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THE APPROXIMATION PROPERTY FOR WEIGHTED SPACES OF DIFFERENTIABLE FUNCTIONS

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Abstract. We study spaces $\mathcal{CV}^k(\Omega,E)$ of k-times continuously partially differentiable functions on an open set $\Omega\subset\mathbb{R}^d$ with values in a locally convex Hausdorff space E. The space $\mathcal{CV}^k(\Omega,E)$ is given a weighted topology generated by a family of weights $\mathcal{V}^k$. For the space $\mathcal{CV}^k(\Omega,E)$ and its subspace $\mathcal{CV}^k_0(\Omega,E)$ of functions that vanish at infinity in the weighted topology we try to answer the question whether their elements can be approximated by functions with values in a finite dimensional subspace. We derive sufficient conditions for an affirmative answer to this question using the theory of tensor products.

1. Introduction. This paper is dedicated to the following problem: *Which vector-valued k-times continuously partially differentiable functions can be approximated in a weighted topology by functions with values in a finite dimensional subspace?* The answer to this question is closely related to the theory of tensor products and the so-called approximation property. A locally convex Hausdorff space X is said to have (Schwartz') approximation property if the identity I_X on X is contained in the closure of $\mathfrak{F}(X)$ in $L_\kappa(X)$ where $L_\kappa(X)$ denotes the space of continuous linear operators from X to X equipped with the topology of uniform convergence on the absolutely convex compact subsets of X and $\mathfrak{F}(X)$ its subspace of operators with finite rank.

The case $k=0$ is well-studied. In [1], [2] and [3] Bierstedt considered the space $\mathcal{CV}(\Omega,E)$ of all continuous functions $f\colon\Omega\to E$ from a completely regular Hausdorff space Ω to a locally convex Hausdorff space $(E,(p_\alpha)_{\alpha\in\mathfrak{A}})$ over a field $\mathbb{K}$ with a topology

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induced by a Nachbin-family $\mathcal{V} := (\nu_j)_{j\in J}$ of weights, i.e. the space

$$\mathcal{CV}(\Omega, E) := \{f \in \mathcal{C}(\Omega, E) \mid \forall\, j \in J,\ \alpha \in \mathfrak{A} : |f|_{j,\alpha} < \infty\}$$

where $\mathcal{C}(\Omega, E) := \mathcal{C}^0(\Omega, E)$ is the space of continuous functions from Ω to E and

$$|f|_{j,\alpha} := \sup_{x\in\Omega} p_\alpha(f(x))\nu_j(x).$$

Recall that a family $\mathcal{V} := (\nu_j)_{j\in J}$ of non-negative functions $\nu_j\colon \Omega \to [0,\infty)$ is called a Nachbin-family of weights if the functions ν_j are upper semi-continuous and the family is directed, i.e. for every $j, i \in J$ there are $k \in J$ and $C > 0$ such that $\max(\nu_i, \nu_j) \leq C\nu_k$. The notion $\mathcal{U} \leq \mathcal{V}$ for two Nachbin-families means that for every $\mu \in \mathcal{U}$ there is $\nu \in \mathcal{V}$ such that $\mu \leq \nu$.

From the perspective of our problem the space $\mathcal{CV}(\Omega, E)$ has an interesting topological subspace, namely, the space $\mathcal{CV}_0(\Omega, E)$ consisting of the functions that vanish at infinity when weighted which is given by

$$\mathcal{CV}_0(\Omega, E) := \{f \in \mathcal{CV}(\Omega, E) \mid \forall\, \varepsilon > 0,\ j \in J,\ \alpha \in \mathfrak{A}\ \exists\, K \subset \Omega \text{ compact} : |f|_{\Omega\setminus K,j,\alpha} < \varepsilon\}$$

where

$$|f|_{\Omega\setminus K,j,\alpha} := \sup_{x\in\Omega\setminus K} p_\alpha(f(x))\nu_j(x).$$

One of the main results from [2] solves our problem for $k = 0$, Nachbin-families of weights and involves $k_{\mathbb{R}}$-spaces. A completely regular space Ω is a $k_{\mathbb{R}}$-space if for any completely regular space Y and any map $f\colon \Omega \to Y$ whose restriction to each compact $K \subset \Omega$ is continuous the map is already continuous on Ω (see [5, (2.3.7) Proposition, p. 22]). Obviously, every locally compact Hausdorff space is a $k_{\mathbb{R}}$-space. Further examples of $k_{\mathbb{R}}$-spaces are metrisable spaces by [13, Proposition 11.5, p. 181] and [8, 3.3.20, 3.3.21 Theorem, p. 152] as well as strong duals of Fréchet–Montel spaces by [9, Proposition 3.27, p. 95] and [16, 4.11 Theorem, p. 39].

Theorem 1.1 ([2, 5.5 Theorem, p. 205–206]). *Let E be a locally convex Hausdorff space, Ω a completely regular Hausdorff space and $\mathcal{V}$ a Nachbin-family on Ω such that one of the following conditions is satisfied.*

(i) $\mathcal{Z} := \{v\colon \Omega \to \mathbb{R} \mid v \text{ constant},\ v \geq 0\} \leq \mathcal{V}$.
(ii) $\mathcal{W} := \{\mu\chi_K \mid \mu > 0,\ K \subset \Omega \text{ compact}\} \leq \mathcal{V}$, *where $\chi_K\colon \Omega \to \mathbb{R}$ is the characteristic function of K, and Ω is a $k_{\mathbb{R}}$-space.*

Then the following holds.

a) $\mathcal{CV}_0(\Omega) \otimes E$ *is dense in* $\mathcal{CV}_0(\Omega, E)$.
b) *If E is complete, then*

$$\mathcal{CV}_0(\Omega, E) \cong \mathcal{CV}_0(\Omega)\varepsilon E \cong \mathcal{CV}_0(\Omega)\widehat{\otimes}_\varepsilon E.$$

c) $\mathcal{CV}_0(\Omega)$ *has the approximation property.*

Here $\mathcal{CV}_0(\Omega) \otimes E$ stands for the tensor product, $\mathcal{CV}_0(\Omega)\widehat{\otimes}_\varepsilon E$ for the completion of the injective tensor product and $\mathcal{CV}_0(\Omega)\varepsilon E := L_e(\mathcal{CV}_0(\Omega)'_\kappa, E)$ for the ε-product of Schwartz of the spaces $\mathcal{CV}_0(\Omega) := \mathcal{CV}_0(\Omega, \mathbb{K})$ and E. Part a) gives an affirmative answer to our question for the space $\mathcal{CV}_0(\Omega, E)$ since it implies that for every $\varepsilon > 0$, $\alpha \in \mathfrak{A}$, $j \in J$ and $f \in \mathcal{CV}_0(\Omega, E)$ there are $m \in \mathbb{N}$, $f_n \in \mathcal{CV}_0(\Omega)$ and $e_n \in E$, $1 \leq n \leq m$, such that

$$\Big| f - \sum_{n=1}^{m} f_n e_n \Big|_{j,\alpha} < \varepsilon.$$

Concerning $\mathcal{CV}(\Omega, E)$, the answer to our question is not that satisfying but still affirmative if we make some restrictions on E. If E has the approximation property, then $E \otimes_\varepsilon \mathcal{CV}(\Omega)$ is dense in $E\varepsilon\mathcal{CV}(\Omega)$. Due to the symmetries $\mathcal{CV}(\Omega) \otimes_\varepsilon E \cong E \otimes_\varepsilon \mathcal{CV}(\Omega)$ and $\mathcal{CV}(\Omega)\varepsilon E \cong E\varepsilon\mathcal{CV}(\Omega)$, we infer that $\mathcal{CV}(\Omega) \otimes_\varepsilon E$ is dense in $\mathcal{CV}(\Omega)\varepsilon E \cong \mathcal{CV}(\Omega, E)$ if E is a semi-Montel space with approximation property and $\mathcal{Z} \leq \mathcal{V}$ or Ω is a $k_{\mathbb{R}}$-space by [3, 2.12 Satz (1), p. 141]. A second condition for an affirmative answer without supposing that E has the approximation property but putting more restrictions on $\mathcal{CV}(\Omega)$ can be found in [3, 2.12 Satz (2), p. 141].

We aim to prove a version of Bierstedt's theorem for spaces of weighted continuously partially differentiable functions. To the best of our knowledge the approximation problem was not considered in a general setting for $k > 0$ and open $\Omega \subset \mathbb{R}^d$, i.e. to derive sufficient conditions on the weights and the spaces such that the answer is positive. For special cases with $\Omega = \mathbb{R}^d$ like the Schwartz space an affirmative answer was already given in e.g. [21, Proposition 9, p. 108] and [21, Théorème 1, p. 111]. For the space of k-times continuously partially differentiable functions on open $\Omega \subset \mathbb{R}^d$ with the topology of uniform convergence of all partial derivatives up to order k on compact sets a positive answer can be found in e.g. [23, Proposition 44.2, p. 448] and [23, Theorem 44.1, p. 449]. Let us consider for a moment the latter space and the corresponding proof given by Trèves in [23]. The space $\mathcal{C}^k(\Omega, E)$ of k-times continuously partially differentiable functions on a locally compact Hausdorff space Ω if $k = 0$, resp. open $\Omega \subset \mathbb{R}^d$ if $k \in \mathbb{N} \cup \{\infty\}$, is equipped with the system of seminorms given by

$$q_{K,l,\alpha}(f) := \sup_{\substack{x \in K \\ \beta \in \mathbb{N}_0^d, |\beta| \leq l}} p_\alpha\big(\partial^\beta f(x)\big), \quad f \in \mathcal{C}^k(\Omega, E), \tag{1}$$

for $K \subset \Omega$ compact, $l \in \mathbb{N}_0$, $0 \leq l \leq k$ if $k < \infty$, and $\alpha \in \mathfrak{A}$. For $E = \mathbb{K}$ we fix the notion $\mathcal{C}^k(\Omega) := \mathcal{C}^k(\Omega, \mathbb{K})$ and denote by $\mathcal{C}^k_c(\Omega)$ the space of all functions in $\mathcal{C}^k(\Omega)$ having compact support. Trèves' affirmative answer to our question has the following form.

THEOREM 1.2 ([23, Proposition 44.2, p. 448] and [23, Theorem 44.1, p. 449]). *Let E be a locally convex Hausdorff space, $k \in \mathbb{N}_0 \cup \{\infty\}$ and Ω a locally compact Hausdorff space if $k = 0$, resp. an open subset of $\mathbb{R}^d$ if $k > 0$. Then the following is true.*

a) *$\mathcal{C}^0_c(\Omega) \otimes E$ is dense in $\mathcal{C}^0(\Omega, E)$.*
b) *$\mathcal{C}^\infty_c(\Omega) \otimes E$ is dense in $\mathcal{C}^k(\Omega, E)$.*
c) *If E is complete, then*

$$\mathcal{C}^k(\Omega, E) \cong \mathcal{C}^k(\Omega) \widehat{\otimes}_\varepsilon E.$$

We observe that $\mathcal{CW}(\Omega, E) = \mathcal{CW}_0(\Omega, E) = \mathcal{C}^0(\Omega, E)$ equipped with the usual topology of uniform convergence on compact subsets of Ω which means that Theorem 1.1 contains the case $k = 0$ of the preceding theorem since locally compact Hausdorff spaces are $k_{\mathbb{R}}$-spaces. The proofs of Theorem 1.1 a) and Theorem 1.2 a) are done by using different partitions of unity, the first uses the partition of unity from [20, 23, Lemma 2, p. 71] and the second the one from [4, Chap. IX, §4.3, Corollary, p. 186]. The key idea for the proof of Theorem 1.2 b) is an approximation in three steps relying on part a) and convolution. First, for every $f \in \mathcal{C}^k(\Omega, E)$ there is an approximation $\widetilde{f} \in \mathcal{C}^k_c(\Omega, E)$ of f by multiplication of f with a suitable cut-off function. Second, for every $\widetilde{f} \in \mathcal{C}^k_c(\Omega, E)$ the convolution $\widetilde{f} * \rho_n$ of $\widetilde{f}$ with a sequence (ρ_n) of mollifiers in $\mathcal{C}^\infty_c(\Omega)$ converges to $\widetilde{f}$ in $\mathcal{C}^k(\Omega, \widehat{E})$ where $\widehat{E}$ denotes the completion of E (approximation by regularisation). Third, for every $\widetilde{f} \in \mathcal{C}^k_c(\Omega, E)$ there is an approximation $g \in \mathcal{C}^0_c(\Omega) \otimes E$ in the topology of $\mathcal{C}^0(\Omega, E)$ by part a). Using the properties of the convolution, one gets that $g * \rho_n \in \mathcal{C}^\infty_c(\Omega) \otimes E$ and approximates $\widetilde{f} * \rho_n$ for n large enough in $\mathcal{C}^k(\Omega, \widehat{E})$ which itself is identical to the completion of $\mathcal{C}^k(\Omega, E)$.

The outline of our paper is along the lines of Trèves' proof. After introducing some notation and preliminaries in Section 2, we define the weighted spaces $\mathcal{CV}^k(\Omega, E)$ and $\mathcal{CV}^k_0(\Omega, E)$ in Section 3 and show that they are complete if the family of weights $\mathcal{V}^k$ is locally bounded away from zero (see Definition 3.6). Then we treat their relation to the space $\mathcal{C}^k_c(\Omega, E)$ of functions in $\mathcal{C}^k(\Omega, E)$ with compact support where the condition of local boundedness of a family of weights comes into play (see Definition 3.8). We formulate a cut-off criterion (see Definition 3.10) which is a sufficient condition for the density of $\mathcal{C}^k_c(\Omega, E)$ in $\mathcal{CV}^k_0(\Omega, E)$ for locally bounded $\mathcal{V}^k$. We close the third section with the relation between tensor products and our problem on finite dimensional approximation. In Section 4 we define the convolution $f * g$ of $f \in \mathcal{C}^k(\mathbb{R}^d, E)$ and $g \in \mathcal{C}^n(\mathbb{R}^d)$ when one of them is compactly supported and prove an approximation by regularisation result. In the last section we verify the corresponding part a) of Theorem 1.2 for $\mathcal{CV}^0_0(\Omega, E)$ with locally compact Ω where we adapt the proof of Theorem 1.1 a) in a way that we can use the partition of unity from [4, Chap. IX, §4.3, Corollary, p. 186] instead and weaken the condition of upper semi-continuity of the weights to being locally bounded and locally bounded away from zero. Then we mix all ingredients to get our main Theorem 5.2 which is a version of Theorem 1.1 and 1.2 for barrelled $\mathcal{CV}^k_0(\Omega)$ with a family of weights $\mathcal{V}^k$ being locally bounded and locally bounded away from zero if $\mathcal{CV}^k_0(\Omega, E)$ fulfils the cut-off criterion.

