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On linearisation and existence of preduals

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Abstract
We study the problem of existence of preduals of locally convex Hausdorff spaces. We derive necessary and sufficient conditions for the existence of a predual with certain properties of a bornological locally convex Hausdorff space X. Then we turn to the case that $X = \mathcal{F}(\Omega)$ is a space of scalar-valued functions on a non-empty set Ω and characterise those among them which admit a special predual, namely a *strong linearisation*, i.e. there are a locally convex Hausdorff space Y, a map $\delta\colon \Omega \to Y$ and a topological isomorphism $T\colon \mathcal{F}(\Omega) \to Y_b'$ such that $T(f) \circ \delta = f$ for all $f \in \mathcal{F}(\Omega)$.

Keywords Dual space · Predual · Linearisation · Mixed topology

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1 Introduction

The present paper is dedicated to preduals of locally convex Hausdorff spaces, in particular their existence. A *predual* of a locally convex Hausdorff space X is a tuple (Y, φ) of a locally convex Hausdorff space Y and a topological isomorphism $\varphi\colon X \to Y_b'$ where Y_b' is the strong dual of Y. The space X is called a *dual space*. This topic is thoroughly studied in the case of Banach spaces X, mostly with regard to isometric preduals. Necessary and sufficient conditions for the existence of a Banach predual of a Banach space are due to Dixmier [8, Théorème 17', p. 1069] and were augmented by Waelbroeck [29, Proposition 1, p. 122], Ng [25, Theorem 1, p. 279] and Kaijser [17, Theorem 1, p. 325].

All of the preceding results on the existence of a Banach predual of a Banach space involve (relative) compactness with respect to an additional locally convex topology on X and were transfered to (ultra)bornological locally convex Hausdorff spaces by Mujica [22, Theorem 1, p. 320–321] and then generalised by Bierstedt and Bonet [1]. Namely, if (X, τ)

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is a bornological locally convex Hausdorff space such that there exists a locally convex Hausdorff topology $\widetilde{\tau}$ on X such that

(BBC) every τ-bounded subset of X is contained in an absolutely convex τ-bounded $\widetilde{\tau}$-compact set, and
(CNC) τ has a 0-neighbourhood basis of absolutely convex $\widetilde{\tau}$-closed sets,

then (X, τ) has a complete barrelled predual by [1, 1. Theorem (Mujica), 2. Corollary, p. 115]. Besides noting that (BBC) and (CNC) are also necessary for the existence of a complete barrelled predual, we derive several other necessary and sufficient conditions for the existence of a complete barrelled predual in Corollary 3.22. In particular, we show that the existence of a *semi-Montel prebidual* (Y, φ), i.e. Y is a semi-Montel space and $\varphi\colon (X, \tau) \to (Y_b')_b'$ a topological isomorphism, such that Y_b' is complete is a necessary and sufficient condition for the existence of a complete barrelled predual. Moreover, we refine such conditions to characterise which Fréchet spaces, complete bornological DF-spaces and completely normable spaces have complete barrelled DF-preduals, Fréchet preduals and Banach preduals, respectively, in Corollaries 3.23, 3.24 and 3.26. We adapt ideas from the theory of Saks spaces and mixed topologies by Cooper [7] to achieve this.

Then we extend these results to obtain necessary and sufficient conditions in Theorem 4.5, Corollaries 4.6, 4.7 and 4.8 for the existence of strong linearisations which are special preduals of locally convex Hausdorff spaces $\mathcal{F}(\Omega)$ of $\mathbb{K}$-valued functions on a non-empty set Ω where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We call a triple (δ, Y, T) of a locally convex Hausdorff space Y over the field $\mathbb{K}$, a map $\delta\colon \Omega \to Y$ and a topological isomorphism $T\colon \mathcal{F}(\Omega) \to Y_b'$ a *strong linearisation of* $\mathcal{F}(\Omega)$ if $T(f) \circ \delta = f$ for all $f \in \mathcal{F}(\Omega)$ (see [6, p. 683], [14, p. 181, 184] and Proposition 2.6). In comparison to the necessary and sufficient conditions (BBC) and (CNC) for the existence of a complete barrelled predual of a bornological space $X = \mathcal{F}(\Omega)$ we only have to add that $\widetilde{\tau}$ is finer than the topology of pointwise convergence τ_{p} to obtain necessary and sufficient conditions for the existence of a strong linearisation with complete barrelled Y (see Theorem 4.5). In Corollaries 4.6, 4.7 and 4.8 we derive corresponding conditions that guarantee that Y is a complete barrelled DF-space, Fréchet space and completely normable space, respectively. Moreover, in Theorem 4.10 we give a result on *continuous* strong linearisations, i.e. where δ is in addition continuous and Ω a topological Hausdorff space, which generalises [14, Theorem 2.2, Corollary 2.3, p. 188–189]. Linearisations are a useful tool since they identify (usually) non-linear functions f with (continuous) linear operators $T(f)$ and thus allow to apply linear functional analysis to non-linear functions. We refer the reader who is also interested in the corresponding results of the present paper in the isometric Banach setting, where $\mathcal{F}(\Omega)$ and Y are Banach spaces and T an isometry, to [20].

2 Notions and preliminaries

In this short section we recall some basic notions from the theory of locally convex spaces and present some prelimary results on dual spaces and their preduals. For a locally convex Hausdorff space X over the field $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$ we denote by X' the topological linear dual space and by U° the *polar set* of a subset $U \subset X$. If we want to emphasize the dependency on the locally convex Hausdorff topology τ of X, we write (X, τ) and $(X, \tau)'$ instead of just X and X', respectively. We denote by $\sigma(X', X)$ the topology on X' of uniform convergence on finite subsets of X, by $\tau_{\mathrm{c}}(X', X)$ the topology on X' of uniform convergence on compact subsets of X and by $\beta(X', X)$ the topology on X' of uniform convergence on bounded subsets of X. Further, we set $X_b' := (X', \beta(X', X))$. Furthermore, we say that a linear map

$T: X \to Y$ between two locally convex Hausdorff spaces X and Y is *(locally) bounded* if it maps bounded sets to bounded sets. Moreover, for two locally convex Hausdorff topologies τ_0 and τ_1 on X we write $\tau_0 \leq \tau_1$ if τ_0 is coarser than τ_1. For a normed space $(X, \|\cdot\|)$ we denote by $B_{\|\cdot\|} := \{x \in X \mid \|x\| \leq 1\}$ the $\|\cdot\|$-closed unit ball of X. Further, we write τ_{co} for the *compact-open topology*, i.e. the topology of uniform convergence on compact subsets of Ω, on the space $\mathcal{C}(\Omega)$ of $\mathbb{K}$-valued continuous functions on a topological Hausdorff space Ω. In addition, we write τ_{p} for the *topology of pointwise convergence* on the space $\mathbb{K}^{\Omega}$ of $\mathbb{K}$-valued functions on a set Ω. By a slight abuse of notation we also use the symbols τ_{co} and τ_{p} for the relative compact-open topology and the relative topology of pointwise convergence on topological subspaces of $\mathcal{C}(\Omega)$ and $\mathbb{K}^{\Omega}$, respectively. For further unexplained notions on the theory of locally convex Hausdorff spaces we refer the reader to [15, 21, 26].

Definition 2.1 Let X be a locally convex Hausdorff space. We call X a *dual space* if there are a locally convex Hausdorff space Y and a topological isomorphism $\varphi\colon X \to Y_b'$. The tuple (Y, φ) is called a *predual* of X.

In the context of dual Banach spaces the preceding definition of a predual is already given e.g. in [4, p. 321]. If X is a dual space with a quasi-barrelled predual, we may consider this predual as a topological subspace of the strong dual of X.

Proposition 2.2 *Let X be a dual space with quasi-barrelled predual (Y, φ). Then the map*

$$\Phi_\varphi\colon Y \to X_b',\ y \longmapsto [x \mapsto \varphi(x)(y)],$$

is a topological isomorphism into, i.e. a topological isomorphism to its range.

Proof Since Y is quasi-barrelled, the evaluation map $\mathcal{J}_Y : Y \to (Y_b')_b', y \longmapsto [y' \mapsto y'(y)]$, is a topological isomorphism into by [15, 11.2.2. Proposition, p. 222]. Furthermore, the map φ gives a ono-to-one correspondence between the bounded subsets of X and the bounded subsets of Y_b'. We observe that for every bounded set $B \subset X$ it holds that

$$\sup_{x\in B} |\Phi_\varphi(y)(x)| = \sup_{x\in B} |\varphi(x)(y)| = \sup_{x\in B} |\mathcal{J}_Y(y)(\varphi(x))| = \sup_{y'\in\varphi(B)} |\mathcal{J}_Y(y)(y')|$$

for all $y \in Y$, which proves the claim. □

Next, we come to linearisations of function spaces.

Definition 2.3 Let $\mathcal{F}(\Omega)$ be a linear space of $\mathbb{K}$-valued functions on a non-empty set Ω.

(a) We call a triple (δ, Y, T) of a locally convex Hausdorff space Y over the field $\mathbb{K}$, a map $\delta\colon \Omega \to Y$ and an algebraic isomorphism $T : \mathcal{F}(\Omega) \to Y'$ a *linearisation of* $\mathcal{F}(\Omega)$ if $T(f) \circ \delta = f$ for all $f \in \mathcal{F}(\Omega)$.
(b) Let Ω be a topological Hausdorff space. We call a linearisation (δ, Y, T) of $\mathcal{F}(\Omega)$ *continuous* if δ is continuous.
(c) Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space. We call a linearisation (δ, Y, T) of $\mathcal{F}(\Omega)$ *strong* if $T : \mathcal{F}(\Omega) \to Y_b'$ is a topological isomorphism.
(d) We call a (strong) linearisation (δ, Y, T) of $\mathcal{F}(\Omega)$ a (strong) *complete barrelled (Fréchet, DF-, Banach) linearisation* if Y is a complete barrelled (Fréchet, DF-, completely normable) space.
(e) We say $\mathcal{F}(\Omega)$ *admits a (continuous, strong, complete barrelled, Fréchet, DF-, Banach) linearisation* if there exists a (continuous, strong, complete barrelled, Fréchet, DF-, Banach) linearisation (δ, Y, T) of $\mathcal{F}(\Omega)$.

