Topology and its Applications 345 (2024) 108843

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

Mixed topologies on Saks spaces of vector-valued functions

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A R T I C L E   I N F O

Article history:
Received 24 September 2023
Received in revised form 20 January 2024
Accepted 26 January 2024
Available online 30 January 2024

MSC:
46A70
46E40
46E10
46E15
47D06
54D55

Keywords:
Saks space
Tight
Mixed topology
C-sequential
Mackey

A B S T R A C T

We study Saks spaces of functions with values in a normed space and the associated mixed topologies. We are interested in properties of such Saks spaces and mixed topologies which are relevant for applications in the theory of bi-continuous semigroups. In particular, we are interested if such Saks spaces are complete, semi-Montel, C-sequential or a (strong) Mackey space with respect to the mixed topology. Further, we consider the question whether the mixed and the submixed topology coincide on such Saks spaces and seek for explicit systems of seminorms that generate the mixed topology.

1. Introduction

This paper is dedicated to Saks spaces of vector-valued functions and their properties. A *Saks space* is a triple $(X, \|\cdot\|, \tau)$ consisting of a normed space $(X, \|\cdot\|)$ and a coarser locally convex Hausdorff topology τ on X such that the norm $\|\cdot\|$ is the supremum taken over some directed system of continuous seminorms that generates the τ-topology, see [13]. Associated to a Saks space is the *mixed topology* $\gamma := \gamma(\|\cdot\|, \tau)$, which was introduced in [60] and is the finest locally convex Hausdorff, even linear, topology between the $\|\cdot\|$-topology and τ. Sequentially complete Saks spaces, i.e. (X, γ) is sequentially complete, are needed in

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https://doi.org/10.1016/j.topol.2024.108843

the theory of bi-continuous semigroups, which were introduced in [47,48], to treat semigroups on Banach spaces $(X, \|\cdot\|)$ which are usually not strongly continuous w.r.t. the norm $\|\cdot\|$ but only strongly continuous w.r.t. the coarser topology τ, e.g. dual semigroups, implemented semigroups or transition semigroups like the Ornstein–Uhlenbeck semigroup on the space of bounded continuous functions on a Polish space.

Besides sequential completeness there are several other properties of Saks spaces that are of importance in applications. The Lumer–Phillips generation theorems for bi-continuous semigroups from [45] need knowledge of explicit systems of seminorms that generate the mixed topology γ because the concept of dissipativity depends on the choice of the system of seminorms. There is another locally convex Hausdorff topology associated to a Saks space, namely the *submixed topology* $\gamma_{\mathrm{s}} := \gamma_{\mathrm{s}}(\|\cdot\|, \tau)$, which is defined by an explicit system of seminorms and is in general coarser than γ but has the same convergent sequences as γ, see Definition 2.1. Therefore one is interested in the question when γ and γ_{s} coincide. Moreover, the generation theorems like [45, Theorems 3.10, 3.17, Corollary 3.15] need the completeness of the Saks space, i.e. that (X, γ) is complete. A sufficient condition for $\gamma = \gamma_{\mathrm{s}}$ is that (X, γ) is a semi-Montel space, which also implies that (X, γ) is a complete semi-reflexive space and semi-reflexivity is needed for [45, Theorems 3.17] as well. On the other hand, the Lumer–Phillips generation theorem [11, Theorem 3.15, p. 75] for bi-continuous semigroups needs that (X, γ_{s}) is complete (see [45, Theorem 3.11, Remark 3.12 (b)]).

The question whether γ and γ_{s} coincide is also important for perturbation results of bi-continuous semigroups. If $\gamma = \gamma_{\mathrm{s}}$ and the Saks space is sequentially complete and *C-sequential*, i.e. every convex sequentially open subset of (X, γ) is already open, then a bi-continuous semigroup on the corresponding Saks space is already locally, even quasi-, equitight by [42, Theorem 3.17 (b), p. 13]. Locally equitight bi-continuous semigroups are sometimes just called "tight" or "local" (see [19,21]) and local equitightness is needed for perturbation theorems like [19, Theorem 1.2, p. 669], [22, Theorems 2.4, 3.2, p. 92, 94–95], [22, Remark 4.1, p. 101], [9, Theorem 5, p. 8] and [10, Theorem 3.3, p. 582]. Equitightness is relevant in ergodic theory for bi-continuous semigroups, see [2, Remark 3.5 (ii), p. 147, Proposition 3.8, p. 150].

Apart from its relation to local equitightness its is also known that every bi-continuous semigroup on a sequentially complete C-sequential Saks space is locally, even quasi-, γ-equicontinuous by [35, Theorem 7.4, p. 180] and [42, Theorem 3.17 (a), p. 13]. Equicontinuity and local equicontinuity are needed for perturbation results like dissipative perturbations or Desch–Schappacher perturbations [1,26] and the infinitesimal description of Markov processes [25]. Sequentially complete C-sequential Saks spaces also play a role in the duality between cost-uniform approximate null-controllability and final state observability, see [44, Theorem 5.18, p. 441]. A sufficient condition for (X, γ) being C-sequential is that (X, γ) is a Mackey–Mazur space by [58, Corollary 7.6, p. 52]. Here, (X, γ) being a *Mackey space* means that γ is the Mackey topology of a dual pairing $\langle X, Y\rangle$ where Y is a Banach space topologically isomorphic to the strong dual $(X, \gamma)_b'$, and being a *Mazur space* means that all sequentially γ-continuous linear functionals are already γ-continuous. The question whether (X, γ) is a Mackey space or even a *strong* Mackey space, i.e. a Mackey space such that $\sigma(Y, X)$-compact subsets of Y are γ-equicontinuous, is interesting in itself, see e.g. [49, p. 553] and [35, Propositions 3.4, 4.9, p. 161, 166]. The condition that (X, γ) is a sequentially complete Mackey–Mazur space is also sufficient for the existence of a dual bi-continuous semigroup of a bi-continuous semigroup in the sun dual theory for bi-continuous semigroups, see [43, 3.8 Theorem (b), p. 9–10].

We are interested in all of the properties listed above in the case of Saks spaces of vector-valued functions. Let us give an outline of our paper. In Section 2 we briefly recall some notions and results from the theory of Saks spaces and give a characterisation of the approximation property of (X, γ) in the case that (X, γ) is a semi-Montel space in Proposition 2.6.

In Section 3 we start with a Saks space $(\mathcal{F}(\Omega), \|\cdot\|, \tau)$ of real- or complex-valued functions on a non-empty set Ω such that $(\mathcal{F}(\Omega), \|\cdot\|)$ is a Banach space and $\gamma = \gamma_{\mathrm{s}}$. We construct a weak E-valued version $(\mathcal{F}(\Omega, E)_\sigma, \|\cdot\|_\sigma^E, \tau_\sigma^E)$ of this space in a canonical way, where E is a normed space (or more general a locally convex Hausdorff space), and show in Theorem 3.3 that this triple is a complete Saks space and even complete when equipped with the submixed topology if $(\mathcal{F}(\Omega), \|\cdot\|, \tau)$ is semi-Montel w.r.t. γ, τ finer than the topology of pointwise convergence and E a Banach space. The proof of this result is based on linearisation via Schwartz' ε-product and as a byproduct we also get a characterisation of $(\mathcal{F}(\Omega), \gamma)$ having the approximation property in Corollary 3.4. We apply this result to weak E-valued versions of the Hardy space, the weighted Bergman space and the Dirichlet space, whose properties we collect in Corollary 3.5

In Section 4 we consider a different way of defining an E-valued version of $(\mathcal{F}(\Omega), \|\cdot\|, \tau)$ in Definition 4.2 which is available for some spaces and often stronger in the sense that is a subspace of $\mathcal{F}(\Omega, E)_\sigma$ and sometimes even a strict subspace, see Proposition 4.4. In Theorem 4.3 we collect some of the properties we are interested in of such strong E-valued Saks function spaces. Then we turn to specific examples. Among them are weighted spaces of continuous functions in Corollary 4.5, weighted space of holomorphic functions in Corollary 4.6, weighted kernels of hypoelliptic linear partial differential operators in spaces of smooth functions in Corollary 4.8, in particular weighted spaces of harmonic functions, weighted Bloch spaces in Corollary 4.10, spaces of Lipschitz continuous functions in Corollary 4.11 and spaces of k-times continuously partially differentiable functions on some open bounded set $\Omega \subset \mathbb{R}^d$ whose partial derivatives extend continuously to the boundary of Ω and whose partial derivatives of order k are α-Hölder continuous for some $0 < \alpha \leq 1$ in Corollary 4.12.

We close our paper with Section 5 where we characterise the dual of the spaces from the preceding section with respect to a certain submixed topology which sometimes coincides with the mixed topology, see Theorem 5.1. The interest in such a characterisation may also be motivated by the Lumer–Phillips generation theorem [45, Corollary 3.15] which involves the dual w.r.t. the mixed topology.

2. Notions and preliminaries

In this short section we recall some basic notions from the theory of locally convex spaces, Saks spaces and mixed topologies. For a locally convex Hausdorff space X over the field $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$ we denote by X' the topological linear dual space of X. If we want to emphasize the dependency on the locally convex Hausdorff topology τ of X, we write (X, τ) and $(X, \tau)'$ instead of just X and X', respectively. We write $(X, \tau)'_t$ for the space $(X, \tau)'$ equipped with the locally convex topology of uniform convergence on the bounded subsets of (X, τ) if $t = b$, and on the absolutely convex compact subsets of (X, τ) if $t = \kappa$. For two locally convex Hausdorff spaces (X, τ) and (E, τ_E) we denote by $L((X, \tau), (E, \tau_E))$ the space of continuous linear maps from (X, τ) to (E, τ_E). By [54, Chap. I, §1, Définition, p. 18] the *ε-product* of Schwartz of (X, τ) and (E, τ_E) is defined by $X\varepsilon E := L_e((X, \tau)'_\kappa, (E, \tau_E))$ where $L_e((X, \tau)'_\kappa, (E, \tau_E))$ is the space $L((X, \tau)'_\kappa, (E, \tau_E))$ equipped with the topology of uniform convergence on the equicontinuous subsets of $(X, \tau)'$. We identify the tensor product $X \otimes E$ with the linear finite rank operators in $X\varepsilon E$ and recap that X has the *approximation property* of Schwartz if and only if $X \otimes E$ is dense in $X\varepsilon E$ for all Banach spaces E (see e.g. [30, Satz 10.17, p. 250]). Moreover, for two locally convex Hausdorff topologies τ_0 and τ_1 on X we write $\tau_0 \leq \tau_1$ if τ_0 is coarser than τ_1. We write τ_{co}^E for the *compact-open topology*, i.e. the topology of uniform convergence on compact subsets of Ω, on the space $\mathcal{C}(\Omega, E)$ of continuous functions on a topological Hausdorff space Ω with values in a locally convex Hausdorff space E. If $E = \mathbb{K}$, we just write $\tau_{\mathrm{co}} := \tau_{\mathrm{co}}^{\mathbb{K}}$. In addition, we write τ_{p} for the *topology of pointwise convergence* on the space $\mathbb{K}^\Omega$ of $\mathbb{K}$-valued functions on a set Ω. By a slight abuse of notation we also use the symbols τ_{co}^E and τ_{p} for the relative compact-open topology and the relative topology of pointwise convergence on topological subspaces of $\mathcal{C}(\Omega, E)$ and $\mathbb{K}^\Omega$, respectively (cf. [40,41, Section 2]).

Let us recall the definition of the mixed topology, [60, Section 2.1], and the notion of a Saks space, [13, I.3.2 Definition, p. 27–28], which will be important for the rest of the paper.

2.1 Definition *([42, Definition 2.2, p. 3]).* Let $(X, \|\cdot\|)$ be a normed space and τ a locally convex Hausdorff topology on X such that $\tau \leq \tau_{\|\cdot\|}$ where $\tau_{\|\cdot\|}$ denotes the topology induced by $\|\cdot\|$. Then

(a) the *mixed topology* $\gamma := \gamma(\|\cdot\|, \tau)$ is the finest linear topology on X that coincides with τ on $\|\cdot\|$-bounded sets and such that $\tau \leq \gamma \leq \tau_{\|\cdot\|}$,
(b) the triple $(X, \|\cdot\|, \tau)$ is called a *Saks space* if there exists a directed system of continuous seminorms Γ_τ that generates the topology τ such that

$$\|x\| = \sup_{q \in \Gamma_\tau} q(x), \quad x \in X. \tag{1}$$

In comparison to [42, Definition 2.2, p. 3] we dropped the assumption that the space $(X, \|\cdot\|)$ should be complete. The mixed topology γ is actually Hausdorff locally convex and the definition given above is equivalent to the one introduced by Wiweger [60, Section 2.1] due to [60, Lemmas 2.2.1, 2.2.2, p. 51].

It is often useful to have a characterisation of the mixed topology by generating systems of continuous seminorms, e.g. the definition of dissipativity in Lumer–Phillips generation theorems for bi-continuous semigroups from [45] mentioned in the introduction depends on the choice of the generating system of seminorms of the mixed topology. For that purpose we introduce the following auxiliary topology.

2.2 Definition *([42, Definition 3.9, p. 9])*. Let $(X, \|\cdot\|, \tau)$ be a Saks space and Γ_τ a directed system of continuous seminorms that generates the topology τ and fulfils (1). We set

$$\mathcal{N} := \{(q_n, a_n)_{n\in\mathbb{N}} \mid (q_n)_{n\in\mathbb{N}} \subset \Gamma_\tau,\ (a_n)_{n\in\mathbb{N}} \in c_0^+\}$$

where c_0^+ is the family of all real non-negative null-sequences. For $(q_n, a_n)_{n\in\mathbb{N}} \in \mathcal{N}$ we define the seminorm

$$|||x|||_{(q_n,a_n)_{n\in\mathbb{N}}} := \sup_{n\in\mathbb{N}} q_n(x) a_n, \quad x \in X.$$

We denote by $\gamma_s := \gamma_s(\|\cdot\|, \tau)$ the locally convex Hausdorff topology that is generated by the system of seminorms $(|||\cdot|||_{(q_n,a_n)_{n\in\mathbb{N}}})_{(q_n,a_n)_{n\in\mathbb{N}} \in \mathcal{N}}$ and call it the *submixed topology*.

We note that the submixed topology γ_s does not depend on the choice of Γ_τ that fulfils (1). By [13, I.1.10 Proposition, p. 9], [13, I.4.5 Proposition, p. 41–42] and [21, Lemma A.1.2, p. 72] we have the following relation between the mixed and the submixed topology.

2.3 Remark *([42, Remark 3.10, p. 9])*. Let $(X, \|\cdot\|, \tau)$ be a Saks space, Γ_τ a directed system of continuous seminorms that generates the topology τ and fulfils (1), $\gamma := \gamma(\|\cdot\|, \tau)$ the mixed and $\gamma_s := \gamma_s(\|\cdot\|, \tau)$ the submixed topology.

(a) We have $\tau \leq \gamma_s \leq \gamma$ and γ_s has the same convergent sequences as γ.
(b) If
  (i) for every $x \in X$, $\varepsilon > 0$ and $q \in \Gamma_\tau$ there are $y, z \in X$ such that $x = y + z$, $q(z) = 0$ and $\|y\| \leq q(x) + \varepsilon$, or
  (ii) the $\|\cdot\|$-closed unit ball $B_{\|\cdot\|} := \{x \in X \mid \|x\| \leq 1\}$ is τ-compact,

  then it holds $\gamma = \gamma_s$.

The submixed topology γ_s was originally introduced in [60, Theorem 3.1.1, p. 62] where a proof of Remark 2.3 (b) can be found, too. The following notions will also be needed, which were introduced in [13, I.3.2 Definition, p. 27–28], [41, 3.3 Definition, p. 7] and [45, Definition 2.2].

2.4 Definition. Let $(X, \|\cdot\|, \tau)$ be a Saks space.

(a) We call $(X, \|\cdot\|, \tau)$ *complete* if (X, γ) is complete.
(b) We call $(X, \|\cdot\|, \tau)$ *semi-Montel* if (X, γ) is a semi-Montel space.
(c) We call $(X, \|\cdot\|, \tau)$ *C-sequential* if (X, γ) is a C-sequential space, i.e. every convex sequentially open subset of (X, γ) is already open (see [56, p. 273]).

2.5 Remark. Let $(X, \|\cdot\|)$ be a normed space and τ a locally convex Hausdorff topology on X. Set $X^* := (X, \|\cdot\|)'$ and denote by $\|\cdot\|_{X^*}$ the dual norm on X^*.

(a) $(X, \|\cdot\|, \tau)$ is a semi-Montel Saks space if and only if $B_{\|\cdot\|}$ is τ-compact by [13, I.1.13 Proposition, p. 11] and [41, 3.6 Remark (c), p. 8].
(b) If $(X, \|\cdot\|, \tau)$ is a semi-Montel Saks space, then $(X, \|\cdot\|, \tau)$ and $(X, \|\cdot\|)$ are complete by [41, 3.6 Remark (a), (b), p. 8].
(c) If $(X, \|\cdot\|, \tau)$ is a semi-Montel Saks space and τ_0 a locally convex Hausdorff topology on X such that $\tau_0 \leq \tau$, then $(X, \|\cdot\|, \tau_0)$ is a semi-Montel Saks space and $\gamma_{\mathrm{s}}(\|\cdot\|, \tau) = \gamma(\|\cdot\|, \tau) = \gamma(\|\cdot\|, \tau_0) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau_0)$ by part (a), condition (ii) of Remark 2.3 (b) and [41, 3.6 Remark (c), p. 8].
(d) Let $(X, \|\cdot\|, \tau)$ be a Saks space, set $X'_\gamma := (X, \gamma)'$ and denote by $\|\cdot\|_{X'_\gamma}$ the restriction of $\|\cdot\|_{X^*}$ to X'_γ. Then $(X, \gamma)'_b = (X'_\gamma, \|\cdot\|_{X'_\gamma})$ and $(X'_\gamma, \|\cdot\|_{X'_\gamma})$ is a Banach space by [13, I.1.18 Proposition, p. 15].

We close this section with the following observation concerning the approximation property of (X, γ) in the semi-Montel case, whose proof is an adaptation of [29, Theorem 4.6 (i)⇔(ii), p. 651–652] where $X = \mathrm{Lip}_0(\Omega)$ is the space of $\mathbb{K}$-valued Lipschitz continuous functions on a metric space Ω that vanish at the origin (see Corollary 4.11).