2. Notation and preliminaries.

We set $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ and $\mathbb{N}_{0,\infty} := \mathbb{N}_0 \cup \{\infty\}$. For $k \in \mathbb{N}_{0,\infty}$ we use the notation $\langle k \rangle := \{n \in \mathbb{N}_0 \,|\, 0 \leq n \leq k\}$ if $k \neq \infty$ and $\langle k \rangle := \mathbb{N}_0$ if $k = \infty$. We equip the spaces $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\mathbb{C}$ with the usual Euclidean norm $|\cdot|$, write $\overline{M}$ for the closure of a subset $M \subset \mathbb{R}^d$ and denote by $\mathbb{B}_r(x) := \{w \in \mathbb{R}^d \,|\, |w - x| < r\}$ the ball around $x \in \mathbb{R}^d$ with radius $r > 0$.

By E we always denote a non-trivial locally convex Hausdorff space, in short lcHs, over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha \in \mathfrak{A}}$. If $E = \mathbb{K}$, then we set $(p_\alpha)_{\alpha \in \mathfrak{A}} := \{|\cdot|\}$. Further, we denote by $\widehat{E}$ the completion of a locally convex Hausdorff space E. For details on the theory of locally convex spaces see [10], [14] or [18].

A function $f\colon \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ to a locally convex Hausdorff space E is called continuously partially differentiable (f is $\mathcal{C}^1$) if for the n-th unit vector $e_n \in \mathbb{R}^d$ the limit

$$(\partial^{e_n})f(x) := (\partial^{e_n})^E f(x) := (\partial_{x_n})^E f(x) := \lim_{\substack{h\to 0\\ h\in\mathbb{R}, h\neq 0}} \frac{f(x+he_n)-f(x)}{h}$$

exists in E for every $x \in \Omega$ and $\partial^{e_n} f$ is continuous on Ω ($\partial^{e_n} f$ is $\mathcal{C}^0$) for every $1 \leq n \leq d$. For $k \in \mathbb{N}$ a function f is said to be k-times continuously partially differentiable (f is $\mathcal{C}^k$) if f is $\mathcal{C}^1$ and all its first partial derivatives are $\mathcal{C}^{k-1}$. A function f is called infinitely continuously partially differentiable (f is $\mathcal{C}^\infty$) if f is $\mathcal{C}^k$ for every $k \in \mathbb{N}$. For $k \in \mathbb{N}_\infty$ the linear space of all functions $f\colon \Omega \to E$ which are $\mathcal{C}^k$ is denoted by $\mathcal{C}^k(\Omega, E)$. Its subspace of functions with compact support is written as $\mathcal{C}^k_c(\Omega, E)$ where we denote the support of $f \in \mathcal{C}^k(\Omega, E)$ by $\operatorname{supp} f$.

Let $f \in \mathcal{C}^k(\Omega, E)$. For $\beta \in \mathbb{N}_0^d$ with $|\beta| := \sum_{n=1}^d \beta_n \leq k$ we set $\partial^{\beta_n} f := (\partial^{\beta_n})^E f := f$ if $\beta_n = 0$, and

$$\partial^{\beta_n} f := (\partial^{\beta_n})^E f := \underbrace{(\partial^{e_n})^E \cdots (\partial^{e_n})^E}_{\beta_n\text{-times}} f$$

if $\beta_n \neq 0$ as well as

$$\partial^\beta f := (\partial^\beta)^E f := \partial^{\beta_1} \cdots \partial^{\beta_d} f.$$

Due to the vector-valued version of Schwarz' theorem $\partial^\beta f$ is independent of the order of the partial derivatives on the right-hand side and we call $|\beta|$ the order of differentiation. Further, we observe that $e' \circ f \in \mathcal{C}^k(\Omega)$ and $(\partial^\beta)^{\mathbb{K}}(e' \circ f) = e' \circ (\partial^\beta)^E f$ for every $e' \in E'$, $f \in \mathcal{C}^k(\Omega, E)$ and $|\beta| \leq k$.

By $L(F, E)$ we denote the space of continuous linear operators from F to E where F and E are locally convex Hausdorff spaces. If $E = \mathbb{K}$, we just write $F' := L(F, \mathbb{K})$ for the dual space. If F and E are (linearly topologically) isomorphic, we write $F \cong E$. The so-called ε-product of Schwartz is defined by

$$F\varepsilon E := L_e(F'_\kappa, E) \tag{2}$$

where F' is equipped with the topology of uniform convergence on absolutely convex compact subsets of F and $L(F'_\kappa, E)$ is equipped with the topology of uniform convergence on equicontinuous subsets of F' (see [22, Chap. I, §1, Définition, p. 18]). It is symmetric which means that $F\varepsilon E \cong E\varepsilon F$ and in the literature the definition of the ε-product is sometimes done the other way around, i.e. $E\varepsilon F$ is defined by the right-hand side of (2). We write $F \widehat{\otimes}_\varepsilon E$ for the completion of the injective tensor product $F \otimes_\varepsilon E$ and denote by $\mathfrak{F}(E)$ the space of linear operators from E to E with finite rank. We recall from the introduction that a locally convex Hausdorff space E is said to have (Schwartz') approximation property if the identity I_E on E is contained in the closure of $\mathfrak{F}(E)$ in $L_\kappa(E) := L_\kappa(E, E)$ which is equipped with the topology of uniform convergence on the absolutely convex compact subsets of E. The space E has the approximation property if and only if $E \otimes F$ is dense in $E\varepsilon F$ for every locally convex Hausdorff space (every Banach space) F by [15, Satz 10.17, p. 250]. For more information on the theory of ε-products and tensor products see [6], [14] and [15].

3. Weighted vector-valued differentiable functions and the ε-product. In this section we introduce the spaces $\mathcal{CV}^k(\Omega, E)$ and $\mathcal{CV}_0^k(\Omega, E)$ we want to consider. Then we turn to the question of completeness of $\mathcal{CV}^k(\Omega, E)$ and $\mathcal{CV}_0^k(\Omega, E)$ and when $\mathcal{C}_c^k(\Omega, E)$ is dense in the latter space. At the end of this section we describe their connection to the ε-product and the (completion of the) injective tensor product and derive sufficient conditions such that they coincide.

Definition 3.1 (weight). Let $k \in \mathbb{N}_{0,\infty}$. We say that $\mathcal{V}^k := (\nu_{j,l})_{j\in J, l\in\langle k\rangle}$ is a (*directed*) *family of weights* on a locally compact Hausdorff space Ω if $\nu_{j,l}\colon \Omega \to [0,\infty)$ for every $j \in J$, $l \in \langle k\rangle$ and

$$\forall\, j_1, j_2 \in J,\ l_1, l_2 \in \langle k\rangle\ \exists\, j_3 \in J,\ l_3 \in \langle k\rangle,\ C > 0\ \forall\, i \in \{1,2\} : \nu_{j_i,l_i} \leq C\nu_{j_3,l_3}$$

as well as

$$\forall\, l \in \langle k\rangle,\ x \in \Omega\ \exists\, j \in J : 0 < \nu_{j,l}(x).$$

Definition 3.2. For $k \in \mathbb{N}_{0,\infty}$ and a (directed) family $\mathcal{V}^k := (\nu_{j,l})_{j\in J, l\in\langle k\rangle}$ of weights on a locally compact Hausdorff space Ω if $k = 0$ or an open set $\Omega \subset \mathbb{R}^d$ if $k \in \mathbb{N}_\infty$ we define the *space of weighted continuous*, resp. *k-times continuously partially differentiable*, functions with values in an lcHs E as

$$\mathcal{CV}^k(\Omega, E) := \{f \in \mathcal{C}^k(\Omega, E) \mid \forall\, j \in J,\ l \in \langle k\rangle,\ \alpha \in \mathfrak{A} : |f|_{j,l,\alpha} < \infty\}$$

where

$$|f|_{j,l,\alpha} := \sup_{\substack{x\in\Omega \\ \beta\in\mathbb{N}_0^d, |\beta|\leq l}} p_\alpha\big((\partial^\beta)^E f(x)\big)\nu_{j,l}(x).$$

We define the topological subspace of $\mathcal{CV}^k(\Omega, E)$ consisting of the functions that vanish with all their derivatives when weighted at infinity by

$$\mathcal{CV}_0^k(\Omega, E) := \{f \in \mathcal{CV}^k(\Omega, E) \mid \forall\, j \in J,\ l \in \langle k\rangle,\ \alpha \in \mathfrak{A},\ \varepsilon > 0\ \exists\, K \subset \Omega \text{ compact} : |f|_{\Omega\setminus K,j,l,\alpha} < \varepsilon\}$$

where

$$|f|_{\Omega\setminus K,j,l,\alpha} := \sup_{\substack{x\in\Omega\setminus K \\ \beta\in\mathbb{N}_0^d, |\beta|\leq l}} p_\alpha\big((\partial^\beta)^E f(x)\big)\nu_{j,l}(x).$$

It is easily seen that these spaces are locally convex Hausdorff spaces with a directed system of seminorms due to our assumptions on the family $\mathcal{V}^k$ of weights.

Remark 3.3. Suppose that in the definition of the space $\mathcal{CV}^k(\Omega, E)$ the weights also depend on $\beta \in \mathbb{N}_0^d$, i.e. the seminorms used to define $\mathcal{CV}^k(\Omega, E)$ are of the form

$$|f|_{j,l,\alpha}^{\sim} := \sup_{\substack{x\in\Omega \\ \beta\in\mathbb{N}_0^d, |\beta|\leq l}} p_\alpha\big((\partial^\beta)^E f(x)\big)\nu_{j,l,\beta}(x).$$

Without loss of generality we may always use weights which are independent of β. Namely, by setting $\nu_{j,l} := \max_{\beta\in\mathbb{N}_0^d, |\beta|\leq l} \nu_{j,l,\beta}$ for $j \in J$ and $l \in \langle k\rangle$, we can switch to the usual system of seminorms $(|f|_{j,l,\alpha})$ induced by the weights $(\nu_{j,l})$ which is equivalent to $(|f|_{j,l,\alpha}^{\sim})$.

The standard structure of a directed family $\mathcal{V}^k$ of weights on a locally compact Hausdorff space Ω is given by the following. Let $(\Omega_j)_{j\in J}$ be a family of sets such that $\Omega_j \subset \Omega_{j+1}$

for all $j \in J$ with $\Omega = \bigcup_{j\in J} \Omega_j$. Let $\widetilde{\nu}_{j,l} \colon \Omega \to (0,\infty)$ be continuous for all $j \in J$ and $l \in \langle k\rangle$, increasing in $j \in J$, i.e. $\widetilde{\nu}_{j,l} \leq \widetilde{\nu}_{j+1,l}$, and in $l \in \langle k\rangle$, i.e. $\widetilde{\nu}_{j,l} \leq \widetilde{\nu}_{j,l+1}$ if $l+1 \in \langle k\rangle$, such that

$$\nu_{j,l}(x) = \chi_{\Omega_j}(x)\widetilde{\nu}_{j,l}(x), \quad x \in \Omega,$$

for every $j \in J$ and $l \in \langle k\rangle$ where χ_{Ω_j} is the indicator function of Ω_j. Further, we remark that the spaces $\mathcal{CV}^k(\Omega, E)$ and $\mathcal{CV}_0^k(\Omega, E)$ might coincide which is already mentioned in [2, 1.3 Bemerkung, p. 189] for $k = 0$.

Remark 3.4. If for every $j \in J$ and $l \in \langle k\rangle$ there are $i \in J$ and $m \in \langle k\rangle$ such that for all $\varepsilon > 0$ there is a compact set $K \subset \Omega$ with $\nu_{j,l}(x) \leq \varepsilon\nu_{i,m}(x)$ for all $x \in \Omega \setminus K$, then $\mathcal{CV}^k(\Omega, E) = \mathcal{CV}_0^k(\Omega, E)$.

Examples of spaces where this happens are $\mathcal{C}^k(\Omega, E)$ with the topology of uniform convergence of all partial derivatives up to order k on compact subsets of Ω and the Schwartz space $\mathcal{S}(\mathbb{R}^d, E)$.

Example 3.5. Let E be an lcHs, $k \in \mathbb{N}_{0,\infty}$ and $\Omega \subset \mathbb{R}^d$ open. Then

a) $\mathcal{C}^k(\Omega, E) = \mathcal{CW}^k(\Omega, E) = \mathcal{CW}_0^k(\Omega, E)$ with $\mathcal{W}^k := \{\nu_{j,l} := \chi_{\Omega_j} \mid j \in \mathbb{N},\ l \in \langle k\rangle\}$ where $(\Omega_j)_{j\in\mathbb{N}}$ is a compact exhaustion of Ω,

b) $\mathcal{S}(\mathbb{R}^d, E) = \mathcal{CV}^\infty(\mathbb{R}^d, E) = \mathcal{CV}_0^\infty(\mathbb{R}^d, E)$ with $\mathcal{V}^\infty := \{\nu_{j,l} \mid j \in \mathbb{N},\ l \in \mathbb{N}_0\}$ where $\nu_{j,l}(x) := (1 + |x|^2)^{l/2}$ for $x \in \mathbb{R}^d$.

Proof.

a) $(\Omega_j)_{j\in\mathbb{N}}$ being a compact exhaustion of Ω means that $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$, Ω_j is compact and $\Omega_j \subset \mathring{\Omega}_{j+1}$ for all $j \in \mathbb{N}$ where $\mathring{\Omega}_{j+1}$ is the set of inner points of Ω_{j+1}. For compact $\Omega_j \subset \Omega$ and $l \in \langle k\rangle$ our claim follows from Remark 3.4 with the choice $i := j$, $m := l$ and $K := \Omega_j$.

b) We recall that the Schwartz space is defined by

$$\mathcal{S}(\mathbb{R}^d, E) := \big\{f \in \mathcal{C}^\infty(\mathbb{R}^d, E) \mid \forall\, l \in \mathbb{N}_0,\ \alpha \in \mathfrak{A} : \|f\|_{l,\alpha} < \infty\big\}$$

where

$$\|f\|_{l,\alpha} := \sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d, |\beta|\leq l}} p_\alpha\big((\partial^\beta)^E f(x)\big)(1 + |x|^2)^{l/2}.$$

Thus $\mathcal{S}(\mathbb{R}^d, E) = \mathcal{CV}^\infty(\mathbb{R}^d, E)$. We note that for every $j \in \mathbb{N}$, $l \in \mathbb{N}_0$ and $\varepsilon > 0$ there is $r > 0$ such that

$$\frac{\nu_{j,l}(x)}{\nu_{j,2(l+1)}(x)} = \frac{(1 + |x|^2)^{l/2}}{(1 + |x|^2)^{l+1}} = (1 + |x|^2)^{-(l/2)-1} < \varepsilon$$

for all $x \notin \overline{\mathbb{B}_r(0)} =: K$ yielding $\mathcal{S}(\mathbb{R}^d, E) = \mathcal{CV}_0^\infty(\mathbb{R}^d, E)$ by Remark 3.4. ∎

The question of finite dimensional approximation from the introduction is closely connected to the property of a family of weights being locally bounded away from zero.

