



Parameter dependence of solutions of the Cauchy–Riemann equation on weighted spaces of smooth functions

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Abstract

Let Ω be an open subset of \mathbb{R}^2 and E a complete complex locally convex Hausdorff space. The purpose of this paper is to find conditions on certain weighted Fréchet spaces $\mathcal{E}\mathcal{V}(\Omega)$ of smooth functions and on the space E to ensure that the vector-valued Cauchy–Riemann operator $\bar{\partial} : \mathcal{E}\mathcal{V}(\Omega, E) \rightarrow \mathcal{E}\mathcal{V}(\Omega, E)$ is surjective. This is done via splitting theory and positive results can be interpreted as parameter dependence of solutions of the Cauchy–Riemann operator.

Keywords Cauchy–Riemann · Parameter dependence · Weight · Smooth · Solvability · Vector-valued

Mathematics Subject Classification 35A01 · 35B30 · 32W05 · 46A63 · 46A32 · 46E40

1 Introduction

Let E be a linear space of functions on a set U and $P(\partial) : \mathcal{F}(\Omega) \rightarrow \mathcal{F}(\Omega)$ be a linear partial differential operator with constant coefficients which acts continuously on a locally convex Hausdorff space of (generalized) differentiable scalar-valued functions $\mathcal{F}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^n$. We call the elements of U parameters and say that a family $(f_\lambda)_{\lambda \in U}$ in $\mathcal{F}(\Omega)$ depends on a parameter w.r.t. E if the map $\lambda \mapsto f_\lambda(x)$ is an element of E for every $x \in \Omega$. The question of parameter dependence is whether for every family $(f_\lambda)_{\lambda \in U}$ in $\mathcal{F}(\Omega)$ depending on a parameter w.r.t. E there is a family $(u_\lambda)_{\lambda \in U}$ in $\mathcal{F}(\Omega)$ with the same kind of parameter dependence which solves the partial differential equation

$$P(\partial)u_\lambda = f_\lambda, \quad \lambda \in U.$$

In particular, it is the question of \mathcal{C}^k -smooth (holomorphic, distributional, etc.) parameter dependence if E is the space $\mathcal{C}^k(U)$ of k -times continuously partially differentiable functions on an open set $U \subset \mathbb{R}^d$ (the space $\mathcal{O}(U)$ of holomorphic functions on an open set $U \subset \mathbb{C}$, the space of distributions $\mathcal{D}(V)'$ on an open set $V \subset \mathbb{R}^d$ where $U = \mathcal{D}(V)$, etc.).

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The question of parameter dependence has been subject of extensive research varying in the choice of the spaces E , $\mathcal{F}(\Omega)$ and the properties of the partial differential operator $P(\partial)$, e.g. being (hypo)elliptic, parabolic or hyperbolic. Even partial differential differential operators $P_\lambda(\partial)$ where the coefficients also depend $C^k([0, 1])$ -smoothly [49], C^∞ -smoothly [61], holomorphically [50,61] or differentiable resp. real analytic [13] on the parameter λ were considered. The case that the coefficients of the partial differential differential operator $P(x, \partial)$ are non-constant functions in $x \in \Omega$ was treated for $\mathcal{F}(\Omega) = \mathcal{A}(\mathbb{R}^n)$, the space of real analytic functions on \mathbb{R}^n , as well [3].

The answer to the question of C^k -smooth (holomorphic, distributional, etc.) parameter dependence is obviously affirmative if $P(\partial)$ has a linear continuous right inverse. The problem to determine those $P(\partial)$ which have such a right inverse was posed by Schwartz in the early 1950s (see [21, p. 680]). In the case that $\mathcal{F}(\Omega)$ is the space of C^∞ -smooth functions or distributions on an open set $\Omega \subset \mathbb{R}^n$ the problem was solved in [51,52] and in the case of ultradifferentiable functions or ultradistributions in [53] by means of Phragmén-Lindelöf type conditions. The case that $\mathcal{F}(\Omega)$ is a weighted space of C^∞ -smooth functions on $\Omega = \mathbb{R}^n$ or its dual was handled in [40], even for some $P(x, \partial)$ with smooth coefficients, the case of tempered distributions in [38] and of Fourier (ultra-)hyperfunctions in [44,45]. For Hörmander's spaces $B_{p,\kappa}^{loc}(\Omega)$ as $\mathcal{F}(\Omega)$ the problem was studied in [25].

The necessary condition of surjectivity of the partial differential operator $P(\partial)$ was studied in many papers, e.g. in [1,23,28,48,67] on C^∞ -smooth functions and distributions, in [9,26,43] on real analytic functions, in [8,14] on Gevrey classes, in [10,12,41,42,55] on ultradifferentiable functions of Roumieu type, in [22] on ultradistributions of Beurling type, in [7,11] on ultradifferentiable functions and ultradistributions and in [47] on the multiplier space \mathcal{O}_M .

However, if $P(\partial): C^\infty(\Omega) \rightarrow C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, is elliptic, then $P(\partial)$ has a linear right inverse (by means of a Hamel basis of $C^\infty(\Omega)$) and it has a continuous right inverse due to Michael's selection theorem [56, Theorem 3.2'', p. 367] and [29, Satz 9.28, p. 217], but $P(\partial)$ has no linear continuous right inverse if $n \geq 2$ by a result of Grothendieck [62, Theorem C.1, p. 109]. Nevertheless, the question of parameter dependence w.r.t. E has a positive answer for several locally convex Hausdorff spaces E due to tensor product techniques. In this case the question of parameter dependence obviously has a positive answer if the topology of E is stronger than the topology of pointwise convergence on U and

$$P(\partial)^E: C^\infty(\Omega, E) \rightarrow C^\infty(\Omega, E)$$

is surjective where $C^\infty(\Omega, E)$ is the space of C^∞ -smooth E -valued functions on Ω and $P(\partial)^E$ the version of $P(\partial)$ for E -valued functions. From Grothendieck's classical theory of tensor products [24] and the surjectivity of $P(\partial)$ it follows that $P(\partial)^E$ is also surjective if E is a Fréchet space. In general, Grothendieck's theory of tensor products can be applied if $P(\partial)$ is surjective and $\mathcal{F}(\Omega)$ a nuclear Fréchet space. However, the surjectivity of $P(\partial)^E$, $n \geq 2$, can even be extended beyond the class of Fréchet spaces E due to the splitting theory of Vogt for Fréchet spaces [64,65] and of Bonet and Domański for PLS-spaces [4,6] if, in addition, $\ker P(\partial)$ has the property (DN) and E is the dual of a Fréchet space with the property (DN) or an ultrabornological PLS-space with the property (PA) . The splitting theory of Bonet and Domański can also be applied if $\mathcal{F}(\Omega)$ is a non-Fréchet PLS-space and for PLH-spaces $\mathcal{F}(\Omega)$, e.g. \mathcal{D}_{L^2} and $B_{2,\kappa}^{loc}(\Omega)$ which are non-PLS-spaces, the splitting theory of Dierolf and Sieg [15,16] is available. For applications we refer the reader to the already mentioned papers [4,6,15,16,64,65] as well as [5,18] where $\mathcal{F}(\Omega)$ is the space of ultradistributions of Beurling type or of ultradifferentiable functions of Roumieu type and E , amongst others, the space

of real analytic functions and to [30] where $\mathcal{F}(\Omega)$ is the space of C^∞ -smooth functions or distributions.

Notably, the preceding results imply that the inhomogeneous Cauchy–Riemann equation with a right-hand side $f \in \mathcal{E}(\Omega, E) := C^\infty(\Omega, E)$, where $\Omega \subset \mathbb{R}^2$ is open and E a locally convex Hausdorff space over \mathbb{C} whose topology is induced by a system of seminorms $(p_\alpha)_{\alpha \in \mathfrak{A}}$, given by

$$\bar{\partial}^E u := (1/2)(\partial_1^E + i\partial_2^E)u = f \tag{1}$$

has a solution $u \in \mathcal{E}(\Omega, E)$ if E is a Fréchet space or $E := F'_b$ where F is a Fréchet space satisfying the condition (DN) or if E is an ultrabornological PLS-space having the property (PA) . Among these spaces E are several spaces of distributions like $\mathcal{D}(V)'$, the space of tempered distributions, the space of ultradistributions of Beurling type etc. In the present paper we study this problem under the constraint that the right-hand side f fulfils additional growth conditions given by an increasing family of positive continuous functions $\mathcal{V} := (v_n)_{n \in \mathbb{N}}$ on an increasing sequence of open subsets $(\Omega_n)_{n \in \mathbb{N}}$ of Ω with $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$, namely,

$$|f|_{n,m,\alpha} := \sup_{\substack{x \in \Omega_n \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha((\partial^\beta)^E f(x))v_n(x) < \infty$$

for every $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$. Let us call the space of such functions $\mathcal{E}\mathcal{V}(\Omega, E)$. Our interest is in conditions on \mathcal{V} and $(\Omega_n)_{n \in \mathbb{N}}$ such that there is a solution $u \in \mathcal{E}\mathcal{V}(\Omega, E)$ of (1), i.e. we search for conditions that guarantee the surjectivity of

$$\bar{\partial}^E : \mathcal{E}\mathcal{V}(\Omega, E) \rightarrow \mathcal{E}\mathcal{V}(\Omega, E).$$

Using Grothendieck’s theory of tensor products, this was already done in [33] in the case that E is a Fréchet space. In the present paper we want to extend this result beyond the class of Fréchet spaces E . Concerning the sequence $(\Omega_n)_{n \in \mathbb{N}}$, we concentrate on the case that it is a sequence of strips along the real axis, i.e. $\Omega_n := \{z \in \mathbb{C} \mid |\text{Im}(z)| < n\}$. The case that this sequence has holes along the real axis is treated in [35].

Let us briefly outline the content of our paper. In Sect. 2 we summarise the necessary definitions and preliminaries which are needed in the subsequent sections. In Sect. 3 we recall the definitions of the topological invariants (Ω) , (DN) and (PA) and provide some examples of spaces E having these invariants. Then we prove our main result on the surjectivity of Cauchy–Riemann operator on $\mathcal{E}\mathcal{V}(\Omega, E)$ which depends on these invariants (see Theorem 5). To apply our main result, the kernel $\ker \bar{\partial}$ needs to have (Ω) and in Sect. 4 we provide sufficient conditions on the weights and the sequence $(\Omega_n)_{n \in \mathbb{N}}$ which guarantee (Ω) (see Theorem 10 and Corollary 13). We close this section with a special case of our main theorem where $(\Omega_n)_{n \in \mathbb{N}}$ is a sequence of strips along the real axis (see Corollary 17) and for example $v_n(z) := \exp(a_n |\text{Re}(z)|^\gamma)$ for some $0 < \gamma \leq 1$ and $a_n \nearrow 0$ (see Corollary 18).

2 Notation and preliminaries

The notation and preliminaries are essentially the same as in [33,36, Sect. 2]. We define the distance of two subsets $M_0, M_1 \subset \mathbb{R}^2$ w.r.t. a norm $\|\cdot\|$ on \mathbb{R}^2 via

$$d^{\|\cdot\|}(M_0, M_1) := \begin{cases} \inf_{x \in M_0, y \in M_1} \|x - y\|, & M_0, M_1 \neq \emptyset, \\ \infty, & M_0 = \emptyset \text{ or } M_1 = \emptyset. \end{cases}$$

Moreover, we denote by $\|\cdot\|_\infty$ the sup-norm, by $|\cdot|$ the Euclidean norm on \mathbb{R}^2 , by $\mathbb{B}_r(x) := \{w \in \mathbb{R}^2 \mid |w - x| < r\}$ the Euclidean ball around $x \in \mathbb{R}^2$ with radius $r > 0$ and identify \mathbb{R}^2 and \mathbb{C} as (normed) vector spaces. We denote the complement of a subset $M \subset \mathbb{R}^2$ by $M^C := \mathbb{R}^2 \setminus M$, the closure of M by \bar{M} and the boundary of M by ∂M . For a function $f: M \rightarrow \mathbb{C}$ and $K \subset M$ we denote by $f|_K$ the restriction of f to K and by

$$\|f\|_K := \sup_{x \in K} |f(x)|$$

the sup-norm on K . By $L^1(\Omega)$ we denote the space of (equivalence classes of) \mathbb{C} -valued Lebesgue integrable functions on a measurable set $\Omega \subset \mathbb{R}^2$ and by $L^q(\Omega)$, $q \in \mathbb{N}$, the space of functions f such that $f^q \in L^1(\Omega)$. If $(a_n)_{n \in \mathbb{N}}$ is a strictly increasing real sequence, we write $a_n \nearrow 0$ resp. $a_n \nearrow \infty$ if $a_n < 0$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} a_n = 0$ resp. $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} a_n = \infty$.

By E we always denote a non-trivial locally convex Hausdorff space over the field \mathbb{C} equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha \in \mathfrak{A}}$. If $E = \mathbb{C}$, then we set $(p_\alpha)_{\alpha \in \mathfrak{A}} := \{|\cdot|\}$. Further, we denote by $L(F, E)$ the space of continuous linear maps from a locally convex Hausdorff space F to E and sometimes write $\langle T, f \rangle := T(f)$, $f \in F$, for $T \in L(F, E)$. If $E = \mathbb{C}$, we write $F' := L(F, \mathbb{C})$ for the dual space of F . If F and E are (linearly topologically) isomorphic, we write $F \cong E$. We denote by $L_t(F, E)$ the space $L(F, E)$ equipped with the locally convex topology of uniform convergence on the finite subsets of F if $t = \sigma$, on the precompact subsets of F if $t = \gamma$, on the absolutely convex, compact subsets of F if $t = \kappa$ and on the bounded subsets of F if $t = b$.