2.6 Proposition. *Let $(X, \|\cdot\|, \tau)$ be a semi-Montel Saks space. Then the following assertions are equivalent.*

(a) *(X, γ) has the approximation property.*
(b) *$(X'_\gamma, \|\cdot\|_{X'_\gamma})$ has the approximation property.*

Proof. (a)⇒(b): Due to Remark 2.5 (a) and [13, I.4.1 Proposition, p. 38] (or [13, I.4.2 Corollary (d), p. 38] in combination with [51, Theorem 4.1, p. 43]) we have $(X, \gamma) = (X'_\gamma, \|\cdot\|_{X'_\gamma})'_\kappa$. Hence $E := (X'_\gamma, \|\cdot\|_{X'_\gamma})$ has the approximation property by [16, Corollary 1.3, p. 144] because $(X, \gamma) = E'_\kappa$ has the approximation property.

(b)⇒(a): We note that $(X'_\gamma, \|\cdot\|_{X'_\gamma}) = (X, \gamma)'_b = (X, \gamma)'_\kappa$ by Remark 2.5 (d) and the semi-Montel property of (X, γ). Thus $E := (X, \gamma)$ has the approximation property by [16, Corollary 1.3, p. 144] because $(X'_\gamma, \|\cdot\|_{X'_\gamma}) = E'_\kappa$ has the approximation property. □

In particular, the preceding proof shows that $(X, \gamma) = (X'_\gamma, \|\cdot\|_{X'_\gamma})'_\kappa = ((X, \gamma)'_b)'_\kappa$, i.e. (X, γ) is a *DFC-space* by [51, Theorem 4.1, p. 43], if $(X, \|\cdot\|, \tau)$ is a semi-Montel Saks space. Sufficient conditions that guarantee that (X, γ) has the approximation property may be found in [13, I.4.20 Proposition, p. 53] and [13, I.4.21, I.4.22 Corollaries, p. 54].

3. Saks spaces of weak vector-valued functions

In this section we introduce Saks spaces of weak vector-valued functions. We use a linearisation based on the ε-product to show that they are complete w.r.t. the mixed and the submixed topology if their scalar-valued version is semi-Montel, $\tau_{\mathrm{p}} \leq \tau$ and they have values in a Banach space.

Let $(\mathcal{F}(\Omega), \|\cdot\|)$ be a Banach space of $\mathbb{K}$-valued functions on a non-empty set Ω such that $\tau_{\mathrm{p}} \leq \tau_{\|\cdot\|}$. We recall from [40, p. 31] a canonical construction of a weak vector-valued version of such a space. For a locally convex Hausdorff space E over $\mathbb{K}$ with directed system of seminorms Γ_E generating its topology we define the *space of weak E-valued $\mathcal{F}$-functions* by

$$\mathcal{F}(\Omega,E)_\sigma := \{f\colon \Omega\to E \mid \forall\, e'\in E'\colon\ e'\circ f\in\mathcal{F}(\Omega)\}.$$

For $p\in\Gamma_E$ we set $U_p := \{x\in E \mid p(x)<1\}$ and denote by U_p° the polar of U_p. Since $(\mathcal{F}(\Omega),\|\cdot\|)$ is a Banach space, thus webbed, the supremum $\|f\|_{\sigma,p} := \sup_{e'\in U_p^\circ}\|e'\circ f\|$ is finite for every $f\in\mathcal{F}(\Omega,E)_\sigma$ and $p\in\Gamma_E$ by [40, 5.1 Remark, p. 31]. Hence the space $\mathcal{F}(\Omega,E)_\sigma$ equipped with the system of seminorms $(\|\cdot\|_{\sigma,p})_{p\in\Gamma_E}$ is a locally convex Hausdorff space. If $(E,\|\cdot\|_E)$ is a normed space with $\Gamma_E := \{\|\cdot\|_E\}$, then $(\mathcal{F}(\Omega,E)_\sigma,\|\cdot\|_\sigma^E)$ is a normed space where $\|\cdot\|_\sigma^E := \|\cdot\|_{\sigma,\|\cdot\|_E}$.

Now, let τ be an additional locally convex Hausdorff topology on $\mathcal{F}(\Omega)$ such that $(\mathcal{F}(\Omega),\|\cdot\|,\tau)$ is a Saks space and $\gamma := \gamma(\|\cdot\|,\tau) = \gamma_{\mathrm{s}}(\|\cdot\|,\tau)$. Then, by Definition 2.2, a directed system of seminorms that generates γ is given by

$$|||f|||_{(q_n,a_n)_{n\in\mathbb{N}}} := \sup_{n\in\mathbb{N}} q_n(f)a_n,\quad f\in\mathcal{F}(\Omega),$$

for $(q_n)_{n\in\mathbb{N}}\subset\Gamma_\tau$, where Γ_τ is a directed system of continuous seminorms that generates the topology τ and fulfils (1), and $(a_n)_{n\in\mathbb{N}}\in c_0^+$. We set

$$|||f|||_{\sigma,(q_n,a_n)_{n\in\mathbb{N}},p} := \sup_{e'\in U_p^\circ}|||e'\circ f|||_{(q_n,a_n)_{n\in\mathbb{N}}} = \sup_{e'\in U_p^\circ}\sup_{n\in\mathbb{N}} q_n(e'\circ f)a_n,\quad f\in\mathcal{F}(\Omega,E)_\sigma, \tag{2}$$

for $p\in\Gamma_E$, $(q_n)_{n\in\mathbb{N}}\subset\Gamma_\tau$ and $(a_n)_{n\in\mathbb{N}}\in c_0^+$. Then for $p\in\Gamma_E$, $(q_n)_{n\in\mathbb{N}}\subset\Gamma_\tau$ and $(a_n)_{n\in\mathbb{N}}\in c_0^+$ it holds

$$|||f|||_{\sigma,(q_n,a_n)_{n\in\mathbb{N}},p} \leq \sup_{n\in\mathbb{N}}|a_n|\|f\|_{\sigma,p}<\infty,\quad f\in\mathcal{F}(\Omega,E)_\sigma.$$

So the system of seminorms $(|||f|||_{\sigma,(q_n,a_n)_{n\in\mathbb{N}},p})_{(q_n,a_n)_{n\in\mathbb{N}}\in\mathcal{N},p\in\Gamma_E}$ induces a locally convex Hausdorff topology on $\mathcal{F}(\Omega,E)_\sigma$ which we denote by $\gamma_{\sigma,\mathrm{s}}^E$.

3.1 Remark. Let $(\mathcal{F}(\Omega),\|\cdot\|,\tau)$ be a Saks space of $\mathbb{K}$-valued functions on a non-empty set Ω such that $(\mathcal{F}(\Omega),\|\cdot\|)$ is a Banach space, $\tau_{\mathrm{p}}\leq\tau_{\|\cdot\|}$ and $\gamma(\|\cdot\|,\tau)=\gamma_{\mathrm{s}}(\|\cdot\|,\tau)$, and E a locally convex Hausdorff space over $\mathbb{K}$ with directed system of seminorms Γ_E generating its topology. Then the topology $\gamma_{\sigma,\mathrm{s}}^E$ does not depend on the choice of the system of seminorms that generates $\gamma := \gamma(\|\cdot\|,\tau)$. Indeed, let Γ_γ be another system of seminorms that generates γ. Then for every $(q_n)_{n\in\mathbb{N}}\subset\Gamma_\tau$, Γ_τ as above, and $(a_n)_{n\in\mathbb{N}}\in c_0^+$ there are $C_0\geq 0$ and $r_0\in\Gamma_\gamma$ such that $|||f|||_{\sigma,(q_n,a_n)_{n\in\mathbb{N}}}\leq C_0 r_0(f)$ for all $f\in\mathcal{F}(\Omega)$. On the other hand, for every and $r_1\in\Gamma_\gamma$ there are $C_1\geq 0$, $(\widetilde{q}_n)_{n\in\mathbb{N}}\subset\Gamma_\tau$ and $(\widetilde{a}_n)_{n\in\mathbb{N}}\in c_0^+$ such that $r_1(f)\leq C_1|||f|||_{\sigma,(\widetilde{q}_n,\widetilde{a}_n)_{n\in\mathbb{N}}}$ for all $f\in\mathcal{F}(\Omega)$. This implies that

$$|||f|||_{\sigma,(q_n,a_n)_{n\in\mathbb{N}},p}\leq C_0\sup_{e'\in U_p^\circ} r_0(e'\circ f)$$

and

$$\sup_{e'\in U_p^\circ} r_1(e'\circ f)\leq C_1|||f|||_{\sigma,(\widetilde{q}_n,\widetilde{a}_n)_{n\in\mathbb{N}},p}$$

for all $f\in\mathcal{F}(\Omega,E)_\sigma$ and $p\in\Gamma_E$. Thus the system of seminorms given by

$$|f|_{\sigma,r,p} := \sup_{e'\in U_p^\circ} r(e'\circ f),\quad f\in\mathcal{F}(\Omega,E)_\sigma,$$

for $r\in\Gamma_\gamma$ and $p\in\Gamma_E$ also generates $\gamma_{\sigma,\mathrm{s}}^E$. Similarly, $\gamma_{\sigma,\mathrm{s}}^E$ does not depend on the choice of Γ_E.

3.2 Remark. Let $(\mathcal{F}(\Omega), \|\cdot\|, \tau)$ be a Saks space of $\mathbb{K}$-valued functions on a non-empty set Ω such that $(\mathcal{F}(\Omega), \|\cdot\|)$ is a Banach space, $\tau_{\mathrm{p}} \leq \tau_{\|\cdot\|}$ and $\gamma(\|\cdot\|, \tau) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau)$, and $(E, \|\cdot\|_E)$ a normed space over $\mathbb{K}$. Then $(\mathcal{F}(\Omega, E)_\sigma, \|\cdot\|_\sigma^E, \tau_\sigma^E)$ is a Saks space where τ_σ^E is the locally convex Hausdorff topology on $\mathcal{F}(\Omega, E)_\sigma$ generated by the system of seminorms given by

$$q_\sigma^E(f) \coloneqq \sup_{e' \in U^\circ_{\|\cdot\|_E}} q(e' \circ f) = \sup_{e^* \in B_{\|\cdot\|_{E^*}}} q(e^* \circ f), \quad f \in \mathcal{F}(\Omega, E)_\sigma,$$

for $q \in \Gamma_\tau$ and Γ_τ as above. Indeed, this follows from Definition 2.1 and the observation

$$\begin{aligned}\sup_{q \in \Gamma_\tau} q_\sigma^E(f) &= \sup_{q \in \Gamma_\tau} \sup_{e^* \in B_{\|\cdot\|_{E^*}}} q(e^* \circ f) = \sup_{e^* \in B_{\|\cdot\|_{E^*}}} \sup_{q \in \Gamma_\tau} q(e^* \circ f) = \sup_{e^* \in B_{\|\cdot\|_{E^*}}} \|e^* \circ f\| \\ &= \|f\|_\sigma^E.\end{aligned}$$

Further, $\gamma_{\sigma,\mathrm{s}}^E = \gamma_{\mathrm{s}}(\|\cdot\|_\sigma^E, \tau_\sigma^E)$ by Definition 2.2 and the definitions of τ_σ^E and $\gamma_{\sigma,\mathrm{s}}^E$.

For a linear space $\mathcal{F}(\Omega)$ of $\mathbb{K}$-valued functions on a non-empty set Ω and $x \in \Omega$ we define the linear functional $\Delta(x)\colon \mathcal{F}(\Omega) \to \mathbb{K}$, $\Delta(x)(f) \coloneqq f(x)$.

3.3 Theorem. *Let $(\mathcal{F}(\Omega), \|\cdot\|, \tau)$ be a semi-Montel Saks space of $\mathbb{K}$-valued functions on a non-empty set Ω such that $\tau_{\mathrm{p}} \leq \tau$, E a complete locally convex Hausdorff space over $\mathbb{K}$ and set $\mathcal{F}(\Omega)_\gamma \coloneqq (\mathcal{F}(\Omega), \gamma)$. Then the map*

$$\chi\colon \mathcal{F}(\Omega)_\gamma \varepsilon E \to (\mathcal{F}(\Omega, E)_\sigma, \gamma_{\mathrm{s},\sigma}^E),\ \chi(u) \coloneqq u \circ \Delta = [x \mapsto u(\Delta(x))],$$

is a topological isomorphism. In particular, $(\mathcal{F}(\Omega, E)_\sigma, \gamma_{\sigma,\mathrm{s}}^E)$ is complete. If E is a Banach space, then $(\mathcal{F}(\Omega, E)_\sigma, \|\cdot\|_\sigma^E, \tau_\sigma^E)$ is a complete Saks space.

Proof. First, we show that χ is well-defined and linear. Since $\tau_{\mathrm{p}} \leq \tau \leq \gamma$, it holds $\Delta(x) \in \mathcal{F}(\Omega)'_\gamma$ for all $x \in \Omega$ (cf. [41, 4.2 Remark, p. 12]). We note that $(\mathcal{F}(\Omega)_\gamma)'_\kappa = (\mathcal{F}(\Omega)_\gamma)'_b$ because $\mathcal{F}(\Omega)_\gamma$ is a semi-Montel space. Hence $\mathcal{F}(\Omega)_\gamma \varepsilon E = L_e((\mathcal{F}(\Omega)_\gamma)'_b, (E, \tau_E))$ where τ_E denotes the locally convex Hausdorff topology of E. Let $u \in \mathcal{F}(\Omega)_\gamma \varepsilon E$ and $e' \in E'$. Using that $e' \circ u \in ((\mathcal{F}(\Omega)_\gamma)'_b)'$ and the semi-reflexivity of the semi-Montel space $\mathcal{F}(\Omega)_\gamma$, we note that there is $f_{e' \circ u} \in \mathcal{F}(\Omega)$ such that $(e' \circ u)(f') = f'(f_{e' \circ u})$ for all $f' \in \mathcal{F}(\Omega)'_\gamma$. This implies with $f' = \Delta(x)$ that $(e' \circ u) \circ \Delta = f_{e' \circ u}$. Thus the map χ is well-defined and it is easily seen to be linear as well.

Second, we show that χ is injective and continuous. Let Γ_τ be a directed system of continuous seminorms that generates the topology τ and fulfils (1). For $(q_n)_{n \in \mathbb{N}} \subset \Gamma_\tau$ and $(a_n)_{n \in \mathbb{N}} \in c_0^+$ we define $U_{q_n a_n} \coloneqq \{f \in \mathcal{F}(\Omega) \mid q_n(f) a_n < 1\}$ and note that the sets

$$V_{(q_n, a_n)_{n \in \mathbb{N}}} \coloneqq \bigcap_{n \in \mathbb{N}} U_{q_n a_n}$$

form a base of γ-neighbourhoods of zero by Definition 2.2 since $\gamma = \gamma_{\mathrm{s}}$ by condition (ii) of Remark 2.3 (b) and Remark 2.5. By the bipolar theorem we have

$$V^\circ_{(q_n, a_n)_{n \in \mathbb{N}}} = \overline{\operatorname{acx}}\Big(\bigcup_{n \in \mathbb{N}} U^\circ_{q_n a_n}\Big) \eqqcolon \overline{\operatorname{acx}}(W_{(q_n, a_n)_{n \in \mathbb{N}}})$$

where $\overline{\operatorname{acx}}(W_{(q_n, a_n)_{n \in \mathbb{N}}})$ denotes the closure in $(\mathcal{F}(\Omega), \gamma)'_\kappa$ of the absolutely convex hull $\operatorname{acx}(W_{(q_n, a_n)_{n \in \mathbb{N}}})$ of $W_{(q_n, a_n)_{n \in \mathbb{N}}} \coloneqq \bigcup_{n \in \mathbb{N}} U^\circ_{q_n a_n}$ (see [28, 8.2.4 Corollary, p. 149]). Due to [28, 8.4, p. 152, 8.5, p. 156–157] the topology of $\mathcal{F}(\Omega)_\gamma \varepsilon E$ is generated by the seminorms

$$|u|_{(q_n,a_n)_{n\in\mathbb{N}},p} \coloneqq \sup_{y\in V^{\circ}_{(q_n,a_n)_{n\in\mathbb{N}}}} p(u(y)) = \sup_{y\in\overline{\operatorname{acx}}(W_{(q_n,a_n)_{n\in\mathbb{N}}})} p(u(y)), \quad u\in\mathcal{F}(\Omega)_\gamma\varepsilon E,$$

for $(q_n)_{n\in\mathbb{N}}\subset\Gamma_\tau$, $(a_n)_{n\in\mathbb{N}}\in c_0^+$ and $p\in\Gamma_E$ where Γ_E denotes a system of seminorms that generates τ_E. By the continuity of $u\in\mathcal{F}(\Omega)_\gamma\varepsilon E$ we have

$$|u|_{(q_n,a_n)_{n\in\mathbb{N}},p} = \sup_{y\in\operatorname{acx}(W_{(q_n,a_n)_{n\in\mathbb{N}}})} p(u(y)) \geq \sup_{y\in W_{(q_n,a_n)_{n\in\mathbb{N}}}} p(u(y)).$$

On the other hand, for $y\in\operatorname{acx}(W_{(q_n,a_n)_{n\in\mathbb{N}}})$ there are $m\in\mathbb{N}$, $\lambda_k\in\mathbb{K}$, $f'_k\in U^{\circ}_{q_k a_k}$, $1\leq k\leq m$, with $\sum_{k=1}^m|\lambda_k|=1$ such that $y=\sum_{k=1}^m\lambda_k f'_k$. It follows that for all $u\in\mathcal{F}(\Omega)_\gamma\varepsilon E$

$$p(u(y))\leq\sum_{k=1}^m|\lambda_k|p(u(f'_k))\leq\sup_{1\leq k\leq m}p(u(f'_k))\leq\sup_{z\in W_{(q_n,a_n)_{n\in\mathbb{N}}}}p(u(z))$$

and we deduce

$$|u|_{(q_n,a_n)_{n\in\mathbb{N}},p} = \sup_{y\in W_{(q_n,a_n)_{n\in\mathbb{N}}}} p(u(y)) = \sup_{n\in\mathbb{N}}\sup_{f'\in U^{\circ}_{q_n a_n}} p(u(f')). \tag{3}$$

By the first part of the proof there is $f_{e'\circ u}\in\mathcal{F}(\Omega)$ such that $(e'\circ u)(f')=f'(f_{e'\circ u})$ and $(e'\circ u)\circ\Delta=f_{e'\circ u}$ for all $u\in\mathcal{F}(\Omega)_\gamma\varepsilon E$, $f'\in\mathcal{F}(\Omega)'_\gamma$ and $e'\in E'$. We conclude that

$$\begin{aligned}|||\chi(u)|||_{\sigma,(q_n,a_n)_{n\in\mathbb{N}},p} &= \sup_{e'\in U_p^{\circ}}\sup_{n\in\mathbb{N}} q_n(e'\circ(u\circ\Delta))a_n = \sup_{e'\in U_p^{\circ}}\sup_{n\in\mathbb{N}}\sup_{f'\in U^{\circ}_{q_n a_n}}|f'(f_{e'\circ u})| \\ &= \sup_{e'\in U_p^{\circ}}\sup_{n\in\mathbb{N}}\sup_{f'\in U^{\circ}_{q_n a_n}}|(e'\circ u)(f')| = \sup_{n\in\mathbb{N}}\sup_{f'\in U^{\circ}_{q_n a_n}} p(u(f')) \\ &\underset{(3)}{=} |u|_{(q_n,a_n)_{n\in\mathbb{N}},p} \end{aligned}\tag{4}$$

for all $u\in\mathcal{F}(\Omega)_\gamma\varepsilon E$. Hence χ is injective and continuous.