Definition 3.6 (locally bounded away from zero). Let Ω be a locally compact Hausdorff space and $k \in \mathbb{N}_{0,\infty}$. A family of weights $\mathcal{V}^k$ is called *locally bounded away from zero* on Ω if

$$\forall\, K \subset \Omega \text{ compact},\ l \in \langle k\rangle\ \exists\, j \in J : \inf_{x\in K} \nu_{j,l}(x) > 0.$$

For $k=0$ (and locally compact Hausdorff Ω) this coincides with condition (ii) of Theorem 1.1. It even guarantees that the spaces $\mathcal{CV}^k(\Omega,E)$ and $\mathcal{CV}^k_0(\Omega,E)$ are complete for complete E.

Proposition 3.7. *Let E be a complete lcHs, $k\in\mathbb{N}_{0,\infty}$ and $\mathcal{V}^k$ be a family of weights which is locally bounded away from zero on a locally compact Hausdorff space Ω ($k=0$) or an open set $\Omega\subset\mathbb{R}^d$ ($k>0$). Then $\mathcal{CV}^k(\Omega,E)$ and $\mathcal{CV}^k_0(\Omega,E)$ are complete locally convex Hausdorff spaces. In particular, they are Fréchet spaces if E is a Fréchet space and J countable.*

Proof. Let $(f_\tau)_{\tau\in\mathcal{T}}$ be a Cauchy net in $\mathcal{CV}^k(\Omega,E)$. The space $\mathcal{C}^k(\Omega,E)$ equipped with the usual system of seminorms $(q_{K,l,\alpha})$ given in (1) is complete by [23, Proposition 44.1, p. 446]. Let $K\subset\Omega$ compact, $l\in\langle k\rangle$ and $\alpha\in\mathfrak{A}$. Since $\mathcal{V}^k$ is locally bounded away from zero, there is $j\in J$ such that

$$q_{K,l,\alpha}(f)\leq\sup_{x\in K}\nu_{j,l}(x)^{-1}|f|_{j,l,\alpha}=\big(\inf_{x\in K}\nu_{j,l}(x)\big)^{-1}|f|_{j,l,\alpha},\quad f\in\mathcal{CV}^k(\Omega,E),$$

implying that the inclusion $\mathcal{CV}^k(\Omega,E)\hookrightarrow\mathcal{C}^k(\Omega,E)$ is continuous. Thus (f_τ) is a Cauchy net in $\mathcal{C}^k(\Omega,E)$ as well and has a limit f in this space due to the completeness. Let $j\in J$, $l\in\langle k\rangle$, $\alpha\in\mathfrak{A}$ and $\varepsilon>0$. As this convergence implies pointwise convergence, we have that for all $x\in\Omega$ and $\beta\in\mathbb{N}_0^d$, $|\beta|\leq l$, there exists $\tau_{j,l,\beta,x}\in\mathcal{T}$ such that for all $\tau\geq\tau_{j,l,\beta,x}$

$$p_\alpha\big((\partial^\beta)^Ef_\tau(x)-(\partial^\beta)^Ef(x)\big)<\frac{\varepsilon}{2\nu_{j,l}(x)}\tag{3}$$

if $\nu_{j,l}(x)>0$. Furthermore, there exists $\tau_0\in\mathcal{T}$ such that for all $\tau,\mu\geq\tau_0$

$$|f_\tau-f_\mu|_{j,l,\alpha}<\frac{\varepsilon}{2}\tag{4}$$

by assumption. Hence we get for all $\tau\geq\tau_0$ by choosing $\mu\geq\tau_{j,l,\beta,x},\tau_0$

$$\begin{aligned}
&p_\alpha\big((\partial^\beta)^Ef(x)\big)\nu_{j,l}(x)-p_\alpha\big((\partial^\beta)^Ef_\tau(x)\big)\nu_{j,l}(x)\\
&\leq p_\alpha\big((\partial^\beta)^Ef_\tau(x)-(\partial^\beta)^Ef(x)\big)\nu_{j,l}(x)\\
&\leq p_\alpha\big((\partial^\beta)^Ef_\tau(x)-(\partial^\beta)^Ef_\mu(x)\big)\nu_{j,l}(x)+p_\alpha\big((\partial^\beta)^Ef_\mu(x)-(\partial^\beta)^Ef(x)\big)\nu_{j,l}(x)\\
&\underset{(3)}{<}\sup_{z\in\Omega}p_\alpha\big((\partial^\beta)^Ef_\tau(z)-(\partial^\beta)^Ef_\mu(z)\big)\nu_{j,l}(z)+\frac{\varepsilon}{2}\\
&\leq\sup_{\substack{z\in\Omega\\ \gamma\in\mathbb{N}_0^d,\,|\gamma|\leq l}}p_\alpha\big((\partial^\gamma)^Ef_\tau(z)-(\partial^\gamma)^Ef_\mu(z)\big)\nu_{j,l}(z)+\frac{\varepsilon}{2}=|f_\tau-f_\mu|_{j,l,\alpha}+\frac{\varepsilon}{2}\underset{(4)}{<}\varepsilon
\end{aligned}$$

if $\nu_{j,l}(x)>0$. We deduce that for all $\tau\geq\tau_0$

$$\begin{aligned}
&p_\alpha\big((\partial^\beta)^Ef(x)\big)\nu_{j,l}(x)-p_\alpha\big((\partial^\beta)^Ef_\tau(x)\big)\nu_{j,l}(x)\\
&\leq p_\alpha\big((\partial^\beta)^Ef_\tau(x)-(\partial^\beta)^Ef(x)\big)\nu_{j,l}(x)<\varepsilon
\end{aligned}$$

if $\nu_{j,l}(x)>0$. If $\nu_{j,l}(x)=0$, then this estimate is also fulfilled and so $|f_\tau-f|_{j,l,\alpha}\leq\varepsilon$ as well as $|f|_{j,l,\alpha}\leq\varepsilon+|f_\tau|_{j,l,\alpha}$ for all $\tau\geq\tau_0$. This means that $f\in\mathcal{CV}^k(\Omega,E)$ and that (f_τ) converges to f in $\mathcal{CV}^k(\Omega,E)$. Therefore $\mathcal{CV}^k(\Omega,E)$ is complete and $\mathcal{CV}^k_0(\Omega,E)$ as well because it is a closed subspace of the complete space $\mathcal{CV}^k(\Omega,E)$. ∎

For $k \in \mathbb{N}_{0,\infty}$ and locally compact Hausdorff Ω ($k = 0$) or open $\Omega \subset \mathbb{R}^d$ ($k > 0$) we define $\mathcal{CV}^k_c(\Omega, E)$ to be the subspace of $\mathcal{CV}^k(\Omega, E)$ of functions with compact support. Obviously we have $\mathcal{CV}^k_c(\Omega, E) \subset \mathcal{CV}^k_0(\Omega, E)$ and $\mathcal{CV}^k_c(\Omega, E) \subset \mathcal{C}^k_c(\Omega, E)$. On the other hand, the space $\mathcal{C}^k_c(\Omega, E)$ is a linear subspace of $\mathcal{CV}^k_c(\Omega, E)$ if the family of weights $\mathcal{V}^k$ fulfils the definition of local boundedness.

Definition 3.8 (locally bounded). Let Ω be a locally compact Hausdorff space and $k \in \mathbb{N}_{0,\infty}$. A family of weights $\mathcal{V}^k$ is called *locally bounded* on Ω if

$$\forall\, K \subset \Omega \text{ compact},\ j \in J,\ l \in \langle k \rangle : \sup_{x \in K} \nu_{j,l}(x) < \infty.$$

Indeed, if $f \in \mathcal{C}^k_c(\Omega, E)$, then we have for $K := \operatorname{supp} f$

$$|f|_{j,l,\alpha} = \sup_{\substack{x \in K \\ \beta \in \mathbb{N}^d_0, |\beta| \le l}} p_\alpha\big((\partial^\beta)^E f(x)\big)\nu_{j,l}(x) \le \Big(\sup_{\substack{z \in K \\ \beta \in \mathbb{N}^d_0, |\beta| \le l}} p_\alpha\big((\partial^\beta)^E f(z)\big)\Big) \sup_{x \in K} \nu_{j,l}(x)$$

for all $j \in J$, $l \in \langle k \rangle$ and $\alpha \in \mathfrak{A}$. Hence we have:

Remark 3.9. Let E be an lcHs and $k \in \mathbb{N}_{0,\infty}$. If $\mathcal{V}^k$ is a family of locally bounded weights, then $\mathcal{C}^k_c(\Omega, E) = \mathcal{CV}^k_c(\Omega, E)$ algebraically.

Next, we phrase a sufficient criterion for the density of $\mathcal{C}^k_c(\Omega, E)$ in $\mathcal{CV}^k_0(\Omega, E)$ for $k \in \mathbb{N}_{0,\infty}$, $\Omega \subset \mathbb{R}^d$ open and locally bounded $\mathcal{V}^k$.

Definition 3.10 (cut-off criterion). Let E be an lcHs, $k \in \mathbb{N}_{0,\infty}$, $\Omega \subset \mathbb{R}^d$ open and $\mathcal{V}^k$ be a family of weights on Ω. We say that $\mathcal{CV}^k_0(\Omega, E)$ satisfies *the cut-off criterion* if

$$\forall\, f \in \mathcal{CV}^k_0(\Omega, E),\ j \in J,\ l \in \langle k \rangle,\ \alpha \in \mathfrak{A}\ \exists\, \delta > 0\ \forall\, \varepsilon > 0\ \exists\, K \subset \Omega \text{ compact}:$$
$$\big(K + \overline{\mathbb{B}_\delta(0)}\big) \subset \Omega \quad \text{and} \quad |f|_{\Omega \setminus K, j,l,\alpha} < \varepsilon.$$

Remark 3.11. If $\Omega = \mathbb{R}^d$, then the cut-off criterion is satisfied for any $\delta > 0$.

Example 3.12. Let E be an lcHs, $k \in \mathbb{N}_{0,\infty}$ and $\Omega \subset \mathbb{R}^d$ open. The space $\mathcal{C}^k(\Omega, E)$ with the usual topology of uniform convergence of all partial derivatives up to order k on compact subsets of Ω and the Schwartz space $\mathcal{S}(\mathbb{R}^d, E)$ fulfil the cut-off criterion.

Proof. For the Schwartz space this follows directly from Example 3.5 b) and Remark 3.11. By Example 3.5 a) we have $\mathcal{C}^k(\Omega, E) = \mathcal{CW}^k_0(\Omega, E)$ with $\mathcal{W}^k := \{\nu_{j,l} := \chi_{\Omega_j} \mid j \in \mathbb{N}, l \in \langle k \rangle\}$ where $(\Omega_j)_{j \in \mathbb{N}}$ is a compact exhaustion of Ω. Choosing $K := \Omega_j$ and $\delta := \inf\{|z - x| \mid z \in \partial\Omega_j,\ x \in \partial\Omega_{j+1}\} > 0$ for $j \in \mathbb{N}$, we note that the cut-off criterion is fulfilled. ■

The proof of the density given below uses cut-off functions and the additional $\delta > 0$ independent of $\varepsilon > 0$ allows us to choose a suitable cut-off function whose derivatives can be estimated independently of ε. But first we recall the following definitions since we need the product rule. Let $\gamma, \beta \in \mathbb{N}^d_0$. We write $\gamma \le \beta$ if $\gamma_n \le \beta_n$ for all $1 \le n \le d$, and define

$$\binom{\beta}{\gamma} := \prod_{n=1}^{d} \binom{\beta_n}{\gamma_n}$$

if $\gamma\leq\beta$ where the right-hand side is defined by ordinary binomial coefficients. Now, we can phrase the product rule whose proof follows by induction (just adapt the proof for scalar-valued functions).

PROPOSITION 3.13 (product rule). *Let E be an lcHs, $k\in\mathbb{N}_{0,\infty}$, $\Omega\subset\mathbb{R}^d$ open, $f\in\mathcal{C}^k(\Omega,E)$ and $g\in\mathcal{C}^k(\Omega)$. Then $gf\in\mathcal{C}^k(\Omega,E)$ and*

$$(\partial^{\beta})^{E}(gf)(x)=\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}(\partial^{\beta-\gamma})^{\mathbb{K}}g(x)(\partial^{\gamma})^{E}f(x),\quad x\in\Omega,\ \beta\in\mathbb{N}_0^d,\ |\beta|\leq k.$$

LEMMA 3.14. *Let E be an lcHs, $k\in\mathbb{N}_{0,\infty}$ and $\mathcal{V}^k$ be a family of locally bounded weights on an open set $\Omega\subset\mathbb{R}^d$. If $\mathcal{CV}_0^k(\Omega,E)$ satisfies the cut-off criterion, then the space $\mathcal{C}_c^k(\Omega,E)$ is dense in $\mathcal{CV}_0^k(\Omega,E)$.*

Proof. The local boundedness of $\mathcal{V}^k$ yields that $\mathcal{C}_c^k(\Omega,E)$ is a linear subspace of $\mathcal{CV}_0^k(\Omega,E)$ by Remark 3.9 which we equip with the induced topology. Let $f\in\mathcal{CV}_0^k(\Omega,E)$, $j\in J$, $l\in\langle k\rangle$ and $\alpha\in\mathfrak{A}$. Due to the cut-off criterion there is $\delta>0$ such that for $\varepsilon>0$ there is $K\subset\Omega$ compact with $\big(K+\overline{\mathbb{B}_\delta(0)}\big)\subset\Omega$ and $|f|_{\Omega\setminus K,j,l,\alpha}<\varepsilon$. We choose a cut-off function $\psi\in\mathcal{C}_c^\infty(\Omega)$ with $0\leq\psi\leq 1$ so that $\psi=1$ in a neighbourhood of K and

$$\big|(\partial^{\beta})^{\mathbb{K}}\psi\big|\leq C_\beta\delta^{-|\beta|}$$

on Ω for all $\beta\in\mathbb{N}_0^d$ where $C_\beta>0$ only depends on β (see [12, Theorem 1.4.1, p. 25]). We set $K_0:=\operatorname{supp}\psi$, note that $\psi f\in\mathcal{C}_c^k(\Omega,E)$ by the product rule and

$$\begin{aligned}
|f-\psi f|_{j,l,\alpha}&=\sup_{\substack{x\in\Omega\setminus K\\ \beta\in\mathbb{N}_0^d,|\beta|\leq l}}p_\alpha\big((\partial^\beta)^E(f-\psi f)(x)\big)\nu_{j,l}(x)\\
&\leq\sup_{\substack{x\in\Omega\setminus K\\ \beta\in\mathbb{N}_0^d,|\beta|\leq l}}p_\alpha\big((\partial^\beta)^E f(x)\big)\nu_{j,l}(x)+\sup_{\substack{x\in\Omega\setminus K\\ \beta\in\mathbb{N}_0^d,|\beta|\leq l}}p_\alpha\big((\partial^\beta)^E(\psi f)(x)\big)\nu_{j,l}(x)\\
&=|f|_{\Omega\setminus K,j,l,\alpha}+\sup_{\substack{x\in(\Omega\setminus K)\cap K_0\\ \beta\in\mathbb{N}_0^d,|\beta|\leq l}}p_\alpha\Big(\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}(\partial^{\beta-\gamma})^{\mathbb{K}}\psi(x)(\partial^\gamma)^E f(x)\Big)\nu_{j,l}(x)\\
&\leq|f|_{\Omega\setminus K,j,l,\alpha}+\sup_{\substack{z\in K_0\\ \beta\in\mathbb{N}_0^d,|\beta|\leq l}}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\big|(\partial^{\beta-\gamma})^{\mathbb{K}}\psi(z)\big|\Big(\sup_{\substack{x\in\Omega\setminus K\\ \tau\in\mathbb{N}_0^d,|\tau|\leq l}}p_\alpha\big((\partial^\tau)^E f(x)\big)\nu_{j,l}(x)\Big)\\
&\leq|f|_{\Omega\setminus K,j,l,\alpha}+\underbrace{\sup_{\beta\in\mathbb{N}_0^d,|\beta|\leq l}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}C_{\beta-\gamma}\delta^{-|\beta-\gamma|}}_{=:C_{l,\delta}<\infty}|f|_{\Omega\setminus K,j,l,\alpha}\\
&=(1+C_{l,\delta})|f|_{\Omega\setminus K,j,l,\alpha}<(1+C_{l,\delta})\varepsilon.
\end{aligned}$$

The independence of $C_{l,\delta}$ from ε implies the statement. ∎

We complete this section by pointing out the link between our question on finite dimensional approximation and the tensor product. If $\mathcal{V}^k$ is locally bounded away from zero, there is a nice relation between our spaces of vector-valued functions and the ε-product which uses that the point-evaluation functionals $\delta_x\colon f\mapsto f(x)$ are continuous on $\mathcal{CV}^k(\Omega)$ by our definition of a weight.

