Then locality of $\hat{\mathfrak{a}}$ implies the locality of \mathfrak{a} .*

In Section 4.2, we give an example to show that the positivity assumption in Theorem 3.2(b) cannot be dropped. In fact, we show that it is not even sufficient that the semigroup operators become (and remain) positive after a large enough time.

Proof. (a) Assume that the form \mathfrak{a} is local and that $D(\mathfrak{a})$ is a sublattice of $L^2_\mu(\Omega)$. Then, by the characterisation (3.1) of locality,

$$\begin{aligned} \mathfrak{a}(u) &= \mathfrak{a}(u^+ - u^-) \\ &= \mathfrak{a}(u^+) + \mathfrak{a}(-u^-) \\ &= \mathfrak{a}(u^+) + \mathfrak{a}(u^-) \\ &= \mathfrak{a}(u^+ + u^-) \\ &= \mathfrak{a}(|u|) \end{aligned}$$

for all $u \in D(\mathfrak{a})$. Thus, the first Beurling-Deny criterion implies that the semigroup T is positive.

(b) Suppose that the form $\hat{\mathfrak{a}}$ generating \hat{T} is local. The positivity of T and the first Beurling-Deny criterion imply, that for every real $u \in D(\mathfrak{a})$,

$$\begin{aligned} \mathfrak{a}(u^+) + \mathfrak{a}(u^-) - 2\text{Re } \mathfrak{a}(u^+, u^-) &= \mathfrak{a}(u) \\ &\geq \mathfrak{a}(|u|) \\ &= \mathfrak{a}(u^+) + \mathfrak{a}(u^-) + 2\text{Re } \mathfrak{a}(u^+, u^-). \end{aligned}$$

Hence, $\operatorname{Re} \mathfrak{a}(u^+, u^-) \leq 0$ for all real $u \in D(\mathfrak{a})$. Thus, if $u, v \in D(\mathfrak{a})$ are both positive and $uv = 0$, then the characterisation of domination from (2.4) implies

$$0 \geq \operatorname{Re} \mathfrak{a}(u, v) \geq \operatorname{Re} \hat{\mathfrak{a}}(u, v).$$

However, because $\hat{\mathfrak{a}}$ is local, the right-hand side of this chain of inequalities is zero. Thus,

$$\operatorname{Re} \mathfrak{a}(u, v) = 0 \text{ for every positive } u, v \in D(\mathfrak{a}) \text{ such that } uv = 0.$$

From the sesquilinearity of the form, we then obtain

$$\operatorname{Re} \mathfrak{a}(u, v) = 0 \text{ for every } u, v \in D(\mathfrak{a}) \text{ such that } uv = 0.$$

Finally, replacing u by $e^{i\theta}u$ yields

$$\mathfrak{a}(u, v) = 0 \text{ for every } u, v \in D(\mathfrak{a}) \text{ such that } uv = 0.$$

Whence, the form \mathfrak{a} is local. \square

4. FINAL REMARKS

4.1. The Laplace operator with Dirichlet and Neumann boundary conditions: what is in between? Let T, \hat{T} , and S be three semigroups generated by closed, accretive, and Hermitian forms $\mathfrak{a}, \hat{\mathfrak{a}}$, and \mathfrak{b} respectively. If S is sandwiched between T and \hat{T} , in the sense that \hat{T} dominates S which in turn dominates T , then S is necessarily positive. Indeed this holds because

$$|T(t)u| \leq S(t)|u|$$

for every $u \in L^2_\mu(\Omega)$. Therefore Theorem 3.2 implies that if $\hat{\mathfrak{a}}$ is local, then \mathfrak{b} is local as well.

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded open set with boundary $\partial\Omega$. By $\mathfrak{a}^N : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ and $\mathfrak{a}^D : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ we denote the sesquilinear and Hermitian forms of the Neumann-Laplace operator and the Dirichlet-Laplace operator, respectively, i.e.,

$$\begin{aligned} \mathfrak{a}^N(u, v) &:= \int_{\Omega} \nabla u \overline{\nabla v} \, dx & (u, v \in H^1(\Omega)) \text{ and} \\ \mathfrak{a}^D(u, v) &:= \int_{\Omega} \nabla u \overline{\nabla v} \, dx & (u, v \in H_0^1(\Omega)). \end{aligned}$$