Third, we show that χ is surjective and note that (4) implies that the inverse of χ is continuous. Due to Remark 2.5 (d) we have

$$\mathcal{F}(\Omega)_\gamma\varepsilon E = L_e((\mathcal{F}(\Omega)_\gamma)'_b,(E,\tau_E)) = L_e((\mathcal{F}(\Omega)'_\gamma,\|\cdot\|_{\mathcal{F}(\Omega)'_\gamma}),(E,\tau_E)).$$

Hence the surjectivity of χ is a consequence of [40, 5.5 Theorem, p. 33], and [41, 4.1 Corollary, p. 12].

Fourth, since $\mathcal{F}(\Omega)_\gamma$ and E are complete, $\mathcal{F}(\Omega)_\gamma\varepsilon E$ is also complete by [30, Satz 10.3, p. 234], implying the completeness of $(\mathcal{F}(\Omega,E)_\sigma,\gamma^E_{\sigma,\mathrm{s}})$. Let E be a Banach space. Due to Remark 3.2 and Remark 2.3 (a) $(\mathcal{F}(\Omega,E)_\sigma,\|\cdot\|^E_\sigma,\tau^E_\sigma)$ is a Saks space and $\gamma^E_{\sigma,\mathrm{s}}=\gamma_\mathrm{s}(\|\cdot\|^E_\sigma,\tau^E_\sigma)\leq\gamma(\|\cdot\|^E_\sigma,\tau^E_\sigma)$, which implies the completeness of the Saks space. □

The proof of the continuity of χ and its inverse in Theorem 3.3 is similar to the proof of [37, Lemma 7, p. 1517]. Moreover, Theorem 3.3 in combination with Remark 3.2, Proposition 4.4 (b) and Corollary 4.11 (a) and (d) generalises [29, Theorem 4.4, p. 648] where $\mathcal{F}(\Omega,E)=\operatorname{Lip}_0(\Omega,E)$, E is a Banach space and $\gamma^E_{\mathrm{s},\sigma}=\gamma_\mathrm{s}(\|\cdot\|_{\operatorname{Lip}_0(\Omega,E)},\tau^E_{\mathcal{N}_{\Omega_{\mathrm{wd}}}})=\gamma\tau_\gamma$. Due to Proposition 4.4 (d) in combination with Corollary 4.5 (b), Corollary 4.6 (d) and (e), Corollary 4.8 (b) and (c), the result of Theorem 3.3 is already contained in [5, 3.1 Bemerkung, p. 141] (cf. [39, 5.2.10 Proposition, p. 77], [39, 5.2.17 Corollary, p. 80] and [52, 4.8 Theorem, p. 878]) for the weighted space $\mathcal{F}(\Omega,E)=\mathcal{C}v(\Omega,E)$ of continuous functions from Corollary 4.5 if Ω is discrete, the weighted space $\mathcal{F}(\Omega,E)=\mathcal{H}v(\Omega,E)$ of holomorphic functions from Corollary 4.6 and the weighted kernel $\mathcal{F}(\Omega, E) = \mathcal{C}_P v(\Omega, E)$ of a hypoelliptic linear partial differential operator from Corollary 4.8 even for quasi-complete locally convex Hausdorff spaces E. However, the proof is different. Theorem 3.3 also allows us to characterise $(\mathcal{F}(\Omega), \gamma)$ having the approximation property by approximation in $(\mathcal{F}(\Omega, E)_\sigma, \gamma^E_{\sigma,\mathrm{s}})$.

3.4 Corollary. *Let $(\mathcal{F}(\Omega), \|\cdot\|, \tau)$ be a semi-Montel Saks space of $\mathbb{K}$-valued functions on a non-empty set Ω such that $\tau_\mathrm{p} \leq \tau$. Then the following assertions are equivalent.*

(a) *$(\mathcal{F}(\Omega), \gamma)$ has the approximation property.*
(b) *$(\mathcal{F}(\Omega)'_\gamma, \|\cdot\|_{\mathcal{F}(\Omega)'_\gamma})$ has the approximation property.*
(c) *$\mathcal{F}(\Omega) \otimes E$ is dense in $(\mathcal{F}(\Omega, E)_\sigma, \gamma^E_{\sigma,\mathrm{s}})$ for every Banach space E over $\mathbb{K}$.*
(d) *$\mathcal{F}(\Omega) \otimes E$ is dense in $(\mathcal{F}(\Omega, E)_\sigma, \gamma^E_{\sigma,\mathrm{s}})$ for every complete locally convex Hausdorff space E over $\mathbb{K}$.*

Proof. The equivalence (a)$\Leftrightarrow$(b) follows from Proposition 2.6. The remaining equivalences are a consequence of Theorem 3.3 and [30, Satz 10.17, p. 250]. $\square$

[29, Theorem 4.6, p. 651–652] is a special case of Corollary 3.4 for $\mathcal{F}(\Omega) = \mathrm{Lip}_0(\Omega)$. For the space $\mathcal{F}(\Omega) = H^\infty(\Omega) = \mathcal{H}v(\Omega)$, $v(z) := 1$ for $z \in \Omega$, of bounded holomorphic $\mathbb{C}$-valued functions on a balanced bounded open subset Ω of a complex Banach space from Corollary 4.6 the statement of Corollary 3.4 is contained in [52, 5.4 Theorem, p. 883]. Further, it is known that the spaces $(H^\infty(\Omega), \gamma)$ and $(H^\infty(\Omega)'_\gamma, \|\cdot\|_{H^\infty(\Omega)'_\gamma})$ with $\gamma = \gamma(\|\cdot\|, \tau_\mathrm{co})$ have the approximation property by [5, Satz 3.9, p. 145] for simply connected open $\Omega \subset \mathbb{C}$ (cf. [13, V.2.4 Proposition, p. 233] for $\Omega = \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$). The same is true for $(\mathcal{C}v(\Omega), \gamma)$ and $(\mathcal{C}v(\Omega)'_\gamma, \|\cdot\|_{\mathcal{C}v(\Omega)'_\gamma})$ by [4, 5.5 Theorem (3), (4), p. 205] if Ω is discrete.

We close this section with an application of Theorem 3.3 to some spaces of integrable holomorphic functions. We denote by $\mathcal{H}(\mathbb{D})$ the space of $\mathbb{C}$-valued holomorphic functions on $\mathbb{D}$. For $1 \leq p < \infty$ the *Hardy space* is defined by

$$H^p := \Big\{ f \in \mathcal{H}(\mathbb{D}) \mid \|f\|_p^p := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{\mathrm{i}\theta})|^p \mathrm{d}\theta < \infty \Big\}$$

and the *weighted Bergman space* for $\alpha > -1$ by

$$A^p_\alpha := \Big\{ f \in \mathcal{H}(\mathbb{D}) \mid \|f\|^p_{\alpha,p} := \frac{\alpha+1}{\pi} \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \mathrm{d}z < \infty \Big\}.$$

The *Dirichlet space* is defined by

$$\mathcal{D} := \Big\{ f \in \mathcal{H}(\mathbb{D}) \mid \|f\|^2_{\mathcal{D}} := |f(0)|^2 + \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \mathrm{d}z < \infty \Big\}.$$

3.5 Corollary. *Let $(E, \|\cdot\|_E)$ be a normed space over $\mathbb{C}$ and $(\mathcal{F}(\mathbb{D}), \|\cdot\|)$ be one of the spaces $(H^p, \|\cdot\|_p)$, $(A^p_\alpha, \|\cdot\|_{\alpha,p})$ for $1 \leq p < \infty$ and $\alpha > -1$, or $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$.*

(a) *$(\mathcal{F}(\mathbb{D}, E)_\sigma, \|\cdot\|^E_\sigma, \tau^E_{\mathrm{co},\sigma})$ is a C-sequential Saks space where $\tau^E_{\mathrm{co},\sigma} := (\tau_\mathrm{co})^E_\sigma$.*
(b) *If E is a Banach space, then $(\mathcal{F}(\mathbb{D}, E)_\sigma, \|\cdot\|^E_\sigma, \tau^E_{\mathrm{co},\sigma})$ is complete.*
(c) *If $E = \mathbb{C}$, then $(\mathcal{F}(\mathbb{D}), \|\cdot\|, \tau_\mathrm{co})$ is semi-Montel and $\gamma(\|\cdot\|, \tau_\mathrm{co}) = \gamma_\mathrm{s}(\|\cdot\|, \tau_\mathrm{co})$.*
(d) *If $\mathcal{F}(\mathbb{D}) = H^p$, then τ_co on H^p is generated by the directed system of continuous seminorms $(|\cdot|_{p,s})_{0<s<1}$ given by*

$$|f|_{p,s}^{p} := \sup_{0<r<s} \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^{p} d\theta, \quad f \in H^{p},$$

for $0 < s < 1$, *which fulfils* $\|f\|_{p} = \sup_{0<s<1} |f|_{p,s}$ *for all* $f \in H^{p}$.

(e) *If* $\mathcal{F}(\mathbb{D}) = A_{\alpha}^{p}$, *then* τ_{co} *on* A_{α}^{p} *is generated by the directed system of continuous seminorms* $(|\cdot|_{\alpha,p,r})_{0<r<1}$ *given by*

$$|f|_{\alpha,p,r}^{p} := \frac{\alpha+1}{\pi} \int_{\mathbb{D}_{r}} |f(z)|^{p} (1-|z|^{2})^{\alpha} dz, \quad f \in A_{\alpha}^{p},$$

for $0 < r < 1$, *which fulfils* $\|f\|_{\alpha,p} = \sup_{0<r<1} |f|_{\alpha,p,r}$ *for all* $f \in A_{\alpha}^{p}$, *where* $\mathbb{D}_{r} := \{z \in \mathbb{C} \mid |z| < r\}$.

(f) *If* $\mathcal{F}(\mathbb{D}) = \mathcal{D}$, *then* τ_{co} *on* $\mathcal{D}$ *is generated by the directed system of continuous seminorms* $(|\cdot|_{\mathcal{D},r})_{0<r<1}$ *given by*

$$|f|_{\mathcal{D},r}^{2} := |f(0)|^{2} + \frac{1}{\pi} \int_{\mathbb{D}_{r}} |f'(z)|^{2} dz, \quad f \in \mathcal{D},$$

for $0 < r < 1$, *which fulfils* $\|f\|_{\mathcal{D}} = \sup_{0<r<1} |f|_{\mathcal{D},r}$ *for all* $f \in \mathcal{D}$.

Proof. (c) In all the cases we note that the τ_{co}-compactness of $B_{\|\cdot\|}$ is obtained from [17, p. 4–5] (which uses that $B_{\|\cdot\|}$ is compact in the Montel space $(\mathcal{H}(\mathbb{D}), \tau_{\mathrm{co}})$ since $B_{\|\cdot\|}$ is relatively compact there and its closedness is a consequence of Fatou's lemma). It follows that $(\mathcal{F}(\mathbb{D}), \|\cdot\|, \tau_{\mathrm{co}})$ is a semi-Montel Saks space by Remark 2.5 (a). Further, condition (ii) of Remark 2.3 (b) yields that $\gamma(\|\cdot\|, \tau_{\mathrm{co}}) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau_{\mathrm{co}})$.

(a)+(b) Due to part (c) and $\tau_{\mathrm{p}} \leq \tau_{\mathrm{co}}$ we have that $(\mathcal{F}(\mathbb{D}, E)_{\sigma}, \|\cdot\|_{\sigma}^{E}, \tau_{\mathrm{co},\sigma}^{E})$ is a Saks space, which is complete if E is a Banach space, by Remark 3.2 and Theorem 3.3. Since the countable system of seminorms

$$|f|_{n}^{E} := \sup_{e^{*} \in B_{\|\cdot\|_{E^{*}}}} \sup_{z \in \overline{\mathbb{D}_{1-(1/n)}}} |e^{*}(f(z))|, \quad f \in \mathcal{F}(\mathbb{D}, E)_{\sigma},$$

for $n \in \mathbb{N}$, $n \geq 2$, generates $\tau_{\mathrm{co},\sigma}^{E}$ on $\mathcal{F}(\mathbb{D}, E)_{\sigma}$, we get that $\tau_{\mathrm{co},\sigma}^{E}$ is metrisable on $\mathcal{F}(\mathbb{D}, E)_{\sigma}$. Hence $(\mathcal{F}(\mathbb{D}, E)_{\sigma}, \|\cdot\|_{\sigma}^{E}, \tau_{\mathrm{co},\sigma}^{E})$ is C-sequential by [46, Proposition 5.7, p. 2681–2682].

(d) For $0 < s < 1$ we have

$$|f|_{p,s}^{p} = \sup_{0<r<s} \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^{p} d\theta \leq \sup_{0 \leq r<s} \sup_{\theta \in [0,2\pi]} |f(re^{i\theta})|^{p} = \big(\sup_{z \in \mathbb{D}_{s}} |f(z)|\big)^{p}$$

as well as $\|f\|_{p} = \sup_{0<s<1} |f|_{p,s}$ for all $f \in H^{p}$. Furthermore, for $0 < s < r < 1$ we remark that

$$\begin{aligned} |f(z)|^{p} = \left|f\left(r\frac{z}{r}\right)\right|^{p} &\leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^{p} d\theta \frac{1}{1-\frac{|z|^{2}}{r^{2}}} \\ &\leq \frac{1}{2\pi\left(1-\frac{s^{2}}{r^{2}}\right)} \int_{0}^{2\pi} |f(re^{i\theta})|^{p} d\theta \leq \frac{1}{2\pi\left(1-\frac{s^{2}}{r^{2}}\right)} |f|_{p,r}^{p} \end{aligned}$$

for all $z \in \mathbb{D}_{s}$ and $f \in H^{p}$ by the proof of [61, Theorem 9.1, p. 253]. It follows that τ_{co} on H^{p} is generated by the directed system of continuous seminorms $(|\cdot|_{p,s})_{0<s<1}$.

(e) We have $\|f\|_{\alpha,p}=\sup_{0<r<1}|f|_{\alpha,p,r}$ and

$$|f|^p_{\alpha,p,r}\leq \frac{\alpha+1}{\pi}\max_{z\in\overline{\mathbb{D}}_r}(1-|z|^2)^{\alpha}\big(\sup_{z\in\mathbb{D}_r}|f(z)|\big)^p$$

for all $f\in A^p_{\alpha}$ and $0<r<1$. Now, for $0<r<1$ we choose $0<s<1-r$. We deduce from the mean value equality for holomorphic functions and Hölder's inequality that

$$\begin{aligned}|f(z)|&=\frac{1}{\pi s^2}\Big|\int_{\mathbb{D}_s(z)}f(w)\mathrm{d}w\Big|\leq\frac{(\pi s^2)^{\frac{1}{q}}}{\pi s^2}\Big(\int_{\mathbb{D}_s(z)}|f(w)|^p\mathrm{d}w\Big)^{\frac{1}{p}}\\&\leq(\pi s^2)^{\frac{1}{q}-1}\Big(\int_{\mathbb{D}_{r+s}}|f(w)|^p\frac{(1-|w|^2)^{\alpha}}{(1-|w|^2)^{\alpha}}\mathrm{d}w\Big)^{\frac{1}{p}}\\&\leq\frac{\pi(\pi s^2)^{\frac{1}{q}-1}}{\alpha+1}\max_{w\in\overline{\mathbb{D}}_{r+s}}(1-|w|^2)^{-\frac{\alpha}{p}}|f|_{\alpha,p,r+s}\end{aligned}$$

for all $z\in\mathbb{D}_r$ and $f\in A^p_{\alpha}$ with $\mathbb{D}_s(z):=\{w\in\mathbb{C}\mid|w-z|<s\}$ and $1<q\leq\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$. Thus τ_{co} on A^p_{α} is generated by the directed system of continuous seminorms $(|\cdot|_{\alpha,p,r})_{0<r<1}$.