Proposition 3.15. *Let E be an lcHs, $k \in \mathbb{N}_{0,\infty}$, $\mathcal{V}^k$ be a family of weights which is locally bounded away from zero on a locally compact Hausdorff space Ω $(k = 0)$ or an open set $\Omega \subset \mathbb{R}^d$ $(k > 0)$.*

a) *In addition, let $\mathcal{CV}_0^k(\Omega)$ be barrelled if $k > 0$. Then*

$$S_{\mathcal{CV}_0^k(\Omega)} \colon \mathcal{CV}_0^k(\Omega)\varepsilon E \to \mathcal{CV}_0^k(\Omega, E), \ u \longmapsto [x \mapsto u(\delta_x)],$$

is an isomorphism into, i.e. an isomorphism to its range.

b) *In addition, let $\mathcal{CV}^k(\Omega)$ be barrelled if $k > 0$. Then*

$$S_{\mathcal{CV}^k(\Omega)} \colon \mathcal{CV}^k(\Omega)\varepsilon E \to \mathcal{CV}^k(\Omega, E), \ u \longmapsto [x \mapsto u(\delta_x)],$$

is an isomorphism into.

Proof. Let $u \in \mathcal{CV}_0^k(\Omega)\varepsilon E$, resp. $\mathcal{CV}^k(\Omega)\varepsilon E$, and as a simplification we omit the index of S. The continuity of $S(u)$ is a consequence of [17, 4.1 Proposition, p. 18] and [17, 4.2 Lemma (i), p. 19] since $\mathcal{V}^k$ is locally bounded away from zero. If $k > 0$, then the continuous partial differentiability of $S(u)$ up to order k follows from [17, 4.12 Proposition, p. 22] as $\mathcal{CV}_0^k(\Omega)$, resp. $\mathcal{CV}^k(\Omega)$, is barrelled and $\mathcal{V}^k$ locally bounded away from zero. If $u \in \mathcal{CV}_0^k(\Omega)\varepsilon E$, then $S(u)$ vanishes together with all its derivatives when weighted at infinity by [17, 4.13 Proposition, p. 23]. Thanks to these observations [17, 3.9 Theorem, p. 9] proves our statement. ∎

In particular, if J is countable and $\mathcal{V}^k$ locally bounded away from zero, then the Fréchet spaces $\mathcal{CV}^k(\Omega)$ and $\mathcal{CV}_0^k(\Omega)$ are barrelled. This result allows us to identify the injective tensor product of $\mathcal{CV}^k(\Omega)$, resp. $\mathcal{CV}_0^k(\Omega)$, and E with a subspace of $\mathcal{CV}^k(\Omega, E)$, resp. $\mathcal{CV}_0^k(\Omega, E)$. Let us use the symbol $\mathcal{F}$ for $\mathcal{CV}^k$ or $\mathcal{CV}_0^k$. We consider $\mathcal{F}(\Omega) \otimes E$ as an algebraic subspace of $\mathcal{F}(\Omega)\varepsilon E$ by means of the linear injection

$$\Theta_{\mathcal{F}(\Omega)} \colon \mathcal{F}(\Omega) \otimes E \to \mathcal{F}(\Omega)\varepsilon E, \quad \sum_{n=1}^{m} f_n \otimes e_n \longmapsto \Big[y \mapsto \sum_{n=1}^{m} y(f_n)e_n\Big].$$

Via $\Theta_{\mathcal{F}(\Omega)}$ the topology of $\mathcal{F}(\Omega)\varepsilon E$ induces a locally convex topology on $\mathcal{F}(\Omega) \otimes E$ and $\mathcal{F}(\Omega) \otimes_\varepsilon E$ denotes $\mathcal{F}(\Omega) \otimes E$ equipped with this topology. From the preceding proposition and the composition $S_{\mathcal{F}(\Omega)} \circ \Theta_{\mathcal{F}(\Omega)}$ we obtain:

Corollary 3.16. *Let E be an lcHs, $k \in \mathbb{N}_{0,\infty}$, $\mathcal{V}^k$ be a family of weights which is locally bounded away from zero on a locally compact Hausdorff space Ω $(k = 0)$ or an open set $\Omega \subset \mathbb{R}^d$ $(k > 0)$. Fix the notation $\mathcal{F} = \mathcal{CV}^k$ or $\mathcal{CV}_0^k$ and let $\mathcal{F}(\Omega)$ be barrelled if $k > 0$.*

a) *We get by identification of isomorphic subspaces*

$$\mathcal{F}(\Omega) \otimes_\varepsilon E \subset \mathcal{F}(\Omega)\varepsilon E \subset \mathcal{F}(\Omega, E)$$

and the embedding $\mathcal{F}(\Omega) \otimes E \hookrightarrow \mathcal{F}(\Omega, E)$ is given by $f \otimes e \mapsto [x \mapsto f(x)e]$.

b) *Let $\mathcal{F}(\Omega)$ and E be complete. If $\mathcal{F}(\Omega) \otimes E$ is dense in $\mathcal{F}(\Omega, E)$, then*

$$\mathcal{F}(\Omega, E) \cong \mathcal{F}(\Omega)\varepsilon E \cong \mathcal{F}(\Omega)\widehat{\otimes}_\varepsilon E.$$

In particular, $\mathcal{F}(\Omega)$ has the approximation property if $\mathcal{F}(\Omega) \otimes E$ is dense in $\mathcal{F}(\Omega, E)$ for every complete E.

Proof.

a) The inclusions hold by Proposition 3.15 and $\mathcal{F}(\Omega)\varepsilon E$ and $\mathcal{F}(\Omega, E)$ induce the same topology on $\mathcal{F}(\Omega) \otimes E$. Further, we have

$$f \otimes e \overset{\Theta_{\mathcal{F}(\Omega)}}{\longmapsto} [y \mapsto y(f)e] \overset{S_{\mathcal{F}(\Omega)}}{\longmapsto} [x \longmapsto [y \mapsto y(f)e](\delta_x)] = [x \mapsto f(x)e].$$

b) If $\mathcal{F}(\Omega)$ and E are complete, then we obtain that $\mathcal{F}(\Omega)\varepsilon E$ is complete by [15, Satz 10.3, p. 234]. In addition, we get the completion of $\mathcal{F}(\Omega) \otimes_\varepsilon E$ as its closure in $\mathcal{F}(\Omega)\varepsilon E$ which coincides with the closure in $\mathcal{F}(\Omega, E)$. The rest follows directly from a). ■

Looking at part a), we derive

$$(S_{\mathcal{F}(\Omega)} \circ \Theta_{\mathcal{F}(\Omega)})\Big(\sum_{n=1}^{m} f_n \otimes e_n\Big) = \sum_{n=1}^{m} f_n e_n$$

for $m \in \mathbb{N}$, $f_n \in \mathcal{F}(\Omega)$ and $e_n \in E$, $1 \leq n \leq m$. Hence we see that the answer to our question is affirmative if $\mathcal{F}(\Omega) \otimes E$ is dense in $\mathcal{F}(\Omega, E)$. For the sake of completeness we remark the following.

PROPOSITION 3.17. *Let E be an lcHs, $k \in \mathbb{N}_{0,\infty}$, $\mathcal{V}^k$ be a family of weights which is locally bounded away from zero on a locally compact Hausdorff space Ω ($k = 0$) or an open set $\Omega \subset \mathbb{R}^d$ ($k > 0$).*

a) *In addition, let $\mathcal{CV}_0^k(\Omega)$ be barrelled if $k > 0$. If E is quasi-complete and $\mathcal{V}^k$ locally bounded on Ω, then*

$$\mathcal{CV}_0^k(\Omega)\varepsilon E \cong \mathcal{CV}_0^k(\Omega, E) \quad \text{via } S_{\mathcal{CV}_0^k(\Omega)}.$$

b) *In addition, let $\mathcal{CV}^k(\Omega)$ be barrelled if $k > 0$. If E is a semi-Montel space, then*

$$\mathcal{CV}^k(\Omega)\varepsilon E \cong \mathcal{CV}^k(\Omega, E) \quad \text{via } S_{\mathcal{CV}^k(\Omega)}.$$

Proof. For $k > 0$ this is [17, 5.10 Example a), p. 28], resp. [17, 3.21 Example a), p. 14]. Statement a) for $k = 0$ is a consequence of [17, 3.20 Corollary, p. 13] in combination with [17, 4.1 Proposition, p. 18], [17, 4.2 Lemma (i), p. 19] and [17, 4.13 Proposition, p. 23]. For $k = 0$ statement b) follows from [17, 3.19 Corollary, p. 13] in combination with [17, 4.1 Proposition, p. 18] and [17, 4.2 Lemma (i), p. 19]. ■

The corresponding results for $k = 0$ and a Nachbin-family $\mathcal{V}^0$ of weights are given in [3, 2.4 Theorem, p. 138–139] and [3, 2.12 Satz, p. 141]. In combination with our preceding observation, we deduce that every element of $\mathcal{CV}_0^k(\Omega, E)$ can be approximated in $\mathcal{CV}_0^k(\Omega, E)$ by functions with values in a finite dimensional subspace if E is a quasi-complete space with approximation property and the assumptions of the proposition above are fulfilled. The same is true for $\mathcal{CV}^k(\Omega, E)$ if E is a semi-Montel space with approximation property. Due to the strong conditions on E this is not really satisfying but actually the best we get for general $\mathcal{CV}^k(\Omega, E)$. For $\mathcal{CV}_0^k(\Omega, E)$ there is a better result available, whose proof we prepare on the next pages.

4. Convolution via the Pettis-integral. In this section we review the notion of the Pettis-integral. Trèves uses the Riemann-integral to define the convolution $f * g$ of a function $f \in \mathcal{C}_c^k(\Omega, E)$ and a function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ in the proof of Theorem 1.2 and states (without a proof) that the convolution defined in this way is a function in $\mathcal{C}_c^\infty(\mathbb{R}^d, \widehat{E})$ and has all the properties known from the convolution of two scalar-valued functions. We use the Pettis-integral instead to define the convolution. The reason is that we can use the dominated convergence theorem for the Pettis-integral [19, Theorem 2, p. 162–163] to get the Leibniz' rule for differentiation under the integral sign which enables us to prove that the convolution has some of the key properties known from the scalar-valued case.

Let us fix some notation first. For a measure space (X, Σ, μ) let

$$\mathfrak{L}^1(X,\mu) := \Big\{ f\colon X \to \mathbb{K} \text{ measurable} \,|\, q_1(f) := \int_X |f(x)| \,\mathrm{d}\mu(x) < \infty \Big\}$$

and define the quotient space of integrable functions with respect to the measure μ by $\mathcal{L}^1(X,\mu) := \mathfrak{L}^1(X,\mu)/\{f \in \mathfrak{L}^1(X,\mu) \,|\, q_1(f) = 0\}$. From now on we do not distinguish between equivalence classes and their representatives anymore. We say that $f\colon X \to \mathbb{K}$ is integrable on $\Lambda \in \Sigma$ and write $f \in \mathcal{L}^1(\Lambda,\mu)$ if $\chi_\Lambda f \in \mathcal{L}^1(X,\mu)$ where χ_Λ is the characteristic function of Λ. Then we set

$$\int_\Lambda f(x)\,\mathrm{d}\mu(x) := \int_X \chi_\Lambda(x) f(x)\,\mathrm{d}\mu(x).$$

DEFINITION 4.1 (Pettis-integral). Let (X, Σ, μ) be a measure space and E an lcHs. A function $f : X \to E$ is called *weakly* (*scalarly*) *measurable* if the function $e' \circ f : X \to \mathbb{K}$, $(e' \circ f)(x) := \langle e', f(x)\rangle := e'(f(x))$, is measurable for all $e' \in E'$. A weakly measurable function is said to be *weakly* (*scalarly*) *integrable* if $e' \circ f \in \mathcal{L}^1(X,\mu)$. A function $f\colon X \to E$ is called *Pettis-integrable* on $\Lambda \in \Sigma$ if it is weakly integrable on Λ and

$$\exists\, e_\Lambda \in E \;\forall\, e' \in E' : \langle e', e_\Lambda\rangle = \int_\Lambda \langle e', f(x)\rangle\,\mathrm{d}\mu(x).$$

In this case e_Λ is unique due to E being Hausdorff and we set

$$\int_\Lambda f(x)\,\mathrm{d}\mu(x) := e_\Lambda.$$

A function f is called Pettis-integrable on Σ if it is Pettis-integrable on all $\Lambda \in \Sigma$.

We write $\mathcal{N}_\mu$ for the set of μ-null sets of a measure space (X, Σ, μ) and for $\Lambda \in \Sigma$ we use the notion $(\Lambda, \Sigma_{\,|\,\Lambda}, \mu_{\,|\,\Lambda})$ for the restricted measure space given by $\Sigma_{\,|\,\Lambda} := \{\omega \in \Sigma \,|\, \omega \subset \Lambda\}$ and $\mu_{\,|\,\Lambda} := \mu_{\,|\,\Sigma_{\,|\,\Lambda}}$. If we consider the measure space $(\mathbb{R}^d, \mathscr{L}(\mathbb{R}^d), \lambda)$ of Lebesgue measurable sets, we just write $\mathrm{d}x := \mathrm{d}\lambda(x)$.