Clearly, a strong linearisation (δ, Y, T) of $\mathcal{F}(\Omega)$ gives us the predual (Y, T) of $\mathcal{F}(\Omega)$. Definition 2.3 (c), (d) and (e) are motivated by the definition of a strong Banach linearisation given in [14, p. 184, 187].

Example 2.4 Let $(\ell^1, \|\cdot\|_1)$ denote the Banach space of complex absolutely summable sequences on $\mathbb{N}$, $(c_0, \|\cdot\|_\infty)$ the Banach space of complex zero sequences and $(c, \|\cdot\|_\infty)$ the Banach space of complex convergent sequences, all three equipped with their usual norms.

(i) We define the topological isomorphism

$$\varphi_0\colon (\ell^1, \|\cdot\|_1) \to (c_0, \|\cdot\|_\infty)'_b,\ \varphi_0(x)(y) := \sum_{k=1}^{\infty} x_k y_k.$$

Then the tuple (c_0, φ) is a predual of ℓ^1 and we note that $x_n = \varphi_0(x)(e_n)$ for all $x \in \ell^1$ and $n \in \mathbb{N}$ where e_n denotes the n-th unit sequence. Setting $\delta\colon \mathbb{N} \to c_0$, $\delta(n) := e_n$, we get the strong Banach linearisation (δ, c_0, φ_0) of ℓ^1.

(ii) We define the topological isomorphism

$$\varphi_c\colon (\ell^1, \|\cdot\|_1) \to (c, \|\cdot\|_\infty)'_b,\ \varphi_c(x)(y) := y_\infty x_1 + \sum_{k=1}^{\infty} y_k x_{k+1},$$

where $y_\infty := \lim_{n\to\infty} y_n$ for $y \in c$. Then the tuple (c, φ_c) is a predual of ℓ^1 and we claim that there is no $\delta_c\colon \mathbb{N} \to c$ such that (δ_c, c, φ_c) is a linearisation of ℓ^1. Suppose the contrary. Then we have $x_n = \varphi_c(x)(\delta_c(n))$ for all $x \in \ell^1$ and $n \in \mathbb{N}$. In particular, we get $1 = \varphi_c(e_1)(\delta_c(1)) = \delta_c(1)_\infty$ and $0 = \varphi_c(e_m)(\delta_c(1)) = \delta_c(1)_{m-1}$ for all $m \geq 2$, which is a contradiction.

(iii) We may fix the problem in (ii) by using a different isomorphism. We define the topological isomorphism

$$\psi\colon\ (\ell^1, \|\cdot\|_1) \to (c, \|\cdot\|_\infty)'_b,\ \psi(x)(y) := y_\infty x_1 + \sum_{k=1}^{\infty}(y_k - y_\infty) x_{k+1}.$$

Then the tuple (c, ψ) is a predual of ℓ^1. To obtain a linearisation of ℓ^1 from this predual, we have to to find $\widetilde{\delta}(n) \in c$ such that $x_n = \psi(x)(\widetilde{\delta}(n))$ for all $x \in \ell^1$ and $n \in \mathbb{N}$. Using the unit sequences $e_m \in \ell^1$, $m \in \mathbb{N}$, we see that $\widetilde{\delta}(n)$ has to fulfil

$$e_{1,n} = \psi(e_1)(\widetilde{\delta}(n)) = \widetilde{\delta}(n)_\infty,$$

which implies $\widetilde{\delta}(1)_\infty = 1$ and $\widetilde{\delta}(n)_\infty = 0$ for all $n \geq 2$. Further, $\widetilde{\delta}(n)$ has to fulfil for $m \geq 2$

$$e_{m,n} = \psi(e_m)(\widetilde{\delta}(n)) = \widetilde{\delta}(n)_{m-1} - \widetilde{\delta}(n)_\infty.$$

This yields $\widetilde{\delta}(1)_{m-1} = 1$ for all for $m \geq 2$, $\widetilde{\delta}(n)_{m-1} = 1$ if $n = m$, $n \geq 2$, and $\widetilde{\delta}(n)_{m-1} = 0$ if $n \neq m$, $n \geq 2$. So setting $\widetilde{\delta}(1) := (1, 1, \ldots) \in c$ and $\widetilde{\delta}(n) := e_{n-1} \in c$ for all $n \geq 2$ and observing that $(e_m)_{m\in\mathbb{N}}$ is a Schauder basis of ℓ^1, we obtain the continuous strong Banach linearisation $(\widetilde{\delta}, c, \psi)$ of ℓ^1 if $\mathbb{N}$ is equipped with the Hausdorff topology induced by the absolute value $|\cdot|$. The tuple $(\widetilde{\delta}, c)$ is also given in [6, Example 9, p. 699–700] as a continuous linearisation of ℓ^1 (see Proposition 2.6), however, without the information which isomorphism was used to derive $\widetilde{\delta}$.

Our next goal is to compare our definition of a linearisation to the one given e.g. in [14, p. 181] and [6, p. 683] and to show that both definitions are equivalent,[1]

[1] Both definitions are equivalent if additional assumptions on δ are neglected. In [14] continuous linearisations are considered and in [6] linearisations such that δ is of the "same type" as the functions in $\mathcal{F}(\Omega)$. The

Proposition 2.5 *Let $\mathcal{F}(\Omega)$ be a linear space of $\mathbb{K}$-valued functions on a non-empty set Ω, Y a locally convex Hausdorff space over the field $\mathbb{K}$ and $\delta\colon \Omega \to Y$. Consider the following conditions for the tuple (δ, Y).*

(i) For every continuous linear functional $y' \in Y'$ it holds $y' \circ \delta \in \mathcal{F}(\Omega)$.
(ii) For every $f \in \mathcal{F}(\Omega)$ there is a unique continuous linear functional $T_f \in Y'$ such that $T_f \circ \delta = f$.

Then the following assertions hold.

(a) If condition (ii) is fulfilled, then the map $T\colon \mathcal{F}(\Omega) \to Y'$, $T(f) := T_f$, is linear and injective.
(b) If conditions (i) and (ii) are fulfilled, then the map T is linear and bijective, $T^{-1}(y') = y' \circ \delta$ for all $y' \in Y'$, (δ, Y, T) is a linearisation of $\mathcal{F}(\Omega)$ and the span of $\{\delta(x) \mid x \in \Omega\}$ dense in Y.

Proof (a) By condition (ii) for every $f \in \mathcal{F}(\Omega)$ there is a unique $T_f \in Y'$ such that $T_f \circ \delta = f$. Thus the map T is well-defined. Let $f, g \in \mathcal{F}(\Omega)$ with $T_f = T_g$, then we get

$$f = T_f \circ \delta = T_g \circ \delta = g,$$

which implies that T is injective. Next, we turn to linearity. Let $f, g \in \mathcal{F}(\Omega)$ and $\lambda \in \mathbb{K}$. Then we have $(T_f + T_g) \circ \delta = f + g = (T_{f+g}) \circ \delta$ and $T_{\lambda f} \circ \delta = \lambda f = (\lambda T_f) \circ \delta$. Due to uniqueness we get that T is linear.

(b) First, we show that T is surjective. Let $y' \in Y'$. Then $f_{y'} := y' \circ \delta \in \mathcal{F}(\Omega)$ by condition (i), $T(f_{y'}) \in Y'$ and

$$T(f_{y'}) \circ \delta = f_{y'} = y' \circ \delta$$

by condition (ii). The uniqueness of the functional in Y' in condition (ii) implies $T(y' \circ \delta) = T(f_{y'}) = y'$. Hence T is surjective, so bijective by part (a), and $T^{-1}(y') = y' \circ \delta$ for all $y' \in Y'$ and (δ, Y, T) is also a linearisation of $\mathcal{F}(\Omega)$.

Second, suppose that the span of $\{\delta(x) \mid x \in \Omega\}$ is not dense in Y. Then there is $u \in Y'$, $u \neq 0$, such that $u(\delta(x)) = 0$ for all $x \in \Omega$ by the bipolar theorem. Since T is bijective, there is $f^u \in \mathcal{F}(\Omega)$, $f^u \neq 0$, such that $T(f^u) = u$. It follows that $f^u(x) = (T(f^u) \circ \delta)(x) = (u \circ \delta)(x) = 0$ for all $x \in \Omega$ by condition (ii), which is a contradiction. □

In [14, p. 181] a linearisation of $\mathcal{F}(\Omega)$ is defined as a tuple (δ, Y) that fulfils conditions (i) and (ii) of Proposition 2.5 (if we ignore the assumption that δ is continuous in [14, p. 181]). The following result shows that our definition of a linearisation is equivalent to the one in [14, p. 181].

Proposition 2.6 *Let $\mathcal{F}(\Omega)$ be a linear space of $\mathbb{K}$-valued functions on a non-empty set Ω, Y a locally convex Hausdorff space over the field $\mathbb{K}$ and $\delta\colon \Omega \to Y$. Then the following assertions are equivalent.*

(a) (δ, Y) fulfils conditions (i) and (ii) of Proposition 2.5.
(b) There is a (unique) algebraic isomorphism $T\colon \mathcal{F}(\Omega) \to Y'$ such that (δ, Y, T) is a linearisation of $\mathcal{F}(\Omega)$.

constructed linearisation in [6] is continuous, see [6, Theorem 2, p. 689] and the type of δ is handled in [6, Proposition 2, p. 688].