(f) We observe that $\|f\|_{\mathcal{D}}=\sup_{0<r<1}|f|_{\mathcal{D},r}$ for all $f\in\mathcal{D}$. Moreover, we obtain by Hölder's inequality that

$$\begin{aligned}|f(z)|&\leq|f(0)|+\Big|\int_0^z f'(w)\mathrm{d}w\Big|\leq|f(0)|+\int_{\mathbb{D}_r}|f'(w)|\mathrm{d}w\\&\leq|f(0)|+(\pi r^2)^{\frac{1}{2}}\Big(\int_{\mathbb{D}_r}|f'(w)|^2\mathrm{d}w\Big)^{\frac{1}{2}}\leq 2^{\frac{1}{2}}\Big(|f(0)|^2+\pi r^2\int_{\mathbb{D}_r}|f'(w)|^2\mathrm{d}w\Big)^{\frac{1}{2}}\\&\leq 2^{\frac{1}{2}}\pi r|f|_{\mathcal{D},r}\end{aligned}$$

for all $z\in\mathbb{D}_r$ and $f\in\mathcal{D}$. Now, for $0<r<1$ we choose $0<s<1-r$. From Cauchy's inequality we deduce the estimate

$$\begin{aligned}|f|^2_{\mathcal{D},r}&\leq|f(0)|^2+\frac{1}{\pi}\int_{\mathbb{D}_r}|f'(z)|^2\mathrm{d}z\leq|f(0)|^2+r^2\Big(\sup_{z\in\mathbb{D}_r}|f'(z)|\Big)^2\\&\leq|f(0)|^2+\frac{r^2}{s^2}\Big(\sup_{z\in\mathbb{D}_r}\max_{w\in\mathbb{C},\,|w-z|=s}|f(w)|\Big)^2\leq\Big(1+\frac{r^2}{s^2}\Big)\Big(\sup_{z\in\mathbb{D}_{r+s}}|f(z)|\Big)^2\end{aligned}$$

for all $f\in\mathcal{D}$, which implies that τ_{co} on $\mathcal{D}$ is generated by the directed system of continuous seminorms $(|\cdot|_{\mathcal{D},r})_{0<r<1}$. □

4. Saks spaces of vector-valued functions

This section is dedicated to Saks spaces $\mathcal{F}(\Omega,E)$ of vector-valued functions which are often stronger than the spaces $\mathcal{F}(\Omega,E)_\sigma$ of weak vector-valued functions from the preceding section (see Proposition 4.4). In order to derive certain systems of seminorms on such spaces which generate the mixed topology we need to recall some results for *completely regular* Hausdorff spaces Ω (see [27, Definition 11.1, p. 180]). Examples of completely regular Hausdorff spaces are metrisable spaces by [27, Proposition 11.5, p. 181] and locally convex Hausdorff spaces by [20, Proposition 3.27, p. 95]. Further, every subspace of a completely regular

Hausdorff space is completely regular and Hausdorff as well. For a completely regular Hausdorff space Ω we denote by $\mathcal{W}^{+}_{\mathrm{b},0}(\Omega)$ the family of all bounded functions $w\colon \Omega \to [0,\infty)$ that *vanish at infinity*, i.e. for every $\varepsilon > 0$ the set $\{x \in \Omega \mid w(x) \geq \varepsilon\}$ is compact. Further, we denote by $\mathcal{W}^{+}_{\mathrm{usc},0}(\Omega)$ resp. $\mathcal{C}^{+}_{0}(\Omega)$ the family of all upper semicontinuous resp. continuous functions $w\colon \Omega \to [0,\infty)$ that vanish at infinity. We note that $\mathcal{C}^{+}_{0}(\Omega) \subset \mathcal{W}^{+}_{\mathrm{usc},0}(\Omega) \subset \mathcal{W}^{+}_{\mathrm{b},0}(\Omega)$ because upper semicontinuous functions are bounded on compact sets. By the proofs of [13, II.1.11 Proposition, p. 82–83] and [12, Proposition 3, p. 590] we have the following proposition.

4.1 Proposition. *Let Ω be a completely regular Hausdorff space, $(K_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of compact subsets of Ω and $(a_n)_{n\in\mathbb{N}}$ a strictly decreasing positive null-sequence. Then there is $w \in \mathcal{W}^{+}_{\mathrm{usc},0}(\Omega)$ such that $\operatorname{supp} w \subset \bigcup_{n\in\mathbb{N}} K_n$ and $w(x) = a_1$ for $x \in K_1$ and $a_{n+1} \leq w(x) \leq a_n$ for $x \in K_{n+1} \setminus K_n$ and $n \in \mathbb{N}$. If Ω is locally compact and $K_n \subset \mathring{K}_{n+1}$ for every $n \in \mathbb{N}$, then we may choose $w \in \mathcal{C}^{+}_{0}(\Omega)$.*

Here, $\operatorname{supp} w$ denotes the support of w and $\mathring{K}_{n+1}$ the set of inner points of K_{n+1}.

4.2 Definition. Let Ω and Λ be non-empty sets, $v\colon \Lambda \to (0,\infty)$, $(E, \|\cdot\|_E)$ a normed space over $\mathbb{K}$, $\mathcal{G}(\Omega, E)$ a linear subspace of E^{Ω}, $q^E\colon \mathcal{G}(\Omega,E) \to [0,\infty)$ a seminorm and $T^E\colon \mathcal{G}(\Omega,E) \to E^{\Lambda}$ a linear map where E^{Λ} denotes the space of functions from Λ to E. We define the space

$$\mathcal{F}v(\Omega, E) := \{f \in \mathcal{G}(\Omega,E) \mid \|f\|^E := q^E(f) + \sup_{x\in\Lambda} \|T^E(f)(x)\|_E v(x) < \infty\}.$$

If $E = \mathbb{K}$, we write $\mathcal{G}(\Omega) := \mathcal{G}(\Omega,\mathbb{K})$, $\mathcal{F}v(\Omega) := \mathcal{F}v(\Omega,\mathbb{K})$, $q := q^{\mathbb{K}}$, $T := T^{\mathbb{K}}$ and $\|\cdot\| := \|\cdot\|^{\mathbb{K}}$. If we want to emphasize dependencies, we write $\|f\|_{\mathcal{F}v(\Omega,E)}$ instead of $\|f\|^E$.

For a non-empty set Λ we denote by $\mathcal{N}_{\Lambda}$ the family of finite subsets of Λ. If Λ is a topological space, we denote by $\mathcal{K}_{\Lambda}$ the family of compact subsets of Λ.

4.3 Theorem. *Let Ω and Λ be non-empty sets, $v\colon \Lambda \to (0,\infty)$, $(E, \|\cdot\|_E)$ a normed space over $\mathbb{K}$, $\mathcal{G}(\Omega,E)$ a linear subspace of E^{Ω}, $q^E\colon \mathcal{G}(\Omega,E) \to [0,\infty)$ a seminorm, $T^E\colon \mathcal{G}(\Omega,E) \to E^{\Lambda}$ a linear map and suppose that $(\mathcal{F}v(\Omega,E), \|\cdot\|^E)$ is normed. Let $\mathcal{S}$ be a family of subsets of Λ such that $\mathcal{S}$ is closed under finite unions, $\Lambda = \bigcup_{S\in\mathcal{S}} S$ and denote by $\tau^E_{\mathcal{S}}$ the locally convex Hausdorff topology generated by the directed system of seminorms*

$$q^E_S(f) := q^E(f) + \sup_{x\in S} \|T^E(f)(x)\|_E v(x), \quad f \in \mathcal{F}v(\Omega,E),$$

for $S \in \mathcal{S}$. Then the following assertions hold.

(a) *$(\mathcal{F}v(\Omega,E), \|\cdot\|^E, \tau^E_{\mathcal{S}})$ is a Saks space.*
(b) *If there is a sequence $(S_n)_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that for every $S \in \mathcal{S}$ there is $N \in \mathbb{N}$ with $S \subset S_N$, then $(\mathcal{F}v(\Omega,E), \|\cdot\|^E, \tau^E_{\mathcal{S}})$ is C-sequential.*
(c) *The submixed topology $\gamma_{\mathrm{s}}(\|\cdot\|^E, \tau^E_{\mathcal{S}})$ is generated by the system of seminorms*

$$|||f|||^E_{(S_n,a_n)_{n\in\mathbb{N}}} := \sup_{n\in\mathbb{N}} \sup_{x\in S_n} (q^E(f) + \|T^E(f)(x)\|_E v(x)) a_n, \quad f \in \mathcal{F}v(\Omega,E),$$

where $(S_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{S}$ and $(a_n)_{n\in\mathbb{N}} \in c^{+}_{0}$.
(d) *If $\mathcal{S} = \mathcal{N}_{\Lambda}$, then $\gamma_{\mathrm{s}}(\|\cdot\|^E, \tau^E_{\mathcal{N}_{\Lambda}})$ is generated by the system of seminorms*

$$|||f|||^E_{(x_n,a_n)_{n\in\mathbb{N}}} := \sup_{n\in\mathbb{N}} (q^E(f) + \|T^E(f)(x_n)\|_E v(x_n)) a_n, \quad f \in \mathcal{F}v(\Omega,E),$$

where $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* Λ *and* $(a_n)_{n\in\mathbb{N}} \in c_0^+$.

(e) *Let* Λ *be a completely regular Hausdorff space and set* $\mathcal{W}_0 := \mathcal{W}^+_{\mathrm{b},0}(\Lambda)$ *or* $\mathcal{W}^+_{\mathrm{usc},0}(\Lambda)$. *If* $\mathcal{S} = \mathcal{K}_\Lambda$ *and* $\gamma(\|\cdot\|^E, \tau^E_{\mathcal{K}_\Lambda}) = \gamma_{\mathrm{s}}(\|\cdot\|^E, \tau^E_{\mathcal{K}_\Lambda})$, *then the mixed topology* $\gamma(\|\cdot\|^E, \tau^E_{\mathcal{K}_\Lambda})$ *is generated by the system of seminorms*

$$|f|^E_w := \sup_{x\in\Lambda}(q^E(f) + \|T^E(f)(x)\|_E v(x))w(x), \quad f \in \mathcal{F}v(\Omega, E),$$

for $w \in \mathcal{W}_0$. *If* Λ *is locally compact, we may replace* $\mathcal{W}_0$ *by* $\mathcal{C}^+_0(\Lambda)$.

Proof. (a) First, we note that the system of seminorms $(q^E_S)_{S\in\mathcal{S}}$ is directed and Hausdorff since $\mathcal{S}$ is closed under finite unions, $\|\cdot\|^E$ a norm by assumption and

$$\|f\|^E = q^E(f) + \sup_{S\in\mathcal{S}} q^E_S(f)$$

for all $f \in \mathcal{F}v(\Omega, E)$ because $\Lambda = \bigcup_{S\in\mathcal{S}} S$. Hence $\tau^E_{\mathcal{S}}$ is a locally convex Hausdorff topology with $\tau^E_{\mathcal{S}} \leq \tau_{\|\cdot\|^E}$ and $(\mathcal{F}v(\Omega, E), \|\cdot\|^E, \tau^E_{\mathcal{S}})$ a Saks space.

(b) The countable system of seminorms $(q_{S_n})_{n\in\mathbb{N}}$ generates $\tau^E_{\mathcal{S}}$ by our assumption on $\mathcal{S}$. Hence $\tau^E_{\mathcal{S}}$ is metrisable and so $(\mathcal{F}v(\Omega, E), \gamma(\|\cdot\|^E, \tau^E_{\mathcal{S}}))$ C-sequential by [46, Proposition 5.7, p. 2681–2682].

(c) This follows from part (a) and the definition of $\gamma_{\mathrm{s}}(\|\cdot\|^E, \tau^E_{\mathcal{S}})$.

(d) Let $(a_n)_{n\in\mathbb{N}} \in c_0^+$ and $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of Λ with cardinality $m_n := |S_n|$ for $n \in \mathbb{N}$. Then every S_n, $n \in \mathbb{N}$, is of the form $S_n = \{s^n_1, \dots, s^n_{m_n}\}$ with distinct elements $s^n_i \in \Lambda$ for $1 \leq i \leq m_n$. We set $x_i := s^1_i$ for $1 \leq i \leq m_1$. Further, for $n \in \mathbb{N}$ we set $j_n := \sum_{l=1}^n m_l$ and $x_{j_n+i} := s^{n+1}_i$ for $1 \leq i \leq m_{n+1}$. Moreover, we set $b_i := a_1$ for $1 \leq i \leq m_1$. For $n \in \mathbb{N}$ we set $b_{j_n+i} := a_{n+1}$ for $1 \leq i \leq m_{n+1}$. Then we have $(b_n)_{n\in\mathbb{N}} \in c_0^+$ and $|||f|||^E_{(S_n,a_n)_{n\in\mathbb{N}}} = |||f|||^E_{(x_n,b_n)_{n\in\mathbb{N}}}$ for all $f \in \mathcal{F}v(\Omega, E)$. On the other hand, $|||f|||^E_{(\{z_n\},a_n)_{n\in\mathbb{N}}} = |||f|||^E_{(z_n,a_n)_{n\in\mathbb{N}}}$ for all $f \in \mathcal{F}v(\Omega, E)$ and every sequence $(z_n)_{n\in\mathbb{N}}$ in Λ. Thus statement (d) follows from part (c).

(e) We denote by ω^E_{b} and ω^E_{usc} the locally convex Hausdorff topologies generated by $(|\cdot|^E_w)_{w\in\mathcal{W}^+_{\mathrm{b},0}(\Lambda)}$ and $(|\cdot|^E_w)_{w\in\mathcal{W}^+_{\mathrm{usc},0}(\Lambda)}$, respectively. First, we prove that the identity map $\mathrm{id}\colon (\mathcal{F}v(\Omega, E), \gamma(\|\cdot\|^E, \tau^E_{\mathcal{K}_\Lambda})) \to (\mathcal{F}v(\Omega, E), \omega^E_{\mathrm{b}})$ is continuous. Due to [13, I.1.7 Corollary, p. 8] and [13, I.1.8 Lemma, p. 8] we only need to prove that its restriction to $B_{\|\cdot\|^E}$ is $\tau^E_{\mathcal{K}_\Lambda}$-continuous at zero. Let $\varepsilon > 0$, $w \in \mathcal{W}^+_{\mathrm{b},0}(\Lambda)$ and set $V := \{f \in \mathcal{F}v(\Omega, E) \mid |f|^E_w \leq \varepsilon\}$. Then there is a compact set $K \subset \Lambda$ such that $w(x) < \frac{\varepsilon}{2}$ for $x \in \Lambda \setminus K$. We define $U := \{f \in \mathcal{F}v(\Omega, E) \mid q^E_K(f) \leq \frac{\varepsilon}{2(1+\|w\|_\infty)}\}$ where $\|w\|_\infty := \sup_{x\in\Lambda} w(x)$ and note that for all $f \in U \cap B_{\|\cdot\|^E}$ it holds that

$$\begin{aligned} |f|^E_w &\leq \sup_{x\in\Lambda\setminus K}(q^E(f) + \|T^E(f)(x)\|_E v(x))w(x) \\ &\quad + \sup_{x\in K}(q^E(f) + \|T^E(f)(x)\|_E v(x))w(x) \\ &\leq \frac{\varepsilon}{2}\|f\|^E + \|w\|_\infty q^E_K(f) \leq \frac{\varepsilon}{2} + \|w\|_\infty \frac{\varepsilon}{2(1+\|w\|_\infty)} \leq \varepsilon, \end{aligned}$$

yielding $(U \cap B_{\|\cdot\|^E}) \subset V$ and so the continuity of id.

Second, we prove that $\mathrm{id}\colon (\mathcal{F}v(\Omega, E), \omega^E_{\mathrm{usc}}) \to (\mathcal{F}v(\Omega, E), \gamma(\|\cdot\|^E, \tau^E_{\mathcal{K}_\Lambda}))$ is continuous. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of Λ and $(a_n)_{n\in\mathbb{N}} \in c_0^+$. W.l.o.g. $K_n \subset K_{n+1}$ and $0 < a_{n+1} < a_n$ for all $n \in \mathbb{N}$. Then there is $w \in \mathcal{W}^+_{\mathrm{usc},0}(\Lambda)$ with $\operatorname{supp} w \subset \bigcup_{n\in\mathbb{N}} K_n$ such that $w(x) = a_1$ for $x \in K_1$ and $a_{n+1} \leq w(x) \leq a_n$ for $x \in K_{n+1} \setminus K_n$ by Proposition 4.1. It follows that

$$|||f|||^E_{(K_n,a_n)_{n\in\mathbb{N}}} \leq \sup_{x\in\Lambda}(q^E(f) + \|T^E(f)(x)\|_E v(x))w(x) = |f|^E_w$$

for all $f \in \mathcal{F}v(\Omega, E)$, which yields the continuity of id by part (c) and the assumption $\gamma(\|\cdot\|^E, \tau^E_{\mathcal{K}_\Lambda}) = \gamma_s(\|\cdot\|^E, \tau^E_{\mathcal{K}_\Lambda})$. Due to $\mathcal{C}_0^+(\Lambda) \subset \mathcal{W}^+_{\mathrm{usc},0}(\Lambda) \subset \mathcal{W}^+_{\mathrm{b},0}(\Lambda)$ and Proposition 4.1 this proves statement (e). □

The proof of Theorem 4.3 (e) is just an adaptation of the proofs of [12, Proposition 3, p. 590] and [13, II.1.11 Proposition, p. 82]. In the next proposition we show that the space $\mathcal{F}v(\Omega, E)$ from Definition 4.2 is a linear subspace of $\mathcal{F}v(\Omega, E)_\sigma$ under some mild assumptions and we use the topology $\tau^E_{\mathcal{S}}$ defined in Theorem 4.3.

4.4 Proposition. *Let Ω and Λ be non-empty sets, $v\colon \Lambda \to (0,\infty)$, $(E, \|\cdot\|_E)$ a normed space over $\mathbb{K}$, $\mathcal{G}(\Omega)$ a linear subspace of $\mathbb{K}^\Omega$, $\mathcal{G}(\Omega, E)$ a linear subspace of E^Ω, $q\colon \mathcal{G}(\Omega) \to [0,\infty)$ and $q^E\colon \mathcal{G}(\Omega, E) \to [0,\infty)$ seminorms, $T\colon \mathcal{G}(\Omega) \to \mathbb{K}^\Lambda$ and $T^E\colon \mathcal{G}(\Omega, E) \to E^\Lambda$ linear maps and $(\mathcal{F}v(\Omega), \|\cdot\|)$ a normed space such that*

(i) *$e^* \circ f \in \mathcal{F}v(\Omega)$ and $(e^* \circ T^E)(f) = T(e^* \circ f)$ for all $e^* \in E^*$ and $f \in \mathcal{F}v(\Omega, E)$,*
(ii) *$q^E(f) = \sup_{e^* \in B_{\|\cdot\|_{E^*}}} q(e^* \circ f)$ for all $f \in \mathcal{F}v(\Omega, E)$.*

Then the following assertions hold.

(a) *$\mathcal{F}v(\Omega, E)$ is a linear subspace of $\mathcal{F}v(\Omega, E)_\sigma$ and $\|\cdot\|^E_\sigma \leq \|\cdot\|^E \leq 2\|\cdot\|^E_\sigma$ on $\mathcal{F}v(\Omega, E)$. In particular, $\|\cdot\|^E$ is a norm on $\mathcal{F}v(\Omega, E)$. If $q = 0$, then $\|\cdot\|^E_\sigma = \|\cdot\|^E$ on $\mathcal{F}v(\Omega, E)$.*

Suppose for (b)–(d) that $\mathcal{S}$ is a family of subsets of Λ such that $\mathcal{S}$ is closed under finite unions and $\Lambda = \bigcup_{S \in \mathcal{S}} S$, and $(\mathcal{F}v(\Omega), \|\cdot\|, \tau_{\mathcal{S}})$ is semi-Montel where $\tau_{\mathcal{S}} := \tau^{\mathbb{K}}_{\mathcal{S}}$.