REMARK 4.2. Let (X, Σ, μ) be a measure space, E an lcHs and f Pettis-integrable on $\Lambda \in \Sigma$. If $\omega \in \Sigma$ such that $\omega \subset \Lambda$ and $(\Lambda \setminus \omega) \subset \{x \in X \,|\, f(x) = 0\}$, then f is Pettis-integrable on ω and

$$\int_\omega f(x)\,\mathrm{d}\mu(x) = \int_\Lambda f(x)\,\mathrm{d}\mu(x). \tag{5}$$

This follows directly from

$$\Big\langle e', \int_\Lambda f(x)\,\mathrm{d}\mu(x) \Big\rangle = \int_\Lambda \langle e', f(x)\rangle\,\mathrm{d}\mu(x) = \int_\omega \langle e', f(x)\rangle\,\mathrm{d}\mu(x), \quad e' \in E'.$$

Lemma 4.3. *Let E be a quasi-complete lcHs, (X,Σ,μ) a measure space, T a metric space and suppose that $f\colon X\times T\to E$ fulfils the following conditions.*

a) *$f(\cdot,t)$ is Pettis-integrable on Σ for all $t\in T$,*
b) *$f(x,\cdot)\colon T\to E$ is continuous in a point $t_0\in T$ for μ-almost all $x\in X$,*
c) *there is a neighbourhood $U\subset T$ of t_0 and a Pettis-integrable function ψ on Σ such that*

$$\forall\, t\in U,\ e'\in E'\ \exists\, N\in\mathcal{N}_\mu\ \forall\, x\in X\setminus N: |\langle e', f(x,t)\rangle| \leq |\langle e',\psi(x)\rangle|.$$

Then $g_\Lambda\colon T\to E$, $g_\Lambda(t):=\int_\Lambda f(x,t)\,\mathrm{d}\mu(x)$, is well-defined and continuous in t_0 for every $\Lambda\in\Sigma$.

Proof. Let $\Lambda\in\Sigma$ and (t_n) be a sequence in U converging to t_0. From the continuous dependency of a scalar integral on a parameter (see [7, 5.6 Satz, p. 147]) we derive

$$\lim_{n\to\infty}\int_\Lambda \langle e', \underbrace{f(x,t_n)}_{=:f_n(x)}\rangle\,\mathrm{d}\mu(x) = \int_\Lambda \langle e', \underbrace{f(x,t_0)}_{=:\widetilde{f}(x)}\rangle\,\mathrm{d}\mu(x). \tag{6}$$

For $n\in\mathbb{N}$ and $e'\in E'$ there is $N\in\mathcal{N}_\mu$ such that

$$|\langle e', f_n(x)\rangle| = |\langle e', f(x,t_n)\rangle| \leq |\langle e',\psi(x)\rangle| \tag{7}$$

for every $x\in X\setminus N$. Due to (6) for every $\Lambda\in\Sigma$ and $e'\in E'$, (7) and the quasi-completeness of E we can apply the dominated convergence theorem for the Pettis-integral [19, Theorem 2, p. 162–163] and deduce

$$\lim_{n\to\infty} g_\Lambda(t_n) = \lim_{n\to\infty}\int_\Lambda f_n(x)\,\mathrm{d}\mu(x) = \int_\Lambda \widetilde{f}(x)\,\mathrm{d}\mu(x) = g_\Lambda(t_0). \ \blacksquare$$

The next lemma is the Leibniz' rule for differentiation under the integral sign for the Pettis-integral.

Lemma 4.4 (Leibniz' rule). *Let E be a quasi-complete lcHs, (X,Σ,μ) a measure space, $T\subset\mathbb{R}^d$ open and suppose that $f\colon X\times T\to E$ fulfils the following conditions.*

a) *$f(\cdot,t)$ is Pettis-integrable on Σ for all $t\in T$,*
b) *there is a μ-null set $N_0\in\mathcal{N}_\mu$ with $f(x,\cdot)\in\mathcal{C}^1(T,E)$ for all $x\in X\setminus N_0$,*
c) *for every $j\in\mathbb{N}$, $1\leq j\leq d$, there is a Pettis-integrable function ψ_j on Σ such that*

$$\forall\, e'\in E'\ \exists\, N\in\mathcal{N}_\mu\ \forall\, x\in X\setminus(N\cup N_0): \left|(\partial_{t_j})^{\mathbb{K}}\langle e', f(x,\cdot)\rangle\right| \leq |\langle e',\psi_j(x)\rangle|.$$

Then $g_\Lambda\colon T\to E$, $g_\Lambda(t):=\int_\Lambda f(x,t)\,\mathrm{d}\mu(x)$, is well-defined for every $\Lambda\in\Sigma$, $g_\Lambda\in\mathcal{C}^1(T,E)$ and

$$(\partial_{t_j})^E g_\Lambda(t) = \int_\Lambda (\partial_{t_j})^E f(x,t)\,\mathrm{d}\mu(x),\quad t\in T.$$

Proof. First, we consider the case $\mathbb{K}=\mathbb{R}$. Let $\Lambda\in\Sigma$, $j\in\mathbb{N}$, $1\leq j\leq d$, $t\in T$ and (h_n) be a real sequence converging to 0 such that $h_n\neq 0$ and $t+h_ne_j\in T$ for all n where e_j is the j-th unit vector in $\mathbb{R}^d$. Then

$$\frac{g_\Lambda(t+h_ne_j)-g_\Lambda(t)}{h_n} = \int_\Lambda \underbrace{\frac{f(x,t+h_ne_j)-f(x,t)}{h_n}}_{=:f_n(x)}\,\mathrm{d}\mu(x).$$

We define the function $\widetilde{f}\colon X \to E$ given by $\widetilde{f}(x) := (\partial_{t_j})^E f(x,t)$ for $x \in X \setminus N_0$ and $\widetilde{f}(x) := 0$ for $x \in N_0$. We observe that

$$\begin{aligned}\lim_{n\to\infty}\int_\Lambda \langle e', f_n(x)\rangle \,\mathrm{d}\mu(x) &= \int_\Lambda (\partial_{t_j})^{\mathbb{K}}\langle e', f(x,t)\rangle \,\mathrm{d}\mu(x)\\ &= \int_\Lambda \langle e', \partial^E_{t_j} f(x,t)\rangle \,\mathrm{d}\mu(x) = \int_\Lambda \langle e', \widetilde{f}(x)\rangle \,\mathrm{d}\mu(x) \qquad (8)\end{aligned}$$

holds for every $e' \in E'$ where we used the scalar Leibniz' rule for differentiation under the integral sign for the first equation which can be applied due to our assumptions (see [7, 5.7 Satz, p. 147–148]). For $e' \in E'$ there is $N \in \mathcal{N}_\mu$ such that for every $x \in X \setminus (N \cup N_0)$ and $n \in \mathbb{N}$ there is $\theta \in [0,1]$ with

$$\langle e', f_n(x)\rangle = \frac{\langle e', f(x,t+h_n e_j)\rangle - \langle e', f(x,t)\rangle}{h_n} = (\partial_{t_j})^{\mathbb{K}}\langle e', f(x,t+\theta h_n e_j)\rangle$$

by the mean value theorem ($\mathbb{K} = \mathbb{R}$) implying

$$|\langle e', f_n(x)\rangle| = |(\partial_{t_j})^{\mathbb{K}}\langle e', f(x,t+\theta h_n e_j)\rangle| \leq |\langle e', \psi_j(x)\rangle|. \qquad (9)$$

Due to (8) for every $\Lambda \in \Sigma$ and $e' \in E'$, (9) and the quasi-completeness of E we can apply the dominated convergence theorem for the Pettis-integral [19, Theorem 2, p. 162–163] again and obtain that $\widetilde{f}$ is Pettis-integrable on Σ plus

$$\begin{aligned}(\partial_{t_j})^E g_\Lambda(t) &= \lim_{n\to\infty}\frac{g_\Lambda(t+h_n e_j) - g_\Lambda(t)}{h_n} = \lim_{n\to\infty}\int_\Lambda f_n(x)\,\mathrm{d}\mu(x)\\ &= \int_\Lambda \widetilde{f}(x)\,\mathrm{d}\mu(x) = \int_\Lambda (\partial_{t_j})^E f(x,t)\,\mathrm{d}\mu(x).\end{aligned}$$

The continuity of $(\partial_{t_j})^E g_\Lambda$ follows from Lemma 4.3 by replacing f with $(\partial_{t_j})^E f$. For $\mathbb{K} = \mathbb{C}$ we just have to substitute $\langle e', \cdot\rangle$ by $\mathrm{Re}\langle e', \cdot\rangle$ (real part) and $\mathrm{Im}\langle e', \cdot\rangle$ (imaginary part) in the considerations above. ∎

Now, we are able to define the convolution of a vector-valued and a scalar-valued continuous function via the Pettis-integral, if one of them has compact support, and to show some of its basic properties which are known from the convolution of scalar-valued functions (scalar convolution). For the properties of the scalar convolution see e.g. [23, Chap. 26, p. 278–283].

LEMMA 4.5. *Let E be a quasi-complete lcHs, $k, n \in \mathbb{N}_{0,\infty}$, $f \in \mathcal{C}^k(\mathbb{R}^d, E)$ and $g \in \mathcal{C}^n(\mathbb{R}^d)$, either one having compact support. The convolution*

$$f * g\colon \mathbb{R}^d \to E, \quad (f*g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y,$$

is well-defined, $\operatorname{supp}(f*g) \subset \operatorname{supp} f + \operatorname{supp} g$, $f*g = g*f$, *where*

$$g * f\colon \mathbb{R}^d \to E, \quad (g*f)(x) := \int_{\mathbb{R}^d} g(y)f(x-y)\,\mathrm{d}y,$$

and $f * g \in \mathcal{C}^n(\mathbb{R}^d, E)$ *plus*

$$(\partial^\beta)^E(f*g) = f * \big((\partial^\beta)^{\mathbb{K}} g\big), \quad |\beta| \leq n, \qquad (10)$$

$$(\partial^\beta)^E(f*g) = \big((\partial^\beta)^E f\big) * g, \quad |\beta| \leq \min(k,n). \qquad (11)$$

Proof. Let $h\colon \mathbb{R}^d\times\mathbb{R}^d\to E$, $h(y,x):=f(y)g(x-y)$. First, we show that $h(\cdot,x)$ is Pettis-integrable on $\mathscr{L}(\mathbb{R}^d)$ for every $x\in\mathbb{R}^d$ implying that $f*g$ is well-defined. We note that $\langle e',h(\cdot,x)\rangle\in\mathcal{L}^1(\mathbb{R}^d,\lambda)$ for every $e'\in E'$ and $x\in\mathbb{R}^d$. Let $x\in\mathbb{R}^d$ and $\Lambda\in\mathscr{L}(\mathbb{R}^d)$. We define the linear map

$$I_{\Lambda,x}\colon E'\to\mathbb{K},\ I_{\Lambda,x}(e'):=\int_{\Lambda}\langle e',h(y,x)\rangle\,\mathrm{d}y.$$

Setting $K_f:=\operatorname{supp} f$ and $K_g:=\operatorname{supp} g$, we observe that

$$I_{\Lambda,x}(e')=\int_{\Lambda\cap K_f}\langle e',f(y)g(x-y)\rangle\,\mathrm{d}y=\int_{\Lambda\cap(x-K_g)}\langle e',f(y)g(x-y)\rangle\,\mathrm{d}y.$$

If $K_f=\operatorname{supp} f$ is compact, we get

$$|I_{\Lambda,x}(e')|\leq\lambda(K_f)\sup\bigl\{|e'(z)|\mid z\in f(K_f)g(x-K_f)\bigr\}.$$

The set $f(K_f)g(x-K_f)$ is compact in E and thus the closure of its absolutely convex hull is compact in E as well by [24, 9-2-10 Example, p. 134] because E is quasi-complete. Hence it follows that $I_{\Lambda,x}\in(E'_\kappa)'\cong E$ by the theorem of Mackey–Arens meaning that there is $e_\Lambda(x)\in E$ such that

$$\langle e',e_\Lambda(x)\rangle=I_{\Lambda,x}(e')=\int_{\Lambda}\langle e',h(y,x)\rangle\,\mathrm{d}y$$

for all $e'\in E'$. Thus $h(\cdot,x)$ is Pettis-integrable on $\mathscr{L}(\mathbb{R}^d)$ and

$$(f*g)(x)=e_{\mathbb{R}^d}(x)\underset{(5)}{=}e_{K_f}(x)=e_{x-K_g}(x)$$

for every $x\in\mathbb{R}^d$ if $K_f=\operatorname{supp} f$ is compact. If $K_g=\operatorname{supp} g$ is compact, then the estimate

$$|I_{\Lambda,x}(e')|\leq\lambda(x-K_g)\sup\bigl\{|e'(z)|\mid z\in f(x-K_g)g(K_g)\bigr\}$$

yields to the Pettis-integrability in the same manner.

Let $x\notin\operatorname{supp} f+\operatorname{supp} g$. If $y\notin\operatorname{supp} f$, then $h(y,x)=0$. If $y\in\operatorname{supp} f$, then $x-y\notin\operatorname{supp} g$ and thus $h(y,x)=0$. Hence we have $h(\cdot,x)=0$ implying $\operatorname{supp}(f*g)\subset\operatorname{supp} f+\operatorname{supp} g$. From

$$\begin{aligned}\langle e',(f*g)(x)\rangle&=\int_{\mathbb{R}^d}\langle e',f(y)g(x-y)\rangle\,\mathrm{d}y=\int_{\mathbb{R}^d}\langle e',f(y)\rangle g(x-y)\,\mathrm{d}y\\&=\bigl((e'\circ f)*g\bigr)(x)=\bigl(g*(e'\circ f)\bigr)(x)=\int_{\mathbb{R}^d}\langle e',g(y)f(x-y)\rangle\,\mathrm{d}y\end{aligned}$$

for every $x\in\mathbb{R}^d$ and $e'\in E'$, where we used the commutativity of scalar convolution for the fourth equation, it follows that

$$(f*g)(x)=e_{\mathbb{R}^d}(x)=(g*f)(x)$$

for every $x\in\mathbb{R}^d$.