The so-called ε -product of Schwartz is defined by

$$F \varepsilon E := L_e(F'_\kappa, E) \tag{2}$$

where $L(F'_\kappa, E)$ is equipped with the topology of uniform convergence on equicontinuous subsets of F' . This definition of the ε -product coincides with the original one by Schwartz [59, Chap. I, Sect. 1, Définition, p. 18].

We recall the following well-known definitions concerning continuous partial differentiability of vector-valued functions (c.f. [34, p. 237]). A function $f: \Omega \rightarrow E$ on an open set $\Omega \subset \mathbb{R}^2$ to E is called continuously partially differentiable (f is \mathcal{C}^1) if for the n -th unit vector $e_n \in \mathbb{R}^2$ the limit

$$(\partial^{e_n})^E f(x) := (\partial_n)^E f(x) := \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}, h \neq 0}} \frac{f(x + he_n) - f(x)}{h}$$

exists in E for every $x \in \Omega$ and $(\partial^{e_n})^E f$ is continuous on Ω ($(\partial^{e_n})^E f$ is \mathcal{C}^0) for every $n \in \{1, 2\}$. For $k \in \mathbb{N}$ a function f is said to be k -times continuously partially differentiable (f is \mathcal{C}^k) if f is \mathcal{C}^1 and all its first partial derivatives are \mathcal{C}^{k-1} . A function f is called infinitely continuously partially differentiable (f is \mathcal{C}^∞) if f is \mathcal{C}^k for every $k \in \mathbb{N}$. The linear space of all functions $f: \Omega \rightarrow E$ which are \mathcal{C}^∞ is denoted by $\mathcal{C}^\infty(\Omega, E)$. Let $f \in \mathcal{C}^\infty(\Omega, E)$. For $\beta = (\beta_n) \in \mathbb{N}_0^2$ we set $(\partial^{\beta_n})^E f := f$ if $\beta_n = 0$, and

$$(\partial^{\beta_n})^E f := \underbrace{(\partial^{e_n})^E \dots (\partial^{e_n})^E}_{\beta_n\text{-times}} f$$

if $\beta_n \neq 0$ as well as

$$(\partial^\beta)^E f := (\partial^{\beta_1})^E (\partial^{\beta_2})^E f.$$

Due to the vector-valued version of Schwarz’ theorem $(\partial^\beta)^E f$ is independent of the order of the partial derivatives on the right-hand side, we call $|\beta| := \beta_1 + \beta_2$ the order of differentiation and write $\partial^\beta f := (\partial^\beta)^{\mathbb{C}} f$.

A function $f : \Omega \rightarrow E$ on an open set $\Omega \subset \mathbb{C}$ to E is called holomorphic if the limit

$$\left(\frac{\partial}{\partial z}\right)^E f(z_0) := \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{C}, h \neq 0}} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists in E for every $z_0 \in \Omega$ and the space of such functions is denoted by $\mathcal{O}(\Omega, E)$. The exact definition of the spaces from the introduction is as follows.

Definition 1 [34, Definition 3.2, p. 238] Let $\Omega \subset \mathbb{R}^2$ be open and $(\Omega_n)_{n \in \mathbb{N}}$ a family of non-empty open sets such that $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. Let $\mathcal{V} := (v_n)_{n \in \mathbb{N}}$ be a countable family of positive continuous functions $v_n : \Omega \rightarrow (0, \infty)$ such that $v_n \leq v_{n+1}$ for all $n \in \mathbb{N}$. We call \mathcal{V} a directed family of continuous weights on Ω and set for $n \in \mathbb{N}$

(a)

$$\mathcal{E}v_n(\Omega_n, E) := \{f \in C^\infty(\Omega_n, E) \mid \forall \alpha \in \mathfrak{A}, m \in \mathbb{N}_0^2 : |f|_{n,m,\alpha} < \infty\}$$

and

$$\mathcal{E}\mathcal{V}(\Omega, E) := \{f \in C^\infty(\Omega, E) \mid \forall n \in \mathbb{N} : f|_{\Omega_n} \in \mathcal{E}v_n(\Omega_n, E)\}$$

where

$$|f|_{n,m,\alpha} := \sup_{\substack{x \in \Omega_n \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha((\partial^\beta)^E f(x))v_n(x).$$

(b)

$$\mathcal{E}v_{n,\bar{\partial}}(\Omega_n, E) := \{f \in \mathcal{E}v_n(\Omega_n, E) \mid f \in \ker \bar{\partial}^E\}$$

and

$$\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega, E) := \{f \in \mathcal{E}\mathcal{V}(\Omega, E) \mid f \in \ker \bar{\partial}^E\}.$$

(c)

$$\mathcal{O}v_n(\Omega_n, E) := \{f \in \mathcal{O}(\Omega_n, E) \mid \forall \alpha \in \mathfrak{A} : |f|_{n,\alpha} < \infty\}$$

and

$$\mathcal{O}\mathcal{V}(\Omega, E) := \{f \in \mathcal{O}(\Omega, E) \mid \forall n \in \mathbb{N} : f|_{\Omega_n} \in \mathcal{O}v_n(\Omega_n, E)\}$$

where

$$|f|_{n,\alpha} := \sup_{x \in \Omega_n} p_\alpha(f(x))v_n(x).$$

The subscript α in the notation of the seminorms is omitted in the \mathbb{C} -valued case. The letter E is omitted in the case $E = \mathbb{C}$ as well, e.g. we write $\mathcal{E}v_n(\Omega_n) := \mathcal{E}v_n(\Omega_n, \mathbb{C})$ and $\mathcal{E}\mathcal{V}(\Omega) := \mathcal{E}\mathcal{V}(\Omega, \mathbb{C})$.

A projective limit F of a sequence of locally convex Hausdorff spaces $(F_n)_{n \in \mathbb{N}}$ is called weakly reduced if for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $\pi_n(F)$ is dense in F_m w.r.t. the topology of F_n where $\pi_n : F \rightarrow F_n$ is the canonical projection. The spaces $\mathcal{FV}(\Omega, E)$, $\mathcal{F} = \mathcal{E}, \mathcal{O}$, are projective limits, namely, we have

$$\mathcal{FV}(\Omega, E) \cong \varprojlim_{n \in \mathbb{N}} \mathcal{FV}_n(\Omega_n, E)$$

where the spectral maps are given by the restrictions

$$\pi_{k,n} : \mathcal{FV}_k(\Omega_k, E) \rightarrow \mathcal{FV}_n(\Omega_n, E), \quad f \mapsto f|_{\Omega_n}, \quad k \geq n.$$

3 Main result

In this section we prove our main result that the surjectivity of the vector-valued Cauchy–Riemann operator on $\mathcal{E}\mathcal{V}(\Omega, E)$ is inherited from the surjectivity on $\mathcal{E}\mathcal{V}(\Omega)$ if the kernel $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ in the scalar-valued case has (Ω) , and $E := F'_b$ where F is a Fréchet space satisfying the condition (DN) or E is an ultrabornological PLS-space having the property (PA) . Therefore we recall the definitions of the topological invariants (Ω) , (DN) and (PA) and give some examples.

A Fréchet space F with an increasing fundamental system of seminorms $(\|\cdot\|_k)_{k \in \mathbb{N}}$ satisfies (Ω) if

$$\forall p \in \mathbb{N} \exists q \in \mathbb{N} \forall k \in \mathbb{N} \exists n \in \mathbb{N}, C > 0 \forall r > 0 : U_q \subset Cr^n U_k + \frac{1}{r} U_p \tag{3}$$

where $U_k := \{x \in F \mid \|x\|_k \leq 1\}$ (see [54, Chap. 29, Definition, p. 367]).

A Fréchet space $(F, (\|\cdot\|_k)_{k \in \mathbb{N}})$ satisfies (DN) by [54, Chap. 29, Definition, p. 359] if

$$\exists p \in \mathbb{N} \forall k \in \mathbb{N} \exists n \in \mathbb{N}, C > 0 \forall x \in F : \|x\|_k^2 \leq C \|x\|_p \|x\|_n.$$

A PLS-space is a projective limit $X = \varprojlim_{N \in \mathbb{N}} X_N$, where the X_N given by inductive limits $X_N = \varinjlim_{n \in \mathbb{N}} (X_{N,n}, \|\cdot\|_{N,n})$ are DFS-spaces (which are also called LS-spaces), and it satisfies (PA) if

$$\forall N \exists M \forall K \exists n \forall m \forall \eta > 0 \exists k, C, r_0 > 0 \forall r > r_0 \forall x' \in X'_N : \\ \left\| x' \circ i_N^M \right\|_{M,m}^* \leq C (r^\eta \left\| x' \circ i_N^K \right\|_{K,k}^* + \frac{1}{r} \|x'\|_{N,n}^*)$$

where $\|\cdot\|^*$ denotes the dual norm of $\|\cdot\|$ and i_N^M, i_N^K the linking maps (see [6, Sect. 4, Eq. (24), p. 577]).

Due to [63, 1.4 Lemma, p. 110] and [6, Proposition 4.2, p. 577] we have the following relation between the properties (DN) and (PA) .

Remark 2 Let F be a Fréchet-Schwartz space. Then F satisfies (DN) if and only if the DFS-space $E := F'_b$ satisfies (PA) .

Let us summarise some examples of ultrabornological PLS-spaces satisfying (PA) and spaces of the form $E := F'_b$ where F is a Fréchet space satisfying (DN) . The majority of them is already contained in [6], [19] and [64].

Example 3 (a) The following spaces are ultrabornological PLS-spaces with property (PA) and also strong duals of a Fréchet space satisfying (DN):

- the strong dual of a power series space of infinite type $\Lambda_\infty(\alpha)'_b$,
- the strong dual of any space of holomorphic functions $\mathcal{O}(U)'_b$ where U is a Stein manifold with the strong Liouville property (for instance, for $U = \mathbb{C}^d$),
- the space of germs of holomorphic functions $\mathcal{O}(K)$ where K is a completely pluripolar compact subset of a Stein manifold (for instance K consists of one point),
- the space of tempered distributions $\mathcal{S}(\mathbb{R}^d)'_b$ and the space of Fourier ultra-hyperfunctions \mathcal{P}'_{**} (with the strong topology),
- the weighted distribution spaces $(K\{pM\})'_b$ of Gelfand and Shilov if the weight M satisfies

$$\sup_{|y| \leq 1} M(x + y) \leq C \inf_{|y| \leq 1} M(x + y), \quad x \in \mathbb{R}^d,$$

- $\mathcal{D}(K)'_b$ for any compact set $K \subset \mathbb{R}^d$ with non-empty interior,
- $\mathcal{C}^\infty(\bar{U})'_b$ for any non-empty open bounded set $U \subset \mathbb{R}^d$ with \mathcal{C}^1 -boundary.

(b) The following spaces are ultrabornological PLS-spaces with property (PA):

- an arbitrary Fréchet-Schwartz space,
- a PLS-type power series space $A_{r,s}(\alpha, \beta)$ whenever $s = \infty$ or $A_{r,s}(\alpha, \beta)$ is a Fréchet space,
- the spaces of distributions $\mathcal{D}(U)'_b$ and ultradistributions of Beurling type $\mathcal{D}_{(\omega)}(U)'_b$ for any open set $U \subset \mathbb{R}^d$,
- the kernel of any linear partial differential operator with constant coefficients in $\mathcal{D}(U)'_b$ or in $\mathcal{D}_{(\omega)}(U)'_b$ when $U \subset \mathbb{R}^d$ is open and convex,
- the space $L_b(X, Y)$ where X has (DN), Y has (Ω) and both are nuclear Fréchet spaces. In particular, $L_b(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$ if both spaces are nuclear.

(c) The following spaces are strong duals of a Fréchet space satisfying (DN):

- the strong dual F'_b of any Banach space F ,
- the strong dual $\lambda^2(A)'_b$ of the Köthe space $\lambda^2(A)$ with a Köthe matrix $A = (a_{j,k})_{j,k \in \mathbb{N}_0}$ satisfying

$$\exists p \in \mathbb{N}_0 \forall k \in \mathbb{N}_0 \exists n \in \mathbb{N}_0, C > 0 : a_{j,k}^2 \leq C a_{j,p} a_{j,n}.$$

Proof The statement for the spaces in (a) and (b) follows from [19, Corollary 4.8, p. 1116], [54, Proposition 31.12, p. 401], [54, Proposition 31.16, p. 402] and Remark 2. The first part of statement (c) is obvious since Banach spaces clearly satisfy the property (DN). The second part on the Köthe space $\lambda^2(A)$ follows from [29, Satz 12.11 a), p. 305]. \square

Since we will use the ε -product $\mathcal{E}\mathcal{V}(\Omega)\varepsilon E$ to pass the surjectivity from $\bar{\partial}$ to $\bar{\partial}^E$, we remark the following which is not hard to prove (see [31, Sect. 39]).

Proposition 4 (a) *Let X be a semi-reflexive locally convex Hausdorff space and Y a Fréchet space. Then $L_b(X'_b, Y'_b) \cong L_b(Y, (X'_b)'_b)$ via taking adjoints.*

(b) *Let X be a Montel space and E a locally convex Hausdorff space. Then $L_b(X'_b, E) \cong X\varepsilon E$ where the topological isomorphism is the identity map.*

Theorem 5 *Let $\mathcal{E}\mathcal{V}(\Omega)$ be a Schwartz space and $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ a nuclear subspace satisfying property (Ω). Assume that the scalar-valued operator $\bar{\partial} : \mathcal{E}\mathcal{V}(\Omega) \rightarrow \mathcal{E}\mathcal{V}(\Omega)$ is surjective. Moreover, if*

- (a) $E := F'_b$ where F is a Fréchet space over \mathbb{C} satisfying (DN) , or
- (b) E is an ultrabornological PLS-space over \mathbb{C} satisfying (PA) ,

then

$$\bar{\partial}^E : \mathcal{E}\mathcal{V}(\Omega, E) \rightarrow \mathcal{E}\mathcal{V}(\Omega, E)$$

is surjective.