(b) *Then $\gamma_s(\|\cdot\|^E, \tau^E_{\mathcal{S}}) = \gamma_s(\|\cdot\|^E_\sigma, \tau^E_{\mathcal{S},\sigma})$ on $\mathcal{F}v(\Omega, E)$ where $\tau^E_{\mathcal{S},\sigma} := (\tau_{\mathcal{S}})^E_\sigma$.*
(c) *If $\tau_{\mathrm{p}} \leq \tau_{\mathcal{S}}$, $\mathcal{F}v(\Omega, E) = \mathcal{F}v(\Omega, E)_\sigma$ (as linear spaces) and E is a Banach space, then the spaces $(\mathcal{F}v(\Omega, E), \gamma_s(\|\cdot\|^E, \tau^E_{\mathcal{S}}))$ and $(\mathcal{F}v(\Omega, E), \|\cdot\|^E, \tau^E_{\mathcal{S}})$ are complete.*
(d) *If Λ is a completely regular Hausdorff space, $\mathcal{S} = \mathcal{K}_\Lambda$ and $\mathcal{W}_0 := \mathcal{W}^+_{\mathrm{b},0}(\Lambda)$ or $\mathcal{W}^+_{\mathrm{usc},0}(\Lambda)$, then the topology $\gamma_s(\|\cdot\|^E, \tau^E_{\mathcal{K}_\Lambda})$ is generated by the system of seminorms $(|\cdot|^E_w)_{w \in \mathcal{W}_0}$. If Λ is locally compact, we may replace $\mathcal{W}_0$ by $\mathcal{C}_0^+(\Lambda)$.*

Proof. (a) Due to the first part of condition (i) we obtain that $\mathcal{F}v(\Omega, E)$ is a linear subspace of $\mathcal{F}v(\Omega, E)_\sigma$. The second part of condition (i) implies that

$$\begin{aligned}\sup_{x\in\Lambda} \|T^E(f)(x)\|_E v(x) &= \sup_{x\in\Lambda} \sup_{e^* \in B_{\|\cdot\|_{E^*}}} |(e^* \circ T^E)(f)(x)| v(x) \\ &= \sup_{e^* \in B_{\|\cdot\|_{E^*}}} \sup_{x\in\Lambda} |T(e^* \circ f)(x)| v(x)\end{aligned}$$

for all $f \in \mathcal{F}v(\Omega, E)$. Together with condition (ii) this yields that $\|\cdot\|^E_\sigma \leq \|\cdot\|^E \leq 2\|\cdot\|^E_\sigma$ on $\mathcal{F}v(\Omega, E)$, and $\|\cdot\|^E_\sigma = \|\cdot\|^E$ on $\mathcal{F}v(\Omega, E)$ if additionally $q = 0$. Since $\|\cdot\|^E_\sigma$ is a norm on $\mathcal{F}v(\Omega, E)_\sigma$, we get that $\|\cdot\|^E$ is a norm on $\mathcal{F}v(\Omega, E)$.

(b) Let $(S_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$ and $(a_n)_{n\in\mathbb{N}} \in c_0^+$. We have by part (a)

$$|||f|||_{\sigma,(S_n,a_n)_{n\in\mathbb{N}},\|\cdot\|_E} \underset{(2)}{=} \sup_{e^* \in B_{\|\cdot\|_{E^*}}} |||e^* \circ f|||^{\mathbb{K}}_{(S_n,a_n)_{n\in\mathbb{N}}} = \sup_{n\in\mathbb{N}} \sup_{e^* \in B_{\|\cdot\|_{E^*}}} q^{\mathbb{K}}_{S_n}(e^* \circ f) a_n$$

for all $f \in \mathcal{F}v(\Omega, E)$. Using the second part of condition (i) we get as in part (a) that

$$\sup_{x\in S_n}\|T^E(f)(x)\|_E v(x)=\sup_{e^*\in B_{\|\cdot\|_{E^*}}}\sup_{x\in S_n}|T(e^*\circ f)(x)|v(x)$$

for all $f\in\mathcal{F}v(\Omega,E)$. In combination with condition (ii) we obtain the estimates $|||\cdot|||_{\sigma,(S_n,a_n)_{n\in\mathbb{N}},\|\cdot\|_E}\leq |||\cdot|||^E_{(S_n,a_n)_{n\in\mathbb{N}}}\leq 2|||\cdot|||_{\sigma,(S_n,a_n)_{n\in\mathbb{N}},\|\cdot\|_E}$ on $\mathcal{F}v(\Omega,E)$, which imply $\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathcal{S}})=\gamma_{\mathrm{s}}(\|\cdot\|^E_\sigma,\tau^E_{\mathcal{S},\sigma})$ on $\mathcal{F}v(\Omega,E)$ by Remark 3.2 and Theorem 4.3 (b).

(c) The completeness of $(\mathcal{F}v(\Omega,E),\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathcal{S}}))$ follows from part (b), Remark 3.2 and Theorem 3.3. The completeness of $(\mathcal{F}v(\Omega,E),\|\cdot\|^E,\tau^E_{\mathcal{S}})$ then follows from $\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathcal{S}})\leq\gamma(\|\cdot\|^E,\tau^E_{\mathcal{S}})$ by Remark 2.3 (a).

(d) As $(\mathcal{F}v(\Omega),\|\cdot\|,\tau_{\mathcal{K}_\Lambda})$ is semi-Montel, we know that $\gamma(\|\cdot\|,\tau_{\mathcal{K}_\Lambda})=\gamma_{\mathrm{s}}(\|\cdot\|,\tau_{\mathcal{K}_\Lambda})$ by Remark 2.5 (a) and condition (ii) of Remark 2.3 (b). Due to Theorem 4.3 (d) $\gamma(\|\cdot\|,\tau_{\mathcal{K}_\Lambda})$ is generated by the system of seminorms

$$|f|^{\mathbb{K}}_w=\sup_{x\in\Lambda}(q(f)+|T(f)(x)|v(x))w(x),\quad f\in\mathcal{F}v(\Omega),$$

for $w\in\mathcal{W}_0$. We deduce that the system of seminorms given by

$$|f|_{\sigma,w,\|\cdot\|_E}:=\sup_{e^*\in B_{\|\cdot\|_{E^*}}}\sup_{x\in\Lambda}(q(e^*\circ f)+|T(e^*\circ f)(x)|v(x))w(x),\quad f\in\mathcal{F}v(\Omega,E)_\sigma,$$

for $w\in\mathcal{W}_0$ generates the topology $\gamma_{\mathrm{s}}(\|\cdot\|^E_\sigma,\tau^E_{\mathcal{K}_\Lambda,\sigma})$ on $\mathcal{F}v(\Omega,E)_\sigma$ by Remark 3.1 and Remark 3.2. Similar to the proofs of parts (a) and (b) we obtain that $|\cdot|_{\sigma,w,\|\cdot\|_E}\leq|\cdot|^E_w\leq 2|\cdot|_{\sigma,w,\|\cdot\|_E}$ on $\mathcal{F}v(\Omega,E)$ for every $w\in\mathcal{W}_0$, yielding that $\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathcal{K}_\Lambda})$ is generated by the system of seminorms $(|\cdot|^E_w)_{w\in\mathcal{W}_0}$ by part (b). □

Condition (i) of Proposition 4.4 means that the tuple (T^E,T) is strong for $(\mathcal{F}v,E)$ in the sense of [38, Definition 2.2 (b), p. 4]. In our first example of this section we consider weighted Saks spaces of continuous vector-valued functions on a completely regular Hausdorff space and a sufficient condition for their completeness involves the notion of a $k_{\mathbb{R}}$-space. A completely regular space Ω is called a *$k_{\mathbb{R}}$-space* if for any completely regular space Y and any map $f\colon\Omega\to Y$, whose restriction to each compact $K\subset\Omega$ is continuous, the map is already continuous on Ω (see [8, (2.3.7) Proposition, p. 22]). Moreover, a topological space Ω is called a *k-space* if it fulfils the following condition: $A\subset\Omega$ is closed if and only if $A\cap K$ is closed in K for every compact $K\subset\Omega$. Examples of Hausdorff $k_{\mathbb{R}}$-spaces are completely regular Hausdorff k-spaces by [18, 3.3.21 Theorem, p. 152]. In particular, metrisable spaces and *DFM-spaces*, i.e. strong duals of Fréchet–Montel spaces, are completely regular Hausdorff k-spaces by [18, 3.3.20 Theorem, p. 152] and [36, 4.11 Theorem (5), p. 39], respectively. For a non-empty completely regular Hausdorff space Ω, a continuous function $v\colon\Omega\to(0,\infty)$ and a normed space $(E,\|\cdot\|_E)$ over $\mathbb{K}$ we set

$$\mathcal{C}v(\Omega,E):=\{f\in\mathcal{C}(\Omega,E)\mid\|f\|^E:=\sup_{x\in\Omega}\|f(x)\|_E v(x)<\infty\}$$

where $\mathcal{C}(\Omega,E)$ is the space of E-valued continuous functions on Ω. Further, we define $\mathcal{C}v(\Omega):=\mathcal{C}v(\Omega,\mathbb{K})$ and $\|\cdot\|:=\|\cdot\|^{\mathbb{K}}$. Setting $\Lambda:=\Omega$, $\mathcal{G}(\Omega,E):=\mathcal{C}(\Omega,E)$, $q^E:=0$ and $T^E(f):=f$ for $f\in\mathcal{C}(\Omega,E)$, we note that $\mathcal{F}v(\Omega,E)=\mathcal{C}v(\Omega,E)$, $\tau^E_{\mathcal{K}_\Omega}=\tau^E_{\mathrm{co}}$ on $\mathcal{C}v(\Omega,E)$ and conditions (i) and (ii) of Proposition 4.4 are fulfilled.

4.5 Corollary. *Let Ω be a non-empty completely regular Hausdorff space, $v\colon\Omega\to(0,\infty)$ continuous and $(E,\|\cdot\|_E)$ a normed space over $\mathbb{K}$. Then the following assertions hold.*

(a) *$(\mathcal{C}v(\Omega,E),\|\cdot\|^E,\tau^E_{\mathrm{co}})$ is a Saks space, $\gamma(\|\cdot\|^E,\tau^E_{\mathrm{co}})=\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathrm{co}})$ and the mixed topology $\gamma(\|\cdot\|^E,\tau^E_{\mathrm{co}})$ is generated by the system of seminorms*

$$|||f|||^E_{(K_n,a_n)_{n\in\mathbb{N}}} := \sup_{n\in\mathbb{N}}\sup_{x\in K_n}\|f(x)\|_E v(x)a_n,\quad f\in \mathcal{C}v(\Omega,E),$$

where $(K_n)_{n\in\mathbb{N}}$ is a sequence of compact subsets of Ω and $(a_n)_{n\in\mathbb{N}}\in c_0^+$.

(b) *If Ω is discrete, then $\mathcal{C}v(\Omega,E)=\mathcal{C}v(\Omega,E)_\sigma$, $\gamma(\|\cdot\|^E,\tau^E_{\mathrm{co}})=\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathcal{N}_\Omega})$ and $\gamma(\|\cdot\|^E,\tau^E_{\mathrm{co}})$ is also generated by the system of seminorms*

$$|||f|||^E_{(x_n,a_n)_{n\in\mathbb{N}}} := \sup_{n\in\mathbb{N}}\|f(x_n)\|_E v(x_n)a_n,\quad f\in \mathcal{C}v(\Omega,E),$$

where $(x_n)_{n\in\mathbb{N}}$ is a sequence in Ω and $(a_n)_{n\in\mathbb{N}}\in c_0^+$.

(c) *If Ω is a $k_{\mathbb{R}}$-space and E a Banach space, then $(\mathcal{C}v(\Omega,E),\|\cdot\|^E,\tau^E_{\mathrm{co}})$ is complete.*

(d) *If Ω is* hemicompact, *i.e. there is a sequence $(K_n)_{n\in\mathbb{N}}$ in $\mathcal{K}_\Omega$ such that for every $K\in\mathcal{K}_\Omega$ there is $N\in\mathbb{N}$ with $K\subset K_N$, then $(\mathcal{C}v(\Omega,E),\|\cdot\|^E,\tau^E_{\mathrm{co}})$ is C-sequential.*

(e) *If Ω is a hemicompact $k_{\mathbb{R}}$-space, or a completely metrisable space, then $(\mathcal{C}v(\Omega,E),\gamma(\|\cdot\|^E,\tau^E_{\mathrm{co}}))$ is a Mackey space. If E is in addition a Banach space, then $(\mathcal{C}v(\Omega,E),\gamma(\|\cdot\|^E,\tau^E_{\mathrm{co}}))$ is a strong Mackey space.*

(f) *Let $\mathcal{W}_0:=\mathcal{W}^+_{\mathrm{b},0}(\Omega)$ or $\mathcal{W}^+_{\mathrm{usc},0}(\Omega)$. Then $\gamma(\|\cdot\|^E,\tau^E_{\mathrm{co}})$ is also generated by the system of seminorms*

$$|f|^E_w := \sup_{x\in\Omega}\|f(x)\|_E v(x)w(x),\quad f\in \mathcal{C}v(\Omega,E),$$

for $w\in\mathcal{W}_0$. If Ω is locally compact, we may replace $\mathcal{W}_0$ by $\mathcal{C}_0^+(\Omega)$.

Proof. Due Theorem 4.3 and our observations above we only need to prove that $\gamma(\|\cdot\|^E,\tau^E_{\mathrm{co}})=\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathrm{co}})$ from part (a), and parts (b), (c) and (e).

(a) We show that condition (i) of Remark 2.3 (b) is fulfilled. We only need to adjust the proof of [60, Example D), p. 65–66] to the weighted vector-valued case, which we do for the sake of the reader. Let $f\in\mathcal{C}v(\Omega,E)$, $\varepsilon>0$ and $K\subset\Omega$ be compact. Since $\|f(\cdot)\|_E v$ is continuous on Ω, there is an open set $G\subset\Omega$ with $K\subset G$ such that

$$\sup_{x\in G}\|f(x)\|_E v(x)\leq \sup_{x\in K}\|f(x)\|_E v(x)+\varepsilon = q^E_K(f)+\varepsilon. \tag{5}$$

The set $\Omega\setminus G$ is closed and disjoint with the compact set $K\subset\Omega$. By [8, (2.1.5) Proposition, p. 17] the complete regularity of Ω implies that there is a continuous function $u\colon\Omega\to[0,1]$ such that $u_{|K}=0$ and $u_{|\Omega\setminus G}=1$. Now, we set $g:=(1-u)f$ and $h:=uf$ and note that $f=g+h$ and $g,h\in\mathcal{C}v(\Omega,E)$. Due to the properties of u we have $q^E_K(h)=0$ and

$$\begin{aligned}\|g\|^E &= \sup_{x\in\Omega}(1-u(x))\|f(x)\|_E v(x) = \sup_{x\in G}(1-u(x))\|f(x)\|_E v(x)\\ &\leq \sup_{x\in G}\|f(x)\|_E v(x) \underset{(5)}{\leq} q^E_K(f)+\varepsilon.\end{aligned}$$

Thus condition (i) of Remark 2.3 (b) is fulfilled, yielding $\gamma(\|\cdot\|^E,\tau^E_{\mathrm{co}})=\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathrm{co}})$.

(b) Since Ω is discrete, every subset of Ω is open, and a subset of Ω is compact if and only if it is finite. Thus $\tau^E_{\mathrm{co}}=\tau^E_{\mathcal{K}_\Omega}=\tau^E_{\mathcal{N}_\Omega}$ and statement (b) follows from part (a), Theorem 4.3 (d) and Mackey's theorem.

(c) Let $\mathcal{C}_b(\Omega,E):=\mathcal{C}\widetilde{v}(\Omega,E)$ for $\widetilde{v}(x):=1$, $x\in\Omega$, and set $\|\cdot\|^E_\infty:=\|\cdot\|_{\mathcal{C}\widetilde{v}(\Omega,E)}$. By part (f) the multiplication operator

$$M^E_v\colon \mathcal{C}v(\Omega,E)\to\mathcal{C}_b(\Omega,E),\ M^E_v(f):=fv,$$

is a topological isomorphism w.r.t. $\gamma(\|\cdot\|^E, \tau_{\mathrm{co}}^E)$ and $\gamma(\|\cdot\|_\infty^E, \tau_{\mathrm{co}}^E)$. The space $(\mathcal{C}_b(\Omega, E), \gamma(\|\cdot\|_\infty^E, \tau_{\mathrm{co}}^E))$ is complete by [13, II.4.2 Proposition 2), p. 113] because Ω is a $k_{\mathbb{R}}$-space (see [13, p. 112] and note that $k_{\mathbb{R}}$-spaces are exactly the $\mathcal{K}$-complete spaces by [13, p. 80]) and $(E, \|\cdot\|_E, \tau_{\|\cdot\|_E})$ a complete Saks space as $\gamma(\|\cdot\|_E, \tau_{\|\cdot\|_E}) = \tau_{\|\cdot\|_E}$. This implies that $(\mathcal{C}v(\Omega, E), \gamma(\|\cdot\|^E, \tau_{\mathrm{co}}^E))$ is also complete since M_v^E is a topological isomorphism.