Next, we show that $f*g\in\mathcal{C}^n(\mathbb{R}^d,E)$ and (10) holds by applying Lemma 4.3 and 4.4. So we have to check that the conditions a)–c) of these lemmas are fulfilled. First, fix $x_0\in\mathbb{R}^d$, let $\varepsilon>0$ and $\beta\in\mathbb{N}_0^d$, $|\beta|\leq n$. If $K_f=\operatorname{supp} f$ is compact, we set $h_{f,\beta}:=(\partial_x^\beta)^E h_{\mid K_f\times\mathbb{B}_\varepsilon(x_0)}$ and observe that $h_{\mid K_f\times\mathbb{B}_\varepsilon(x_0)}(y,\cdot)\in\mathcal{C}^n(\mathbb{B}_\varepsilon(x_0),E)$ for every $y\in K_f$

(condition b)). It follows from the theorem of Mackey–Arens and

$$\Big|\int_{\omega}\langle e', h_{f,\beta}(y,x)\rangle \,\mathrm{d}Y\Big| \leq \lambda(K_f)\sup\big\{|e'(z)| \mid z\in f(K_f)(\partial^{\beta})^{\mathbb{K}}g(\overline{\mathbb{B}_{\varepsilon}(x_0)}-K_f)\big\}$$

for every $e'\in E'$, $\omega\in\mathscr{L}(\mathbb{R}^d)_{\mid K_f}$ and $x\in\mathbb{B}_{\varepsilon}(x_0)$ that $h_{f,\beta}(\cdot,x)$ is Pettis-integrable on $\mathscr{L}(\mathbb{R}^d)_{\mid K_f}$ for every $x\in\mathbb{B}_{\varepsilon}(x_0)$ (condition a)). Now, we check that condition c) is satisfied. We observe that the estimate

$$\Big|\int_{\omega}\langle e', f(y)\rangle \,\mathrm{d}y\Big| \leq \lambda(K_f)\sup\big\{|e'(z)| \mid z\in f(K_f)\big\}$$

for every $e'\in E'$ and $\omega\in\mathscr{L}(\mathbb{R}^d)_{\mid K_f}$ implies that $f_{\mid K_f}$ is Pettis-integrable on $\mathscr{L}(\mathbb{R}^d)_{\mid K_f}$ due to the theorem of Mackey–Arens again. The inequality

$$\begin{aligned}|\langle e', h_{f,\beta}(y,x)\rangle| &= \big|\langle e', f(y)(\partial^{\beta}_{x})^{\mathbb{K}}[x\mapsto g(x-y)]\rangle\big|\\ &\leq |\langle e', f(y)\rangle|\sup\big\{|(\partial^{\beta})^{\mathbb{K}}g(z)| \mid z\in\overline{\mathbb{B}_{\varepsilon}(x_0)}-K_f\big\}\\ &\leq \big|\langle e', q_{\overline{\mathbb{B}_{\varepsilon}(x_0)}-K_f,n}(g)\cdot f(y)\rangle\big|\end{aligned}$$

for every $e'\in E'$ and $(y,x)\in K_f\times\mathbb{B}_{\varepsilon}(x_0)$ with the seminorm $q_{\overline{\mathbb{B}_{\varepsilon}(x_0)}-K_f,n}$ from (1) yields to condition c) being satisfied. Hence $f*g\in\mathcal{C}^n(\mathbb{B}_{\varepsilon}(x_0),E)$ by Lemma 4.3 if $n=0$ and by Lemma 4.4 if $n=1$ as well as

$$\begin{aligned}\partial^{E}_{x_j}(f*g)(x) &= \partial^{E}_{x_j}\Big[x\mapsto\int_{\mathbb{R}^d}f(y)g(x-y)\,\mathrm{d}y\Big] \underset{(5)}{=} \partial^{E}_{x_j}\Big[x\mapsto\int_{K_f}f(y)g(x-y)\,\mathrm{d}y\Big]\\ &= \int_{K_f}f(y)(\partial_{x_j})^{\mathbb{K}}[x\mapsto g(x-y)]\,\mathrm{d}y \underset{(5)}{=} \int_{\mathbb{R}^d}f(y)(\partial^{e_j})^{\mathbb{K}}g(x-y)\,\mathrm{d}y\\ &= \big(f*\big((\partial^{e_j})^{\mathbb{K}}g\big)\big)(x)\end{aligned}$$

for every $x\in\mathbb{B}_{\varepsilon}(x_0)$. Letting $\varepsilon\to\infty$, we obtain the result for $n=0$ and $n=1$ if $K_f=\operatorname{supp}f$ is compact. For $n\geq 2$ it follows from induction on the order $|\beta|$. If $K_g=\operatorname{supp}g$ is compact, the same approach with $h_{g,\beta}:=(\partial^{\beta}_{x})^{E}h_{\mid K_g\times\mathbb{B}_{\varepsilon}(x_0)}$ instead of $h_{f,\beta}$ proves the statement. Furthermore, for $|\beta|\leq\min(k,n)$ we get

$$\begin{aligned}&\langle e', (\partial^{\beta})^{E}(f*g)(x)\rangle\\ &= \int_{\mathbb{R}^d}\langle e', f(y)(\partial^{\beta})^{\mathbb{K}}g(x-y)\rangle\,\mathrm{d}y = \int_{\mathbb{R}^d}(e'\circ f)(y)(\partial^{\beta})^{\mathbb{K}}g(x-y)\,\mathrm{d}y\\ &= \big((e'\circ f)*\big((\partial^{\beta})^{\mathbb{K}}g\big)\big)(x) = \big((\partial^{\beta})^{\mathbb{K}}(e'\circ f)*g\big)(x)\\ &= \big((e'\circ(\partial^{\beta})^{E}f)*g\big)(x) = \int_{\mathbb{R}^d}\langle e', (\partial^{\beta})^{E}f(y)g(x-y)\rangle\,\mathrm{d}y\end{aligned}$$

for every $e'\in E'$ and $x\in\mathbb{R}^d$, where we used the corresponding result for the scalar convolution for the fourth equation, implying $(\partial^{\beta})^{E}(f*g)=\big((\partial^{\beta})^{E}f\big)*g$. ■

Looking at the lemma above, we see that it differs a bit from the properties known from the convolution of two scalar-valued functions. It is an open problem whether we actually have $f*g\in\mathcal{C}^{\max(k,n)}(\mathbb{R}^d,E)$ and (11) for $|\beta|\leq k$ under the assumptions of the lemma. But since we only apply the lemma above in the case $n=\infty$, this does not affect us.

We recall the construction of a mollifier from [23, p. 155–156]. Let

$$\rho\colon \mathbb{R}^d \to \mathbb{R}, \quad \rho(x) := \begin{cases} C\exp(-1/(1-|x|^2)), & |x|<1,\\ 0, & |x|\geq 1,\end{cases}$$

where $C := \left(\int_{\mathbb{B}_1(0)} \exp(-\frac{1}{1-|x|^2})\,\mathrm{d}x\right)^{-1}$. For $n\in\mathbb{N}$ we define the mollifier ρ_n given by $\rho_n(x) := n^d\rho(nx)$, $x\in\mathbb{R}^d$. Then we have $\rho_n \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\rho_n\geq 0$, $\operatorname{supp}\rho_n = \overline{\mathbb{B}_{1/n}(0)}$ and $\int_{\mathbb{R}^d}\rho_n(x)\,\mathrm{d}x = 1$.

We can extend a function $f\in\mathcal{C}_c^k(\Omega,E)$, $k\in\mathbb{N}_{0,\infty}$ and $\Omega\subset\mathbb{R}^d$, to a function $f_{\mathrm{ex}}\in\mathcal{C}_c^k(\mathbb{R}^d,E)$ by setting $f_{\mathrm{ex}}:=f$ on Ω and $f_{\mathrm{ex}}:=0$ on $\mathbb{R}^d\setminus\Omega$. In this way the convolution $f*g := (f_{\mathrm{ex}}*g)_{\mid\Omega}$ with a function $g\in\mathcal{C}(\mathbb{R}^d)$ is a well-defined function on Ω if E is quasi-complete, and we have the following approximation by regularisation in analogy to the scalar-valued case (see e.g. [23, Chap. 15, Corollary 1, p. 158]).

Lemma 4.6. *Let E be a quasi-complete lcHs, $k\in\mathbb{N}_{0,\infty}$, $\mathcal{V}^k$ be a family of locally bounded weights on an open set $\Omega\subset\mathbb{R}^d$ and $f\in\mathcal{C}_c^k(\Omega,E)$. Then $(f*\rho_n)$ converges to f in $\mathcal{CV}_0^k(\Omega,E)$ as $n\to\infty$.*

Proof. Due to Lemma 4.5 we obtain that $f_{\mathrm{ex}}*\rho_n\in\mathcal{C}_c^\infty(\mathbb{R}^d,E)$ for every $n\in\mathbb{N}$. Since $\mathcal{V}^k$ is locally bounded on Ω, we derive $f*\rho_n\in\mathcal{CV}_0^k(\Omega,E)$. Let $\varepsilon>0$, $j\in J$, $l\in\langle k\rangle$ and $\alpha\in\mathfrak{A}$. For $\beta\in\mathbb{N}_0^d$, $|\beta|\leq l$, there is $\delta_\beta>0$ such that for all $x\in\Omega$ and $y\in\mathbb{R}^d$ with $|y|=|(x-y)-x|\leq\delta_\beta$ we have

$$p_\alpha\big((\partial^\beta)^E f_{\mathrm{ex}}(x-y)-(\partial^\beta)^E f(x)\big)<\varepsilon \tag{12}$$

because the function $(\partial^\beta)^E f_{\mathrm{ex}}$ is uniformly continuous on whole $\mathbb{R}^d$ as it is continuous with compact support. Therefore we deduce for all $n>1/\delta_\beta$ that $\operatorname{supp}\rho_n=\overline{\mathbb{B}_{1/n}(0)}\subset\overline{\mathbb{B}_{\delta_\beta}(0)}$ and hence

$$\begin{aligned}
&p_\alpha\big((\partial^\beta)^E(f*\rho_n-f)(x)\big)\\
&\underset{(11)}{=} p_\alpha\big(\big((\partial^\beta)^E f\big)*\rho_n(x)-(\partial^\beta)^E f(x)\big)\\
&= p_\alpha\big(\rho_n*\big((\partial^\beta)^E f\big)(x)-(\partial^\beta)^E f(x)\big)\\
&= p_\alpha\Big(\int_{\mathbb{R}^d}(\partial^\beta)^E f_{\mathrm{ex}}(x-y)\rho_n(y)\,\mathrm{d}y-(\partial^\beta)^E f(x)\Big)\\
&= p_\alpha\Big(\int_{\mathbb{R}^d}(\partial^\beta)^E f_{\mathrm{ex}}(x-y)\rho_n(y)-(\partial^\beta)^E f(x)\rho_n(y)\,\mathrm{d}y\Big)\\
&\underset{(5)}{=} p_\alpha\Big(\int_{\overline{\mathbb{B}_{1/n}(0)}}(\partial^\beta)^E f_{\mathrm{ex}}(x-y)\rho_n(y)-(\partial^\beta)^E f(x)\rho_n(y)\,\mathrm{d}y\Big)\\
&\underset{(12)}{\leq} \varepsilon\int_{\mathbb{R}^d}\rho_n(y)\,\mathrm{d}y=\varepsilon
\end{aligned}$$

by Lemma 4.5 for every $x\in\Omega$. As $0\in\operatorname{supp}\rho_n$, we get

$$\operatorname{supp}(\partial^\beta)^E(f*\rho_n-f)\subset(\operatorname{supp}f+\operatorname{supp}\rho_n)=\big(\operatorname{supp}f+\overline{\mathbb{B}_{1/n}(0)}\big)$$

for every $|\beta|\leq l$ and $n\in\mathbb{N}$ by virtue of Lemma 4.5. Since $\operatorname{supp}f\subset\Omega$ is compact and Ω open, there is $r>0$ such that $\big(\operatorname{supp}f+\overline{\mathbb{B}_r(0)}\big)\subset\Omega$ yielding

$$\operatorname{supp}(\partial^\beta)^E(f*\rho_n-f)\subset\big(\operatorname{supp}f+\overline{\mathbb{B}_r(0)}\big)=:K$$

for all $n \geq 1/r$. Choosing $\delta := \min\{\delta_\beta \mid \beta \in \mathbb{N}_0^d, |\beta| \leq l\} > 0$, we obtain for all $n > \max\{1/\delta, 1/r\}$ that

$$\left|f * \rho_n - f\right|_{j,l,\alpha} = \sup_{\substack{x\in K\\ \beta\in\mathbb{N}_0^d, |\beta|\leq l}} p_\alpha\big((\partial^\beta)^E(f*\rho_n - f)(x)\big)\nu_{j,l}(x) \leq \varepsilon \sup_{x\in K} \nu_{j,l}(x)$$

which implies our statement since $\mathcal{V}^k$ is locally bounded on Ω and $K \subset \Omega$ is compact. ∎

5. Approximation property. Finally, we dedicate our last section to our main theorem. We start with the case $k = 0$.

PROPOSITION 5.1. *Let E be an lcHs and $\mathcal{V}^0$ a family of locally bounded weights which is locally bounded away from zero on a locally compact Hausdorff space Ω. Then the following statements hold.*

a) *$\mathcal{C}_c^0(\Omega) \otimes E$ is dense in $\mathcal{CV}_0^0(\Omega, E)$.*
b) *For any $f \in \mathcal{C}_c^0(\Omega, E)$ and any open neighbourhood V of* $\operatorname{supp} f$*, for every $\varepsilon > 0$, $j \in J$ and $\alpha \in \mathfrak{A}$, there is $g \in \mathcal{C}_c^0(\Omega) \otimes E$ such that* $\operatorname{supp} g \subset V$ *and $|f - g|_{j,0,\alpha} \leq \varepsilon$.*
c) *If E is complete, then*

$$\mathcal{CV}_0^0(\Omega, E) \cong \mathcal{CV}_0^0(\Omega)\varepsilon E \cong \mathcal{CV}_0^0(\Omega)\widehat{\otimes}_\varepsilon E.$$

d) *$\mathcal{CV}_0^0(\Omega)$ has the approximation property.*

Proof. First, we consider part a). Due to Corollary 3.16 a) and Remark 3.9 $\mathcal{C}_c^0(\Omega) \otimes E$ can be identified with a subspace of $\mathcal{CV}_0^0(\Omega, E)$ equipped with the induced topology since $\mathcal{V}^0$ is locally bounded and locally bounded away from zero.