Proof (a)⇒(b) The existence of the algebraic isomorphism T in (b) follows from Proposition 2.5 (b). Let $\widetilde{T}$ be another algebraic isomorphism such that $(\delta, Y, \widetilde{T})$ is a linearisation of $\mathcal{F}(\Omega)$. Let $f \in \mathcal{F}(\Omega)$. Then we have $\widetilde{T}(f)(\delta(x)) = f(x) = T(f)(\delta(x))$ for all $x \in \Omega$. So $\widetilde{T}(f)$ and $T(f)$ coincide on the span of $\{\delta(x) \mid x \in \Omega\}$, which is dense in Y by Proposition 2.5 (b). Hence the continuity of $\widetilde{T}(f)$ and $T(f)$ implies that $\widetilde{T}(f) = T(f)$ on Y, which settles the uniqueness.

(b)⇒(a) Let $T : \mathcal{F}(\Omega) \to Y'$ be an algebraic isomorphism such that (δ, Y, T) is a linearisation of $\mathcal{F}(\Omega)$. Condition (i) follows from the surjectivity of T and $T(f) \circ \delta = f$ for all $f \in \mathcal{F}(\Omega)$. The second part of the proof of Proposition 2.5 (b) shows that the span of $\{\delta(x) \mid x \in \Omega\}$ is dense in Y since T is bijective and $T(f) \circ \delta = f$ for all $f \in \mathcal{F}(\Omega)$. Let $T_f \in Y'$ such that $T_f \circ \delta = f$ for all $f \in \mathcal{F}(\Omega)$. Then we obtain as above that $T_f = T(f)$ on Y. Hence condition (ii) is fulfilled as well. □

Next, we give a precise characterisation of the surjectivity of the map T from a triple (δ, Y, T) that is almost a linearisation.

Proposition 2.7 *Let $\mathcal{F}(\Omega)$ be a linear space of $\mathbb{K}$-valued functions on a non-empty set Ω, Y a locally convex Hausdorff space over the field $\mathbb{K}$, $\delta : \Omega \to Y$ a map and $T : \mathcal{F}(\Omega) \to Y'$ a linear injective map such that $T(f) \circ \delta = f$ for all $f \in \mathcal{F}(\Omega)$. Then the following assertions are equivalent.*

(a) T is surjective.
(b) The tuple (δ, Y) fulfils condition (i) of Proposition 2.5 and the span of $\{\delta(x) \mid x \in \Omega\}$ is dense in Y.

Proof (a)⇒(b) If T is surjective, then (δ, Y, T) is a linearisation of $\mathcal{F}(\Omega)$. The proof of the implication (b)⇒(a) of Proposition 2.6 shows that the span of $\{\delta(x) \mid x \in \Omega\}$ is dense in Y and condition (i) is fulfilled.

(b)⇒(a) Let $y' \in Y'$. Then $y' \circ \delta \in \mathcal{F}(\Omega)$ by condition (i), $T(y' \circ \delta) \in Y'$ and

$$T(y' \circ \delta)(\delta(x)) = (y' \circ \delta)(x) = y'(\delta(x))$$

for all $x \in \Omega$. Thus the continuous linear functionals $T(y' \circ \delta)$ and y' coincide on a dense subspace of Y and so on the whole space Y. Hence T is surjective. □

3 Existence of preduals and prebiduals

In this section we study necessary and sufficient conditions that guarantee the existence of a predual. We recall the conditions (BBC) and (CNC) from [1, p. 114] and [1, 2. Corollary, p. 115], which were introduced in a slightly less general form in [22, Theorem 1, p. 320–321]. Further, we introduce the condition (BBCl), which is a generalisation of the compatibility condition in [7, p. 6].

Definition 3.1 Let (X, τ) be a locally convex Hausdorff space.

(a) We say that (X, τ) satisfies condition (BBCl) if there exists a locally convex Hausdorff topology $\widetilde{\tau}$ on X such that every τ-bounded subset of X is contained in an absolutely convex τ-bounded $\widetilde{\tau}$-closed set.
(b) We say that (X, τ) satisfies condition (BBC) if there exists a locally convex Hausdorff topology $\widetilde{\tau}$ on X such that every τ-bounded subset of X is contained in an absolutely convex τ-bounded $\widetilde{\tau}$-compact set.

(c) We say that (X, τ) satisfies condition (CNC) if there exists a locally convex Hausdorff topology $\widetilde{\tau}$ on X such that τ has a 0-neighbourhood basis $\mathcal{U}_0$ of absolutely convex $\widetilde{\tau}$-closed sets.

If we want to emphasize the dependency on $\widetilde{\tau}$ we say that (X, τ) satisfies (BBC), (BBCl) resp. (CNC) for $\widetilde{\tau}$. We say (X, τ) satisfies (BBC) (or (BBCl)) and (CNC) for $\widetilde{\tau}$ if it satisfies both conditions for the same $\widetilde{\tau}$.

Obviously, (BBC) implies (BBCl). Let us collect some other useful observations concerning these conditions. The observations (b), (c), (e) and (f) of Remark 3.2 below are taken from [1, p. 114, 116], (a) is clearly valid by the definition of (BBCl), (d) follows from (b) and the remarks above [12, Chap. 3, §9, Proposition 2, p. 231], and (f) follows from the definition of (CNC), [15, 8.2.5 Proposition, p. 148] and the Mackey–Arens theorem.

Remark 3.2 Let (X, τ) be a bornological locally convex Hausdorff space and $\mathcal{B}$ be the family of τ-bounded sets.

(a) Let $\widetilde{\tau}$ be a locally convex Hausdorff topology on X. Then (X, τ) satisfies (BBCl) for $\widetilde{\tau}$ if and only if $\mathcal{B}$ has a basis $\mathcal{B}_0$ of absolutely convex $\widetilde{\tau}$-closed sets.
(b) Let $\widetilde{\tau}$ be a locally convex Hausdorff topology on X. Then (X, τ) satisfies (BBC) for $\widetilde{\tau}$ if and only if $\mathcal{B}$ has a basis $\mathcal{B}_1$ of absolutely convex $\widetilde{\tau}$-compact sets.
(c) If (X, τ) satisfies (BBC) for some $\widetilde{\tau}$, then $\widetilde{\tau} \leq \tau$ and (X, τ) is ultrabornological.
(d) If (X, τ) satisfies (BBC) for some $\widetilde{\tau}$, then (X, τ) satisfies (BBC) for all locally convex Hausdorff topologies $\widetilde{\tau}_0$ on X such that $\widetilde{\tau}_0 \leq \widetilde{\tau}$ since $\widetilde{\tau}_0$ and $\widetilde{\tau}$ coincide on all $B \in \mathcal{B}_1$.
(e) If (X, τ) satisfies (BBC) and (CNC) for some $\widetilde{\tau}$, then (X, τ) is quasi-complete.
(f) If (X, τ) satisfies (CNC) for some $\widetilde{\tau}$, then (X, τ) satisfies (CNC) for all locally convex Hausdorff topologies $\widetilde{\tau}_0$ on X such that $\sigma(X, (X, \widetilde{\tau})') \leq \widetilde{\tau}_0 \leq \mu(X, (X, \widetilde{\tau})')$.

Let Ω be a non-empty topological Hausdorff space. We call $\mathcal{V}$ a *directed family of continuous weights* if $\mathcal{V}$ is a family of continuous functions $v\colon \Omega \to [0, \infty)$ such that for every $v_1, v_2 \in \mathcal{V}$ there are $C \geq 0$ and $v_0 \in \mathcal{V}$ with $\max(v_1, v_2) \leq C v_0$ on Ω. We call a directed family of continuous weights $\mathcal{V}$ *point-detecting* if for every $x \in \Omega$ there is $v \in \mathcal{V}$ such that $v(x) > 0$. For an open set $\Omega \subset \mathbb{R}^d$ we denote by $\mathcal{C}^\infty(\Omega)$ the space of $\mathbb{K}$-valued infinitely continuously partially differentiable functions on Ω. The next example is a slight generalisation of [3, p. 34] where the weighted space $\mathcal{HV}(\Omega)$ of holomorphic functions on an open connected set $\Omega \subset \mathbb{C}^d$ is considered and $\mathcal{V}$ is a point-detecting Nachbin family of continuous weights.

Example 3.3 Let $\Omega \subset \mathbb{R}^d$ be open, $P(\partial)$ a hypoelliptic linear partial differential operator on $\mathcal{C}^\infty(\Omega)$ and $\mathcal{V}$ a point-detecting directed family of continuous weights. We define the space

$$\mathcal{C}_P\mathcal{V}(\Omega) := \{f \in \mathcal{C}_P(\Omega) \mid \forall\, v \in \mathcal{V}:\ \|f\|_v := \sup_{x\in\Omega} |f(x)|v(x) < \infty\},$$

where $\mathcal{C}_P(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) \mid f \in \ker P(\partial)\}$, and equip $\mathcal{C}_P\mathcal{V}(\Omega)$ with the locally convex Hausdorff topology $\tau_\mathcal{V}$ induced by the seminorms $(\|\cdot\|_v)_{v\in\mathcal{V}}$. The space $(\mathcal{C}_P\mathcal{V}(\Omega), \tau_\mathcal{V})$ is complete and $\tau_{\mathrm{co}} := \tau_{\mathrm{co}|\mathcal{C}_P\mathcal{V}(\Omega)} \leq \tau_\mathcal{V}$. Further, the absolutely convex sets

$$U_v := \{f \in \mathcal{C}_P\mathcal{V}(\Omega) \mid \|f\|_v \leq 1\}, \quad v \in \mathcal{V},$$

are τ_{co}-closed and form a 0-neighbourhood basis of $\tau_\mathcal{V}$ (cf. [3, p. 34] where $\mathcal{V}$ is a Nachbin family). Therefore $(\mathcal{C}_P\mathcal{V}(\Omega), \tau_\mathcal{V})$ satisfies (CNC) for τ_{co}. If follows similarly to [1, p. 123] or the proof of [2, Proposition 1.2 (c), p. 274–275] that every $\tau_\mathcal{V}$-bounded set B is contained in an absolutely convex τ_{co}-closed set B_1. Since $(\mathcal{C}_P(\Omega), \tau_{co})$ is a Fréchet–Schwartz space (see e.g. [11, p. 690]), the set B_1 is also τ_{co}-compact. Therefore $(\mathcal{C}_P\mathcal{V}(\Omega), \tau_{\mathcal{V}})$ satisfies (BBC) for τ_{co}. If $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ is countable and *increasing*, i.e. $v_n \leq v_{n+1}$ for all $n \in \mathbb{N}$, then $(\mathcal{C}_P\mathcal{V}(\Omega), \tau_{\mathcal{V}})$ is a Fréchet space, in particular bornological. If $\mathcal{V} = \{v\}$, we set $\mathcal{C}_P v(\Omega) := \mathcal{C}_P\mathcal{V}(\Omega)$ and note that $(\mathcal{C}_P v(\Omega), \|\cdot\|_v)$ is a Banach space.