(e) Let Ω be a hemicompact $k_{\mathbb{R}}$-space. Then Ω is a k-space by [50, Lemma 5.1, p. 884] and we have that $(\mathcal{C}_b(\Omega, E), \gamma(\|\cdot\|_\infty^E, \tau_{\mathrm{co}}^E))$ is a Mackey space, which is strong if E is a Banach space, by part (f) and [33, Theorem 3.4, p. 165]. Let Ω be a completely metrisable space. Then $(\mathcal{C}_b(\Omega, E), \gamma(\|\cdot\|_\infty^E, \tau_{\mathrm{co}}^E))$ is a Mackey space, which is strong if E is a Banach space, by part (f), [32, Theorem 2, p. 35] and [31, Theorem 3.7, p. 202]. Using the topological isomorphism M_v^E from part (c), we note that both statements remain valid if we replace $\mathcal{C}_b(\Omega, E)$ by $\mathcal{C}v(\Omega, E)$ and $\|\cdot\|_\infty^E$ by $\|\cdot\|^E$. □

We remark that $(\mathcal{C}v(\Omega), \|\cdot\|, \tau_{\mathrm{co}})$ is semi-Montel by [41, 3.9 Example (i), p. 10] if Ω is discrete. In the case $\mathcal{C}_b(\Omega, E)$ the statement from Corollary 4.5 (a) that $(\mathcal{C}_b(\Omega, E), \|\cdot\|_\infty^E, \tau_{\mathrm{co}}^E)$ is a Saks space and Corollary 4.5 (f) for $\mathcal{W}_0 = \mathcal{W}_{\mathrm{usc},0}^+(\Omega)$ (see [13, p. 81–82]) are contained in [13, II.4.1 Definition, p. 113] and [13, II.4.2 Proposition 2), 6), p. 113] (see also [23, 1.1 Remark, p. 844]).[1] In the case $\mathcal{C}_b(\Omega)$ Corollary 4.5 (f) is contained in [12, Proposition 3, p. 590] for locally compact Ω and $\mathcal{W}_0 = \mathcal{C}_0^+(\Omega)$. In the case $\mathcal{C}v(\Omega)$ the statement from Corollary 4.5 (a) that $\gamma(\|\cdot\|, \tau_{\mathrm{co}}) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau_{\mathrm{co}})$, the inverse of the topological isomorphism $M_v^{\mathbb{K}}$ from the proof of part (c) and Corollary 4.5 (c) are contained in [25, Lemmas A.1, A.4, p. 44] and [25, Theorem A.5, p. 44]. Moreover, we note that Corollary 4.5 (b) does not hold for general Ω by [30, Beispiel, p. 232]. Furthermore, we remark that $(\mathcal{C}v(\Omega, E), \gamma(\|\cdot\|^E, \tau_{\mathrm{co}}^E))$ is C-sequential by [42, Remark 3.19 (a), p. 14] combined with the topological isomorphism M_v^E if $E = \mathbb{K}$ and Ω a *Polish space*, i.e. separably completely metrisable. It is an open question whether this remains valid if E is a Banach space. Due to Corollary 4.5 (e) and [58, Corollary 7.6, p. 52] it would be sufficient to prove that $(\mathcal{C}v(\Omega, E), \gamma(\|\cdot\|^E, \tau_{\mathrm{co}}^E))$ is a *Mazur space* if Ω is Polish and E a Banach space, i.e. that every sequentially $\gamma(\|\cdot\|^E, \tau_{\mathrm{co}}^E)$-continuous linear functional on $\mathcal{C}v(\Omega, E)$ is already $\gamma(\|\cdot\|^E, \tau_{\mathrm{co}}^E)$-continuous.

Next, we consider subspaces of $\mathcal{C}v(\Omega, E)$. Let $(E, \|\cdot\|_E)$ be a normed space over $\mathbb{C}$. For a non-empty open subset Ω of a complex locally convex Hausdorff space let $\mathcal{H}(\Omega, E)$ be the space of holomorphic functions $f\colon \Omega \to E$, i.e. the space of Gâteaux-holomorphic and continuous functions $f\colon \Omega \to E$ (see [15, Definition 3.6, p. 152]), and for a continuous function $v\colon \Omega \to (0, \infty)$ we set

$$\mathcal{H}v(\Omega, E) \coloneqq \{f \in \mathcal{H}(\Omega, E) \mid \|f\|^E \coloneqq \sup_{z\in\Omega} \|f(z)\|_E v(z) < \infty\}.$$

Further, we define $\mathcal{H}v(\Omega) \coloneqq \mathcal{H}v(\Omega, \mathbb{C})$ and $\|\cdot\| \coloneqq \|\cdot\|^{\mathbb{C}}$. Setting $\Lambda \coloneqq \Omega$, $\mathcal{G}(\Omega, E) \coloneqq \mathcal{H}(\Omega, E)$, $q^E \coloneqq 0$ and $T^E(f) \coloneqq f$ for $f \in \mathcal{H}(\Omega, E)$, we observe that $\mathcal{F}v(\Omega, E) = \mathcal{H}v(\Omega, E)$, $\tau_{\mathcal{K}_\Omega}^E = \tau_{\mathrm{co}}^E$ on $\mathcal{H}v(\Omega, E)$ and conditions (i) and (ii) of Proposition 4.4 are fulfilled.

4.6 Corollary. *Let Ω be a non-empty open subset of a locally convex Hausdorff space X over $\mathbb{C}$, $v\colon \Omega \to (0, \infty)$ continuous and $(E, \|\cdot\|_E)$ a normed space over $\mathbb{C}$. Then the following assertions hold.*

(a) *$(\mathcal{H}v(\Omega, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ is a Saks space.*
(b) *If X is a $k_{\mathbb{R}}$-space and E a Banach space, then $(\mathcal{H}v(\Omega, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ is complete.*
(c) *If Ω is hemicompact, then $(\mathcal{H}v(\Omega, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ is C-sequential.*
(d) *If X is a k-space and $E = \mathbb{C}$, then $(\mathcal{H}v(\Omega), \|\cdot\|, \tau_{\mathrm{co}})$ is semi-Montel and $\gamma(\|\cdot\|, \tau_{\mathrm{co}}) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau_{\mathcal{S}})$ for $\mathcal{S} \in \{\mathcal{N}_\Omega, \mathcal{K}_\Omega\}$.*

[1] The condition of upper semicontinuity or boundedness for the weights w is missing in [23] (and [31–33]) even though it is contained in its reference [55, Theorem 2.4, p. 316] for the proof.

(e) *If X is metrisable or a DFM-space, and E a Banach space, then it holds $\mathcal{H}v(\Omega,E)=\mathcal{H}v(\Omega,E)_\sigma$.*

Proof. (a) and (c) follow from Theorem 4.3 and our observations above.

(b) We set $\gamma_{\mathcal{C}v(\Omega,E)} := \gamma(\|\cdot\|_{\mathcal{C}v(\Omega,E)}, \tau^E_{\mathrm{co}|\mathcal{C}v(\Omega,E)})$ and note $\gamma_{\mathcal{C}v(\Omega,E)|\mathcal{H}v(\Omega,E)} \leq \gamma(\|\cdot\|_{\mathcal{H}v(\Omega,E)}, \tau^E_{\mathrm{co}|\mathcal{H}v(\Omega,E)})$ by [13, p. 39]. The space $(\mathcal{C}v(\Omega,E),\gamma_{\mathcal{C}v(\Omega,E)})$ is complete by Corollary 4.5 (c) and $\mathcal{H}v(\Omega,E)$ a closed subspace. This implies statement (b).

(d) By [41, 3.9 Example (iii), p. 10–11] $(\mathcal{H}v(\Omega),\|\cdot\|,\tau_{\mathrm{co}})$ is semi-Montel. Furthermore, the observations $\tau_{\mathcal{K}_\Omega}=\tau_{\mathrm{co}}$ and $\tau_{\mathcal{N}_\Omega}\leq\tau_{\mathrm{co}}$ on $\mathcal{H}v(\Omega)$ imply that $\gamma(\|\cdot\|,\tau_{\mathrm{co}})=\gamma_{\mathrm{s}}(\|\cdot\|,\tau_{\mathcal{S}})$ for $\mathcal{S}\in\{\mathcal{N}_\Omega,\mathcal{K}_\Omega\}$ by Remark 2.5 (c).

(e) This follows from [15, Example 3.8 (g), p. 159]. □

Regarding Corollary 4.6 (c), we remark that Ω is hemicompact by [14, Example 2.47, p. 79–81] if X is a DFM-space. Theorem 4.3 (e) and Corollary 4.6 (d) imply [6, Proposition 3.1, p. 77] where $X=\mathbb{C}^d$. Theorem 4.3 (d) and Corollary 4.6 (d) imply [52, 4.5 Theorem, p. 875] where X is a Banach space and $v(z):=1$ for all $z\in\Omega$. Further, we remark that $(H^\infty(\mathbb{D}),\gamma(\|\cdot\|_\infty,\tau_{\mathrm{co}}))$ is not a Mackey space by [13, V.2.7 Corollary, p. 235].

4.7 Remark. Let $(E,\|\cdot\|)$ be a Banach space over $\mathbb{C}$ and $1\leq p<\infty$. We may also define a strong E-valued version of the Hardy H^p from Corollary 3.5. Let

$$H^p(E) := \Big\{ f\in\mathcal{H}(\mathbb{D},E) \mid (\|f\|^E_p)^p := \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\|f(re^{\mathrm{i}\theta})\|^p_E \mathrm{d}\theta < \infty \Big\}.$$

However, in contrast to the case $p=\infty$, we only have the strict inclusion $H^p(E)\subsetneq H^p(E)_\sigma$ for $1\leq p<\infty$ by [24, Corollary 12, p. 359] if E is infinite-dimensional.

Let us turn to another subspace of $\mathcal{C}v(\Omega,E)$. For a non-empty open set $\Omega\subset\mathbb{R}^d$ and a normed space $(E,\|\cdot\|_E)$ over $\mathbb{K}$ we denote by $\mathcal{C}^\infty(\Omega,E)$ the space of infinitely continuously partially differentiable E-valued functions on Ω and by $(\partial^\beta)^E f$ the β-th partial derivative of $f\in\mathcal{C}^\infty(\Omega,E)$ for a multi-index $\beta\in\mathbb{N}_0^d$. If $E=\mathbb{K}$, we set $\mathcal{C}^\infty(\Omega):=\mathcal{C}^\infty(\Omega,\mathbb{K})$ and $\partial^\beta f := (\partial^\beta)^{\mathbb{K}} f$ for $f\in\mathcal{C}^\infty(\Omega)$ and $\beta\in\mathbb{N}_0^d$. For $\mathbb{K}=\mathbb{C}$ and a polynomial P on $\mathbb{R}^d$ with constant complex coefficients we define the linear partial differential operator $P(\partial)^E := P((\partial)^E)$ on $\mathcal{C}^\infty(\Omega,E)$ in the usual way and its kernel

$$\mathcal{C}_P(\Omega,E) := \{ f\in\mathcal{C}^\infty(\Omega,E) \mid f\in\ker P(\partial)^E \}.$$

For a continuous function $v\colon\Omega\to(0,\infty)$ we define the weighted kernel

$$\mathcal{C}_P v(\Omega,E) := \{ f\in\mathcal{C}_P(\Omega,E) \mid \|f\|^E := \sup_{x\in\Omega}\|f(x)\|_E v(x) < \infty \}.$$

Further, we define $\mathcal{C}_P v(\Omega) := \mathcal{C}_P v(\Omega,\mathbb{C})$ and $\|\cdot\| := \|\cdot\|^{\mathbb{C}}$. Setting $\Lambda:=\Omega$, $\mathcal{G}(\Omega,E):=\mathcal{C}_P(\Omega,E)$, $q^E:=0$ and $T^E(f):=f$ for $f\in\mathcal{C}_P(\Omega,E)$, we observe that $\mathcal{F}v(\Omega,E)=\mathcal{C}_P v(\Omega,E)$, $\tau^E_{\mathcal{K}_\Omega}=\tau^E_{\mathrm{co}}$ on $\mathcal{C}_P v(\Omega,E)$ and conditions (i) and (ii) of Proposition 4.4 are fulfilled.

4.8 Corollary. *Let $\Omega\subset\mathbb{R}^d$ be non-empty and open, $v\colon\Omega\to(0,\infty)$ continuous, $(E,\|\cdot\|_E)$ a normed space over $\mathbb{C}$ and $P(\partial)^E$ a linear partial differential operator such that $P(\partial)^{\mathbb{C}}$ is hypoelliptic. Then the following assertions hold.*

(a) *$(\mathcal{C}_P v(\Omega,E),\|\cdot\|^E,\tau^E_{\mathrm{co}})$ is a C-sequential Saks space.*

(b) *If* $E=\mathbb{C}$, *then* $(\mathcal{C}_P v(\Omega), \|\cdot\|, \tau_{\mathrm{co}})$ *is semi-Montel and* $\gamma(\|\cdot\|, \tau_{\mathrm{co}}) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau_{\mathcal{S}})$ *for* $\mathcal{S} \in \{\mathcal{N}_\Omega, \mathcal{K}_\Omega\}$.
(c) *If* E *is a Banach space, then* $\mathcal{C}_P v(\Omega, E) = \mathcal{C}_P v(\Omega, E)_\sigma$ *and the Saks space* $(\mathcal{C}_P v(\Omega, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ *is complete.*

Proof. (a) This follows from Theorem 4.3, our observations above and the fact that open subsets of $\mathbb{R}^d$ are hemicompact.

(b) By [41, 3.9 Example (ii), p. 10] $(\mathcal{C}_P v(\Omega), \|\cdot\|, \tau_{\mathrm{co}})$ is semi-Montel. The rest of the proof is analogous to the proof of Corollary 4.6 (d).

(c) It holds $\mathcal{C}_P v(\Omega, E) = \mathcal{C}_P v(\Omega, E)_\sigma$ by the weak-strong principle [7, Theorem 9, p. 232] and Mackey's theorem. Due to part (b), $\tau_{\mathrm{p}} \leq \tau_{\mathcal{K}_\Omega}$, $\tau_{\mathcal{K}_\Omega}^E = \tau_{\mathrm{co}}^E$ and Proposition 4.4 (c) we get that the Saks space $(\mathcal{C}_P v(\Omega, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ is complete. □

4.9 Remark. Let $(X, \|\cdot\|, \tau)$ be a Saks space, Y a linear subspace of X. Then the subtriple $(Y, \|\cdot\|_{|Y}, \tau_{|Y})$ is also a Saks space and $\gamma(\|\cdot\|, \tau)_{|Y} \leq \gamma(\|\cdot\|_{|Y}, \tau_{|Y})$ by [13, p. 39]. In general, $\gamma(\|\cdot\|, \tau)_{|Y} = \gamma(\|\cdot\|_{|Y}, \tau_{|Y})$ does not hold by [3, p. 133]. However, suppose that $(Y, \|\cdot\|_{|Y}, \tau_{|Y})$ fulfils condition (i) or (ii) of Remark 2.3 (b) so that $\gamma(\|\cdot\|_{|Y}, \tau_{|Y}) = \gamma_{\mathrm{s}}(\|\cdot\|_{|Y}, \tau_{|Y})$. Then it follows from [13, I.4.6 Lemma, p. 44] that $\gamma(\|\cdot\|, \tau)_{|Y} = \gamma(\|\cdot\|_{|Y}, \tau_{|Y})$. This observation is an alternative way to derive $\gamma(\|\cdot\|, \tau_{\mathrm{co}}) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau_{\mathcal{K}_\Omega})$ in Corollary 4.6 (d) and Corollary 4.8 (b) since condition (ii) of Remark 2.3 (b) is fulfilled for the subtriple.[2]

For a normed space $(E, \|\cdot\|_E)$ over $\mathbb{C}$ and a continuous function $v\colon \mathbb{D} \to (0, \infty)$ with $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ we define the Bloch type space

$$\mathcal{B}v(\mathbb{D}, E) := \{f \in \mathcal{H}(\mathbb{D}, E) \mid \|f\|^E := \|f(0)\|_E + \sup_{z\in\mathbb{D}} \|\partial_{\mathbb{C}}^E f(z)\|_E v(z) < \infty\}$$

where

$$\partial_{\mathbb{C}}^E f(z) := \lim_{\substack{h\to 0\\ h\in\mathbb{C}, h\neq 0}} \frac{f(z+h)-f(z)}{h}, \quad z \in \mathbb{D},$$

Further, we define $\mathcal{B}v(\mathbb{D}) := \mathcal{B}v(\mathbb{D}, \mathbb{C})$ and $\|\cdot\| := \|\cdot\|^{\mathbb{C}}$. Setting $\Lambda := \mathbb{D}$, $\mathcal{G}(\mathbb{D}, E) := \mathcal{H}(\mathbb{D}, E)$, $q^E(f) := \|f(0)\|_E$ and $T^E(f) := \partial_{\mathbb{C}}^E f(z)$ for $f \in \mathcal{H}(\mathbb{D}, E)$, we observe that $\mathcal{F}v(\Omega, E) = \mathcal{B}v(\mathbb{D}, E)$ and conditions (i) and (ii) of Proposition 4.4 are fulfilled.

4.10 Corollary. *Let* $v\colon \mathbb{D} \to (0, \infty)$ *be continuous and* $(E, \|\cdot\|_E)$ *a normed space over* $\mathbb{C}$. *Then the following assertions hold.*

(a) $(\mathcal{B}v(\mathbb{D}, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ *is a C-sequential Saks space and* $\tau_{\mathrm{co}}^E = \tau_{\mathcal{K}_\mathbb{D}}^E$ *on* $\mathcal{B}v(\mathbb{D}, E)$.
(b) *If* $E = \mathbb{C}$, *then* $(\mathcal{B}v(\mathbb{D}), \|\cdot\|, \tau_{\mathrm{co}})$ *is semi-Montel and* $\gamma(\|\cdot\|, \tau_{\mathrm{co}}) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau_{\mathcal{S}})$ *for* $\mathcal{S} \in \{\mathcal{N}_\mathbb{D}, \mathcal{K}_\mathbb{D}\}$.
(c) *If* E *is a Banach space, then* $\mathcal{B}v(\mathbb{D}, E) = \mathcal{B}v(\mathbb{D}, E)_\sigma$ *and the Saks space* $(\mathcal{B}v(\mathbb{D}, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ *is complete.*

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[2] Since [13, I.4.6 Lemma, p. 44] needs that condition (i) or (ii) of Remark 2.3 (b) is fulfilled for the subtriple $(Y, \|\cdot\|_{|Y}, \tau_{|Y})$ and not for the triple $(X, \|\cdot\|, \tau)$, the proof of [53, 5. Proposition, p. 291] seems to be doubtful where $X = \mathcal{C}_b(U)$, U an open connected subset of a complex Banach space, $\|\cdot\| = \|\cdot\|_\infty$, $\tau = \tau_b$ is the topology of uniform convergence on U-bounded subsets of U and $Y = H^\infty(U)$. With the definition of the strict topology β on $\mathcal{C}_b(U)$ in [53, p.290] it is shown that $\beta = \gamma(\|\cdot\|_\infty, \tau_b)$ in [53, 3. Theorem, p. 291]. Then [53, 5. Proposition, p. 291] says that $\beta_{|H^\infty(U)} = \gamma(\|\cdot\|_{|H^\infty(U)}, \tau_{b|H^\infty(U)})$. However, in its proof it is only shown that the triple $(\mathcal{C}_b(U), \|\cdot\|_\infty, \tau_b)$ fulfils condition (i) of Remark 2.3 (b), not the subtriple $(H^\infty(U), \|\cdot\|_{|H^\infty(U)}, \tau_{b|H^\infty(U)})$. At least if U is a subset of a finite-dimensional complex Banach space, then one can use $\tau_b = \tau_{\mathrm{co}}$ which gives that condition (ii) of Remark 2.3 (b) is fulfilled for the subtriple in this case.