Let $f \in \mathcal{CV}_0^0(\Omega, E)$, $\varepsilon > 0$, $j \in J$ and $\alpha \in \mathfrak{A}$ and fix the notation $\nu_j := \nu_{j,0}$. Then there is a compact set $\widetilde{K} \subset \Omega$ such that

$$|f|_{\Omega\setminus\widetilde{K},j,0,\alpha} = \sup_{x\in\Omega\setminus\widetilde{K}} p_\alpha(f(x))\nu_j(x) < \varepsilon.$$

Let $K := \widetilde{K}$. Since Ω is locally compact, every $w \in K$ has an open, relatively compact neighbourhood $U_w \subset \Omega$. As K is compact and $K \subset \bigcup_{w\in K} U_w$, there are $m \in \mathbb{N}$ and $w_i \in K$, $1 \leq i \leq m$, such that

$$K \subset \bigcup_{i=1}^m U_{w_i} =: W \subset \Omega.$$

The set W is open and relatively compact because it is a finite union of open, relatively compact sets. The local boundedness of $\mathcal{V}^0$ and relative compactness of W imply that

$$N := 1 + \sup_{x\in\overline{W}} \nu_j(x) < \infty.$$

For $x \in K$ we define $V_x := \{y \in \Omega \mid p_\alpha(f(y) - f(x)) < \frac{\varepsilon}{N}\}$. Then $V_x = f^{-1}\big(B_\alpha(f(x), \frac{\varepsilon}{N})\big)$, where $B_\alpha(f(x), \frac{\varepsilon}{N}) := \{e \in E \mid p_\alpha(e - f(x)) < \frac{\varepsilon}{N}\}$, implying that V_x is open in Ω since f is continuous. Hence we get $K \subset \bigcup_{x\in K} V_x$ and conclude that there are $n \in \mathbb{N}$ and $x_i \in K$, $1 \leq i \leq n$, such that $K \subset \bigcup_{i=1}^n V_{x_i}$ from the compactness of K. We note that

$$K = (K \cap \overline{W}) \subset \bigcup_{i=1}^n (V_{x_i} \cap \overline{W}). \tag{13}$$

The sets $V_{x_i}\cap\overline{W}$ are open in the compact Hausdorff space $\overline{W}$ with respect to the topology induced by Ω. Since the compact Hausdorff space $\overline{W}$ is normal by [4, Chap. IX, §4.1, Proposition 1, p. 181] and K is closed in $\overline{W}$, there is a family of non-negative real-valued continuous functions (φ_i) with $\operatorname{supp}\varphi_i\subset(V_{x_i}\cap\overline{W})$ such that $\sum_{i=1}^{n}\varphi_i=1$ on K and $\sum_{i=1}^{n}\varphi_i\leq 1$ on $\overline{W}$ by [4, Chap. IX, §4.3, Corollary, p. 186]. By trivially extending φ_i on $\Omega\setminus\overline{W}$, we obtain $\varphi_i\in\mathcal{C}^0_c(\Omega)$ because $\overline{W}$ is compact. We define

$$g:=\sum_{i=1}^{n}\varphi_i\otimes f(x_i)\in\mathcal{C}^0_c(\Omega)\otimes E$$

and observe $\operatorname{supp}g\subset\bigcup_{i=1}^{n}(V_{x_i}\cap\overline{W})$. If $x\in K$, then $\varphi_i(x)p_\alpha(f(x)-f(x_i))=0$ if $x\notin V_{x_i}\cap\overline{W}$, and

$$\begin{aligned}p_\alpha(f(x)-g(x))&=p_\alpha\Big(\sum_{i=1}^{n}\varphi_i(x)(f(x)-f(x_i))\Big)\leq\sum_{i=1}^{n}\varphi_i(x)p_\alpha(f(x)-f(x_i))\\&\leq\sum_{i=1}^{n}\varphi_i(x)\frac{\varepsilon}{N}=\frac{\varepsilon}{N}\end{aligned}$$

yielding to

$$\sup_{x\in K}p_\alpha((f-g)(x))\nu_j(x)\leq\sup_{x\in K}\frac{\varepsilon}{N}\nu_j(x)\leq\sup_{x\in\overline{W}}\frac{\varepsilon}{N}\nu_j(x)=\frac{\varepsilon}{N}\cdot(N-1)<\varepsilon.$$

If $x\notin K$, then $\varphi_i(x)f(x_i)=0$ if $x\notin(V_{x_i}\cap\overline{W})\setminus K$. If $x\in(V_{x_i}\cap\overline{W})\setminus K$, then

$$p_\alpha(\varphi_i(x)f(x_i))\leq\varphi_i(x)\big(p_\alpha(f(x_i)-f(x))+p_\alpha(f(x))\big)\leq\varphi_i(x)\Big(\frac{\varepsilon}{N}+p_\alpha(f(x))\Big)$$

yielding to

$$\begin{aligned}&|f-g|_{\Omega\setminus K,j,0,\alpha}\\&=\sup_{x\in\Omega\setminus K}p_\alpha((f-g)(x))\nu_j(x)\leq\sup_{x\in\Omega\setminus K}\big(p_\alpha(f(x))+p_\alpha(g(x))\big)\nu_j(x)\\&\leq\varepsilon+\sup_{x\in\Omega\setminus K}\sum_{i=1}^{n}p_\alpha(\varphi_i(x)f(x_i))\nu_j(x)\leq\varepsilon+\sup_{x\in\Omega\setminus K}\sum_{i=1}^{n}\varphi_i(x)\Big(\frac{\varepsilon}{N}+p_\alpha(f(x))\Big)\nu_j(x)\\&\leq 2\varepsilon+\frac{\varepsilon}{N}\sup_{x\in\Omega\setminus K}\sum_{i=1}^{n}\varphi_i(x)\nu_j(x)\leq 2\varepsilon+\frac{\varepsilon}{N}\sup_{x\in\overline{W}}\sum_{i=1}^{n}\varphi_i(x)\nu_j(x)\\&\leq 2\varepsilon+\frac{\varepsilon}{N}\cdot(N-1)<3\varepsilon\end{aligned}$$

implying

$$|f-g|_{j,0,\alpha}<4\varepsilon$$

which proves part a).

Part c) follows from a) and Corollary 3.16 b) because $\mathcal{CV}^0_0(\Omega)$ is complete by Proposition 3.7. Part d) is implied by part c). Let us turn to part b). Let $f\in\mathcal{C}^0_c(\Omega,E)$ and V be an open neighbourhood of $\widetilde{K}:=\operatorname{supp}f$. Then we can replace (13) by

$$K=(K\cap V\cap\overline{W})\subset\bigcup_{i=1}^{n}(V_{x_i}\cap V\cap\overline{W})$$

and then the open sets V_{x_i} by the open sets $V_{x_i} \cap V$ in what follows (13). This gives

$$\operatorname{supp} g \subset \Big(\bigcup_{i=1}^{n}(V_{x_i} \cap V \cap \overline{W})\Big) \subset V$$

proving b). ∎

If Ω is an open subset of $\mathbb{R}^d$, we can choose a smooth partition of unity (see e.g. [12, Theorem 1.4.5, p. 28]) and even deduce that $\mathcal{C}_c^\infty(\Omega) \otimes E$ is dense in $\mathcal{CV}_0^0(\Omega, E)$ under the assumptions of the proposition above.

The proof of part a) is a modification of the proof of [2, 5.1 Satz, p. 204] by Bierstedt. Since Ω is locally compact and not just a completely regular Hausdorff space, we can use the partition of unity from [4, Chap. IX, §4.1, Proposition 1, p. 181]. Bierstedt has to use the partition of unity from [20, 23, Lemma 2, p. 71] and due to the assumptions of this lemma he cannot choose $K = \widetilde{K}$ but has to use

$$K' := \{x \in \Omega \mid p_\alpha(f(x))\nu_j(x) \geq \varepsilon\} \subset \widetilde{K}.$$

Bierstedt's assumption that ν_j is upper semi-continuous guarantees that K' is closed and thus compact as a closed subset of the compact set $\widetilde{K}$. Choosing $K := K'$, the proof above works as well where the existence of the open set $W \subset \Omega$ is a consequence of the upper semi-continuity of ν_j again. Comparing Theorem 1.1 and Proposition 5.1, we see that Theorem 1.1 is far more general concerning the spaces Ω involved but the condition of $\mathcal{V}^0$ being a locally bounded family in Proposition 5.1 is weaker than the condition of being a family of upper semi-continuous weights in Theorem 1.1. Let us phrase our main theorem.

THEOREM 5.2. *Let E be an lcHs, $k \in \mathbb{N}_\infty$ and $\mathcal{V}^k$ be a family of locally bounded weights which is locally bounded away from zero on an open set $\Omega \subset \mathbb{R}^d$. Let $\mathcal{CV}_0^k(\Omega)$ be barrelled and $\mathcal{C}_c^k(\Omega, E)$ dense in $\mathcal{CV}_0^k(\Omega, E)$. Then the following statements hold.*

a) *$\mathcal{C}_c^\infty(\Omega) \otimes E$ is dense in $\mathcal{CV}_0^k(\Omega, E)$.*
b) *If E is complete, then*

$$\mathcal{CV}_0^k(\Omega, E) \cong \mathcal{CV}_0^k(\Omega)\varepsilon E \cong \mathcal{CV}_0^k(\Omega)\widehat{\otimes}_\varepsilon E.$$

c) *$\mathcal{CV}_0^k(\Omega)$ has the approximation property.*

Proof. It suffices to prove part a) because part b) follows from a) and Corollary 3.16 b) since $\mathcal{CV}_0^k(\Omega)$ is complete by Proposition 3.7. Then part c) is a consequence of b). Let us turn to part a). Since $\mathcal{CV}_0^k(\Omega)$ is barrelled, $\mathcal{V}^k$ locally bounded and locally bounded away from zero, the space $\mathcal{C}_c^\infty(\Omega) \otimes E$ can be considered as a topological subspace of $\mathcal{CV}_0^k(\Omega) \otimes_\varepsilon E$ by Corollary 3.16 a) and Remark 3.9 when equipped with the induced topology.

Let $f \in \mathcal{CV}_0^k(\Omega, E)$, $\varepsilon > 0$, $j \in J$, $l \in \langle k \rangle$ and $\alpha \in \widehat{\mathfrak{A}}$ where $(p_\alpha)_{\alpha \in \widehat{\mathfrak{A}}}$ is the system of seminorms describing the locally convex topology of the completion $\widehat{E}$ of E. In the following we consider functions with values in E also as functions with values in $\widehat{E}$ and note that $\mathcal{CV}_0^k(\Omega, \widehat{E})$ is the completion of $\mathcal{CV}_0^k(\Omega, E)$ by Proposition 3.7. Thus the topologies of $\mathcal{CV}_0^k(\Omega, E)$ and $\mathcal{CV}_0^k(\Omega, \widehat{E})$ coincide on $\mathcal{CV}_0^k(\Omega, E)$. The density of $\mathcal{C}_c^k(\Omega, E)$ in $\mathcal{CV}_0^k(\Omega, E)$ yields that there is $\widetilde{f} \in \mathcal{C}_c^k(\Omega, E)$ such that $|f - \widetilde{f}|_{j,l,\alpha} < \varepsilon/3$. Further, there is $N_0 \in \mathbb{N}$ with $|\widetilde{f} - \widetilde{f} * \rho_n|_{j,l,\alpha} < \varepsilon/3$ for all $n \geq N_0$ by Lemma 4.6 as $\widehat{E}$ is complete. Let $K_1 := \operatorname{supp} \widetilde{f}$ and choose an open neighbourhood V of K_1 such that V is relatively compact in Ω which is possible since K_1 is compact and $\Omega \subset \mathbb{R}^d$ open. Since $\mathcal{V}^k$ is locally bounded away from zero, there is $i \in J$ such that

$$C_1 := \sup_{x \in \overline{V}} \nu_{i,0}(x)^{-1} = \big(\inf_{x \in \overline{V}} \nu_{i,0}(x)\big)^{-1} < \infty.$$

From the relative compactness of V in Ω it follows that there is $N_1 \in \mathbb{N}$ such that

$$\overline{V} + \overline{\mathbb{B}_{1/n}(0)} \subset \Omega$$

for all $n \geq N_1$. Choosing $N_2 := \max\{N_0, N_1\}$ and defining the compact set $K_2 := \overline{V} + \overline{\mathbb{B}_{1/N_2}(0)} \subset \Omega$, we get that

$$C_2 := \sup_{x \in K_2} \nu_{j,l}(x) < \infty$$

because $\mathcal{V}^k$ is locally bounded. Further, we estimate

$$C_3 := \sup_{\beta \in \mathbb{N}_0^d, |\beta| \leq l} \int_{\mathbb{R}^d} \big|\partial^\beta \rho_{N_2}(y)\big| \,\mathrm{d}y \leq (N_2)^l \sup_{\beta \in \mathbb{N}_0^d, |\beta| \leq l} \int_{\mathbb{R}^d} \big|\partial^\beta \rho(y)\big| \,\mathrm{d}y < \infty.$$

By virtue of Proposition 5.1 b) there is $g = \sum_{m=1}^q g_m \otimes e_m \in \mathcal{C}_c^0(\Omega) \otimes E$ such that $\operatorname{supp} g \subset V$ and

$$|\widetilde{f} - g|_{i,0,\alpha} < \frac{\varepsilon}{3C_1C_2C_3}.$$

By Lemma 4.5 we observe that $g * \rho_{N_2} \in \mathcal{C}_c^\infty(\Omega, E)$ with

$$\operatorname{supp}(g * \rho_{N_2}) \subset \overline{V} + \overline{\mathbb{B}_{1/N_2}(0)} = K_2 \subset \Omega$$

and

$$g * \rho_{N_2} = \sum_{m=1}^q (g_m * \rho_{N_2}) \otimes e_m \in \mathcal{C}_c^\infty(\Omega) \otimes E.$$

Thus we have by Lemma 4.5

$$\operatorname{supp}(\widetilde{f} * \rho_{N_2}) \subset \overline{V} + \overline{\mathbb{B}_{1/N_2}(0)} = K_2$$

yielding

$$\operatorname{supp}(\widetilde{f} * \rho_{N_2} - g * \rho_{N_2}) \subset \overline{V} + \overline{\mathbb{B}_{1/N_2}(0)} = K_2 \subset \Omega$$

and

$$\begin{aligned}
|\widetilde{f} * \rho_{N_2} - g * \rho_{N_2}|_{j,l,\alpha} &\underset{(10)}{=} \sup_{\substack{x \in K_2\\ \beta \in \mathbb{N}_0^d, |\beta| \leq l}} p_\alpha\big((\widetilde{f} - g) * (\partial^\beta \rho_{N_2})(x)\big)\nu_{j,l}(x) \\
= \sup_{\substack{x \in K_2\\ \beta \in \mathbb{N}_0^d, |\beta| \leq l}} & p_\alpha\Big(\int_{\mathbb{R}^d} (\partial^\beta \rho_{N_2})(x-y)\big(\widetilde{f}_{\mathrm{ex}}(y) - g_{\mathrm{ex}}(y)\big)\,\mathrm{d}y\Big)\nu_{j,l}(x) \\
\leq \sup_{\substack{x \in K_2\\ \beta \in \mathbb{N}_0^d, |\beta| \leq l}} & \int_{\mathbb{R}^d} \big|(\partial^\beta \rho_{N_2})(x-y)\big|\,\mathrm{d}y \sup_{\substack{z \in \operatorname{supp}(\widetilde{f})\\ \cup \operatorname{supp}(g)}} p_\alpha(\widetilde{f}(z) - g(z))\nu_{j,l}(x) \\
= \sup_{\substack{x \in K_2\\ \beta \in \mathbb{N}_0^d, |\beta| \leq l}} & \int_{\mathbb{R}^d} \big|(\partial^\beta \rho_{N_2})(y)\big|\,\mathrm{d}y \sup_{z \in \overline{V}} p_\alpha(\widetilde{f}(z) - g(z))\nu_{j,l}(x)
\end{aligned}$$

$$\le C_3\big(\sup_{x\in K_2}\nu_{j,l}(x)\big)\big(\sup_{z\in\overline{V}}p_\alpha(\widetilde{f}(z)-g(z))\big)$$
$$= C_3C_2\sup_{z\in\overline{V}}p_\alpha(\widetilde{f}(z)-g(z))\nu_{i,0}(z)\nu_{i,0}(z)^{-1}$$
$$\le C_3C_2C_1|\widetilde{f}-g|_{i,0,\alpha}<\frac{\varepsilon}{3}.$$