Proposition 3.4 *Let (X, τ) be a normable locally convex Hausdorff space. Then the following assertions are equivalent.*

(a) (X, τ) *satisfies* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$.
(b) (X, τ) *is quasi-complete and satisfies* (BBC) *for some* $\widetilde{\tau}$.
(c) *There are a norm $|||\cdot|||$ on X which induces τ, and a locally convex Hausdorff topology $\widetilde{\tau}$ such that $B_{|||\cdot|||}$ $\widetilde{\tau}$-compact.*

Proof First, we note that normable spaces are bornological by [21, Proposition 24.10, p. 282].

(a)⇒(b) Using Remark 3.2 (e), this implication is obvious.

(b)⇒(c) Since (X, τ) is quasi-complete and normable, there is a norm $\|\cdot\|$ on X such that the identity map $\mathrm{id}\colon (X, \tau) \to (X, \|\cdot\|)$ is a topological isomorphism and $(X, \|\cdot\|)$ quasi-complete, thus complete. Hence a subset of X is τ-bounded if and only if it is $\|\cdot\|$-bounded. Since (X, τ) satisfies (BBC) for some $\widetilde{\tau}$, there is an absolutely convex $\|\cdot\|$-bounded $\widetilde{\tau}$-compact subset B of X such that $B_{\|\cdot\|} \subset B$. This implies that the Minkowski functional (or gauge) of B given by

$$|||x||| := \inf\{t > 0 \mid x \in tB\}, \quad x \in X,$$

defines a norm on X such that there is $C \geq 0$ with $\|x\| \leq C|||x|||$ for all $x \in X$ by [15, p. 151]. Due to the $\widetilde{\tau}$-compactness of B and Remark 3.2 (c) the set B is τ-closed and thus $\|\cdot\|$-closed. Further, the completeness of $(X, \|\cdot\|)$ yields that B is sequentially $\|\cdot\|$-complete and hence a Banach disk by [21, Corollary 23.14, p. 268]. Therefore $(X, |||\cdot|||)$ is complete and $\mathrm{id}\colon (X, |||\cdot|||) \to (X, \|\cdot\|)$ a topological isomorphism by [21, Open mapping theorem 24.30, p. 289]. Since B is $\|\cdot\|$-closed, it is also $|||\cdot|||$-closed and hence $B_{|||\cdot|||} = B$ by [21, Remark 6.6 (b), p. 47]. We conclude that $B_{|||\cdot|||}$ is $\widetilde{\tau}$-compact.

(c)⇒(a) We set $\mathcal{B}_1 := \mathcal{U}_0 := \{tB_{|||\cdot|||} \mid t > 0\}$. Then $\mathcal{B}_1$ is a basis of τ-bounded sets which are absolutely convex and $\widetilde{\tau}$-compact, implying that (X, τ) satisfies (BBC) for $\widetilde{\tau}$ by Remark 3.2 (b). $\mathcal{B}_1 = \mathcal{U}_0$ is also a 0-neighbourhood basis w.r.t. τ of absolutely convex $\widetilde{\tau}$-closed sets, yielding that (X, τ) satisfies (CNC) for $\widetilde{\tau}$. □

Similarly, we get by analysing the proof above a corresponding result with (BBCl) instead of (BBC).

Proposition 3.5 *Let (X, τ) be a completely normable locally convex Hausdorff space. Then the following assertions are equivalent.*

(a) (X, τ) *satisfies* (BBCl) *and* (CNC) *for some* $\widetilde{\tau} \leq \tau$.
(b) (X, τ) *is complete and satisfies* (BBCl) *for some* $\widetilde{\tau} \leq \tau$.
(c) *There are a norm $|||\cdot|||$ on X which induces τ, and a locally convex Hausdorff topology $\widetilde{\tau} \leq \tau$ such that $(X, |||\cdot|||)$ is complete and $B_{|||\cdot|||}$ $\widetilde{\tau}$-closed.*

Now, we recall the candidate for a predual from e.g. [1, 22, 25]. Let (X, τ) be a locally convex Hausdorff space, $\mathcal{B}$ the family of τ-bounded sets and $\widetilde{\tau}$ another locally convex Hausdorff topology on X. We denote by $X^{\star}$ the algebraic dual space of X and define

$$X'_{\mathcal{B},\widetilde{\tau}} := \{x^{\star} \in X^{\star} \mid x^{\star}_{|B} \text{ is } \widetilde{\tau}\text{-continuous for all } B \in \mathcal{B}\}$$

and observe that $(X, \widetilde{\tau})' \subset X'_{\mathcal{B},\widetilde{\tau}}$ as linear spaces. We equip $X'_{\mathcal{B},\widetilde{\tau}}$ with the topology $\beta := \beta_{\mathcal{B},\widetilde{\tau}} := \beta(X'_{\mathcal{B},\widetilde{\tau}}, (X, \tau))$ of uniform convergence on the τ-bounded subsets of X. If (X, τ) is bornological and satisfies (BBC) for $\widetilde{\tau}$, then we have $\widetilde{\tau} \leq \tau$ by Remark 3.2 (c),

$$X'_{\mathcal{B},\widetilde{\tau}} = \{x^\star \in X^\star \mid x^\star_{|B} \text{ is } \widetilde{\tau}\text{-continuous for all } B \in \mathcal{B}_1\}$$

with $\mathcal{B}_1$ from Remark 3.2 (b), $X'_{\mathcal{B},\widetilde{\tau}} \subset (X, \tau)'$ and

$$\beta = \beta_{\mathcal{B},\widetilde{\tau}} = \beta(X'_{\mathcal{B},\widetilde{\tau}}, (X, \tau)) = \beta((X, \tau)', (X, \tau))_{|X'_{\mathcal{B},\widetilde{\tau}}} = \widetilde{\beta}_{\mathcal{B}_1,\widetilde{\tau}}$$

by [1, p. 115] where $\widetilde{\beta}_{\mathcal{B}_1,\widetilde{\tau}}$ denotes the topology on $X'_{\mathcal{B},\widetilde{\tau}}$ of uniform convergence on the sets $B \in \mathcal{B}_1$.

Remark 3.6 Let (X, τ) be a bornological locally convex Hausdorff space satisfying (BBC) for some $\widetilde{\tau}$ and $\mathcal{B}$ the family of τ-bounded sets. If $\widetilde{\tau}_0$ is a locally convex Hausdorff topology on X such that $\widetilde{\tau}_0 \leq \widetilde{\tau}$, then $X'_{\mathcal{B},\widetilde{\tau}} = X'_{\mathcal{B},\widetilde{\tau}_0}$ and $\beta_{\mathcal{B},\widetilde{\tau}} = \beta_{\mathcal{B},\widetilde{\tau}_0}$. Indeed, this observation follows directly from the considerations above and Remark 3.2 (d).

Proposition 3.7 *Let (X, τ) be a bornological locally convex Hausdorff space satisfying* (BBC) *for some $\widetilde{\tau}$ and $\mathcal{B}$ the family of τ-bounded sets. Then the following assertions hold.*

(a) $X'_{\mathcal{B},\widetilde{\tau}}$ is a closed subspace of the complete space $(X, \tau)'_b$. In particular, $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is complete.
(b) If (X, τ) is a DF-space, then $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is a Fréchet space.
(c) If (X, τ) is normable, then $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is completely normable.

Proof (a) This follows from the proof of [1, 1. Theorem (Mujica), p. 115].

(b) Since (X, τ) is a DF-space, its strong dual $(X, \tau)'_b$ is a Fréchet space by [15, 12.4.2 Theorem, p. 258]. It follows from part (a) that $X'_{\mathcal{B},\widetilde{\tau}}$ is a closed subspace of $(X, \tau)'_b$ and thus $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ a Fréchet space as well.

(c) Since (X, τ) is a normable space, its strong dual $(X, \tau)'_b$ is a completely normable space. The rest follows as in (b). □

The conditions (BBC) and (CNC) for some $\widetilde{\tau}$ guarantee that $X'_{\mathcal{B},\widetilde{\tau}}$ (equipped with a suitable topological isomorphism) is a complete barrelled predual of a bornological locally convex Hausdorff space (X, τ).

Theorem 3.8 ([1, 1. Theorem (Mujica), 2. Corollary, p. 115]) *Let (X, τ) be a bornological locally convex Hausdorff space satisfying* (BBC) *and* (CNC) *for some $\widetilde{\tau}$ and $\mathcal{B}$ the family of τ-bounded sets. Then $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is a complete barrelled locally convex Hausdorff space and the evaluation map*

$$\mathcal{I}\colon (X, \tau) \to (X'_{\mathcal{B},\widetilde{\tau}}, \beta)'_b,\ x \longmapsto [x' \mapsto x'(x)],$$

is a topological isomorphism. In particular, (X, τ) is a dual space with complete barrelled predual $(X'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$.