Proof Throughout this proof we use the notation $X'' := (X'_b)'_b$ for a locally convex Hausdorff space X . In both cases, (a) and (b), the space E is a complete locally convex Hausdorff space. The space $\mathcal{E}\mathcal{V}(\Omega)$ is a Fréchet space by [34, Proposition 3.7, p. 240] and so its closed subspace $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ as well. Further, $\mathcal{E}\mathcal{V}(\Omega)$ is a Schwartz space and $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ nuclear, thus both spaces are reflexive. As the Fréchet-Schwartz space $\mathcal{E}\mathcal{V}(\Omega)$ is a Montel space,

$$S : \mathcal{E}\mathcal{V}(\Omega) \varepsilon E \rightarrow \mathcal{E}\mathcal{V}(\Omega, E), \quad u \mapsto [z \mapsto u(\delta_z)],$$

is a topological isomorphism by [36, 3.21 Example b), p. 14] where δ_z is the point-evaluation at $z \in \Omega$. We denote by $\mathcal{J} : E \rightarrow E'^*$ the canonical injection in the algebraic dual E'^* of the topological dual E' and for $f \in \mathcal{E}\mathcal{V}(\Omega, E)$ we set

$$R_f^t : \mathcal{E}\mathcal{V}(\Omega)' \rightarrow E'^*, \quad y \mapsto [e' \mapsto y(e' \circ f)].$$

Then the map $f \mapsto \mathcal{J}^{-1} \circ R_f^t$ is the inverse of S by [36, 3.17 Theorem, p. 12]. The sequence

$$0 \rightarrow \mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega) \xrightarrow{i} \mathcal{E}\mathcal{V}(\Omega) \xrightarrow{\bar{\partial}} \mathcal{E}\mathcal{V}(\Omega) \rightarrow 0, \tag{4}$$

where i means the inclusion, is a topologically exact sequence of Fréchet spaces because $\bar{\partial}$ is surjective by assumption. Let us denote by $J_0 : \mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega) \rightarrow \mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)''$ and $J_1 : \mathcal{E}\mathcal{V}(\Omega) \rightarrow \mathcal{E}\mathcal{V}(\Omega)''$ the canonical embeddings which are topological isomorphisms since $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ and $\mathcal{E}\mathcal{V}(\Omega)$ are reflexive. Then the exactness of (4) implies that

$$0 \rightarrow \mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)'' \xrightarrow{i_0} \mathcal{E}\mathcal{V}(\Omega)'' \xrightarrow{\bar{\partial}_1} \mathcal{E}\mathcal{V}(\Omega)'' \rightarrow 0, \tag{5}$$

where $i_0 := J_0 \circ i \circ J_0^{-1}$ and $\bar{\partial}_1 := J_1 \circ \bar{\partial} \circ J_1^{-1}$, is an exact topological sequence. Topological as the (strong) bidual of a Fréchet space is again a Fréchet space by [54, Corollary 25.10, p. 298].

(a) Let $E := F'_b$ where F is a Fréchet space with (DN) . Then $\text{Ext}^1(F, \mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)'') = 0$ by [65, 5.1 Theorem, p. 186] since $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ satisfies (Ω) and therefore $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)''$ as well. Combined with the exactness of (5) this implies that the sequence

$$0 \rightarrow L(F, \mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)'') \xrightarrow{i_0^*} L(F, \mathcal{E}\mathcal{V}(\Omega)'') \xrightarrow{\bar{\partial}_1^*} L(F, \mathcal{E}\mathcal{V}(\Omega)'') \rightarrow 0$$

is exact by [57, Proposition 2.1, p. 13-14] where $i_0^*(B) := i_0 \circ B$ and $\bar{\partial}_1^*(D) := \bar{\partial}_1 \circ D$ for $B \in L(F, \mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)'')$ and $D \in L(F, \mathcal{E}\mathcal{V}(\Omega)'')$. In particular, we obtain that

$$\bar{\partial}_1^* : L(F, \mathcal{E}\mathcal{V}(\Omega)'') \rightarrow L(F, \mathcal{E}\mathcal{V}(\Omega)'') \tag{6}$$

is surjective. Via $E = F'_b$ and Proposition 4 ($X = \mathcal{E}\mathcal{V}(\Omega)$ and $Y = F$) we have the topological isomorphism

$$\psi := S \circ {}^t(\cdot) : L(F, \mathcal{E}\mathcal{V}(\Omega)'') \rightarrow \mathcal{E}\mathcal{V}(\Omega, E), \quad \psi(u) = (S \circ {}^t(\cdot))(u) = [z \mapsto {}^t u(\delta_z)],$$

and the inverse

$$\psi^{-1}(f) = (S \circ {}^t(\cdot))^{-1}(f) = {}^t(\cdot) \circ S^{-1}(f) = {}^t(\mathcal{J}^{-1} \circ R'_f), \quad f \in \mathcal{EV}(\Omega, E).$$

Let $g \in \mathcal{EV}(\Omega, E)$. Then $\psi^{-1}(g) \in L(F, \mathcal{EV}(\Omega)'')$ and by the surjectivity of (6) there is $u \in L(F, \mathcal{EV}(\Omega)'')$ such that $\bar{\partial}_1^* u = \psi^{-1}(g)$. So we get $\psi(u) \in \mathcal{EV}(\Omega, E)$. Next, we show that $\bar{\partial}^E \psi(u) = g$ is valid. Let $x \in F, z \in \Omega$ and $h \in \mathbb{R}, h \neq 0$, and e_k denote the k th unit vector in \mathbb{R}^2 . From

$$\left(\frac{\delta_{z+he_k} - \delta_z}{h}\right)(f) = \frac{f(z + he_k) - f(z)}{h} \xrightarrow{h \rightarrow 0} \partial^{e_k} f(z),$$

for every $f \in \mathcal{EV}(\Omega)$ it follows that $\frac{\delta_{z+he_k} - \delta_z}{h}$ converges to $\delta_z \circ \partial^{e_k}$ in $\mathcal{EV}(\Omega)'_\sigma$. Since the Fréchet–Schwartz space $\mathcal{EV}(\Omega)$ is in particular a Montel space, we deduce that $\frac{\delta_{z+he_k} - \delta_z}{h}$ converges to $\delta_z \circ \partial^{e_k}$ in $\mathcal{EV}(\Omega)'_\gamma = \mathcal{EV}(\Omega)'_b$ by the Banach–Steinhaus theorem. Let $B \subset F$ be bounded. As ${}^t u \in L(\mathcal{EV}(\Omega)'_b, F'_b)$, there are a bounded set $B_0 \subset \mathcal{EV}(\Omega)$ and $C > 0$ such that

$$\begin{aligned} & \sup_{x \in B} \left| \left(\frac{{}^t u(\delta_{z+he_k}) - {}^t u(\delta_z)}{h}\right)(x) - {}^t u(\delta_z \circ \partial^{e_k})(x) \right| \\ &= \sup_{x \in B} \left| {}^t u \left(\frac{\delta_{z+he_k} - \delta_z}{h} - \delta_z \circ \partial^{e_k}\right)(x) \right| \leq C \sup_{f \in B_0} \left| \left(\frac{\delta_{z+he_k} - \delta_z}{h} - \delta_z \circ \partial^{e_k}\right)(f) \right| \xrightarrow{h \rightarrow 0} 0, \end{aligned}$$

yielding to $(\partial^{e_k})^E(\psi(u))(z) = {}^t u(\delta_z \circ \partial^{e_k})$. This implies $\bar{\partial}^E(\psi(u))(z) = {}^t u(\delta_z \circ \bar{\partial})$. So for all $x \in F$ and $z \in \Omega$ we have

$$\begin{aligned} \bar{\partial}^E(\psi(u))(z)(x) &= {}^t u(\delta_z \circ \bar{\partial})(x) = u(x)(\delta_z \circ \bar{\partial}) = \langle \delta_z \circ \bar{\partial}, J_1^{-1}(u(x)) \rangle \\ &= \langle \delta_z, \bar{\partial} J_1^{-1}(u(x)) \rangle = \langle [J_1 \circ \bar{\partial} \circ J_1^{-1}](u(x)), \delta_z \rangle = \langle (\bar{\partial}_1 \circ u)(x), \delta_z \rangle \\ &= \langle (\bar{\partial}_1^* u)(x), \delta_z \rangle = \psi^{-1}(g)(x)(\delta_z) = {}^t(\mathcal{J}^{-1} \circ R'_g)(x)(\delta_z) \\ &= (\mathcal{J}^{-1} \circ R'_g)(\delta_z)(x) = \mathcal{J}^{-1}(\mathcal{J}(g(z)))(x) = g(z)(x). \end{aligned}$$

Thus $\bar{\partial}^E(\psi(u))(z) = g(z)$ for every $z \in \Omega$, which proves the surjectivity.

(b) Let E be an ultrabornological PLS-space satisfying (PA). Since the nuclear Fréchet space $\mathcal{EV}_{\bar{\partial}}(\Omega)$ is also a Schwartz space, its strong dual $\mathcal{EV}_{\bar{\partial}}(\Omega)'_b$ is a DFS-space. By [6, Theorem 4.1, p. 577] we obtain $\text{Ext}^1_{PLS}(\mathcal{EV}_{\bar{\partial}}(\Omega)'_b, E) = 0$ as the bidual $\mathcal{EV}_{\bar{\partial}}(\Omega)''$ satisfies $(\Omega), E$ is a PLS-space satisfying (PA) and condition (c) in the theorem is fulfilled because $\mathcal{EV}_{\bar{\partial}}(\Omega)'_b$ is the strong dual of a nuclear Fréchet space. Moreover, we have $\text{Proj}^1 E = 0$ due to [66, Corollary 3.3.10, p. 46] because E is an ultrabornological PLS-space. Then the exactness of the sequence (5), [6, Theorem 3.4, p. 567] and [6, Lemma 3.3, p. 567] (in the lemma the same condition (c) as in [6, Theorem 4.1, p. 577] is fulfilled and we choose $H = \mathcal{EV}_{\bar{\partial}}(\Omega)''$ and $F = G = \mathcal{EV}(\Omega)''$), imply that the sequence

$$0 \rightarrow L(E'_b, \mathcal{EV}_{\bar{\partial}}(\Omega)'') \xrightarrow{i_0^*} L(E'_b, \mathcal{EV}(\Omega)'') \xrightarrow{\bar{\partial}_1^*} L(E'_b, \mathcal{EV}(\Omega)'') \rightarrow 0$$

is exact. The maps i_0^* and $\bar{\partial}_1^*$ are defined like in part (a). Especially, we get that

$$\bar{\partial}_1^*: L(E'_b, \mathcal{EV}(\Omega)'') \rightarrow L(E'_b, \mathcal{EV}(\Omega)'') \tag{7}$$

is surjective.

By [19, Remark 4.4, p. 1114] we have $L_b(\mathcal{E}\mathcal{V}(\Omega)'_b, E'') \cong L_b(E'_b, \mathcal{E}\mathcal{V}(\Omega)'')$ via taking adjoints since $\mathcal{E}\mathcal{V}(\Omega)$, being a Fréchet–Schwartz space, is a PLS-space and hence its strong dual an LFS-space, which is regular by [66, Corollary 6.7, 10. \Leftrightarrow 11., p. 114], and E is an ultrabornological PLS-space, in particular, reflexive by [17, Theorem 3.2, p. 58]. In addition, the map

$$T : L_b(\mathcal{E}\mathcal{V}(\Omega)'_b, E'') \rightarrow L_b(\mathcal{E}\mathcal{V}(\Omega)'_b, E),$$

defined by $T(u)(y) := \mathcal{J}^{-1}(u(y))$ for $u \in L_b(\mathcal{E}\mathcal{V}(\Omega)'_b, E'')$ and $y \in \mathcal{E}\mathcal{V}(\Omega)'$, is a topological isomorphism because E is reflexive. Due to Proposition 4 (b) we obtain the topological isomorphism

$$\begin{aligned} \psi &:= S \circ \mathcal{J}^{-1} \circ {}^t(\cdot) : L_b(E'_b, \mathcal{E}\mathcal{V}(\Omega)'') \rightarrow \mathcal{E}\mathcal{V}(\Omega, E), \\ \psi(u) &= [S \circ \mathcal{J}^{-1} \circ {}^t(\cdot)](u) = [z \mapsto \mathcal{J}^{-1}({}^t u(\delta_z))], \end{aligned}$$

with the inverse given by

$$\psi^{-1}(f) = (S \circ \mathcal{J}^{-1} \circ {}^t(\cdot))^{-1}(f) = [{}^t(\cdot) \circ \mathcal{J} \circ S^{-1}](f) = {}^t(\mathcal{J} \circ \mathcal{J}^{-1} \circ R'_f) = {}^t(R'_f)$$

for $f \in \mathcal{E}\mathcal{V}(\Omega, E)$.

Let $g \in \mathcal{E}\mathcal{V}(\Omega, E)$. Then $\psi^{-1}(g) \in L_b(E'_b, \mathcal{E}\mathcal{V}(\Omega)'')$ and by the surjectivity of (7) there exists $u \in L_b(E'_b, \mathcal{E}\mathcal{V}(\Omega)'')$ such that $\bar{\partial}_1^* u = \psi^{-1}(g)$. So we have $\psi(u) \in \mathcal{E}\mathcal{V}(\Omega, E)$. The last step is to show that $\bar{\partial}^E \psi(u) = g$. Like in part (a) we gain for every $z \in \Omega$

$$\bar{\partial}^E(\psi(u))(z) = \mathcal{J}^{-1}({}^t u(\delta_z \circ \bar{\partial}))$$

and for every $x \in E'$

$$\begin{aligned} {}^t u(\delta_z \circ \bar{\partial})(x) &= u(x)(\delta_z \circ \bar{\partial}) = (\bar{\partial}_1^* u)(x)(\delta_z) = \psi^{-1}(g)(x)(\delta_z) = {}^t(R'_g)(x)(\delta_z) \\ &= \delta_z(x \circ g) = x(g(z)) = \mathcal{J}(g(z))(x). \end{aligned}$$

Thus we have ${}^t u(\delta_z \circ \bar{\partial}) = \mathcal{J}(g(z))$ and therefore $\bar{\partial}^E(\psi(u))(z) = g(z)$ for all $z \in \Omega$. \square

By Remark 2 case (a) is included in case (b) if F is a Fréchet–Schwartz space. Therefore (a) is only interesting for Fréchet spaces F which are not Schwartz spaces. In the next more technical section we will present sufficient conditions for $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ to have (Ω) as well as concrete examples of such spaces.