Proof. (a) For every $0 < r < 1$ we have

$$\max_{|z|\leq r}\|f(z)\|_E \leq \|f(0)\|_E + \max_{|z|\leq r}\|\int_0^z \partial_{\mathbb{C}}^E f(\zeta)\mathrm{d}\zeta\|_E$$
$$\leq \Big(1+\frac{r}{\min_{|\zeta|\leq r} v(\zeta)}\Big)\big(\|f(0)\|_E + \sup_{|\zeta|\leq r}\|\partial_{\mathbb{C}}^E(\zeta)\|_E v(\zeta)\big)$$

for all $f \in \mathcal{B}v(\mathbb{D}, E)$ where the integral in the estimate above is a Bochner integral (cf. [38, Corollary 3.8, p. 9–10] for the case $E = \mathbb{C}$), and for every $0 < s < r < 1$

$$\|f(0)\|_E + \max_{|z|\leq s}\|\partial_{\mathbb{C}}^E f(z)\|_E v(z) \leq \|f(0)\|_E + \frac{1}{r}\max_{|z|\leq s} v(z)\max_{|\zeta|\leq r}\|f(\zeta)\|_E$$
$$\leq \Big(1+\frac{1}{r}\max_{|z|\leq s} v(z)\Big)\max_{|\zeta|\leq r}\|f(\zeta)\|_E$$

for all $f \in \mathcal{B}v(\mathbb{D}, E)$ by Cauchy's inequality, which proves $\tau_{\mathrm{co}}^E = \tau_{\mathcal{K}_{\mathbb{D}}}^E$ on $\mathcal{B}v(\mathbb{D}, E)$. The rest of part (a) follows from Theorem 4.3, our observations above and the fact that $\mathbb{D}$ is hemicompact.

(b) By [41, 3.9 Example (iv), p. 11] $(\mathcal{B}v(\mathbb{D}), \|\cdot\|, \tau_{\mathrm{co}})$ is semi-Montel. The rest of the proof is analogous to the proof of Corollary 4.6 (d).

(c) It holds $\mathcal{B}v(\mathbb{D}, E) = \mathcal{B}v(\mathbb{D}, E)_\sigma$ by the weak-strong principle [7, Theorem 9, p. 232] and Mackey's theorem. Due to part (b), $\tau_{\mathrm{p}} \leq \tau_{\mathcal{K}_{\mathbb{D}}}$, $\tau_{\mathcal{K}_{\mathbb{D}}}^E = \tau_{\mathrm{co}}^E$ and Proposition 4.4 (c) we get that the Saks space $(\mathcal{B}v(\mathbb{D}, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ is complete. □

For a normed space $(E, \|\cdot\|_E)$ over $\mathbb{K}$ and a metric space (Ω, d) with a base point denoted by $\mathbf{0}$, i.e. a *pointed metric space* in the sense of [57, p. 1], we define the space of E-valued Lipschitz continuous on (Ω, d) that vanish at $\mathbf{0}$ by

$$\mathrm{Lip}_0(\Omega, E) := \Big\{f\colon \Omega \to E \mid f(\mathbf{0}) = 0 \text{ and } \|f\|^E := \sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{\|f(x)-f(y)\|_E}{\mathrm{d}(x,y)} < \infty\Big\}.$$

Further, we define $\mathrm{Lip}_0(\Omega) := \mathrm{Lip}_0(\Omega, \mathbb{K})$ and $\|\cdot\| := \|\cdot\|^{\mathbb{K}}$. Setting $\Lambda := \Omega_{\mathrm{wd}} := \{(x, y) \in \Omega^2 \mid x \neq y\}$, $v\colon \Lambda \to (0, \infty)$, $v(x, y) := \frac{1}{\mathrm{d}(x,y)}$, $\mathcal{G}(\Omega, E) := \{f\colon \Omega \to E \mid f(\mathbf{0}) = 0\}$, $q^E := 0$ and $T^E(f)(x, y) := f(x) - f(y)$ for $(x, y) \in \Lambda$ and $f \in \mathcal{G}(\Omega, E)$, we observe that $\mathcal{F}v(\Omega, E) = \mathrm{Lip}_0(\Omega, E)$ and conditions (i) and (ii) of Proposition 4.4 are fulfilled.

4.11 Corollary. *Let (Ω, d) be a pointed metric space and $(E, \|\cdot\|_E)$ a normed space over $\mathbb{K}$. Then the following assertions hold.*

(a) $(\mathrm{Lip}_0(\Omega, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ *is a Saks space,* $\tau_{\mathrm{co}}^E = \tau_{\mathcal{K}_{\Omega_{\mathrm{wd}}}}^E$ *on* $\|\cdot\|^E$*-bounded sets,* $\gamma(\|\cdot\|^E, \tau_{\mathrm{co}}^E) = \gamma(\|\cdot\|^E, \tau_{\mathcal{K}_{\Omega_{\mathrm{wd}}}}^E)$ *and* $\mathrm{Lip}_0(\Omega, E) = \mathrm{Lip}_0(\Omega, E)_\sigma$.
(b) *If E is a Banach space, then* $(\mathrm{Lip}_0(\Omega, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ *is complete.*
(c) *If Ω is hemicompact, then* $(\mathrm{Lip}_0(\Omega, E), \|\cdot\|^E, \tau_{\mathrm{co}}^E)$ *is C-sequential.*
(d) *If $E = \mathbb{K}$, then* $(\mathrm{Lip}_0(\Omega), \|\cdot\|, \tau_{\mathrm{co}})$ *is semi-Montel and* $\gamma(\|\cdot\|, \tau_{\mathrm{co}}) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau_{\mathcal{S}})$ *for* $\mathcal{S} \in \{\mathcal{N}_{\Omega_{\mathrm{wd}}}, \mathcal{K}_{\Omega_{\mathrm{wd}}}\}$.

Proof. (a) First, we note that for compact $K \subset \Omega_{\mathrm{wd}}$ the projections $\pi_1(K)$ and $\pi_2(K)$ on the first and second component, respectively, are compact in Ω and

$$q_K^E(f) = \sup_{(x,y)\in K}\frac{\|f(x)-f(y)\|_E}{\mathrm{d}(x,y)} \leq 2\max_{(x,y)\in K}\frac{1}{\mathrm{d}(x,y)}\sup_{x\in(\pi_1(K)\cup\pi_2(K))}\|f(x)\|_E$$

for all $f \in \operatorname{Lip}_0(\Omega, E)$, which implies $\tau^E_{\mathcal{K}_{\Omega_{\mathrm{wd}}}} \leq \tau^E_{\mathrm{co}}$ on $\operatorname{Lip}_0(\Omega, E)$. On the other hand, for every $\varepsilon > 0$ and compact $K \subset \Omega$ we have with $\mathbb{B}_\varepsilon := \{x \in \Omega \mid \mathrm{d}(x, \mathbf{0}) < \varepsilon\}$ that

$$\begin{aligned}\sup_{x\in K} \|f(x)\|_E &= \sup_{x\in K, x\neq \mathbf{0}} \frac{\|f(x) - f(\mathbf{0})\|_E}{\mathrm{d}(x,\mathbf{0})} \mathrm{d}(x,\mathbf{0}) \\ &\leq \max_{x\in K} \mathrm{d}(x,\mathbf{0}) \sup_{x\in K\setminus \mathbb{B}_\varepsilon} \frac{\|f(x) - f(\mathbf{0})\|_E}{\mathrm{d}(x,\mathbf{0})} + \varepsilon \sup_{x\in \mathbb{B}_\varepsilon, x\neq \mathbf{0}} \frac{\|f(x) - f(\mathbf{0})\|_E}{\mathrm{d}(x,\mathbf{0})} \\ &\leq \max_{x\in K} \mathrm{d}(x,\mathbf{0}) q^E_{(K\setminus \mathbb{B}_\varepsilon)\times\{\mathbf{0}\}}(f) + \varepsilon \|f\|^E\end{aligned}$$

for all $f \in \operatorname{Lip}_0(\Omega, E)$. Thus the topologies $\tau^E_{\mathcal{K}_{\Omega_{\mathrm{wd}}}}$ and τ^E_{co} coincide on $\|\cdot\|^E$-bounded sets. Due to [13, I.3.1 Lemma, p. 27] and Theorem 4.3 (a) this yields that $(\operatorname{Lip}_0(\Omega, E), \|\cdot\|^E, \tau^E_{\mathrm{co}})$ is a Saks space. In addition, we deduce that $\gamma(\|\cdot\|^E, \tau^E_{\mathrm{co}}) = \gamma(\|\cdot\|^E, \tau^E_{\mathcal{K}_{\Omega_{\mathrm{wd}}}})$ by the definition of the mixed topology. Moreover, it holds that $\operatorname{Lip}_0(\Omega, E) = \operatorname{Lip}_0(\Omega, E)_\sigma$ by Mackey's theorem.

(b) This follows from parts (a) and (d), $\tau_{\mathrm{p}} \leq \tau_{\mathcal{K}_{\Omega_{\mathrm{wd}}}}$ and Proposition 4.4 (c).

(c) If Ω is hemicompact, then τ^E_{co} is metrisable and so $(\operatorname{Lip}_0(\Omega, E), \|\cdot\|^E, \tau^E_{\mathrm{co}})$ C-sequential by [46, Proposition 5.7, p. 2681–2682].

(d) By [41, 3.9 Example (v), p. 11] $(\operatorname{Lip}_0(\Omega), \|\cdot\|, \tau_{\mathrm{co}})$ is semi-Montel. Furthermore, the observations $\tau_{\mathcal{N}_{\Omega_{\mathrm{wd}}}} \leq \tau_{\mathcal{K}_{\Omega_{\mathrm{wd}}}} \leq \tau_{\mathrm{co}}$ on $\operatorname{Lip}_0(\Omega)$ imply that $\gamma(\|\cdot\|, \tau_{\mathrm{co}}) = \gamma_{\mathrm{s}}(\|\cdot\|, \tau_{\mathcal{S}})$ for $\mathcal{S} \in \{\mathcal{N}_{\Omega_{\mathrm{wd}}}, \mathcal{K}_{\Omega_{\mathrm{wd}}}\}$ by Remark 2.5 (c). □

Regarding Corollary 4.11 (c), we remark that a metric space is hemicompact if and only if it is separable and locally compact by [18, Exercises 3.4.E (a), (c), p. 165], [18, Exercises 3.8.C (b), p. 194–195] and [59, 16.11 Theorem, p. 112]. The statement that $(\operatorname{Lip}_0(\Omega), \|\cdot\|, \tau_{\mathrm{co}})$ is a complete semi-Montel Saks space from Corollary 4.11 (b) and (d) for $E = \mathbb{K}$ is already contained in [29, Theorem 2.1 (7), p. 642]. Corollary 4.11 (c) and Theorem 4.3 (d) imply [29, Theorem 3.4, p. 647]. Further, Corollary 4.11 (c) and Theorem 4.3 (e) imply [29, Theorem 3.3, p. 645] where Ω is compact and $\mathcal{W}_0 = \mathcal{C}_0^+(\Omega_{\mathrm{wd}})$.

For a normed space $(E, \|\cdot\|_E)$ over $\mathbb{K}$ and $k \in \mathbb{N}_0$ we denote by $\mathcal{C}^k(\Omega, E)$ the space of k-times continuously partially differentiable E-valued functions on a non-empty open bounded set $\Omega \subset \mathbb{R}^d$. We define the space of k-times continuously partially differentiable E-valued functions on Ω whose partial derivatives up to order k are continuously extendable to the boundary of Ω by

$$\mathcal{C}^k(\overline{\Omega}, E) := \{f \in \mathcal{C}^k(\Omega, E) \mid (\partial^\beta)^E f \text{ cont. extendable on } \overline{\Omega} \text{ for all } \beta \in \mathbb{N}_0^d,\ |\beta| \leq k\}$$

which we equip with the norm given by

$$|f|^E_{\mathcal{C}^k(\overline{\Omega})} := \sup_{\substack{x\in\Omega \\ \beta\in\mathbb{N}_0^d, |\beta|\leq k}} \|(\partial^\beta)^E f(x)\|_E = \sup_{\substack{x\in\overline{\Omega} \\ \beta\in\mathbb{N}_0^d, |\beta|\leq k}} \|(\partial^\beta)^E f(x)\|_E, \quad f \in \mathcal{C}^k(\overline{\Omega}, E),$$

where we use the same symbol for the unique continuous extension of $(\partial^\beta)^E f$ to $\overline{\Omega}$. The space of functions in $\mathcal{C}^k(\overline{\Omega}, E)$ such that all its k-th partial derivatives are α-Hölder continuous with $0 < \alpha \leq 1$ is given by

$$\mathcal{C}^{k,\alpha}(\overline{\Omega}, E) := \left\{f \in \mathcal{C}^k(\overline{\Omega}, E) \mid \|f\|^E < \infty\right\}$$

where

$$\|f\|^E := |f|^E_{\mathcal{C}^k(\overline{\Omega})} + \sup_{\beta\in\mathbb{N}_0^d, |\beta|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|(\partial^\beta)^E f(x) - (\partial^\beta)^E f(y)\|_E}{|x-y|^\alpha}.$$

Further, we define $\mathcal{C}^k(\overline{\Omega}) := \mathcal{C}^k(\overline{\Omega},\mathbb{K})$, $\mathcal{C}^{k,\alpha}(\overline{\Omega}) := \mathcal{C}^{k,\alpha}(\overline{\Omega},\mathbb{K})$ and $|\cdot|_{\mathcal{C}^k(\overline{\Omega})} := |\cdot|^{\mathbb{K}}_{\mathcal{C}^k(\overline{\Omega})}$ as well as $\|\cdot\| := \|\cdot\|^{\mathbb{K}}$.

Let E be a Banach space. Then for every $\beta\in\mathbb{N}_0^d$ with $|\beta|=k$ and $f\in\mathcal{C}^{k,\alpha}(\overline{\Omega},E)$ the unique continuous extension of the partial derivative $(\partial^\beta)^E f$ to $\overline{\Omega}$ is α-Hölder continuous and the extension has the same Hölder constant by [57, Proposition 1.6, p. 5] and [57, Proposition 2.50, p. 66], i.e.

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{\|(\partial^\beta)^E f(x)-(\partial^\beta)^E f(y)\|_E}{|x-y|^\alpha} = \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{\|(\partial^\beta)^E f(x)-(\partial^\beta)^E f(y)\|_E}{|x-y|^\alpha}.$$

Setting $\overline{\Omega}_{\mathrm{wd}} := \{(x,y)\in\overline{\Omega}^2 \mid x\neq y\}$, $\Lambda := \{\beta\in\mathbb{N}_0^d \mid |\beta|=k\}\times\overline{\Omega}_{\mathrm{wd}}$, $v\colon\Lambda\to(0,\infty)$, $v(\beta,x,y) := \frac{1}{|x-y|^\alpha}$, $\mathcal{G}(\Omega,E) := \mathcal{C}^{k,\alpha}(\overline{\Omega},E)$, $q^E := |\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}$ and $T^E(f)(\beta,x,y) := (\partial^\beta)^E f(x)-(\partial^\beta)^E f(y)$ for $(\beta,x,y)\in\Lambda$ and $f\in\mathcal{G}(\Omega,E)$, we remark that $\mathcal{F}v(\Omega,E)=\mathcal{C}^{k,\alpha}(\overline{\Omega},E)$ and conditions (i) and (ii) of Proposition 4.4 are fulfilled. Furthermore, we denote by $\mathcal{N}_{k,\overline{\Omega}_{\mathrm{wd}}}$ the family of subsets of Λ of the form $\{\beta\in\mathbb{N}_0^d \mid |\beta|=k\}\times N$ for $N\in\mathcal{N}_{\overline{\Omega}_{\mathrm{wd}}}$. Similarly, we denote by $\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}$ the family of subsets of Λ of the form $\{\beta\in\mathbb{N}_0^d \mid |\beta|=k\}\times K$ for $K\in\mathcal{K}_{\overline{\Omega}_{\mathrm{wd}}}$. Since $\{\beta\in\mathbb{N}_0^d \mid |\beta|=k\}$ is a finite set, we have $\tau^E_{\mathcal{N}_\Lambda}=\tau^E_{\mathcal{N}_{k,\overline{\Omega}_{\mathrm{wd}}}}$ and $\tau^E_{\mathcal{K}_\Lambda}=\tau^E_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}}$.

4.12 Corollary. *Let $\Omega\subset\mathbb{R}^d$ be a non-empty open bounded set, $k\in\mathbb{N}_0$, $0<\alpha\leq 1$ and $(E,\|\cdot\|_E)$ a Banach space over $\mathbb{K}$. Then the following assertions hold.*

(a) *$(\mathcal{C}^{k,\alpha}(\overline{\Omega},E),\|\cdot\|^E,\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}})$ is a C-sequential Saks space, $\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}}=\tau^E_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}}$ on $\|\cdot\|^E$-bounded sets and $\gamma(\|\cdot\|^E,\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}})=\gamma(\|\cdot\|^E,\tau^E_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}})$.*

Suppose for (b)–(c) that Ω has Lipschitz boundary if $k\geq 1$.