Next, we show that under suitable assumptions (X, τ) is also a bidual of a locally convex Hausdorff space, more precisely topologically isomorphic to a bidual. In contrast to the strategy in [1, Section 2, p. 118–122], we do not achieve this by replacing $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ by the strong dual H'_b of a suitable topological subspace H of (X, τ), but by finding a locally convex Hausdorff topology $\widetilde{\gamma}$ on X such that $(X, \widetilde{\gamma})'_b = (X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ under suitable assumptions. This is the strategy that is also used in [2, 1.3 Proposition, p. 276] and [23, 4.7 Proposition (b), p. 877–878] in the case that X is a weighted space of holomorphic functions. This needs a bit of preparation.

Definition 3.9 Let X be a locally convex Hausdorff space. We call X a *bidual space* if there are a locally convex Hausdorff space Y and a topological isomorphism $\varphi\colon X \to (Y_b')_b'$. The tuple (Y, φ) is called a *prebidual* of X.

We note that if X is a bidual space with prebidual (Y, φ), then X is a dual space with predual (Y_b', φ). In particular, every reflexive locally convex Hausdorff space is a bidual space.

Proposition 3.10 *Let (X, τ) be a locally convex Hausdorff space, $B \subset X$ and $\mathcal{J}\colon (X, \tau) \to ((X, \tau)_b')'$, $x \longmapsto [x' \mapsto x'(x)]$, the canonical linear injection. Then B is τ-bounded if and only if $\mathcal{J}(B) \subset ((X, \tau)_b')'$ is equicontinuous.*

Proof $\Rightarrow$ Let B be τ-bounded. Then we have

$$|\mathcal{J}(x)(x')| = |x'(x)| \leq \sup_{z \in B} |x'(z)|$$

for all $x \in B$ and $x' \in (X, \tau)'$, meaning that $\mathcal{J}(B)$ is equicontinuous.

$\Leftarrow$ Let $\mathcal{J}(B)$ be equicontinuous. Then there are a τ-bounded set $\widetilde{B} \subset X$ and $C \geq 0$ such that

$$|x'(x)| = |\mathcal{J}(x)(x')| \leq C \sup_{z \in \widetilde{B}} |x'(z)| < \infty$$

for all $x \in B$ and $x' \in (X, \tau)'$. This yields that B is $\sigma(X, (X, \tau)')$-bounded and thus τ-bounded by the Mackey theorem. □

Proposition 3.11 *Let (X, τ) be a bornological locally convex Hausdorff space and $\widetilde{\tau}$ a locally convex Hausdorff topology on X. Then $\widetilde{\tau} \leq \tau$ if and only if every τ-bounded set is $\widetilde{\tau}$-bounded.*

Proof The implication $\Rightarrow$ is obvious. Let us turn to $\Leftarrow$. We note that the identity map $\mathrm{id}\colon (X, \tau) \to (X, \widetilde{\tau})$ is bounded since every τ-bounded set is $\widetilde{\tau}$-bounded. This yields that id is continuous by [21, Proposition 24.13, p. 283] as (X, τ) is bornological. □

Definition 3.12 Let (X, τ) be a locally convex Hausdorff space and $\widetilde{\tau}$ a locally convex Hausdorff topology on X. Let $\widetilde{\gamma} := \widetilde{\gamma}(\tau, \widetilde{\tau})$ denote the finest locally convex Hausdorff topology on X which coincides with $\widetilde{\tau}$ on τ-bounded sets. We say that $(X, \widetilde{\gamma})$ satisfies (B τ B $\widetilde{\gamma}$) if a subset of X is τ-bounded if and only if it is $\widetilde{\gamma}$-bounded.

For a linear space X we denote by $\mathrm{acx}(U)$ the *absolutely convex hull* of a subset $U \subset X$.

Remark 3.13 Let (X, τ) be a locally convex Hausdorff space, $\mathcal{B}$ the family of τ-bounded sets and $\widetilde{\tau}$ a locally convex Hausdorff topology on X.

(a) The sets $\mathrm{acx}(\bigcup_{B \in \mathcal{B}} U_B \cap B)$ where each U_B is a 0-neighbourhood in $(X, \widetilde{\tau})$ for $B \in \mathcal{B}$ form a basis of absolutely convex 0-neighbourhoods for $\widetilde{\gamma}$ since $\widetilde{\gamma}$ is the finest locally convex Hausdorff topology on X which coincides with $\widetilde{\tau}$ on τ-bounded sets. Further, we note that we may restrict to absolutely convex τ-bounded sets B and absolutely convex $\widetilde{\tau}$-closed 0-neighbourhoods U_B in $(X, \widetilde{\tau})$ as (X, τ) and $(X, \widetilde{\tau})$ are locally convex Hausdorff spaces.

(b) If $\widetilde{\tau}_0$ is another locally convex Hausdorff topology on X which coincides with $\widetilde{\tau}$ on τ-bounded sets, then $\widetilde{\gamma}(\tau, \widetilde{\tau}) = \widetilde{\gamma}(\tau, \widetilde{\tau}_0)$.

(c) If (X, τ) is bornological and $(X, \widetilde{\gamma})$ satisfies (B τ B $\widetilde{\gamma}$), then $\widetilde{\tau} \leq \widetilde{\gamma} \leq \tau$. Indeed, $\widetilde{\tau} \leq \widetilde{\gamma}$ follows from the definition of $\widetilde{\gamma}$, and $\widetilde{\gamma} \leq \tau$ from Proposition 3.11.

(d) If (X, τ) is bornological, $\widetilde{\tau} = \tau$ and $(X, \widetilde{\gamma})$ satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$, then $\widetilde{\gamma} = \widetilde{\gamma}(\tau, \tau) = \tau$. This follows directly from part (c).
(e) Let $(X, \widetilde{\gamma})$ satisfy $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$. Then a subset of X is $\widetilde{\gamma}$-compact (precompact, relatively compact) if and only if it is τ-bounded and $\widetilde{\tau}$-compact (precompact, relatively compact). This follows directly from the definitions of $\widetilde{\gamma}$ and $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$.

Remark 3.13 (e) is a generalisation of [7, I.1.12 Proposition, p. 10] and yields the following corollary, which itself generalises [7, I.1.13 Proposition, p. 11] by Proposition 3.16 below.

Corollary 3.14 *Let (X, τ) be a locally convex Hausdorff space, $\widetilde{\tau}$ a locally convex Hausdorff topology on X and $(X, \widetilde{\gamma})$ satisfy $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$. Then $(X, \widetilde{\gamma})$ is a semi-Montel space if and only if (X, τ) satisfies* (BBC) *for $\widetilde{\tau}$.*

Example 3.15 Let (X, τ) be a complete barrelled locally convex Hausdorff space. Then

$$\widetilde{\gamma} := \widetilde{\gamma}(\beta(X', X), \sigma(X', X)) = \tau_{\mathrm{c}}(X', X)$$

and $(X', \widetilde{\gamma})$ satisfies $(\mathrm{B}\,\beta(X', X)\,\mathrm{B}\,\widetilde{\gamma})$ and is a semi-Montel space. Further, the evaluation map

$$\mathcal{J}_X \colon (X, \tau) \to (X', \widetilde{\gamma})_b', \; x \longmapsto [x' \mapsto x'(x)],$$

is a topological isomorphism.

Proof Since (X, τ) is barrelled, a subset of $X' = (X, \tau)'$ is $\beta(X', X)$-bounded if and only it is τ-equicontinuous by [28, Theorem 33.2, p. 349]. Now, the completeness of (X, τ) and [18, §21.9, (7), p. 271] imply that $\tau_{\mathrm{c}}(X', X)$ is the finest locally convex Hausdorff topology that coincides with $\sigma(X', X)$ on the τ-equicontinuous subsets of X'. This yields $\widetilde{\gamma}(\beta(X', X), \sigma(X', X)) = \tau_{\mathrm{c}}(X', X)$ by the definition of $\widetilde{\gamma}$. Again, due to [28, Theorem 33.2, p. 349] a subset of X' is $\beta(X', X)$-bounded if and only it is $\tau_{\mathrm{c}}(X', X)$-bounded since $\sigma(X', X) \leq \tau_{\mathrm{c}}(X', X) \leq \beta(X', X)$. Hence $(X', \widetilde{\gamma})$ satisfies $(\mathrm{B}\,\beta(X', X)\,\mathrm{B}\,\widetilde{\gamma})$. It also follows from [28, Theorem 33.2, p. 349] that $(X', \beta(X', X))$ satisfies (BBC) for $\sigma(X', X)$, implying that $(X', \widetilde{\gamma})$ is a semi-Montel space by Corollary 3.14.

Further, as a subset of X' is $\beta(X', X)$-bounded if and only if it is $\tau_{\mathrm{c}}(X', X)$-bounded and $\widetilde{\gamma} = \tau_{\mathrm{c}}(X', X)$, the evaluation map $\mathcal{J}_X$ is a topological isomorphism by [15, 11.2.2 Proposition, p. 222] and the Mackey–Arens theorem. □

In view of Proposition 3.16 below, Example 3.15 improves [7, I.2.A Examples, p. 20–21] and [30, Example E), p. 66] where (X, τ) is a Fréchet resp. Banach space. Example 3.15 also shows that $(X', \tau_{\mathrm{c}}(X', X))$ is a predual for any complete barrelled locally convex Hausdorff space (X, τ). However, this predual may not have the properties of a predual one is looking for. For instance, if (X, τ) is completely normable, one is naturally looking for a completely normable predual as well. But the predual $(X', \tau_{\mathrm{c}}(X', X))$ is a semi-Montel space, so not normable unless X is finite-dimensional.