4 (Ω) for $\mathcal{O}\mathcal{V}$ -spaces on strips and applications of the main result

In this section we give some sufficient conditions such that the assumptions of our main result Theorem 5 are fulfilled. The outline is as follows. First, we show that $\mathcal{O}\mathcal{V}(\Omega)$ and $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ coincide topologically under mild assumptions on the weights \mathcal{V} and the sequence of sets (Ω_n) . These mild conditions also imply that $\mathcal{E}\mathcal{V}(\Omega)$ is nuclear, in particular Schwartz, and thus its subspace $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega) = \mathcal{O}\mathcal{V}(\Omega)$ too. Second, we reduce the problem whether the projective limit $\mathcal{O}\mathcal{V}(\Omega)$ has (Ω) to the problem whether it is weakly reduced in the case that the Ω_n are strips along the real axis and the weights have a certain structure. Third, we use a similar result for $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ which was obtained in [33] to prove the weak reducibility of $\mathcal{O}\mathcal{V}(\Omega)$. For corresponding results in the case that $\Omega_n = \Omega$ for all $n \in \mathbb{N}$ see [20, Theorem 3, p. 56], [39, 1.3 Lemma, p. 418] and [58, Theorem 1, p. 145]. We close this section with some examples of our main result. Let us start with the sufficient conditions, guaranteeing

that the projective limit $\mathcal{E}\mathcal{V}(\Omega)$ is nuclear (if $q = 1$). They also allow to switch from sup- to weighted L^q -seminorms which is important for the proof of surjectivity of the scalar-valued $\bar{\partial}$ -operator given in [33], using Hörmander’s L^2 -machinery (if $q = 2$).

Condition (PN) ([33, 3.3 Condition, p. 7]) Let $\mathcal{V} := (v_n)_{n \in \mathbb{N}}$ be a directed family of continuous weights on an open set $\Omega \subset \mathbb{R}^2$ and $(\Omega_n)_{n \in \mathbb{N}}$ a family of non-empty open sets such that $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. For every $k \in \mathbb{N}$ let there be $\rho_k \in \mathbb{R}$ such that $0 < \rho_k < d^{\|\cdot\|_\infty}(\{x\}, \partial\Omega_{k+1})$ for all $x \in \Omega_k$ and let there be $q \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ there is $\psi_n \in L^q(\Omega_k)$, $\psi_n > 0$, and $\mathbb{N} \ni J_i(n) \geq n$ and $C_i(n) > 0$ such that for any $x \in \Omega_k$:

$$(PN.1) \quad \sup_{\zeta \in \mathbb{R}^2, \|\zeta\|_\infty \leq \rho_k} v_n(x + \zeta) \leq C_1(n) \inf_{\zeta \in \mathbb{R}^2, \|\zeta\|_\infty \leq \rho_k} v_{J_1(n)}(x + \zeta)$$

$$(PN.2)^q \quad v_n(x) \leq C_2(n) \psi_n(x) v_{J_2(n)}(x)$$

Example 6 Let $\Omega := \mathbb{R}^2$ and $\Omega_n := \{x = (x_i) \in \mathbb{R}^2 \mid |x_2| < n\}$. Let $0 < \gamma \leq 1$ and $(a_n)_{n \in \mathbb{N}}$ be strictly increasing such that $a_n \geq 0$ for all $n \in \mathbb{N}$ or $a_n \leq 0$ for all $n \in \mathbb{N}$. The family $\mathcal{V} := (v_n)_{n \in \mathbb{N}}$ of positive continuous functions on Ω given by

$$v_n : \Omega \rightarrow (0, \infty), \quad v_n(x) := e^{a_n|x_1|^\gamma},$$

fulfils $v_n \leq v_{n+1}$ all $n \in \mathbb{N}$ and (PN) for every $q \in \mathbb{N}$ with $\psi_n(x) := (1 + |x|^2)^{-2}$, $x \in \mathbb{R}^2$, for every $n \in \mathbb{N}$.

The space $\mathcal{O}\mathcal{V}(\mathbb{C})$ with this kind of weights consists of functions which are entire and exponentially growing ($a_n < 0$) resp. decreasing ($a_n > 0$) with order γ on strips along the real axis. This example of weights and many more are included in [33, 3.7 Example, p. 9]. We restrict to this particular weights because we use it in an example for our main result.

Proposition 7 Let $\mathcal{V} := (v_n)_{n \in \mathbb{N}}$ be a directed family of continuous weights on an open set $\Omega \subset \mathbb{R}^2$ and $(\Omega_n)_{n \in \mathbb{N}}$ a family of non-empty open sets such that $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. If (PN.1) is fulfilled, then

(a) for every $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ there is $C > 0$ such that

$$|f|_{n,m} \leq C |f|_{2J_1(n)}, \quad f \in \mathcal{O}v_{2J_1(n)}(\Omega_{2J_1(n)}).$$

(b) $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega) = \mathcal{O}\mathcal{V}(\Omega)$ as Fréchet spaces.

Proof (a) Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. We note that $\Omega_{n+1} \subset \Omega_{2J_1(n)}$ and $\partial^\beta f(x) = i^{|\beta|} f^{(|\beta|)}(x)$, $x \in \Omega_{2J_1(n)}$, holds for all $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$ and $f \in \mathcal{O}v_{2J_1(n)}(\Omega_{2J_1(n)})$ where $f^{(|\beta|)}$ is the $|\beta|$ th complex derivative of f . Then we obtain via (PN.1) and Cauchy’s inequality

$$\begin{aligned} |f|_{n,m} &= \sup_{\substack{x \in \Omega_n \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} |\partial^\beta f(x)| v_n(x) \leq \sup_{\substack{x \in \Omega_n \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \frac{|\beta|!}{\rho_n^{|\beta|}} \max_{\substack{\zeta \in \mathbb{R}^2 \\ |\zeta - x| = \rho_n}} |f(\zeta)| v_n(x) \\ &\stackrel{(PN.1)}{\leq} C_1 \sup_{\substack{x \in \Omega_n \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \frac{|\beta|!}{\rho_n^{|\beta|}} \max_{\substack{\zeta \in \mathbb{R}^2 \\ |\zeta - x| = \rho_n}} |f(\zeta)| v_{J_1(n)}(\zeta) \\ &\leq C_1 \sup_{\beta \in \mathbb{N}_0^2, |\beta| \leq m} \frac{|\beta|!}{\rho_n^{|\beta|}} \sup_{\zeta \in \Omega_{n+1}} |f(\zeta)| v_{J_1(n)}(\zeta) \leq C_1 \sup_{\beta \in \mathbb{N}_0^2, |\beta| \leq m} \frac{|\beta|!}{\rho_n^{|\beta|}} |f|_{2J_1(n)}. \end{aligned}$$

(b) The space $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ is a Fréchet space since it is a closed subspace of the Fréchet space $\mathcal{E}\mathcal{V}(\Omega)$ by [34, Proposition 3.7, p. 240]. From part (a) and $|f|_n = |f|_{n,0}$ for all $n \in \mathbb{N}$ and $f \in \mathcal{E}\mathcal{V}_{\bar{\partial}}(\Omega)$ follows the statement. \square

Let us come to the second part. Using special weight functions, strips along the real axis as Ω_n and a decomposition theorem of Langenbruch, we will see that answering the question whether $\mathcal{OV}(\Omega)$ satisfies the property (Ω) of Vogt boils down to answering whether the projective limit $\mathcal{OV}(\Omega)$ is weakly reduced. The special weights we want to consider are generated by a function μ with the following properties.

Definition 8 (strong weight generator) A continuous function $\mu: \mathbb{C} \rightarrow [0, \infty)$ is called a *weight generator* if $\mu(z) = \mu(|\operatorname{Re}(z)|)$ for all $z \in \mathbb{C}$, the restriction $\mu|_{[0, \infty)}$ is strictly increasing,

$$\lim_{\substack{x \rightarrow \infty \\ x \in \mathbb{R}}} \frac{\ln(1 + |x|)}{\mu(x)} = 0$$

and

$$\exists \Gamma > 1, C > 0 \forall x \in [0, \infty) : \mu(x + 1) \leq \Gamma \mu(x) + C.$$

If μ is a weight generator which fulfils the stronger condition

$$\exists \Gamma > 1 \forall n \in \mathbb{N} \exists C > 0 \forall x \in [0, \infty) : \mu(x + n) \leq \Gamma \mu(x) + C,$$

then μ is called a *strong weight generator*.

Weight generators are introduced in [46, Definition 2.1, p. 225] and strong weight generators in [60, Definition 2.2.2, p. 43] where they are simply called weight functions resp. strong weight functions. For a weight generator μ we define the space

$$H_\tau(S_t) := \{f \in \mathcal{O}(S_t) \mid \|f\|_{\tau,t} := \sup_{z \in S_t} |f(z)|e^{\tau\mu(z)} < \infty\}$$

for $t > 0$ and $\tau \in \mathbb{R}$ with the strip $S_t := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < t\}$.

Theorem 9 [46, Theorem 2.2, p. 225]¹ *Let μ be a weight generator. There are $\tilde{t}, K_1, K_2 > 0$ such that for any $\tau_0 < \tau < \tau_2$ there is $C_0 = C_0(\operatorname{sign}(\tau))$ such that for any $0 < 2t_0 < t < t_2 < \tilde{t}$ with*

$$t_0 \leq \min \left[K_1, K_2 \sqrt{\frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}} \right]$$

there is $C_1 \geq 1$ such that for any $r \geq 0$ and any $f \in H_\tau(S_t)$ with $\|f\|_{\tau,t} \leq 1$ the following holds: there are $f_2 \in \mathcal{O}(S_{t_2})$ and $f_0 \in \mathcal{O}(S_{t_0})$ such that $f = f_0 + f_2$ on S_{t_0} and

$$\|f_0\|_{C_0\tau_0,t_0} \leq C_1 e^{-Gr} \quad \text{and} \quad \|f_2\|_{\tau_2,t_2} \leq e^r$$

where

$$G := K_1 \min \left[1, \frac{t - t_0}{2\tilde{t}}, \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0} \right].$$

To apply this theorem, we have to know the constants involved. In the following the notation of [46] is used and it is referred to the corresponding positions resp. conditions for these constants. We have

$$\tilde{t} := \frac{1}{4 \ln(\Gamma)}$$

¹ A superfluous constant depending on $\operatorname{sign}(\tau_0)$ is omitted.

by [46, Lemma 2.4, (2.15), p. 228] with Γ from Definition 8 such that $\Gamma \geq e^{1/4}$. The choice $\Gamma \geq e^{1/4}$ comes from wanting $\tilde{t} \leq 1$ in [46, Lemma 2.4, p. 228]. By [46, Corollary 2.6, p. 230–231] we have

$$C_0 := \begin{cases} 4\Gamma B_3 = \frac{64 \cosh(1)}{\cos(1/2)} \Gamma^2 > 1 & , \tau < 0, \\ \frac{1}{4\Gamma B_3} = \frac{\cos(1/2)}{64 \cosh(1)\Gamma^2} < 1 & , \tau \geq 0, \end{cases}$$

where $B_3 := \frac{16 \cosh(1)}{\cos(1/2)} \Gamma$ by [46, Lemma 2.4, p. 228–229].² To get the constants K_1 and K_2 , we have to analyze the conditions for t_0 in the proof of [46, Theorem 2.2, p. 225]. By the assumptions on τ_0, τ and τ_2 and the choice of C_0 we obtain

$$\tau_2 - C_0\tau_0 > \tau_2 - C_0\tau \geq \tau_2 - \tau > 0 \tag{8}$$

and

$$\tau - C_0\tau_0 > \tau - C_0\tau = \tau(1 - C_0) > 0. \tag{9}$$

By choosing $D > 0$ in the proof of [46, Theorem 2.2, (2.22), p. 232–233] as $D := \frac{\tau - C_0\tau_0}{(\tau_2 - C_0\tau_0)2\Gamma_0}$, the estimate

$$D = \frac{\tau - C_0\tau_0}{(\tau_2 - C_0\tau_0)2\Gamma_0} = \min\left(\frac{1}{2\tilde{\Gamma}}, \frac{1}{2\hat{\Gamma}}\right) \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0} \stackrel{(8), (9)}{\leq} \min\left(\frac{1}{2\tilde{\Gamma}}, \frac{1}{2\hat{\Gamma}}\right) \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau}$$

holds where $\Gamma_0 := \max(\tilde{\Gamma}, \hat{\Gamma})$ with $\tilde{\Gamma}, \hat{\Gamma} > 1$ from the proof. With $\theta \geq \frac{t-t_0}{2\tilde{t}}$ (p. 232) we get on p. 233, below (2.24), due to the condition $t_0 \leq T_0 := \min(\frac{t}{2}, \frac{1}{4a^2 B_1 \tilde{t}})$,

$$\begin{aligned} \min\left(\frac{\theta}{2}, D, 1\right) &\geq \min\left(\frac{1}{2}, \frac{1}{2\Gamma_0}\right) \min\left(\theta, \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}, 1\right) \geq \frac{1}{2\Gamma_0} \min\left(\frac{t-t_0}{2\tilde{t}}, \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}, 1\right) \\ &\geq \min\left(\frac{1}{2\Gamma_0}, \frac{1}{4a^2 B_1 \tilde{t}}\right) \min\left(\frac{t-t_0}{2\tilde{t}}, \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}, 1\right) \\ &= \underbrace{\min\left(\frac{1}{2\Gamma_0}, \frac{1}{2 \cosh(1) \ln(\Gamma)}\right)}_{=:K_1} \min\left(\frac{t-t_0}{2\tilde{t}}, \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}, 1\right) =: G \end{aligned}$$

where $a := \ln(\Gamma)$ (in the middle of p. 231) and $B_1 := 2 \cosh(1)$ by the proof of [46, Lemma 2.3, p. 226–227]. The assumptions $2t_0 < t$ and $t_0 \leq K_1$ in Theorem 9 guarantee that the condition $t_0 \leq T_0$ is satisfied. Looking at the condition $t_0 \leq T_1 := \sqrt{\frac{D}{a^2 B_1}}$ (p. 232), we derive

$$T_1 = \frac{1}{\sqrt{2\Gamma_0 a^2 B_1}} \sqrt{\frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}} = \frac{1}{\underbrace{2\sqrt{\cosh(1)\Gamma_0 \ln(\Gamma)}}_{=:K_2}} \sqrt{\frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}}.$$

For the subsequent theorem we merge and modify the proofs of [60, Satz 2.2.3, p. 44]³ ($a_n = n, n \in \mathbb{N}$, and μ a strong weight generator) and [32, 5.20 Theorem, p. 84] ($a_n = -1/n, n \in \mathbb{N}$, and $\mu = |\operatorname{Re}(\cdot)|$).