Both forms \mathfrak{a}^N and \mathfrak{a}^D are local and T^N dominates T^D , where T^N and T^D are the associated semigroups. Let S be sandwiched between T^N and T^D , generated by a closed, accretive, and Hermitian form \mathfrak{b} . Then (as remarked above), \mathfrak{b} is necessarily local by Theorem 3.2. If Ω has Lipschitz boundary, so that \mathfrak{b} is also $C(\overline{\Omega})$ -regular, then the form is associated to a Laplace operator with local Robin boundary conditions [4, Theorem 4.1]. As shown in Theorem 3.2 (b) above, and as already shown by Akhlil in [1], the assumption of locality in [4, Theorem 4.1] is superfluous. To sum up, [4, Theorem 4.1] can be simplified as follows:

Proposition 4.1. *Let S be a semigroup generated by a closed, accretive, and Hermitian form \mathfrak{b} on $L^2_\mu(\Omega)$. Also, let T^D and T^N denote the semigroups generated by the Dirichlet Laplacian and Neumann Laplacian respectively.*

If Ω has Lipschitz boundary, then S is sandwiched between T^D and T^N if and only if S is generated by a Laplace operator with local Robin boundary conditions.

A similar representation theorem was also proved in a nonlinear setting in which closed, accretive, and Hermitian forms are replaced by convex, lower semi-continuous energy functions. For example, [7] characterises all nonlinear semigroups which are sandwiched between the semigroups generated by the p -Laplace operator with Neumann boundary conditions and the p -Laplace operator with Dirichlet boundary conditions, if the energy function generating the sandwiched semigroup is local. Later, in [9], such a characterisation was generalised to semigroups generated by nonlinear and local Dirichlet forms. It is not clear whether locality is a necessary assumption in the nonlinear situation.

4.2. Eventual positivity. Let $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \rightarrow \mathbb{C}$ be a sesquilinear Hermitian form on $L^2_\mu(\Omega)$, where $\Omega \subseteq \mathbb{R}^N$ is a bounded open set. As a consequence of Theorem 3.2(b), we have that, if the semigroup T associated to \mathfrak{a} is dominated by the semigroup generated by the Neumann Laplacian (see above), then the positivity of T implies locality of \mathfrak{a} . However, there are non-positive semigroups that are dominated by the semigroup generated by the Neumann Laplacian. Of course, due to Theorem 3.2(a), they are necessarily non-local. We give an example: Let $\Omega = (0, 1)$ and let T be the semigroup associated with the non-local form

$$\begin{aligned} \mathfrak{a}(u, v) &= \int_0^1 u' \bar{v}' dx + \langle Bu|_{\{0,1\}}, v|_{\{0,1\}} \rangle \\ &= \int_0^1 u' \bar{v}' dx + \lambda(u(0)\overline{v(0)} + u(1)\overline{v(1)} + u(0)\overline{v(1)} + u(1)\overline{v(0)}) \end{aligned}$$

for $u, v \in D(\mathfrak{a}) := H^1(0, 1)$; where $B = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix}$ and λ is a positive non-zero real number. The aforementioned domination is a consequence of Theorem 1.1. Indeed, by Remark 2.1, it suffices to show that the measure is diagonally dominant on $H^1(0, 1)$. Note that the measure here is just the Dirac measure on the four corners of the unit square $[0, 1] \times [0, 1]$ and hence diagonally dominant. In fact, the domination of the semigroup T by the Neumann Laplacian is also mentioned in [1, Section 3] for the case $\lambda = 1$.

While non-positivity of the semigroup T is a consequence of the first Beurling-Deny criterion, it can alternatively be deduced by Theorem 3.2(b). Nevertheless, the semigroup T is *uniformly eventually positive*, i.e., there exists a time $t_0 \geq 0$ such that $T(t)$ leaves the positive cone invariant for all $t \geq t_0$. Indeed, this was shown for $\lambda = 1$ in [12, Theorem 4.2] and the proof for other values of λ remains same.

The first example of an eventually positive semigroup in infinite dimensions was given by Daners in [10] which led to a systematic study in [14, 15]. The theory has been further developed in [5, 11–13]. Recently, the first author along with Glück showed that the semigroup T above is *eventually sandwiched* between the semigroups generated by the Dirichlet Laplacian and Neumann Laplacian [6, Theorem 4.5].

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