Next, we present a non-trivial sufficient condition that guarantees that $(X, \widetilde{\gamma})$ satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$. Let us recall the definition of the mixed topology in the sense of [7, p. 5–6]. Let (X, τ) be a bornological locally convex Hausdorff space, $\mathcal{B}$ the family of τ-bounded sets and $\widetilde{\tau}$ a locally convex Hausdorff topology on X such that $\widetilde{\tau} \leq \tau$ (see Proposition 3.11). Suppose that $\mathcal{B}$ is of *countable type*, i.e. it has a countable basis, and that (X, τ) satisfies (BBCl) for $\widetilde{\tau}$. Then it follows that $\mathcal{B}$ has a countable basis $(B_n)_{n \in \mathbb{N}}$ consisting of absolutely convex

$\widetilde{\tau}$-closed sets such that $2B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$ (see [7, p. 6]). For a sequence $(U_n)_{n\in\mathbb{N}}$ of absolutely convex 0-neighbourhoods in $(X, \widetilde{\tau})$ we set

$$\mathcal{U}((U_n)_{n\in\mathbb{N}}) := \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} (U_k \cap B_k).$$

The family of such sets forms a basis of 0-neighbourhoods of a locally convex Hausdorff topology on X, which is called the *mixed topology* and denoted by $\gamma := \gamma(\tau, \widetilde{\tau})$. We note that our definition of the mixed topology is slightly less general than the one given in [7, p. 5–6] since we only consider the case where the bornology $\mathcal{B}$ is induced by a topology, namely τ, i.e. we only consider the case of von Neumann bornologies. If τ is induced by a norm $\|\cdot\|$, then $\mathcal{B}$ is of countable type and the mixed topology defined above coincides with the mixed topology in the sense of [30, p. 49]. Besides normable spaces, examples of spaces (X, τ) with $\mathcal{B}$ of countable type are df-spaces, in particular gDF-spaces and DF-spaces (see [15, p. 257]).

Proposition 3.16 ([7, I.1.5 Proposition (iii), I.1.11 Proposition, p. 7, 10]) *Let (X, τ) be a bornological locally convex Hausdorff space, the family $\mathcal{B}$ of τ-bounded sets be of countable type and (X, τ) satisfy* (BBCl) *for some $\widetilde{\tau} \leq \tau$. Then $\widetilde{\gamma} = \gamma$ and $(X, \widetilde{\gamma})$ satisfies* ($\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma}$).

However, we note that the condition that $\mathcal{B}$ is of countable type is not a necessary condition for $(X, \widetilde{\gamma})$ satisfying ($\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma}$) by Example 3.15 or by [2, p. 272, 274, 276] where in the latter case $X = \mathcal{HV}(\Omega)$ is a weighted space of holomorphic functions on a (balanced) open set $\Omega \subset \mathbb{C}^N$, $\tau = \tau_{\mathcal{V}}$ is the weighted topology w.r.t. to a family $\mathcal{V}$ of non-negative upper semicontinuous functions on Ω such that $\widetilde{\tau} := \tau_{\mathrm{co}} \leq \tau_{\mathcal{V}}$ and $\widetilde{\gamma} = \overline{\tau}$ (see Example 3.3 as well with $N = 1$, i.e. $d = 2$ there, and $P(\partial)$ being the Cauchy–Riemann operator).

Now, we show that $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ coincides with the strong dual of $(X, \widetilde{\gamma})$ under suitable assumptions. The proof is an adaptation of the proofs of [7, I.1.7 Corollary, p. 8], [7, I.1.18 Proposition, p. 15] and [7, I.1.20 Proposition, p. 16].

Proposition 3.17 *Let (X, τ) be a bornological locally convex Hausdorff space, $\mathcal{B}$ the family of τ-bounded sets, $\widetilde{\tau}$ a locally convex Hausdorff topology on X and $(X, \widetilde{\gamma})$ satisfy* ($\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma}$). *Then the following assertions hold.*

(a) $(X, \widetilde{\gamma})'$ is a closed subspace of $(X, \tau)'_b$, and $(X, \widetilde{\gamma})' = X'_{\mathcal{B},\widetilde{\tau}}$. Further, it holds

$$\beta((X, \widetilde{\gamma})', (X, \widetilde{\gamma})) = \beta = \beta(X'_{\mathcal{B},\widetilde{\tau}}, (X, \tau)) = \beta((X, \tau)', (X, \tau))_{|(X,\widetilde{\gamma})'}.$$

In particular, $(X, \widetilde{\gamma})'_b$ is complete.

(b) If (X, τ) satisfies (BBCl) *for $\widetilde{\tau}$, then $(X, \widetilde{\gamma})'$ is the closure of $(X, \widetilde{\tau})'$ in $(X, \tau)'_b$.*

(c) Let (X, τ) satisfy (BBCl) *for $\widetilde{\tau}$. A subset of X is $\sigma(X, (X, \widetilde{\gamma})')$-compact if and only if it is τ-bounded and $\sigma(X, (X, \widetilde{\tau})')$-compact.*

Proof (a) We start with the proof of $(X, \widetilde{\gamma})' = X'_{\mathcal{B},\widetilde{\tau}}$. We have $(X, \widetilde{\gamma})' \subset X'_{\mathcal{B},\widetilde{\tau}}$ because $\widetilde{\gamma}$ coincides with $\widetilde{\tau}$ on τ-bounded sets. Let us consider the other inclusion. Let $u \in X'_{\mathcal{B},\widetilde{\tau}}$ and V be an absolutely convex 0-neighbourhood in $\mathbb{K}$. For every $B \in \mathcal{B}$ we have

$$u^{-1}(V) \cap B = u_{|B}^{-1}(V) = u_{|B}^{-1}(V) \cap B$$

and $U_B := u_{|B}^{-1}(V)$ is a 0-neighbourhood for $\widetilde{\tau}$ since $u \in X'_{\mathcal{B},\widetilde{\tau}}$. Thus the set $\operatorname{acx}(\bigcup_{B\in\mathcal{B}} U_B \cap B)$ is an absolutely convex 0-neighbourhood for $\widetilde{\gamma}$ contained in $u^{-1}(V)$, implying $u \in (X, \widetilde{\gamma})'$.

Due to Remark 3.13 (c) we have $\widetilde{\gamma} \leq \tau$. This implies that $(X, \widetilde{\gamma})' \subset (X, \tau)'$. Further, it is easily seen that the $\beta((X, \tau)', (X, \tau))$-limit $f \in (X, \tau)'$ of a net $(f_\iota)_{\iota \in I}$ in $(X, \widetilde{\gamma})'$ belongs to $X'_{\mathcal{B}, \widetilde{\tau}}$. Due to the first part of (a) we get that $f \in (X, \widetilde{\gamma})'$, which means that $(X, \widetilde{\gamma})'$ is closed in $(X, \tau)'_b$. The rest follows from the completeness of $(X, \tau)'_b$ and the assumption that $(X, \widetilde{\gamma})$ satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$.

(b) Next, we show that $(X, \widetilde{\tau})'$ is dense in $(X, \widetilde{\gamma})'$. We know that $(X, \widetilde{\tau})'$ is a subspace of $(X, \widetilde{\gamma})'$ by Remark 3.13 (c). Due to (BBCl) for $\widetilde{\tau}$ the family $\mathcal{B}$ has a basis $\mathcal{B}_0$ of τ-bounded sets which are absolutely convex and $\widetilde{\tau}$-closed (see Remark 3.2(a)). Let $u \in (X, \widetilde{\gamma})'$, $B \in \mathcal{B}_0$ and $\varepsilon > 0$. Then there is an absolutely convex $\widetilde{\tau}$-closed 0-neighbourhood U in $(X, \widetilde{\tau})$ such that $|u(x)| \leq \varepsilon$ for all $x \in U \cap B$. This means that $u \in \varepsilon(U \cap B)^\circ$ where the polar set is taken w.r.t. the dual pairing $\langle X, (X, \widetilde{\tau})' \rangle$. The set U is absolutely convex and $\widetilde{\tau}$-closed and thus $\sigma(X, (X, \widetilde{\tau})')$-closed by [15, 8.2.5 Proposition, p. 149]. By the same reasoning the set B is $\sigma(X, (X, \widetilde{\tau})')$-closed. Hence we get

$$(U \cap B)^\circ = \overline{\operatorname{acx}(U^\circ \cup B^\circ)} \subset \overline{(U^\circ + B^\circ)}$$

by [15, 8.2.4 Corollary, p. 149] where the closures are taken w.r.t. $\sigma((X, \widetilde{\tau})', X)$. The polar set U° is $\sigma((X, \widetilde{\tau})', X)$-compact by the Alaoglu–Bourbaki theorem and the polar set B° is $\sigma((X, \widetilde{\tau})', X)$-closed by [15, 8.2.1 Proposition (a), p. 148]. Therefore the sum $U^\circ + B^\circ$ is $\sigma((X, \widetilde{\tau})', X)$-closed and so $(U \cap B)^\circ \subset (U^\circ + B^\circ)$. We deduce that $u \in \varepsilon(U^\circ + B^\circ)$, which yields that there is $v \in \varepsilon U^\circ \subset (X, \widetilde{\tau})'$ such that $u - v \in \varepsilon B^\circ$, i.e. $|u(x) - v(x)| \leq \varepsilon$ for all $x \in B$.

(c) $\Rightarrow$ This implication follows from $\sigma(X, (X, \widetilde{\tau})') \leq \sigma(X, (X, \widetilde{\gamma})')$, the Mackey theorem and that a $\widetilde{\gamma}$-bounded set is τ-bounded because of $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$.