² An error in part b) of this lemma, p. 229, is corrected here such that the term $\cos(1/2) = \min_{|y| \leq \tilde{t}=1/(2C_1)} \cos(C_1 y)$ appears.

³ The proof of [60, Satz 2.2.3, p. 44] relies on [60, Satz 2.2.1, p. 43] which is an announced version (without a proof) of our result Corollary 13 on weak reducibility.

Theorem 10 *Let μ be a strong weight generator, $a_n \nearrow 0$ or $a_n \nearrow \infty$, $\mathcal{V} := (\exp(a_n \mu))_{n \in \mathbb{N}}$ and $\Omega_n := S_n$ for all $n \in \mathbb{N}$. If $\mathcal{O}\mathcal{V}(\mathbb{C})$ is weakly reduced, then $\mathcal{O}\mathcal{V}(\mathbb{C})$ satisfies (Ω) .*

Proof Since $\mathcal{O}\mathcal{V}(\mathbb{C})$ is weakly reduced, for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $\pi_n(\mathcal{O}\mathcal{V}(\mathbb{C}))$ is dense in $\mathcal{O}v_{m_n}(\Omega_{m_n})$ w.r.t. the topology of $\mathcal{O}v_n(\Omega_n)$ where

$$\pi_n : \mathcal{O}\mathcal{V}(\mathbb{C}) \rightarrow \mathcal{O}v_n(\Omega_n), \quad \pi_n(f) := f|_{\Omega_n},$$

is the canonical projection. Let $p, k \in \mathbb{N}$. As $(a_n)_{n \in \mathbb{N}}$ is strictly increasing and $\lim_{n \rightarrow \infty} a_n = 0$ or $\lim_{n \rightarrow \infty} a_n = \infty$, we may choose $q \in \mathbb{N}$ such that $a_{m_p}/C_0 < a_q$ and $2m_p < q$. To use the decomposition from Theorem 9, we need a linear transformation between strips to get the decomposition on the desired strip S_{m_p} . We choose $\Gamma \geq e^{1/4}$ and $T \in \mathbb{R}$ such that

$$0 < T < \frac{1}{4 \max(q + 1, m_k) \ln(\Gamma)} \tag{10}$$

which also fulfils

$$T \leq \frac{1}{m_p} \min \left(\frac{1}{2\Gamma_0}, \frac{1}{2 \cosh(1) \ln(\Gamma)}, \frac{1}{2\sqrt{\cosh(1)\Gamma_0} \ln(\Gamma)} \sqrt{\frac{a_q - a_{m_p}}{\max(a_{q+1}, a_{m_k}) - a_{m_p}}} \right). \tag{11}$$

Let

$$\begin{aligned} \tau_0 &:= \frac{a_{m_p}}{C_0}, & \tau &:= a_q, & \tau_2 &:= \max(a_{q+1}, a_{m_k}), \\ t_0 &:= m_p T, & t &:= qT, & t_2 &:= \max(q + 1, m_k)T. \end{aligned}$$

By the choice of q we have

$$\tau_0 = \frac{a_{m_p}}{C_0} < a_q = \tau < \max(a_{q+1}, a_{m_k}) = \tau_2.$$

By the choice of q and (10) we get

$$0 < 2t_0 = 2m_p T < qT = t < \max(q + 1, m_k)T = t_2 < \frac{1}{4 \ln(\Gamma)} = \tilde{t}.$$

Further, we deduce from (11) that

$$t_0 = m_p T \leq \min \left[K_1, K_2 \sqrt{\frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}} \right].$$

Let $r \geq 0$ and $f \in \mathcal{O}\mathcal{V}(\mathbb{C})$ such that $\|f\|_q = \|f\|_{a_q, q} \leq 1$. We set $\tilde{f} : S_{qT} \rightarrow \mathbb{C}$, $\tilde{f}(z) := f(z/T)$, and define

$$H_\tau^\sim(S_r) := \{g \in \mathcal{O}(S_r) \mid \|g\|_{\tau, r}^\sim := \sup_{z \in S_r} |g(z)| e^{\tau \tilde{\mu}(z)} < \infty\}$$

where $\tilde{\mu} := \mu(\cdot/T)$. We note that for $\tilde{n} := \lceil 1/T \rceil$, where $\lceil \cdot \rceil$ is the ceiling function, there is $C > 0$ such that for all $x \geq 0$

$$\tilde{\mu}(x + 1) = \mu\left(\frac{x + 1}{T}\right) \leq \mu\left(\frac{x}{T} + \left\lceil \frac{1}{T} \right\rceil\right) = \mu\left(\frac{x}{T} + \tilde{n}\right) \leq \Gamma \mu\left(\frac{x}{T}\right) + C = \Gamma \tilde{\mu}(x) + C$$

because μ is a strong weight generator. We conclude that $\tilde{\mu}$ is also a weight generator with the same Γ as μ which is independent of T . Moreover, from

$$\|\tilde{f}\|_{\tau,t} = \sup_{z \in S_q} |\tilde{f}(z)|e^{a_q \tilde{\mu}(z)} = \sup_{z \in S_q} |f(z)|e^{a_q \mu(z)} = |f|_q \leq 1$$

it follows by Theorem 9 that there are $\tilde{f}_j \in \mathcal{O}(S_{t_j})$, $j \in \{0, 2\}$, such that

$$\tilde{f}(z) = \tilde{f}_0(z) + \tilde{f}_2(z), \quad z \in S_{t_0}, \tag{12}$$

and

$$C_1 e^{-Gr} \geq \|\tilde{f}_0\|_{C_0 \tau_0, t_0} = \sup_{z \in S_{t_0/T}} \underbrace{|\tilde{f}_0(Tz)|}_{=: f_0(z)} e^{C_0 \tau_0 \tilde{\mu}(Tz)} = \sup_{z \in S_{m_p}} |f_0(z)|e^{a_{m_p} \mu(z)} = |f_0|_{m_p}, \tag{13}$$

where $f_0 \in \mathcal{O}(S_{m_p})$, as well as

$$e^r \geq \|\tilde{f}_2\|_{\tau_2, t_2} = \sup_{z \in S_{t_2/T}} \underbrace{|\tilde{f}_2(Tz)|}_{=: f_2(z)} e^{\tau_2 \tilde{\mu}(Tz)} \geq \sup_{z \in S_{m_k}} |f_2(z)|e^{a_{m_k} \mu(z)} = |f_2|_{m_k} \tag{14}$$

where $f_2 \in \mathcal{O}(S_{t_2/T}) \subset \mathcal{O}(S_{m_k})$ and the inclusion is justified by the identity theorem. Furthermore, for $z \in S_{t_0/T} = S_{m_p}$ the equation

$$f(z) = \tilde{f}(Tz) \stackrel{(12)}{=} \tilde{f}_0(Tz) + \tilde{f}_2(Tz) = f_0(z) + f_2(z)$$

holds, thus $f = f_0 + f_2$ on S_{m_p} . By virtue of the weak reducibility of $\mathcal{O}\mathcal{V}(\mathbb{C})$ and the choice of m_p, m_k the following is valid:

$$\forall \varepsilon > 0 \exists \widehat{f}_0, \widehat{f}_2 \in \mathcal{O}\mathcal{V}(\mathbb{C}) : (i) \quad |\widehat{f}_0 - f_0|_p < \varepsilon \quad \text{and} \quad (ii) \quad |\widehat{f}_2 - f_2|_k < \varepsilon. \tag{15}$$

Now, we have to consider two cases. Let $\varepsilon := C_1 e^{-Gr}$. For $k \leq p$ we get via (15) (i) that $f = \widehat{f}_0 + (f_2 + f_0 - \widehat{f}_0)$ on S_{m_p} so

$$f_2 + f_0 - \widehat{f}_0 = f - \widehat{f}_0 =: \overline{f}_2 \quad \text{on } S_{m_p} \tag{16}$$

where the function $\overline{f}_2 \in \mathcal{O}\mathcal{V}(\mathbb{C})$ and thus is a holomorphic extension of the left-hand side on \mathbb{C} . Hence we clearly have $f = \widehat{f}_0 + \overline{f}_2$ and

$$|\widehat{f}_0|_p \leq |\widehat{f}_0 - f_0|_p + |f_0|_p \stackrel{(15)(i)}{\leq} \varepsilon + |f_0|_p \leq \varepsilon + |f_0|_{m_p} \stackrel{(13)}{\leq} 2C_1 e^{-Gr} =: C_2 e^{-Gr} \tag{17}$$

as well as

$$\begin{aligned} |\overline{f}_2|_k &\leq |\overline{f}_2 - f_2|_k + |f_2|_k \stackrel{(16), k \leq p}{\leq} |f_0 - \widehat{f}_0|_p + |f_2|_{m_k} \stackrel{(15)(i)}{\leq} \varepsilon + |f_2|_{m_k} \\ &\stackrel{(14)}{\leq} C_1 e^{-Gr} + e^r \leq (C_1 + 1)e^r =: C_3 e^r. \end{aligned} \tag{18}$$

Analogously, for $k > p$ we obtain via (15) (ii) that $f = \widehat{f}_2 + (f_0 + f_2 - \widehat{f}_2)$ on S_{m_p} so

$$f_0 + f_2 - \widehat{f}_2 = f - \widehat{f}_2 =: \overline{f}_0 \quad \text{on } S_{m_p} \tag{19}$$

where the function $\bar{f}_0 \in \mathcal{O}\mathcal{V}(\mathbb{C})$ and thus is a holomorphic extension of the left-hand side on \mathbb{C} . Hence we clearly have $f = \bar{f}_0 + \widehat{f}_2$ and

$$\begin{aligned} |\bar{f}_0|_p &= |f - \widehat{f}_2|_p \stackrel{(19)}{=} |f_0 + f_2 - \widehat{f}_2|_p \leq |f_2 - \widehat{f}_2|_p + |f_0|_p \stackrel{k > p}{\leq} |f_2 - \widehat{f}_2|_k + |f_0|_{m_p} \\ &\stackrel{(15)(ii)}{\leq} \varepsilon + |f_0|_{m_p} \stackrel{(13)}{\leq} 2C_1 e^{-Gr} = C_2 e^{-Gr} \end{aligned} \tag{20}$$

as well as

$$|\widehat{f}_2|_k \leq |\widehat{f}_2 - f_2|_k + |f_2|_k \stackrel{(15)(ii)}{\leq} \varepsilon + |f_2|_{m_k} \stackrel{(14)}{\leq} C_1 e^{-Gr} + e^r \leq C_3 e^r. \tag{21}$$

Next, we set $n := \lceil 1/G \rceil$ and $C := C_3 e^{\ln(C_2)/G}$. Let $\tilde{r} > 0$. For $\tilde{r} \geq 1$ there is $r \geq 0$ such that

$$\tilde{r} = e^{Gr - \ln(C_2)} = \frac{e^{Gr}}{C_2}$$

and we have by (17) and (18) for $k \leq p$

$$|\widehat{f}_0|_p \leq C_2 e^{-Gr} = \frac{1}{\tilde{r}}, \quad |\bar{f}_2|_k \leq C_3 e^r = C_3 e^{\frac{1}{G} \ln(C_2)} e^{\frac{1}{G}(Gr - \ln(C_2))} = C \tilde{r}^{\frac{1}{G}} \leq C \tilde{r}^n,$$

as well as by (20) and (21) for $k > p$

$$|\bar{f}_0|_p \leq \frac{1}{\tilde{r}}, \quad |\widehat{f}_2|_k \leq C \tilde{r}^n.$$

For $0 < \tilde{r} < 1$ we have, since $q \geq p$,

$$|f|_p \leq |f|_q \leq 1 < \frac{1}{\tilde{r}}.$$

Thus our statement is proved. □

Let us remark that the choice of the sequence $(a_n)_{n \in \mathbb{N}}$ in the preceding theorem does not really matter.