Therefore we deduce

$$|f-g*\rho_{N_2}|_{j,l,\alpha}\le|f-\widetilde{f}|_{j,l,\alpha}+|\widetilde{f}-\widetilde{f}*\rho_{N_2}|_{j,l,\alpha}+|\widetilde{f}*\rho_{N_2}-g*\rho_{N_2}|_{j,l,\alpha}<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

If we keep in mind that $f\in\mathcal{CV}_0^k(\Omega,E)$ and $g*\rho_{N_2}\in\mathcal{C}_c^\infty(\Omega)\otimes E$, it follows that $\mathcal{C}_c^\infty(\Omega)\otimes E$ is dense in $\mathcal{CV}_0^k(\Omega,E)$ with respect to the topology of $\mathcal{CV}_0^k(\Omega,\widehat{E})$. However, the latter space is just the completion of $\mathcal{CV}_0^k(\Omega,E)$ and thus the topologies of $\mathcal{CV}_0^k(\Omega,E)$ and $\mathcal{CV}_0^k(\Omega,\widehat{E})$ coincide on $\mathcal{CV}_0^k(\Omega,E)$. Hence $\mathcal{C}_c^\infty(\Omega)\otimes E$ is dense in $\mathcal{CV}_0^k(\Omega,E)$. ∎

$\mathcal{C}_c^k(\Omega,E)$ is dense in $\mathcal{CV}_0^k(\Omega,E)$ by Lemma 3.14 if the latter space fulfils the cut-off criterion and the family $\mathcal{V}^k$ is locally bounded. $\mathcal{CV}_0^k(\Omega)$ is a Fréchet space and thus barrelled by Proposition 3.7 if the J in $\mathcal{V}^k=(\nu_{j,l})_{j\in J,l\in\langle k\rangle}$ is countable. Let us complement what we said about the standard structure of a family of weights (see the remarks below Definition 3.2) by our additional conditions on the weights collected so far. The standard structure of a (countable) locally bounded family $\mathcal{V}^k$ which is bounded away from zero on a locally compact Hausdorff space Ω, resp. on an open set $\Omega\subset\mathbb{R}^d$, is given by the following. Let $J:=\mathbb{N}$, $(\Omega_j)_{j\in J}$, be a family of sets such that $\Omega_j\subset\Omega_{j+1}$ for all $j\in J$ with $\Omega=\bigcup_{j\in J}\Omega_j$ and

$$\forall\, K\subset\Omega\text{ compact }\exists\, j\in J: K\subset\Omega_j.$$

Let $\widetilde{\nu}_{j,l}\colon\Omega\to(0,\infty)$ be continuous for all $j\in J$, $l\in\langle k\rangle$ and increasing in $j\in J$ and in $l\in\langle k\rangle$ such that

$$\nu_{j,l}(x)=\chi_{\Omega_j}(x)\widetilde{\nu}_{j,l}(x),\quad x\in\Omega, \tag{14}$$

for every $j\in J$ and $l\in\langle k\rangle$ where χ_{Ω_j} is the indicator function of Ω_j. If $\Omega\neq\mathbb{R}^d$, then the cut-off criterion may add some restrictions on the structure of the sequence (Ω_j), e.g. a positive distance from the boundary $\partial\Omega_j$ of Ω_j to the boundary $\partial\Omega_{j+1}$ of Ω_{j+1} for all j.

EXAMPLE 5.3. Let E be an lcHs, $k\in\mathbb{N}_\infty$ and $\Omega\subset\mathbb{R}^d$ open. Theorem 5.2 can be applied to the following spaces:

a) $\mathcal{C}^k(\Omega,E)$ with the topology of uniform convergence of all partial derivatives up to order k on compact subsets of Ω,
b) the Schwartz space $\mathcal{S}(\mathbb{R}^d,E)$,
c) the space $\mathcal{O}_M(\mathbb{R}^d,E)$ of multipliers of $\mathcal{S}(\mathbb{R}^d)$,
d) let $\Omega_j:=\{x=(x_1,x_2)\in\mathbb{R}^2\mid 1/(j+1)<|x_2|<j+1\}$ for all $j\in\mathbb{N}$ and

$$\mathcal{C}_{\exp}^k(\mathbb{R}^2\setminus\mathbb{R},E):=\big\{f\in\mathcal{C}^k(\mathbb{R}^2\setminus\mathbb{R},E)\mid\forall\, j\in\mathbb{N},\ l\in\langle k\rangle,\ \alpha\in\mathfrak{A}:|f|_{j,l,\alpha}<\infty\big\}$$

where

$$|f|_{j,l,\alpha}:=\sup_{\substack{(x_1,x_2)\in\Omega_j\\ \beta\in\mathbb{N}_0^2,|\beta|\le l}}p_\alpha\big((\partial^\beta)^Ef(x_1,x_2)\big)e^{-|x_1|/(j+1)}.$$

Proof. a) From Example 3.5 a) we obtain $\mathcal{C}^k(\Omega, E) = \mathcal{CW}^k_0(\Omega, E)$ with $\mathcal{W}^k := \{\nu_{j,l} := \chi_{\Omega_j} \mid j \in \mathbb{N},\ l \in \langle k\rangle\}$ where $(\Omega_j)_{j\in\mathbb{N}}$ is a compact exhaustion of Ω. The family of weights $\mathcal{W}^k$ is locally bounded and locally bounded away from zero. The Fréchet space $\mathcal{C}^k(\Omega)$ is barrelled and the cut-off criterion is fulfilled by Example 3.12.

b) Due to Example 3.5 b) we have $\mathcal{S}(\mathbb{R}^d, E) = \mathcal{CV}^\infty_0(\mathbb{R}^d, E)$ with $\mathcal{V}^\infty := \{\nu_{j,l} \mid j \in \mathbb{N},\ l \in \mathbb{N}_0\}$ where $\nu_{j,l}(x) := (1+|x|^2)^{l/2}$ for $x \in \mathbb{R}^d$. The family of weights is locally bounded and bounded away from zero, the Fréchet space $\mathcal{S}(\mathbb{R}^d)$ is barrelled and $\mathcal{S}(\mathbb{R}^d, E)$ fulfils the cut-off criterion by Example 3.12.

c) The space of multipliers is defined by

$$\mathcal{O}_M(\mathbb{R}^d, E) := \{f \in \mathcal{C}^\infty(\mathbb{R}^d, E) \mid \forall\, g \in \mathcal{S}(\mathbb{R}^d),\ l \in \mathbb{N}_0,\ \alpha \in \mathfrak{A} : \|f\|_{g,l,\alpha} < \infty\}$$

where

$$\|f\|_{g,l,\alpha} := \sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d, |\beta|\leq l}} p_\alpha\big((\partial^\beta)^E f(x)\big)|g(x)|$$

(see [21, 3^0), p. 97]). The space $\mathcal{O}_M(\mathbb{R}^d)$ is barrelled by [11, Chap. II, §4, n°4, Théorème 16, p. 131]. Let $J := \{j \subset \mathcal{S}(\mathbb{R}^d) \mid j \text{ finite}\}$ and define the family $\mathcal{V}^\infty$ of weights given by $\nu_{j,l}(x) := \max_{g\in j} |g(x)|$, $x \in \mathbb{R}^d$, for $j \in J$ and $l \in \mathbb{N}_0$. It is easily seen that the system of seminorms generated by

$$|f|_{j,l,\alpha} := \sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d, |\beta|\leq l}} p_\alpha\big((\partial^\beta)^E f(x)\big)\nu_{j,l}(x), \quad f \in \mathcal{O}_M(\mathbb{R}^d, E),$$

for $j \in J$, $l \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$ induces the same topology on $\mathcal{O}_M(\mathbb{R}^d, E)$. However, the family $\mathcal{V}^\infty$ is directed, locally bounded and bounded away from zero. Further, for every $\varepsilon > 0$ there is $r > 0$ such that $(1 + |x|^2)^{-1} < \varepsilon$ for all $x \notin \overline{\mathbb{B}_r(0)} =: K$ which implies for $j \in J$ and $l \in \mathbb{N}_0$ that

$$\nu_{j,l}(x) \leq \varepsilon \max_{g\in j} |g(x)(1 + |x|^2)| = \varepsilon\nu_{i,l}(x), \quad x \notin K,$$

where $i := \{g \cdot (1 + |\cdot|^2) \mid g \in j\}$ is a finite subset of $\mathcal{S}(\mathbb{R}^d)$. From Remark 3.4 we conclude that $\mathcal{O}_M(\mathbb{R}^d, E) = \mathcal{CV}^\infty(\mathbb{R}^d, E) = \mathcal{CV}^\infty_0(\mathbb{R}^d, E)$. Due to Remark 3.11 we note that $\mathcal{O}_M(\mathbb{R}^d, E)$ satisfies the cut-off criterion.

d) The family $\mathcal{V}^k$ given by $\nu_{j,l}(x_1, x_2) := \chi_{\Omega_j}(x_1, x_2)e^{-|x_1|/(j+1)}$, $(x_1, x_2) \in \mathbb{R}^2\setminus\mathbb{R}$, for $j \in \mathbb{N}$ and $l \in \langle k\rangle$ is locally bounded and bounded away from zero. For $j \in \mathbb{N}$ and $l \in \mathbb{N}_0$ we set $i := 2j+1$, $m := l$, $\delta := 1/(2j+2)$ and for $0 < \varepsilon < 1$ we choose $K := \{x = (x_1, x_2) \in \overline{\Omega_j} \mid |x_1| \leq -(\ln\varepsilon)(2j+2)\}$. This yields $\mathcal{C}^k_{\exp}(\mathbb{R}^2\setminus\mathbb{R}, E) = \mathcal{CV}^k(\mathbb{R}^2\setminus\mathbb{R}, E) = \mathcal{CV}^k_0(\mathbb{R}^2\setminus\mathbb{R}, E)$ by Remark 3.4 and that the cut-off criterion is fulfilled. In addition, the Fréchet space $\mathcal{C}^k_{\exp}(\mathbb{R}^2 \setminus \mathbb{R})$ is barrelled. ∎

Together with Proposition 5.1 we get from example a) one of our starting points, namely Theorem 1.2, back. Example b) and c) are covered by [21, Proposition 9, p. 108] and [21, Théorème 1, p. 111]. The results b) and c) for the Schwartz space in example b) can also be found in [11, Chap. II, §3, n°3, Exemples, p. 80–81] with a different proof using the nuclearity of $\mathcal{S}(\mathbb{R}^d)$. We complete this paper with a comparison of our conditions in Theorem 5.2 with the ones stated by Schwartz in [21] to get the same result for the spaces in example a)–c) but only for $\Omega = \mathbb{R}^d$.

REMARK 5.4. Schwartz treats the case $k>0$ and $\Omega=\mathbb{R}^d$ in [21]. He assumes similar conditions H_1–H_4 for the space $\mathcal{H}^k(\mathbb{R}^d):=\mathcal{H}^k(\mathbb{R}^d,\mathbb{K})$ as we do (see [21, p. 97–98]). In H_1 the members of his family of weights Γ are continuous and for every compact set $K\subset\mathbb{R}^d$ there is a weight in Γ which is non-zero on K. $\mathcal{H}^k(\mathbb{R}^d)$ is the space of functions $f\in\mathcal{C}^k(\mathbb{R}^d)$ such that $\gamma\partial^\beta f$ is bounded on $\mathbb{R}^d$ for every $\gamma\in\Gamma$ and $|\beta|\leq k$. This yields to $\mathcal{C}^k_c(\mathbb{R}^d)\subset\mathcal{H}^k(\mathbb{R}^d)\subset\mathcal{C}^k(\mathbb{R}^d)$ algebraically. In H_2 he demands that $\mathcal{H}^k(\mathbb{R}^d)$ is a locally convex Hausdorff space and that the inclusions $\mathcal{C}^k_c(\mathbb{R}^d)\hookrightarrow\mathcal{H}^k(\mathbb{R}^d)\hookrightarrow\mathcal{C}^k(\mathbb{R}^d)$ are continuous where $\mathcal{C}^k(\mathbb{R}^d)$ has its usual topology and $\mathcal{C}^k_c(\mathbb{R}^d)$ its inductive limit topology. In H_3 he supposes that a subset $B\subset\mathcal{H}^k(\mathbb{R}^d)$ is bounded if and only if for every $\gamma\in\Gamma$ and $|\beta|\leq k$ the set $\{\gamma(x)\partial^\beta f(x)\mid x\in\mathbb{R}^d,\ f\in B\}$ is bounded in $\mathbb{K}$. In H_4 he assumes that on every bounded subset of $\mathcal{H}^k(\mathbb{R}^d)$ the topology of $\mathcal{H}^k(\mathbb{R}^d)$ and the induced topology of $\mathcal{C}^k(\mathbb{R}^d)$ coincide.

He defines the E-valued version $\mathcal{H}^k(\mathbb{R}^d,E)$ which corresponds to the space $\mathcal{H}^k(\mathbb{R}^d)$ for $\mathcal{H}^k=\mathcal{C}^k_c$, $\mathcal{C}^k$, $\mathcal{S}$ and $\mathcal{O}_M$ and shows that the statements of Theorem 5.2 hold for all of them but $\mathcal{H}^k=\mathcal{C}^k_c$ (see [21, p. 94–97], [21, Proposition 9, p. 108] and [21, Théorème 1, p. 111]).

In comparison, our conditions of local boundedness of $\mathcal{V}^k$ and being locally bounded away from zero on $\Omega=\mathbb{R}^d$ imply H_1 and H_2 if the members of $\mathcal{V}^k$ are continuous. The assumption that the members of $\mathcal{V}^k$ are continuous is not a big difference if the members of the family $\mathcal{V}^k$ have a structure like in (14). Then one may replace the indicator functions χ_{Ω_j} by a smoothed version, e.g. by convolution of the indicator function with a suitable mollifier, and then one gets a family of continuous weights which generates the same topology. The condition H_3 is clearly fulfilled for the spaces $\mathcal{CV}^k(\mathbb{R}^d)$ and the topology on them is called 'topologie naturelle' by Schwartz (see [21, p. 98]). The condition H_4 implies that $\mathcal{C}^k_c(\mathbb{R}^d,E)$ is dense in $\mathcal{H}^k(\mathbb{R}^d,E)$ for $\mathcal{H}^k=\mathcal{C}^k$, $\mathcal{S}$ and $\mathcal{O}_M$ and quasi-complete E (see [21, p. 106] and [21, Théorème 1, p. 111]). The same follows in our case from local boundedness and the cut-off criterion.

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