$\Leftarrow$ Let $B \subset X$ be τ-bounded. Then B is $\widetilde{\gamma}$-bounded because of $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$, and $\mathcal{J}(B)$ an equicontinuous subset of $((X, \widetilde{\gamma})'_b)'$ by Proposition 3.10. The topologies $\sigma(((X, \widetilde{\gamma})'_b)', (X, \widetilde{\gamma})')$ and $\sigma(((X, \widetilde{\gamma})'_b)', (X, \widetilde{\tau})')$ coincide on the equicontinuous set $\mathcal{J}(B)$ by [16, Satz 1.4, p. 16] since $(X, \widetilde{\tau})'$ is dense in $(X, \widetilde{\gamma})'_b$ by part (b). Hence $\sigma(X, (X, \widetilde{\gamma})')$ and $\sigma(X, (X, \widetilde{\tau})')$ coincide on B. Thus, if B is in addition $\sigma(X, (X, \widetilde{\tau})')$-compact, we get that B is also $\sigma(X, (X, \widetilde{\gamma})')$-compact. □

Proposition 3.17 (c) in combination with [21, 23.18 Proposition, p. 270] and [15, 8.2.5 Proposition, p. 149] directly implies the following statement, which is a generalisation of [7, I.1.21 Corollary, p. 16].

Corollary 3.18 *Let (X, τ) be a bornological locally convex Hausdorff space, $\mathcal{B}$ the family of τ-bounded sets, $\widetilde{\tau}$ a locally convex Hausdorff topology on X and $(X, \widetilde{\gamma})$ satisfy* $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$. *Then $(X, \widetilde{\gamma})$ is semi-reflexive and (X, τ) satisfies* (BBCl) *for $\widetilde{\tau}$ if and only if $\mathcal{B}$ has a basis of absolutely convex $\sigma(X, (X, \widetilde{\tau})')$-compact sets.*

Corollary 3.19 *Let (X, τ) be a bornological locally convex Hausdorff space, $\widetilde{\tau}$ a locally convex Hausdorff topology on X and $(X, \widetilde{\gamma})$ satisfy* $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$ *with $\widetilde{\gamma} := \widetilde{\gamma}(\tau, \widetilde{\tau})$. Then the following assertions are equivalent.*

(a) (X, τ) satisfies (BBCl) *and* (CNC) *for $\widetilde{\tau}$ and $(X, \widetilde{\gamma})$ is semi-reflexive.*
(b) (X, τ) satisfies (BBC) *and* (CNC) *for $\sigma(X, (X, \widetilde{\tau})')$.*

If one, thus both, of the preceding assertions holds, then we have

$$\widetilde{\gamma}(\tau, \sigma(X, (X, \widetilde{\tau})')) = \widetilde{\gamma}(\tau, \sigma(X, (X, \widetilde{\gamma})')) = \tau_{\mathrm{c}}(X, (X, \widetilde{\gamma})'_b)$$

and $(X, \widetilde{\gamma}(\tau, \sigma(X, (X, \widetilde{\tau})')))$ satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma}(\tau, \sigma(X, (X, \widetilde{\tau})')))$ *as well as*

$$(X, \widetilde{\gamma})'_b = (X, \widetilde{\gamma}(\tau, \sigma(X, (X, \widetilde{\tau})')))'_b.$$

Proof First, we show that (X,τ) satisfies (CNC) for $\widetilde{\tau}$ if and only if (X,τ) satisfies (CNC) for $\sigma(X,(X,\widetilde{\tau})')$. Let (X,τ) satisfy (CNC) for $\widetilde{\tau}$. Then (X,τ) satisfies (CNC) for $\sigma(X,(X,\widetilde{\tau})')$ by Remark 3.2 (f) with $\widetilde{\tau}_0 := \sigma(X,(X,\widetilde{\tau})')$. On the other hand, let (X,τ) satisfy (CNC) for $\sigma(X,(X,\widetilde{\tau})')$. Then τ has a 0-neighbourhood basis $\mathcal{U}_0$ of absolutely convex $\sigma(X,(X,\widetilde{\tau})')$-closed sets. Since $\sigma(X,(X,\widetilde{\tau})') \leq \widetilde{\tau}$, the elements of $\mathcal{U}_0$ are also $\widetilde{\tau}$-closed. Thus (X,τ) satisfies (CNC) for $\widetilde{\tau}$.

Now, the equivalence (a) $\Leftrightarrow$ (b) follows from the observation above and Corollary 3.18. Let one of the assertions (a) or (b), thus both, hold. By the proof of Proposition 3.17 (c) we know that $\sigma(X,(X,\widetilde{\tau})')$ and $\sigma(X,(X,\widetilde{\gamma})')$ coincide on τ-bounded sets. Due to Remark 3.13 (b) we obtain that

$$\widetilde{\gamma}(\tau,\sigma(X,(X,\widetilde{\tau})')) = \widetilde{\gamma}(\tau,\sigma(X,(X,\widetilde{\gamma})')).$$

Since $(X,\widetilde{\gamma})$ is semi-reflexive, all equicontinuous subsets of $((X,\widetilde{\gamma})_b')'$ are of the form $\mathcal{J}(B)$ for some $\widetilde{\gamma}$-bounded set $B \subset X$, and all $\widetilde{\gamma}$-bounded subsets of X are of the form $\mathcal{J}^{-1}(A)$ for some equicontinuous set $A \subset ((X,\widetilde{\gamma})_b')'$ by Proposition 3.10. Now, the completeness of $(X,\widetilde{\gamma})_b'$ by Proposition 3.17 (a) and [18, §21.9, (7), p. 271] imply that $\tau_{\mathrm{c}}(\mathcal{J}(X),(X,\widetilde{\gamma})_b')$ is the finest locally convex Hausdorff topology that coincides with $\sigma(\mathcal{J}(X),(X,\widetilde{\gamma})')$ on the equicontinuous subsets of $\mathcal{J}(X) = ((X,\widetilde{\gamma})_b')'$. Due to $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$ this yields

$$\widetilde{\gamma}(\tau,\sigma(X,(X,\widetilde{\gamma})')) = \tau_{\mathrm{c}}(X,(X,\widetilde{\gamma})_b').$$

Further, the Mackey theorem in combination with $(X,\widetilde{\gamma})$ satisfying $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$ yields that a subset of X is $\widetilde{\gamma}(\tau,\sigma(X,(X,\widetilde{\tau})'))$-bounded if and only if it is $\widetilde{\gamma}$-bounded if and only if it is τ-bounded. This implies that the space $(X,\widetilde{\gamma}(\tau,\sigma(X,(X,\widetilde{\tau})')))$ satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma}(\tau,\sigma(X,(X,\widetilde{\tau})')))$. The rest of the statement follows from the Mackey–Arens theorem. □

Combining Theorem 3.8, Corollary 3.19 and Proposition 3.17 (a), we get the biduality we were aiming for.

Corollary 3.20 *Let (X,τ) be a bornological locally convex Hausdorff space satisfying* (BBCl) *and* (CNC) *for some $\widetilde{\tau}$ and $(X,\widetilde{\gamma})$ a semi-reflexive space satisfying* $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$. *Then $(X,\widetilde{\gamma})_b'$ is a complete barrelled locally convex Hausdorff space and the evaluation map*

$$\mathcal{I}\colon (X,\tau) \to ((X,\widetilde{\gamma})_b')_b',\ x \longmapsto [x' \mapsto x'(x)],$$

is a topological isomorphism. In particular, (X,τ) is a bidual space with semi-reflexive prebidual $((X,\widetilde{\gamma}),\mathcal{I})$.

Next, we show that (BBC) and (CNC) in Theorem 3.8 are also necessary conditions for the existence of a complete barrelled predual. Let X be a dual space with predual (Y,φ). We define two locally convex Hausdorff topology on X w.r.t. the dual paring $\langle X,Y,\varphi\rangle$. We define the systems of seminorms

$$p_N(x) := \sup_{y\in N} |\varphi(x)(y)|, \quad x \in X,$$

for finite resp. compact sets $N \subset Y$, which induce two locally convex Hausdorff topologies on X w.r.t. the dual paring $\langle X,Y,\varphi\rangle$ and we denote these topologies by $\sigma_\varphi(X,Y)$ resp. $\tau_{\mathrm{c},\varphi}(X,Y)$. We recall from Example 3.15 that the evaluation map $\mathcal{J}_Y\colon Y \to (Y',\tau_{\mathrm{c}}(Y',Y))_b'$, $y \longmapsto [y' \mapsto y'(y)]$, is a topological isomorphism if Y is complete and barrelled.

Proposition 3.21 *Let (X, τ) be a dual space with complete barrelled predual (Y, φ) and $\mathcal{B}$ the family of τ-bounded sets. Then the following assertions hold.*

(a) (X, τ) satisfies (BBC) *and* (CNC) *for $\sigma_\varphi(X, Y)$,*

$$\widetilde{\gamma}_\varphi := \widetilde{\gamma}(\tau, \sigma_\varphi(X, Y)) = \tau_{c,\varphi}(X, Y)$$

and $(X, \widetilde{\gamma}_\varphi)$ satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma}_\varphi)$.

(b) The three maps

$$\kappa_\varphi : (Y', \tau_c(Y', Y))'_b \to (X, \widetilde{\gamma}_\varphi)'_b, \;\; y'' \mapsto y'' \circ \varphi,$$

and $\kappa_\varphi \circ \mathcal{J}_Y : Y \to (X, \widetilde{\gamma}_\varphi)'_b$ as well as

$$\mathcal{I}_\varphi : (X, \tau) \to ((X, \widetilde{\gamma}_\varphi)'_b)'_b, \;\; x \longmapsto [x' \mapsto x'(x)],$$

are topological isomorphisms with $(\kappa_\varphi \circ \mathcal{J}_Y)^t = \varphi \circ \mathcal{I}_\varphi^{-1}$ and

$$(X, \widetilde{\gamma}_\varphi)'_b = (X'_{\mathcal{B},\sigma_\varphi(X,Y)}, \beta(X'_{\mathcal{B},\sigma_\varphi(X,Y)}, (X, \tau))).$$

Here, $(\kappa_\varphi \circ \mathcal{J}_Y)^t$ denotes the dual map of $\kappa_\varphi \circ \mathcal{J}_Y$.

(c) Let $x' : X \to \mathbb{K}$. Then $x' \in (X, \widetilde{\gamma}_\varphi)'$ if and only if $x' \in (X, \sigma_\varphi(X, Y))'$ if and only if there is a (unique) $y \in Y$ such that $x' = \varphi(\cdot)(y)$.