Remark 11 Let $\mu : \mathbb{C} \rightarrow [0, \infty)$ be continuous, $a_n \nearrow 0$ or $a_n \nearrow \infty$, $\mathcal{V} := (\exp(a_n \mu))_{n \in \mathbb{N}}$ and $\Omega_n := S_n$ for all $n \in \mathbb{N}$. Set $\mathcal{V}_- := (\exp((-1/n)\mu))_{n \in \mathbb{N}}$ and $\mathcal{V}_+ := (\exp(n\mu))_{n \in \mathbb{N}}$. Then

$$\mathcal{O}\mathcal{V}(\mathbb{C}) \cong \mathcal{O}\mathcal{V}_-(\mathbb{C}), \quad \text{if } a_n \nearrow 0, \quad \text{and } \mathcal{O}\mathcal{V}(\mathbb{C}) \cong \mathcal{O}\mathcal{V}_+(\mathbb{C}), \quad \text{if } a_n \nearrow \infty,$$

which is easily seen. Thus one may choose the most suitable sequence $(a_n)_{n \in \mathbb{N}}$ for one's purpose without changing the space.

Let us turn to the third part. The following quite technical conditions guarantee a kind of weak reducibility of the projective limit $\mathcal{E}\mathcal{V}(\Omega)$ and in combination with (PN.1) the weak reducibility of $\mathcal{O}\mathcal{V}(\Omega)$ too.

Condition (WR) Let $\mathcal{V} := (v_n)_{n \in \mathbb{N}}$ be a directed family of continuous weights on an open set $\Omega \subset \mathbb{R}^2$ and $(\Omega_n)_{n \in \mathbb{N}}$ a family of non-empty open sets such that $\Omega_n \neq \mathbb{R}^2$, $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$, $d_{n,k} := d^{|\cdot|}(\Omega_n, \partial\Omega_k) > 0$ for all $n, k \in \mathbb{N}$, $k > n$, and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$.

(WR.1) For every $n \in \mathbb{N}$ let there be $g_n \in \mathcal{O}(\mathbb{C})$ with $g_n(0) = 1$ and $\mathbb{N} \ni I_j(n) > n$ such that

(a) for every $\varepsilon > 0$ there is a compact set $K \subset \overline{\Omega}_n$ with $v_n(x) \leq \varepsilon v_{I_1(n)}(x)$ for all $x \in \Omega_n \setminus K$.

- (b) there is an open set $X_{I_2(n)} \subset \mathbb{R}^2 \setminus \overline{\Omega}_{I_2(n)}$ such that there are $R_n, r_n \in \mathbb{R}$ with $0 < 2R_n < d^{| \cdot |}(X_{I_2(n)}, \overline{\Omega}_{I_2(n)}) := d_{X, I_2(n)}$ and $R_n < r_n < d_{X, I_2(n)} - R_n$ as well as $A_2(\cdot, n): X_{I_2(n)} + \mathbb{B}_{R_n}(0) \rightarrow (0, \infty)$, $A_2(\cdot, n)|_{X_{I_2(n)}}$ locally bounded, satisfying

$$\max\{|g_n(\zeta)|v_{I_2(n)}(z) \mid \zeta \in \mathbb{R}^2, |\zeta - (z - x)| = r_n\} \leq A_2(x, n)$$

for all $z \in \Omega_{I_2(n)}$ and $x \in X_{I_2(n)} + \mathbb{B}_{R_n}(0)$.

- (c) for every compact set $K \subset \mathbb{R}^2$ there is $A_3(n, K) > 0$ with

$$\int_K \frac{|g_n(x - y)|v_n(x)}{|x - y|} dy \leq A_3(n, K), \quad x \in \Omega_n.$$

(WR.2) Let (WR.1a) be fulfilled. For every $n \in \mathbb{N}$ let there be $\mathbb{N} \ni I_4(n) > n$ and $A_4(n) > 0$ such that

$$\int_{\Omega_{I_4(n)}} \frac{|g_{I_4(n)}(x - y)|v_p(x)}{|x - y|v_k(y)} dy \leq A_4(n), \quad x \in \Omega_p,$$

for $(k, p) = (I_4(n), n)$ and $(k, p) = (I_{14}(n), I_{14}(n))$ where $I_{14}(n) := I_1(I_4(n))$.

(WR.3) Let (WR.1a), (WR.1b) and (WR.2) be fulfilled. For every $n \in \mathbb{N}$, every closed subset $M \subset \overline{\Omega}_n^C$ and every component N of M^C we have

$$N \cap \overline{\Omega}_n^C \neq \emptyset \Rightarrow N \cap X_{I_{214}(n)} \neq \emptyset$$

where $I_{214}(n) := I_2(I_{14}(n))$, fulfilling $I_{214}(n) \geq I_{14}(n + 1)$.

(WR) is [33, 4.2 Condition, p. 10] combined with the assumption $I_{214}(n) \geq I_{14}(n + 1)$, $n \in \mathbb{N}$. We will see that $\Omega_n := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < n\}$ and $v_n(z) := \exp(a_n |\operatorname{Re}(z)|^\gamma)$ for some $0 < \gamma \leq 1$ and $a_n \nearrow 0$ or $a_n \nearrow \infty$ fulfil the conditions above with $g_n(z) := \exp(-z^2)$.

Theorem 12 [33, 4.3 Theorem, p. 10] *Let $n \in \mathbb{N}$. Then $\pi_{I_{214}(n), n}(\mathcal{E}v_{I_{214}(n), \overline{\delta}}(\Omega_{I_{214}(n)}))$ is dense in $\pi_{I_{14}(n), n}(\mathcal{E}v_{I_{14}(n), \overline{\delta}}(\Omega_{I_{14}(n)}))$ w.r.t. $(|\cdot|_n, m)_{m \in \mathbb{N}_0}$ if (WR) is fulfilled.*

As a consequence of this theorem, whose proof does not need the assumption $I_{214}(n) \geq I_{14}(n + 1)$, we obtain that the projective limit $\mathcal{O}\mathcal{V}(\Omega)$ is weakly reduced, which is a generalisation of [32, 5.6 Corollary, p. 69] and [32, 5.11 Corollary, p. 75].

Corollary 13 $\mathcal{O}\mathcal{V}(\Omega)$ is weakly reduced if (WR) and (PN.1) are satisfied.

Proof Let $n \in \mathbb{N}$. We show that $\pi_n(\mathcal{O}\mathcal{V}(\Omega))$ is dense in $\pi_{2J_1 I_{14}(n), n}(\mathcal{O}v_{2J_1 I_{14}(n)}(\Omega_{2J_1 I_{14}(n)}))$ w.r.t. $|\cdot|_n$ where $J_1 I_{14}(n) := J_1(I_{14}(n))$ and

$$\pi_n : \mathcal{O}\mathcal{V}(\Omega) \rightarrow \mathcal{O}v_n(\Omega_n), \quad \pi_n(f) := f|_{\Omega_n}.$$

We omit the restriction maps in our proof. Due to Proposition 7 (a) the restrictions to $\Omega_{I_{14}(n)}$ of functions from $\mathcal{O}v_{2J_1 I_{14}(n)}(\Omega_{2J_1 I_{14}(n)})$ are elements of $\mathcal{E}v_{I_{14}(n), \overline{\delta}}(\Omega_{I_{14}(n)})$. Let $\varepsilon > 0$ and $f_0 \in \mathcal{O}v_{2J_1 I_{14}(n)}(\Omega_{2J_1 I_{14}(n)})$. For every $j \in \mathbb{N}$ there exists

- (i) $f_j \in \mathcal{E}v_{I_{214}(n+j-1), \overline{\delta}}(\Omega_{I_{214}(n+j-1)})$ with
- (ii) $f_j|_{\Omega_{I_{14}(n+j)}} \in \mathcal{E}v_{I_{14}(n+j), \overline{\delta}}(\Omega_{I_{14}(n+j)}) \subset \mathcal{O}v_{I_{14}(n+j)}(\Omega_{I_{14}(n+j)})$

such that

$$|f_j - f_{j-1}|_{n+j-1} = |f_j - f_{j-1}|_{n+j-1, 0} < \frac{\varepsilon}{2^{j+1}} \tag{22}$$

by Theorem 12 and the condition $I_{214}(k) \geq I_{14}(k + 1)$ for all $k \in \mathbb{N}$ from (WR). Therefore we obtain for every $k \in \mathbb{N}$

$$\begin{aligned}
 |f_k - f_0|_n &= \left| \sum_{j=1}^k f_j - f_{j-1} \right|_n \leq \sum_{j=1}^k |f_j - f_{j-1}|_n \leq \sum_{j=1}^k |f_j - f_{j-1}|_{n+j-1} \\
 &\stackrel{(22)}{\leq} \sum_{j=1}^k \frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2} \left(1 - \frac{1}{2^k} \right) < \frac{\varepsilon}{2}.
 \end{aligned} \tag{23}$$

Now, let $\varepsilon_0 > 0$ and $l \in \mathbb{N}$. We choose $l_0 \in \mathbb{N}$, $l_0 \geq l$, such that $\frac{\varepsilon}{2^{l_0+1}} < \varepsilon_0$. Similarly, we get for all $p \geq k \geq l_0$

$$\begin{aligned}
 |f_p - f_k|_l &\leq |f_p - f_k|_{l_0} = \left| \sum_{j=k+1}^p f_j - f_{j-1} \right|_{l_0} \leq \sum_{j=k+1}^p |f_j - f_{j-1}|_{l_0} \\
 &\stackrel{(22)}{\leq} \sum_{\substack{l_0 \leq k \leq j-1 \\ < n+j-1}}^p |f_j - f_{j-1}|_{n+j-1} \stackrel{(22)}{\leq} \sum_{j=k+1}^p \frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2} \left(\frac{1}{2^k} - \frac{1}{2^p} \right) < \frac{\varepsilon}{2^{k+1}} \leq \frac{\varepsilon}{2^{l_0+1}} < \varepsilon_0.
 \end{aligned}$$

Hence $(f_k)_{k \geq n_0}$ is a Cauchy sequence in the Banach space $\mathcal{O}v_{I_{14}(n+n_0)}(\mathcal{O}\Omega_{I_{14}(n+n_0)})$ for every $n_0 \in \mathbb{N}_0$ and thus has a limit $F_{n_0} \in \mathcal{O}v_{I_{14}(n+n_0)}(\mathcal{O}\Omega_{I_{14}(n+n_0)})$. These limits coincide on their common domain because for every $n_1, n_2 \in \mathbb{N}_0$ with $I_{14}(n+n_1) < I_{14}(n+n_2)$ and $\varepsilon_1 > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$\begin{aligned}
 |F_{n_1} - F_{n_2}|_{I_{14}(n+n_1)} &\leq |F_{n_1} - f_k|_{I_{14}(n+n_1)} + |f_k - F_{n_2}|_{I_{14}(n+n_1)} \\
 &\leq |F_{n_1} - f_k|_{I_{14}(n+n_1)} + |f_k - F_{n_2}|_{I_{14}(n+n_2)} < \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1.
 \end{aligned}$$

We deduce that the glued limit function f given by $f := F_{n_0}$ on $\Omega_{I_{14}(n+n_0)}$ for all $n_0 \in \mathbb{N}_0$ is well-defined and we have $f \in \bigcap_{n_0 \in \mathbb{N}_0} \mathcal{O}v_{I_{14}(n+n_0)}(\mathcal{O}\Omega_{I_{14}(n+n_0)}) = \mathcal{O}\mathcal{V}(\Omega)$ since $I_{14}(n+n_0) \geq n+n_0$. By the definition of f there exists $N \in \mathbb{N}$ such that for every $k \geq N$

$$|f - f_0|_n \leq |f - f_k|_n + |f_k - f_0|_n \stackrel{(22)}{\leq} \frac{\varepsilon}{2} + |f_k - f_0|_n \stackrel{(23)}{\leq} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves our statement. □

Combining Theorem 10 and Corollary 13, we obtain the following corollary.

Corollary 14 *Let $a_n \nearrow 0$ or $a_n \nearrow \infty$, $\mathcal{V} := (\exp(a_n \mu))_{n \in \mathbb{N}}$ and $\Omega_n := S_n$ for all $n \in \mathbb{N}$ where*

$$\mu: \mathbb{C} \rightarrow [0, \infty), \quad \mu(z) := |\operatorname{Re}(z)|^\gamma,$$

for some $0 < \gamma \leq 1$. Then $\mathcal{O}\mathcal{V}(\mathbb{C})$ satisfies (Ω) .

Proof We only need to check that the conditions of Theorem 10 are fulfilled. Obviously, $\mu(z) = \mu(|\operatorname{Re}(z)|)$ for all $z \in \mathbb{C}$, μ is strictly increasing on $[0, \infty)$ and $\lim_{x \rightarrow \infty, x \in \mathbb{R}} \frac{\ln(1+|x|)}{\mu(x)} = 0$. The observation

$$\mu(x+n) - \mu(x) = |x+n|^\gamma - |x|^\gamma \leq |x+n-n|^\gamma = n^\gamma, \quad n \in \mathbb{N}, \quad x \in [0, \infty),$$

implies that μ is a strong weight generator with any $\Gamma > 1$ and $C := n^\gamma$ by Definition 8. Let us turn to the conditions (WR) and (PN.1) which we need for the weak reducibility of $\mathcal{O}\mathcal{V}(\mathbb{C})$ by Corollary 13. Condition (PN.1) is fulfilled by Example 6. If $a_n < 0$ for all

$n \in \mathbb{N}$, then (WR) is fulfilled by [33, 4.10 Example a), p. 22] where we used $\tilde{\mu}(z) := |z|^\gamma$ instead of μ , which does not make a difference since

$$|\operatorname{Re}(z)|^\gamma \leq |z|^\gamma \leq |\operatorname{Re}(z)|^\gamma + n^\gamma, \quad z \in \Omega_n = S_n.$$

If $a_n \geq 0$ for all $n \in \mathbb{N}$, we only have to modify [33, 4.10 Example a), p. 22] a bit. We choose $I_j(n) := 2n$ for $j \in \{1, 2, 4\}$ and define the open set $X_{I_2(n)} := \overset{C}{S}_{4n}$. Then we have

$$I_{214}(n) = 8n \geq 4n + 4 = I_{14}(n + 1), \quad n \in \mathbb{N}.$$

Furthermore, we have $d_{n,k} = |n - k|$ for all $n, k \in \mathbb{N}$.