(b) *If $E=\mathbb{K}$, then $(\mathcal{C}^{k,\alpha}(\overline{\Omega}),\|\cdot\|,\tau_{|\cdot|_{\mathcal{C}^k(\overline{\Omega})}})$ is semi-Montel and $\gamma(\|\cdot\|,\tau_{|\cdot|_{\mathcal{C}^k(\overline{\Omega})}})=\gamma_{\mathrm{s}}(\|\cdot\|,\tau_{\mathcal{S}})$ for $\mathcal{S}\in\{\mathcal{N}_{k,\overline{\Omega}_{\mathrm{wd}}},\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}\}$.*
(c) *$\mathcal{C}^{k,\alpha}(\overline{\Omega},E)=\mathcal{C}^{k,\alpha}(\overline{\Omega},E)_\sigma$ and $(\mathcal{C}^{k,\alpha}(\overline{\Omega},E),\|\cdot\|^E,\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}})$ is complete.*

Proof. (a) We set $M:=\{\beta\in\mathbb{N}_0^d \mid |\beta|=k\}$ and note that for compact $K\subset\overline{\Omega}_{\mathrm{wd}}$ the projections $\pi_1(K)$ and $\pi_2(K)$ on the first and second component, respectively, are compact in $\overline{\Omega}$ and

$$\begin{aligned} q^E_{M\times K}(f) &= \sup_{\substack{(x,y)\in K\\ \beta\in\mathbb{N}_0^d,|\beta|=k}} |f|^E_{\mathcal{C}^k(\overline{\Omega})} + \frac{\|(\partial^\beta)^E f(x)-(\partial^\beta)^E f(y)\|_E}{|x-y|^\alpha} \\ &\leq |f|^E_{\mathcal{C}^k(\overline{\Omega})} + 2\max_{(x,y)\in K}\frac{1}{|x-y|^\alpha}\sup_{\substack{x\in(\pi_1(K)\cup\pi_2(K))\\ \beta\in\mathbb{N}_0^d,|\beta|=k}}\|(\partial^\beta)^E f(x)\|_E \\ &\leq \Big(1+2\max_{(x,y)\in K}\frac{1}{|x-y|^\alpha}\Big)|f|^E_{\mathcal{C}^k(\overline{\Omega})} \end{aligned}$$

for all $f\in\mathcal{C}^{k,\alpha}(\overline{\Omega})$, which implies $\tau^E_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}}\leq\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}}$. On the other hand, fix some $y_0\in\Omega$. For every $\varepsilon>0$ we have with $\mathbb{B}_\varepsilon := \{x\in\overline{\Omega} \mid |x-y_0|^\alpha<\varepsilon\}$ that

$$\begin{aligned} |f|^E_{\mathcal{C}^k(\overline{\Omega})} &\leq |f|^E_{\mathcal{C}^k(\overline{\Omega})} + \sup_{\substack{x\in\Omega,x\neq y_0\\ \beta\in\mathbb{N}_0^d,|\beta|=k}}\|(\partial^\beta)^E f(x)-(\partial^\beta)^E f(y_0)\|_E \\ &\leq |f|^E_{\mathcal{C}^k(\overline{\Omega})} + \sup_{\substack{x\in\overline{\Omega},x\neq y_0\\ \beta\in\mathbb{N}_0^d,|\beta|=k}}\frac{\|(\partial^\beta)^E f(x)-(\partial^\beta)^E f(y_0)\|_E}{|x-y_0|^\alpha}|x-y_0|^\alpha \end{aligned}$$

$$\leq |f|^E_{\mathcal{C}^k(\overline{\Omega})} + \max_{x\in\overline{\Omega}} |x-y_0|^\alpha \sup_{\substack{x\in\overline{\Omega}\setminus\mathbb{B}_\varepsilon\\ \beta\in\mathbb{N}_0^d,|\beta|=k}} \frac{\|(\partial^\beta)^E f(x)-(\partial^\beta)^E f(y_0)\|_E}{|x-y_0|^\alpha}$$
$$+\varepsilon \sup_{\substack{x\in\mathbb{B}_\varepsilon, x\neq y_0\\ \beta\in\mathbb{N}_0^d,|\beta|=k}} \frac{\|(\partial^\beta)^E f(x)-(\partial^\beta)^E f(y_0)\|_E}{|x-y_0|^\alpha}$$
$$\leq \Big(1+\max_{x\in\overline{\Omega}}|x-y_0|^\alpha\Big) q^E_{M\times((\overline{\Omega}\setminus\mathbb{B}_\varepsilon)\times\{y_0\})}(f) + \varepsilon\|f\|^E$$

for all $f\in\mathcal{C}^{k,\alpha}(\overline{\Omega},E)$. Thus the topologies $\tau^E_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}}$ and $\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}}$ coincide on $\|\cdot\|^E$-bounded sets. Due to [13, I.3.1 Lemma, p. 27] and Theorem 4.3 (a) this yields that $(\mathcal{C}^{k,\alpha}(\overline{\Omega},E),\|\cdot\|^E,\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}})$ is a Saks space. Furthermore, we deduce that $\gamma(\|\cdot\|^E,\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}})=\gamma(\|\cdot\|^E,\tau^E_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}})$ by the definition of the mixed topology. Since $\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}}$ is metrisable, $(\mathcal{C}^{k,\alpha}(\overline{\Omega},E),\|\cdot\|^E,\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}})$ is C-sequential by [46, Proposition 5.7, p. 2681–2682].

(b) By [41, 3.9 Example (vi), p. 11] $(\mathcal{C}^{k,\alpha}(\overline{\Omega}),\|\cdot\|,\tau_{|\cdot|_{\mathcal{C}^k(\overline{\Omega})}})$ is semi-Montel. Moreover, the observations $\tau_{\mathcal{N}_{k,\overline{\Omega}_{\mathrm{wd}}}}\leq\tau_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}}\leq\tau_{|\cdot|_{\mathcal{C}^k(\overline{\Omega})}}$ on $\mathcal{C}^{k,\alpha}(\overline{\Omega})$ imply that $\gamma(\|\cdot\|,\tau_{|\cdot|_{\mathcal{C}^k(\overline{\Omega})}})=\gamma_{\mathrm{s}}(\|\cdot\|,\tau_{\mathcal{S}})$ for $\mathcal{S}\in\{\mathcal{N}_{k,\overline{\Omega}_{\mathrm{wd}}},\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}\}$ by Remark 2.5 (c).

(c) We have $\mathcal{C}^{k,\alpha}(\overline{\Omega},E)=\mathcal{C}^{k,\alpha}(\overline{\Omega},E)_\sigma$ by [39, 5.3.3 Corollary, p. 106]. From parts (a) and (b), $\tau_{\mathrm{p}}\leq\tau_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}}$ and Proposition 4.4 (c) we deduce that the Saks space $(\mathcal{C}^{k,\alpha}(\overline{\Omega},E),\|\cdot\|^E,\tau_{|\cdot|^E_{\mathcal{C}^k(\overline{\Omega})}})$ is complete. □

4.13 Remark. Let the assumptions of Corollary 4.12 (b) be fulfilled. Since the set $\{\beta\in\mathbb{N}_0^d\mid|\beta|=k\}$ is finite and $\tau_{\mathcal{N}_\Lambda}=\tau_{\mathcal{N}_{k,\overline{\Omega}_{\mathrm{wd}}}}$, we obtain analogously to the proof of Theorem 4.3 (d) that $\gamma(\|\cdot\|,\tau_{|\cdot|_{\mathcal{C}^k(\overline{\Omega})}})=\gamma(\|\cdot\|,\tau_{\mathcal{N}_{k,\overline{\Omega}_{\mathrm{wd}}}})$ is generated by the system of seminorms

$$|||f|||_{(x_n,y_n,a_n)_{n\in\mathbb{N}}} := \sup_{\substack{n\in\mathbb{N}\\ \beta\in\mathbb{N}_0^d,|\beta|=k}} \Big(|f|_{\mathcal{C}^k(\overline{\Omega})}+\frac{|(\partial^\beta)f(x_n)-(\partial^\beta)f(y_n)|}{|x_n-y_n|^\alpha}\Big)a_n,\quad f\in\mathcal{C}^{k,\alpha}(\overline{\Omega}),$$

where $(x_n,y_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{\Omega}_{\mathrm{wd}}$ and $(a_n)_{n\in\mathbb{N}}\in c_0^+$.

Let $\mathcal{W}_0:=\mathcal{W}^+_{\mathrm{b},0}(\overline{\Omega}_{\mathrm{wd}})$ or $\mathcal{W}^+_{\mathrm{usc},0}(\overline{\Omega}_{\mathrm{wd}})$ or $\mathcal{C}^+_0(\overline{\Omega}_{\mathrm{wd}})$. Since $\{\beta\in\mathbb{N}_0^d\mid|\beta|=k\}$ is a finite set, $\tau_{\mathcal{K}_\Lambda}=\tau_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}}$ and $\overline{\Omega}_{\mathrm{wd}}$ locally compact, we may also modify the system of seminorms in Theorem 4.3 (e) and obtain that $\gamma(\|\cdot\|,\tau_{|\cdot|_{\mathcal{C}^k(\overline{\Omega})}})=\gamma(\|\cdot\|,\tau_{\mathcal{K}_{k,\overline{\Omega}_{\mathrm{wd}}}})$ is generated by the system of seminorms

$$|f|^\sim_w := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \sup_{\beta\in\mathbb{N}_0^d,|\beta|=k} \Big(|f|_{\mathcal{C}^k(\overline{\Omega})}+\frac{|\partial^\beta f(x)-\partial^\beta f(y)|}{|x-y|^\alpha}\Big)w(x,y),\quad f\in\mathcal{C}^{k,\alpha}(\overline{\Omega}),$$

for $w\in\mathcal{W}_0$ (without the modification the weights w depend on β as well).

5. The dual space of $(\mathcal{F}v(\Omega,E),\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathcal{N}_\Lambda}))$

In our closing section we give a characterisation of the dual space of the space $\mathcal{F}v(\Omega,E)$ from Definition 4.2 w.r.t. the submixed topology $\gamma_{\mathrm{s}}(\|\cdot\|^E,\tau^E_{\mathcal{N}_\Lambda})$. We know from the preceding section that this submixed topology often coincides with the mixed topology, at least if Ω is discrete or $E=\mathbb{K}$. Our proof is an adaptation of the proof of the corresponding result [29, Theorem 5.1, p. 652] for the case $\mathcal{F}v(\Omega,E)=\mathrm{Lip}_0(\Omega,E)$. For a normed space $(E,\|\cdot\|_E)$ we denote by $E\oplus_1 E$ the space $E\times E$ equipped with the norm $\|\cdot\|_{E\oplus_1 E}$ given by $\|(x,y)\|_{E\oplus_1 E}:=\|x\|_E+\|y\|_E$ for $x,y\in E$. By $\ell^1(\mathbb{N},(E\oplus_1 E)^*)$ we denote the space of $(E\oplus_1 E)^*$-valued sequences $y=(y_n)_{n\in\mathbb{N}}$ such that $\|y\|_1:=\sum_{n=1}^\infty\|y_n\|_{(E\oplus_1 E)^*}<\infty$.

5.1 Theorem. *Let Ω and Λ be non-empty sets, $v\colon \Lambda \to (0,\infty)$, $(E, \|\cdot\|_E)$ a normed space over $\mathbb{K}$, $\mathcal{G}(\Omega, E)$ a linear subspace of E^Ω, $q^E\colon \mathcal{G}(\Omega,E) \to [0,\infty)$ a seminorm, $T^E\colon \mathcal{G}(\Omega,E)\to E^\Lambda$ a linear map. Suppose that $(\mathcal{F}v(\Omega,E), \|\cdot\|^E)$ is normed, $q^E = \|T_0^E(\cdot)\|_E$ for some linear map $T_0^E\colon \mathcal{G}(\Omega,E)\to E$ and $f'\colon \mathcal{F}v(\Omega,E)\to\mathbb{K}$. Then $f' \in (\mathcal{F}v(\Omega,E), \gamma_{\mathrm{s}}(\|\cdot\|^E, \tau^E_{\mathcal{N}_\Lambda}))'$ if and only if there are $(\lambda_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}, (E\oplus_1 E)^*)$ and $(x_n)_{n\in\mathbb{N}}$ in Λ such that*

$$f'(f) = \sum_{n=1}^{\infty} \lambda_n\big(T_0^E(f), T^E(f)(x_n)v(x_n)\big), \quad f\in\mathcal{F}v(\Omega,E).$$

Proof. $\Leftarrow$ Let there be sequences $(\lambda_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}, (E\oplus_1 E)^*)$ and $(x_n)_{n\in\mathbb{N}}$ in Λ such that

$$f'(f) = \sum_{n=1}^{\infty} \lambda_n\big(T_0^E(f), T^E(f)(x_n)v(x_n)\big)$$

for all $\mathcal{F}v(\Omega,E)$. So f' is linear. Since $\sum_{n=1}^{\infty}\|\lambda_n\|_{(E\oplus_1 E)^*} < \infty$, there is a positive sequence $(\mu_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\mu_n = \infty$ and $C := \sum_{n=1}^{\infty}\mu_n\|\lambda_n\|_{(E\oplus_1 E)^*} < \infty$ by [34, Chap. IX, §39, Theorem of Dini, p. 293]. It follows that

$$\begin{aligned} |f'(f)| &\leq \sum_{n=1}^{\infty}\|\lambda_n\|_{(E\oplus_1 E)^*}(\|T_0^E(f)\|_E + \|T^E(f)(x_n)\|_E v(x_n)) \\ &\leq C\sup_{n\in\mathbb{N}}(\|T_0^E(f)\|_E + \|T^E(f)(x_n)\|_E v(x_n))\mu_n^{-1} = C\|f\|^E_{(x_n,\mu_n^{-1})_{n\in\mathbb{N}}} \end{aligned}$$

for all $f\in\mathcal{F}v(\Omega,E)$, implying $f' \in (\mathcal{F}v(\Omega,E), \gamma_{\mathrm{s}}(\|\cdot\|^E, \tau^E_{\mathcal{N}_\Lambda}))'$ by Theorem 4.3 (d).

$\Rightarrow$ Let $f' \in (\mathcal{F}v(\Omega,E), \gamma_{\mathrm{s}}(\|\cdot\|^E, \tau^E_{\mathcal{N}_\Lambda}))'$. Then there are a sequence $(x_n)_{n\in\mathbb{N}}$ in Λ, $(a_n)_{n\in\mathbb{N}}\in c_0^+$ and $C\geq 0$ such that

$$|f'(f)| \leq C\|f\|^E_{(x_n,a_n)_{n\in\mathbb{N}}} = \sup_{n\in\mathbb{N}}(\|T_0^E(f)\|_E + \|T^E(f)(x_n)\|_E v(x_n))\widetilde{a}_n \tag{6}$$

for all $f\in\mathcal{F}v(\Omega,E)$ by Theorem 4.3 (d) where $\widetilde{a}_n := Ca_n$ for all $n\in\mathbb{N}$. Let $c_0(\mathbb{N}, E\oplus_1 E)$ denote the space of $(E\oplus_1 E)$-valued null-sequences on $\mathbb{N}$. We define the linear subspace

$$X := \{(T_0^E(f)\widetilde{a}_n, T^E(f)(x_n)v(x_n)\widetilde{a}_n)_{n\in\mathbb{N}} \mid f\in\mathcal{F}v(\Omega,E)\}$$

of $c_0(\mathbb{N}, E\oplus_1 E)$ and the functional $g^*\colon X\to\mathbb{K}$ given by

$$g^*\big((T_0^E(f)\widetilde{a}_n, T^E(f)(x_n)v(x_n)\widetilde{a}_n)_{n\in\mathbb{N}}\big) := f'(f).$$

The functional g^* is well-defined and linear by (6) combined with the linearity of T_0^E and T^E. The estimate

$$\begin{aligned} &\big|g^*\big((T_0^E(f)\widetilde{a}_n, T^E(f)(x_n)v(x_n)\widetilde{a}_n)_{n\in\mathbb{N}}\big)\big| \\ &= |f'(f)| \underset{(6)}{\leq} \sup_{n\in\mathbb{N}}(\|T_0^E(f)\|_E + \|T^E(f)(x_n)\|_E v(x_n))\widetilde{a}_n \\ &= \sup_{n\in\mathbb{N}}\|(T_0^E(f)\widetilde{a}_n, T^E(f)(x_n)v(x_n)\widetilde{a}_n)\|_{E\oplus_1 E} \\ &= \|(T_0^E(f)\widetilde{a}_n, T^E(f)(x_n)v(x_n)\widetilde{a}_n)_{n\in\mathbb{N}}\|_\infty \end{aligned}$$

for all $f\in\mathcal{F}v(\Omega,E)$ implies that g^* is continuous on $(X, \|\cdot\|_{\infty|X})$ where $\|\cdot\|_\infty$ denotes the supremum norm on $c_0(\mathbb{N}, E\oplus_1 E)$. Due to the Hahn–Banach theorem there exists an extension $\widehat{g}^* \in (c_0(\mathbb{N}, E\oplus_1 E), \|\cdot\|_\infty)'$ of g^*. Since the map

$$\Theta\colon (\ell^1(\mathbb{N},(E\oplus_1 E)^*),\|\cdot\|_1)\to (c_0(\mathbb{N},E\oplus_1 E),\|\cdot\|_\infty)',\ \Theta(y)(z):=\sum_{n=1}^{\infty}y_n(z_n),$$

is an isometric isomorphism, there is $(\kappa_n)_{n\in\mathbb{N}}\in\ell^1(\mathbb{N},(E\oplus_1 E)^*)$ such that $\widehat{g}^*(z)=\sum_{n=1}^{\infty}\kappa_n(z_n)$ for all $z\in c_0(\mathbb{N},E\oplus_1 E)$. We set $\lambda_n:=\kappa_n\widetilde{a}_n$ for all $n\in\mathbb{N}$ and note that

$$\sum_{n=1}^{\infty}\|\lambda_n\|_{(E\oplus_1 E)^*}\leq\|(\kappa_n)_{n\in\mathbb{N}}\|_1(\sup_{m\in\mathbb{N}}|\widetilde{a}_m|)<\infty,$$

implying $(\lambda_n)_{n\in\mathbb{N}}\in\ell^1(\mathbb{N},(E\oplus_1 E)^*)$. Finally, we conclude that

$$\begin{aligned}f'(f)&=\widehat{g}^*\big((T_0^E(f)\widetilde{a}_n,T^E(f)(x_n)v(x_n)\widetilde{a}_n)_{n\in\mathbb{N}}\big)\\&=\sum_{n=1}^{\infty}\lambda_n\big(T_0^E(f),T^E(f)(x_n)v(x_n)\big)\end{aligned}$$

for all $f\in\mathcal{F}v(\Omega,E)$. □

5.2 Remark. If $q^E=0$, then we may take $(\lambda_n)_{n\in\mathbb{N}}\in\ell^1(\mathbb{N},(\{0\}\oplus_1 E)^*)=\ell^1(\mathbb{N},E^*)$ in Theorem 5.1.

We observe that Theorem 5.1 is applicable to the spaces $\mathcal{F}v(\Omega,E)=\mathcal{C}v(\Omega,E)$, $\mathcal{H}v(\Omega,E)$, $\mathcal{C}_Pv(\Omega,E)$ and $\mathrm{Lip}_0(\Omega,E)$ with $T_0^E(f):=0$ for $f\in\mathcal{F}v(\Omega,E)$, implying $q^E=0$, and to the space $\mathcal{F}v(\Omega,E)=\mathcal{B}v(\mathbb{D},E)$ with $T_0^E(f):=f(0)$ for $f\in\mathcal{B}v(\mathbb{D},E)$.

Acknowledgements

We acknowledge the support by the Deutsche Forschungsgemeinschaft (DFG) within the Research Training Group GRK 2583 “Modeling, Simulation and Optimization of Fluid Dynamic Applications”. Further, we are grateful to the anonymous reviewer for their thorough review and helpful suggestions.

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