(WR.1a) and (WR.3): Verbatim as in [33, 4.10 Example a), p. 22].

(WR.1b): We have $d_{X,I_2} = 2n$. We choose $g_n : \mathbb{C} \rightarrow \mathbb{C}$, $g_n(z) := \exp(-z^2)$, as well as $r_n := 1/(4n)$ and $R_n := 1/(6n)$ for $n \in \mathbb{N}$. Let $z = z_1 + iz_2 \in \Omega_{I_2(n)} = S_{2n}$ and $x \in X_{I_2(n)} + \mathbb{B}_{R_n}(0)$. For $\zeta = \zeta_1 + i\zeta_2 \in \mathbb{C}$ with $|\zeta - (z - x)| = r_n$ we have

$$\begin{aligned} |g_n(\zeta)|e^{a_{2n}\mu(z)} &= e^{-\operatorname{Re}(\zeta^2)}e^{a_{2n}|\operatorname{Re}(z)|^\gamma} \leq e^{-\zeta_1^2 + \zeta_2^2}e^{a_{2n}(1+|z_1|)} \\ &\leq e^{(r_n+|z_2|+|x_2|)^2+a_{2n}(1+r_n+|x_1|)}e^{-|\zeta_1|^2+a_{2n}|\zeta_1|} \\ &\leq e^{(r_n+2n+|x_2|)^2+a_{2n}(1+r_n+|x_1|)} \sup_{t \in \mathbb{R}} e^{-t^2+a_{2n}t} \\ &= e^{(r_n+2n+|x_2|)^2+a_{2n}(1+r_n+|x_1|)+a_{2n}^2/4} =: A_2(x, n) \end{aligned}$$

and observe that $A_2(\cdot, n)$ is continuous and thus locally bounded on $X_{I_2(n)}$.

(WR.1c): Let $K \subset \mathbb{C}$ be compact and $x = x_1 + ix_2 \in \Omega_n$. Then there is $b > 0$ such that $|y| \leq b$ for all $y = y_1 + iy_2 \in K$ and from polar coordinates and Fubini’s theorem it follows that

$$\begin{aligned} &\int_K \frac{|g_n(x - y)|}{|x - y|} dy \\ &\leq \underbrace{\sup_{w \in K} e^{a_{2n}|\operatorname{Re}(w)|}}_{=: C_1} \int_K \frac{e^{-\operatorname{Re}((x-y)^2)}}{|x - y|} e^{-a_{2n}|y_1|} dy \\ &\leq C_1 \left(\int_{\mathbb{B}_1(x)} \frac{e^{-\operatorname{Re}((x-y)^2)}}{|x - y|} e^{-a_{2n}|\operatorname{Re}(y)|} dy + \int_{K \setminus \mathbb{B}_1(x)} \frac{e^{-\operatorname{Re}((x-y)^2)}}{|x - y|} e^{-a_{2n}|\operatorname{Re}(y)|} dy \right) \\ &\leq C_1 \left(\int_0^{2\pi} \int_0^1 \frac{e^{-r^2 \cos(2\varphi)}}{r} e^{-a_{2n}|x_1+r \cos(\varphi)|} r dr d\varphi + \int_{K \setminus \mathbb{B}_1(x)} e^{-\operatorname{Re}((x-y)^2)} e^{-a_{2n}|\operatorname{Re}(y)|} dy \right) \\ &\leq C_1 (2\pi e^{1+a_{2n}} e^{-a_{2n}|x_1|} + \int_{-b}^b e^{(x_2-y_2)^2} dy_2 \int_{\mathbb{R}} e^{-(x_1-y_1)^2+a_{2n}|x_1-y_1|} dy_1 e^{-a_{2n}|x_1|}) \\ &\leq C_1 (2\pi e^{1+a_{2n}} + 2be^{(|x_2|+b)^2} \int_{\mathbb{R}} e^{-y_1^2+a_{2n}|y_1|} dy_1) e^{-a_{2n}|x_1|} \\ &= C_1 (2\pi e^{1+a_{2n}} + 2be^{(|x_2|+b)^2} e^{a_{2n}^2/4} \int_{\mathbb{R}} e^{-(|y_1|-a_{2n}/2)^2} dy_1) e^{-a_{2n}|x_1|} \\ &= C_1 (2\pi e^{1+a_{2n}} + 4be^{(|x_2|+b)^2} e^{a_{2n}^2/4} \int_{-a_{2n}/2}^\infty e^{-y_1^2} dy_1) e^{-a_{2n}|x_1|} \\ &\leq C_1 (2\pi e^{1+a_{2n}} + 4\sqrt{\pi} b e^{(n+b)^2+a_{2n}^2/4}) e^{-a_{2n}|x_1|}. \end{aligned}$$

We conclude that (WR.1c) holds since

$$e^{-a_{2n}|x_1|} e^{a_n |\operatorname{Re}(x)|^\gamma} \leq e^{(a_n - a_{2n})|x_1| + a_n} \leq e^{a_n}.$$

(WR.2): Let $p, k \in \mathbb{N}$ with $p \leq k$. For all $x = x_1 + ix_2 \in \Omega_p$ and $y = y_1 + iy_2 \in \Omega_{I_4(n)}$ we note that

$$a_p |\operatorname{Re}(x)|^\gamma - a_k |\operatorname{Re}(y)|^\gamma \leq a_k |x_1 - y_1|^\gamma \leq a_k (1 + |x_1 - y_1|)$$

because $(a_n)_{n \in \mathbb{N}}$ is non-negative and increasing and $0 < \gamma \leq 1$. Like before we deduce that

$$\begin{aligned} & \int_{\Omega_{I_4(n)}} \frac{|g_n(x-y)| v_p(x)}{|x-y| v_k(y)} dy \\ &= \int_{\Omega_{2n}} \frac{e^{-\operatorname{Re}((x-y)^2)}}{|x-y|} e^{a_p |\operatorname{Re}(x)|^\gamma - a_k |\operatorname{Re}(y)|^\gamma} dy \leq \int_{\Omega_{2n}} \frac{e^{-\operatorname{Re}((x-y)^2)}}{|x-y|} e^{a_k |\operatorname{Re}(x) - \operatorname{Re}(y)|^\gamma} dy \\ &\leq \int_0^{2\pi} \int_0^1 \frac{e^{-r^2 \cos(2\varphi)}}{r} e^{a_k r^\gamma} r dr d\varphi + \int_{\Omega_{2n} \setminus \mathbb{B}_1(x)} e^{-\operatorname{Re}((x-y)^2)} e^{a_k |\operatorname{Re}(x) - \operatorname{Re}(y)|^\gamma} dy \\ &\leq 2\pi e^{1+a_k} + e^{a_k} \int_{-2n}^{2n} e^{(x_2-y_2)^2} dy_2 \int_{\mathbb{R}} e^{-(x_1-y_1)^2 + a_k |x_1-y_1|} dy_1 \\ &\leq 2\pi e^{1+a_k} + 8\sqrt{\pi} n e^{a_k + (|x_2|+2n)^2 + a_k^2/4} \\ &\leq 2\pi e^{1+a_{I_4(n)}} + 8\sqrt{\pi} n e^{a_{I_4(n)} + (I_{14}(n)+2n)^2 + a_{I_4(n)}^2/4} \end{aligned}$$

for $(k, p) = (I_4(n), n)$ and $(k, p) = (I_{14}(n), I_{14}(n))$ as $(a_n)_{n \in \mathbb{N}}$ is non-negative and increasing. □

We close this section with a special case of our main result on the surjectivity of the Cauchy–Riemann operator on $\mathcal{E}\mathcal{V}(\Omega, E)$. We recall the corresponding result for $E = \mathbb{C}$ which we will need for the application of our main result. It is a consequence of the approximation Theorem 12 in combination with Hörmander’s solution of the $\bar{\partial}$ -problem in weighted L^2 -spaces [27, Theorem 4.4.2, p. 94] and the Mittag–Leffler procedure.

Theorem 15 [33, 4.8 Theorem, p. 20] *Let (PN) with $\psi_n(z) := (1 + |z|^2)^{-2}$, $z \in \Omega$, and (WR) be fulfilled and $-\ln v_n$ be subharmonic on Ω for every $n \in \mathbb{N}$. Then*

$$\bar{\partial}: \mathcal{E}\mathcal{V}(\Omega) \rightarrow \mathcal{E}\mathcal{V}(\Omega)$$

is surjective.

A consequence of this theorem is the following corollary.

Corollary 16 [33, 4.10 Example a), p. 22] *Let $(a_n)_{n \in \mathbb{N}}$ be strictly increasing, $a_n < 0$ for all $n \in \mathbb{N}$, $\mathcal{V} := (\exp(a_n \mu))_{n \in \mathbb{N}}$ and $\Omega_n := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < n\}$ for all $n \in \mathbb{N}$ where*

$$\mu: \mathbb{C} \rightarrow [0, \infty), \mu(z) := |\operatorname{Re}(z)|^\gamma,$$

for some $0 < \gamma \leq 1$. Then

$$\bar{\partial}: \mathcal{E}\mathcal{V}(\mathbb{C}) \rightarrow \mathcal{E}\mathcal{V}(\mathbb{C})$$

is surjective.

The restriction to negative a_n comes from the condition that $-\ln v_n$ should be subharmonic. We note that the E -valued versions of Theorem 15 and Corollary 16 where E is a Fréchet space over \mathbb{C} hold as well by the classical theory of tensor products for nuclear Fréchet spaces (see [33, 4.9 Corollary, p. 21]). Now, we use the results obtained so far to obtain a special case of our main result.

Corollary 17 *Let μ be a subharmonic strong weight generator and $\mathcal{V} := (\exp(a_n \mu))_{n \in \mathbb{N}}$ with $a_n \nearrow 0$. Let (PN) with $\psi_n(z) := (1 + |z|^2)^{-2}$, $z \in \mathbb{C}$, and (WR) with $\Omega_n := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < n\}$ for all $n \in \mathbb{N}$ be fulfilled. If*

- (a) $E := F'_b$ where F is a Fréchet space over \mathbb{C} satisfying (DN) , or
- (b) E is an ultrabornological PLS-space over \mathbb{C} satisfying (PA) ,

then

$$\bar{\partial}^E : \mathcal{E}\mathcal{V}(\mathbb{C}, E) \rightarrow \mathcal{E}\mathcal{V}(\mathbb{C}, E)$$

is surjective.

Proof The space $\mathcal{E}\mathcal{V}(\mathbb{C})$ is nuclear, in particular Schwartz, by [37, Theorem 3.1, p. 188], [37, Remark 2.7, p. 178-179] and [37, Remark 2.3 (b), p. 177] because $(PN.1)$ and $(PN.2)^1$ from (PN) are fulfilled. Hence the subspace $\mathcal{E}\mathcal{V}_{\bar{\partial}}(\mathbb{C}) = \mathcal{O}\mathcal{V}(\mathbb{C})$ is nuclear by Proposition 7 (b) as well. Further, $\mathcal{O}\mathcal{V}(\mathbb{C})$ is weakly reduced by Corollary 13 due to (WR) and thus satisfies (\mathcal{Q}) by Theorem 10. Therefore, the assertion is a consequence of the surjectivity of $\bar{\partial}$ in the \mathbb{C} -valued case by Theorem 15 and our main result Theorem 5. □

Corollary 17 generalises a part of [32, 5.24 Theorem, p. 95] ($K = \emptyset$) which is the case $\gamma = 1$ of the next corollary.

Corollary 18 *Let $a_n \nearrow 0$, $\mathcal{V} := (\exp(a_n \mu))_{n \in \mathbb{N}}$ and $\Omega_n := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < n\}$ for all $n \in \mathbb{N}$ where*

$$\mu : \mathbb{C} \rightarrow [0, \infty), \mu(z) := |\operatorname{Re}(z)|^\gamma,$$

for some $0 < \gamma \leq 1$. If

- (a) $E := F'_b$ where F is a Fréchet space over \mathbb{C} satisfying (DN) , or
- (b) E is an ultrabornological PLS-space over \mathbb{C} satisfying (PA) ,

then

$$\bar{\partial}^E : \mathcal{E}\mathcal{V}(\mathbb{C}, E) \rightarrow \mathcal{E}\mathcal{V}(\mathbb{C}, E)$$

is surjective.

Proof Follows from Corollary 17, (the proof of) Corollary 14 and Example 6. □

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References

1. Agranovich, M.S.: Partial differential equations with constant coefficients. *Rus. Math. Surv.* **16**(2), 23 (1961). <https://doi.org/10.1070/rm1961v016n02abeh004104>
2. Bierstedt, K.D., Pietsch, A., Ruess, W.M., Vogt, D. (eds.): *Functional Analysis (Proc., Essen, 1991)*, Lect. Notes in Pure and Appl. Math. 150. Dekker, New York (1994)
3. Bonet, J., Domański, P.: Parameter dependence of solutions of partial differential equations in spaces of real analytic functions. *Proc. Am. Math. Soc.* **129**(2), 495–503 (2001). <https://doi.org/10.1090/S0002-9939-00-05867-6>
4. Bonet, J., Domański, P.: Parameter dependence of solutions of differential equations on spaces of distributions and the splitting of short exact sequences. *J. Funct. Anal.* **230**(2), 329–381 (2006). <https://doi.org/10.1016/j.jfa.2005.06.007>
5. Bonet, J., Domański, P.: The structure of spaces of quasianalytic functions of Roumieu type. *Arch. Math. (Basel)* **89**(5), 430–441 (2007). <https://doi.org/10.1007/s00013-007-2073-y>
6. Bonet, J., Domański, P.: The splitting of exact sequences of PLS-spaces and smooth dependence of solutions of linear partial differential equations. *Adv. Math.* **217**, 561–585 (2008). <https://doi.org/10.1016/j.aim.2007.07.010>
7. Bonet, J., Galbis, A., Meise, R.: On the range of convolution operators on non-quasianalytic ultradifferentiable functions. *Studia Math.* **126**(2), 171–198 (1997). <https://doi.org/10.4064/sm-126-2-171-198>
8. Braun, R.W.: A partial differential operator which is surjective on Gevrey classes $\Gamma^d(\mathbb{R}^3)$ with $1 \leq d < 2$ and $d \geq 6$ but not for $2 \leq d < 6$. *Studia Math.* **107**(2), 157–169 (1993). <https://doi.org/10.4064/sm-107-2-157-169>
9. Braun, R.W., Meise, R., Taylor, B.A.: Surjectivity of constant coefficient partial differential operators on $\mathcal{A}(\mathbb{R}^4)$ and Whitney's C_4 -cone. *Bull. Soc. R. Sci. Liège* **70**(4–6), 195–206 (2001)
10. Braun, R.W., Meise, R., Vogt, D.: Applications of the projective limit functor to convolution and partial differential equations. In: T. Terzioğlu (ed.) *Advances in the Theory of Fréchet Spaces (Proc., Istanbul, 1988)*, *NATO Sci. Ser. C Math. Phys. Sci.*, vol. 287, pp. 29–46. Kluwer, Dordrecht (1989). https://doi.org/10.1007/978-94-009-2456-7_4
11. Braun, R.W., Meise, R., Vogt, D.: Existence of fundamental solutions and surjectivity of convolution operators on classes of ultra-differentiable functions. *Proc. Lond. Math. Soc.* (3) **61**(2), 344–370 (1990). <https://doi.org/10.1112/plms/s3-61.2.344>
12. Braun, R.W., Meise, R., Vogt, D.: Characterization of the linear partial differential operators with constant coefficients which are surjective on non-quasianalytic classes of Roumieu type on \mathbb{R}^N . *Math. Nachr.* **168**(1), 19–54 (1994). <https://doi.org/10.1002/mana.19941680103>
13. Browder, F.E.: Analyticity and partial differential equations. I. *Am. J. Math.* **84**(4), 666–710 (1962). <https://doi.org/10.2307/2372872>
14. Cattabriga, L.: Solutions in Gevrey spaces of partial differential equations with constant coefficients. In: *Analytic solutions of partial differential equations (Proc., Trento, 1981)*, *Astérisque*, vol. 89–90, pp. 129–151. Soc. Math. France, Paris (1981)
15. Dierolf, B.: Splitting theory for PLH spaces. Ph.D. thesis, Universität Trier, Trier (2014). <https://doi.org/10.25353/ubtr-xxxx-4b2b-53a5>
16. Dierolf, B., Sieg, D.: Splitting and parameter dependence in the category of PLH spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.)* **113**, 59–93 (2019). <https://doi.org/10.1007/s13398-017-0424-5>
17. Domański, P.: Classical PLS-spaces: spaces of distributions, real analytic functions and their relatives. In: Z. Ciesielski, A. Pełczyński, L. Skrzypczak (eds.) *Orlicz Centenary Volume, Banach Center Publications*, vol. 64, pp. 51–70. Polish Acad. Sci., Warsaw (2004). <https://doi.org/10.4064/bc64-0-5>
18. Domański, P.: Real analytic parameter dependence of solutions of differential equations over Roumieu classes. *Funct. Approx. Comment. Math.* **44**(1), 79–109 (2011). <https://doi.org/10.7169/facm/1301497748>

19. Domański, P., Langenbruch, M.: Vector valued hyperfunctions and boundary values of vector valued harmonic and holomorphic functions. Publ. RIMS Kyoto Univ. **44**, 1097–1142 (2008)
20. Epifanov, O.V.: On solvability of the nonhomogeneous Cauchy–Riemann equation in classes of functions that are bounded with weights or systems of weights. Math. Notes **51**(1), 54–60 (1992). <https://doi.org/10.1007/BF01229435>
21. Farkas, H., Kawai, T., Kuchment, P., Quinto, T.E., Sternberg, S., Struppa, D., Taylor, B.A.: Remembering leon ehrenpreis (1930–2010). Not. Am. Math. Soc. **58**(5), 674–681 (2011)
22. Franken, U., Meise, R.: Generalized Fourier expansions for zero-solutions of surjective convolution operators on $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}'_w(\mathbb{R})$. Note Mat. **X**(1), 251–272 (1990). <https://doi.org/10.1285/i15900932v10supn1p251>
23. Frerick, L., Kalmes, T.: Some results on surjectivity of augmented semi-elliptic differential operators. Math. Ann. **347**(1), 81–94 (2010). <https://doi.org/10.1007/s00208-009-0418-5>
24. Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires, 4th edn. Mem. Am. Math. Soc. 16. AMS, Providence (1966). <https://doi.org/10.1090/memo/0016>
25. Hermanns, V.: Zur Existenz von Rechtsinversen linearer partieller Differentialoperatoren mit konstanten Koeffizienten auf $B_{p,\kappa}^{loc}(\Omega)$ -Räumen. Ph.D. thesis, Universität Wuppertal, Wuppertal (2005)
26. Hörmander, L.: On the existence of real analytic solutions of partial differential equations with constant coefficients. Invent. Math. **21**, 151–182 (1973). <https://doi.org/10.1007/BF01390194>
27. Hörmander, L.: An introduction to complex analysis in several variables, 3rd edn. North-Holland, Amsterdam (1990)
28. Hörmander, L.: The analysis of linear partial differential operators II. Classics Math. Springer, Berlin (2005). <https://doi.org/10.1007/b138375>
29. Kabbalo, W.: Aufbaukurs funktionalanalysis und operatortheorie. Springer, Berlin (2014). <https://doi.org/10.1007/978-3-642-37794-5>
30. Kalmes, T.: Surjectivity of differential operators and linear topological invariants for spaces of zero solutions. Rev. Mat. Complut. **32**, 37–55 (2019). <https://doi.org/10.1007/s13163-018-0266-5>
31. Köthe, G.: Topological vector spaces II Grundlehren Math. Wiss., vol. 237. Springer, Berlin (1979). <https://doi.org/10.1007/978-1-4684-9409-9>
32. Kruse, K.: Vector-valued fourier hyperfunctions. Ph.D. thesis, Universität Oldenburg, Oldenburg (2014)
33. Kruse, K.: Surjectivity of the $\bar{\partial}$ -operator between spaces of weighted smooth vector-valued functions (2018). Arxiv preprint <https://arxiv.org/abs/1810.05069v1>
34. Kruse, K.: The approximation property for weighted spaces of differentiable functions. In: M. Kosek (ed.) Function Spaces XII (Proc., Kraków, 2018), *Banach Center Publ.*, vol. 119, pp. 233–258. Inst. Math., Polish Acad. Sci., Warszawa (2019). <https://doi.org/10.4064/bc119-14>
35. Kruse, K.: The inhomogeneous Cauchy–Riemann equation for weighted smooth vector-valued functions on strips with holes (2019). Arxiv preprint <https://arxiv.org/abs/1901.02093v2>
36. Kruse, K.: Weighted vector-valued functions and the ε -product (2019). Arxiv preprint <https://arxiv.org/abs/1712.01613v6>
37. Kruse, K.: On the nuclearity of weighted spaces of smooth functions. Ann. Polon. Math. **124**(2), 173–196 (2020). <https://doi.org/10.4064/ap190728-17-11>
38. Langenbruch, M.: Real roots of polynomials and right inverses for partial differential operators in the space of tempered distributions. Proc. R. Soc. Edinburgh Sect. A **114**(3–4), 169–179 (1990). <https://doi.org/10.1017/S0308210500024367>
39. Langenbruch, M.: Differentiable functions and the $\bar{\partial}$ -complex. In: Bierstedt et al. [2], pp. 415–434
40. Langenbruch, M.: Continuous linear right inverses for differential operators. In: D. Przeworska-Rolewicz (ed.) Different Aspects of Differentiability (Proc., Warsaw, 1993), *Dissertationes Math.*, vol. 340, pp. 163–181. Polish Acad. Sci., Warsaw (1995)
41. Langenbruch, M.: Surjective partial differential operators on spaces of ultradifferentiable functions of Roumieu type. Results Math. **29**(3), 254–275 (1996). <https://doi.org/10.1007/BF033222300>
42. Langenbruch, M.: Surjectivity of partial differential operators on Gevrey classes and extension of regularity. Math. Nachr. **196**(1), 103–140 (1998). <https://doi.org/10.1002/mana.19981960106>
43. Langenbruch, M.: Inheritance of surjectivity for partial differential operators on spaces of real analytic functions. J. Math. Anal. Appl. **297**(2), 696–719 (2004). <https://doi.org/10.1016/j.jmaa.2004.04.035>
44. Langenbruch, M.: Right inverses for partial differential operators on Fourier hyperfunctions. Studia Math. **183**(3), 273–299 (2007). <https://doi.org/10.4064/sm183-3-5>
45. Langenbruch, M.: Right inverses for differential operators on Fourier ultra-hyperfunctions and the property (DN). In: A. Aytuna, R. Meise, T. Terzioğlu, D. Vogt (eds.) Functional Analysis and Complex Analysis (Proc., Istanbul, 2007), *Contemp. Math.*, vol. 481, pp. 81–104. AMS, Providence (2009). <https://doi.org/10.1090/conm/481>

46. Langenbruch, M.: Bases in spaces of analytic germs. *Ann. Polon. Math.* **106**, 223–242 (2012). <https://doi.org/10.4064/ap106-0-18>
47. Larcher, J.: Surjectivity of differential operators and the division problem in certain function and distribution spaces. *J. Math. Anal. Appl.* **409**(1), 91–99 (2014). <https://doi.org/10.1016/j.jmaa.2013.05.078>
48. Malgrange, B.: Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. *Ann. Inst. Fourier (Grenoble)* **6**, 271–355 (1956). <https://doi.org/10.5802/aif.65>
49. Mantlik, F.: Linear equations depending differentiably on a parameter. *Integral Equ. Oper. Theory* **13**(2), 231–250 (1990). <https://doi.org/10.1007/BF01193758>
50. Mantlik, F.: Partial differential operators depending analytically on a parameter. *Ann. Inst. Fourier (Grenoble)* **41**(3), 577–599 (1991). <https://doi.org/10.5802/aif.1266>
51. Meise, R., Taylor, B.A., Vogt, D.: Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse. *Ann. Inst. Fourier (Grenoble)* **40**(3), 619–655 (1990). <https://doi.org/10.5802/aif.1226>
52. Meise, R., Taylor, B.A., Vogt, D.: Continuous linear right inverses for partial differential operators with constant coefficients and Phragmén-Lindelöf conditions. In: Bierstedt et al. [2], pp. 357–389
53. Meise, R., Taylor, B.A., Vogt, D.: Continuous linear right inverses for partial differential operators on non-quasianalytic classes and on ultradistributions. *Math. Nachr.* **180**(1), 213–242 (1996). <https://doi.org/10.1002/mana.3211800110>
54. Meise, R., Vogt, D.: Introduction to functional analysis. *Oxf. Grad. Texts Math*, vol. 2. Clarendon Press, Oxford (1997)
55. Meyer, T.: Surjectivity of convolution operators on spaces of ultradifferentiable functions of Roumieu type. *Studia Math.* **125**(2), 101–129 (1997). <https://doi.org/10.4064/sm-125-2-101-129>
56. Michael, E.: Continuous selections. I. *Ann. Math. (2)* **63**(2), 361–382 (1956). <https://doi.org/10.2307/1969615>
57. Palamodov, V.P.: Homological methods in the theory of locally convex spaces. *Rus. Math. Surv.* **26**, 1–64 (1971). <https://doi.org/10.1070/RM1971v026n01ABEH003815>
58. Polyakova, D.A.: Solvability of the inhomogeneous Cauchy–Riemann equation in projective weighted spaces. *Sib. Math. J.* **58**(1), 142–152 (2017). <https://doi.org/10.1134/S0037446617010189>
59. Schwartz, L.: Théorie des distributions à valeurs vectorielles. I. *Ann. Inst. Fourier (Grenoble)* **7**, 1–142 (1957). <https://doi.org/10.5802/aif.68>
60. Tönjes, K.: Linear topologische Invarianten für Räume holomorpher Funktionen. Master’s thesis, Universität Oldenburg, Oldenburg (2012)
61. Trèves, F.: Fundamental solutions of linear partial differential equations with constant coefficients depending on parameters. *Am. J. Math.* **84**(4), 561–577 (1962). <https://doi.org/10.2307/2372862>
62. Trèves, F.: Locally convex spaces and linear partial differential equations. Springer, New York (1967). <https://doi.org/10.1007/978-3-642-87371-3>
63. Vogt, D.: Charakterisierung der Unterräume von s . *Math. Z.* **155**, 109–118 (1977). <https://doi.org/10.1007/BF01214210>
64. Vogt, D.: On the solvability of $P(D)f = g$ for vector valued functions. In: H. Komatsu (ed.) Generalized functions and linear differential equations 8 (Proc., Kyoto, 1982), *RIMS Kôkyûroku*, vol. 508, pp. 168–181. RIMS, Kyoto (1983)
65. Vogt, D.: On the functors $\text{Ext}^1(E, F)$ for Fréchet spaces. *Studia Math.* **85**(2), 163–197 (1987). <https://doi.org/10.4064/sm-85-2-163-197>
66. Wengenroth, J.: Derived functors in functional analysis. *Lecture Notes in Math*, vol. 1810. Springer, Berlin (2003). <https://doi.org/10.1007/b80165>
67. Wengenroth, J.: Surjectivity of partial differential operators with good fundamental solutions. *J. Math. Anal. Appl.* **379**(2), 719–723 (2011). <https://doi.org/10.1016/j.jmaa.2011.01.074>

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