Proof (a) The space Y'_b satisfies (BBC) and (CNC) for $\sigma(Y', Y)$ by [1, p. 116], which directly yields that (X, τ) satisfies (BBC) and (CNC) for $\sigma_\varphi(X, Y)$. Due to Example 3.15 the evaluation map $\mathcal{J}_Y$ is a topological isomorphism and we have

$$\widetilde{\gamma} := \widetilde{\gamma}(\beta(Y', Y), \sigma(Y', Y)) = \tau_c(Y', Y)$$

as well as that $(Y', \widetilde{\gamma})$ satisfies $(\mathrm{B}\,\beta(Y', Y)\,\mathrm{B}\,\widetilde{\gamma})$, which implies

$$\widetilde{\gamma}_\varphi = \widetilde{\gamma}(\tau, \sigma_\varphi(X, Y)) = \tau_{c,\varphi}(X, Y)$$

and that $(X, \widetilde{\gamma}_\varphi)$ satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma}_\varphi)$.

(b) The map κ_φ is a topological isomorphism since φ is a topological isomorphism and $\widetilde{\gamma}_\varphi = \tau_{c,\varphi}(X, Y)$ by part (a). Thus $\kappa_\varphi \circ \mathcal{J}_Y$ is a topological isomorphism as well. It follows that the map $\psi \circ \varphi : (X, \tau) \to ((X, \widetilde{\gamma}_\varphi)'_b)'_b$ with

$$\psi : Y'_b \to ((X, \widetilde{\gamma}_\varphi)'_b)'_b, \;\; \psi(y') := y' \circ (\kappa_\varphi \circ \mathcal{J}_Y)^{-1},$$

is a topological isomorphism. Let $x' \in (X, \widetilde{\gamma}_\varphi)'$. Then we have $(\kappa_\varphi \circ \mathcal{J}_Y)^{-1}(x') = \mathcal{J}_Y^{-1}(x' \circ \varphi^{-1})$ and

$$\begin{aligned}(\psi \circ \varphi)(x)(x') &= \varphi(x)(\mathcal{J}_Y^{-1}(x' \circ \varphi^{-1})) = \mathcal{J}_Y(\mathcal{J}_Y^{-1}(x' \circ \varphi^{-1}))(\varphi(x)) \\ &= (x' \circ \varphi^{-1})(\varphi(x)) = x'(x) = \mathcal{I}_\varphi(x)(x')\end{aligned}$$

for all $x \in X$, proving that the map $\mathcal{I}_\varphi$ is a topological isomorphism. Looking at the proof of Proposition 3.17 (a), we see that

$$(X, \widetilde{\gamma}_\varphi)'_b = (X'_{\mathcal{B},\sigma_\varphi(X,Y)}, \beta(X'_{\mathcal{B},\sigma_\varphi(X,Y)}, (X, \tau))).$$

holds due to $(X, \widetilde{\gamma}_\varphi)$ satisfying $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma}_\varphi)$ even without the assumption that (X, τ) is bornological.

Next, we show that $(\kappa_\varphi \circ \mathcal{J}_Y)^t = \varphi \circ \mathcal{I}_\varphi^{-1}$. We have

$$(\kappa_\varphi \circ \mathcal{J}_Y)(y)(x) = (\mathcal{J}_Y(y) \circ \varphi)(x) = \mathcal{J}_Y(y)(\varphi(x)) = \varphi(x)(y) \tag{1}$$

for all $y \in Y$ and $x \in X$. Let $x'' \in ((X, \widetilde{\gamma}_{\varphi})_b')'$. Then there is a unique $x \in X$ such that $x'' = \mathcal{I}_{\varphi}(x)$ and

$$(\kappa_{\varphi} \circ \mathcal{J}_Y)^t(x'')(y) = x''((\kappa_{\varphi} \circ \mathcal{J}_Y)(y)) \underset{(1)}{=} x''(\varphi(\cdot)(y)) = \mathcal{I}_{\varphi}(x)(\varphi(\cdot)(y))$$
$$= \varphi(x)(y) = \varphi(\mathcal{I}_{\varphi}^{-1}(x''))(y)$$

for all $y \in Y$.

(c) This part follows from part (b), (1) and [27, Chap. IV, §1, 1.2, p. 124]. □

Due to Theorem 3.8 and Proposition 3.21 (a) we have that (BBC) and (CNC) are necessary and sufficient conditions for the existence of a complete barrelled predual of a bornological space.

Corollary 3.22 *Let (X, τ) be a bornological locally convex Hausdorff space. Then the following assertions are equivalent.*

(a) (X, τ) *has a complete barrelled predual.*
(b) (X, τ) *has a semi-Montel prebidual* (Y, φ) *such that* Y_b' *is complete.*
(c) (X, τ) *satisfies* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *and* $(X, \widetilde{\gamma})$ *satisfies* (B τ B $\widetilde{\gamma}$).
(d) (X, τ) *satisfies* (BBCl) *and* (CNC) *for some* $\widetilde{\tau}$ *and* $(X, \widetilde{\gamma})$ *is semi-reflexive and satisfies* (B τ B $\widetilde{\gamma}$).
(e) (X, τ) *satisfies* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$.

Proof (b)⇒(a) This implication follows from the observations that semi-Montel space are semi-reflexive and semi-reflexive locally convex Hausdorff spaces are distinguished by [15, 11.4.1 Proposition, p. 227]. Hence the tuple (Y_b', φ) is a complete barrelled predual of (X, τ).

(a)⇒(c) This implication follows from Proposition 3.21 (a) with $\widetilde{\tau} := \sigma_{\varphi}(X, Y)$ for a complete barrelled predual (Y, φ) of (X, τ).

(c)⇒(b) Due to Corollary 3.14 $(X, \widetilde{\gamma})$ is a semi-Montel space and thus semi-reflexive. Hence the implication follows from Corollary 3.20 with $Y := (X, \widetilde{\gamma})$ and $\varphi := \mathcal{I}$.

(c)⇒(d) This implication follows from Corollary 3.14 and the observation that semi-Montel space are semi-reflexive.

(d)⇒(a) This implication follows from Corollary 3.20 with $Y := (X, \widetilde{\gamma})$ and $\varphi := \mathcal{I}$.

(c)⇒(e) This implication is obvious.

(e)⇒(a) This implication follows from Theorem 3.8. □

Corollary 3.23 *Let (X, τ) be a bornological locally convex Hausdorff space. Then the following assertions are equivalent.*

(a) (X, τ) *has a complete barrelled DF-predual.*
(b) (X, τ) *has a semi-Montel prebidual* (Y, φ) *such that* Y_b' *is a complete DF-space.*
(c) (X, τ) *is a Fréchet space satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *and* $(X, \widetilde{\gamma})$ *satisfies* (B τ B $\widetilde{\gamma}$).
(d) (X, τ) *is a Fréchet space satisfying* (BBCl) *and* (CNC) *for some* $\widetilde{\tau}$ *and* $(X, \widetilde{\gamma})$ *is semi-reflexive and satisfies* (B τ B $\widetilde{\gamma}$).
(e) (X, τ) *is a Fréchet space satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$.

Proof (b)⇒(a), (c)⇒(d), (c)⇒(e) These implications follow from Corollary 3.22.

(a)⇒(c) This implication follows from the proof of the implication (a)⇒(c) of Corollary 3.22 and the observation that the strong dual of a DF-space is a Fréchet space by [15, 12.4.2 Theorem, p. 258].

(c)⇒(b), (d)⇒(a), (e)⇒(a) These implications follow from the proof of the corresponding implications of Corollary 3.22 and the observation that $F := (X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is a complete barrelled DF-space by the proof of [1, 5. Corollary (b), p. 117–118], which coincides with $(X, \widetilde{\gamma})'_b$ by Proposition 3.17 (a) if $(X, \widetilde{\gamma})$ satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$. □

Corollary 3.24 *Let (X, τ) be a bornological locally convex Hausdorff space and $\mathcal{B}$ the family of τ-bounded sets. Then the following assertions are equivalent.*

(a) *(X, τ) has a Fréchet predual.*
(b) *(X, τ) has a semi-Montel prebidual (Y, φ) such that Y'_b is a Fréchet space.*
(c) *(X, τ) is a complete DF-space satisfying* (BBC) *and* (CNC) *for some $\widetilde{\tau}$ and (X, γ) satisfies* $(\mathrm{B}\,\tau\,\mathrm{B}\,\gamma)$.
(d) *(X, τ) is a complete DF-space satisfying* (BBCl) *and* (CNC) *for some $\widetilde{\tau} \leq \tau$ and (X, γ) is semi-reflexive.*
(e) *(X, τ) is a complete DF-space satisfying* (BBC) *and* (CNC) *for some $\widetilde{\tau}$.*
(f) *(X, τ) is a complete DF-space satisfying* (BBC) *for some $\widetilde{\tau}$ and the Fréchet space $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is distinguished.*

Proof (c)⇒(e) This implication is obvious.

(b)⇒(a), (c)⇒(d) These implications follow from the proof of the corresponding implications of Corollary 3.22 by noting that $\gamma = \widetilde{\gamma}$ and that (X, γ) satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\gamma)$ by Proposition 3.16 and Remark 3.2 (c).

(a)⇒(c) This implication follows from the proof of the implication (a)⇒(c) of Corollary 3.22, the observation that the strong dual of a Fréchet predual (Y, φ) of (X, τ) is a DF-space by [15, 12.4.5 Theorem, p. 260] and by noting that $\widetilde{\gamma}_\varphi = \gamma(\tau, \sigma_\varphi(X, Y))$ by Proposition 3.16 and Remark 3.2 (c).

(c)⇒(b), (d)⇒(a), (e)⇒(a) These implications follow from the proof of the corresponding implications of Corollary 3.22 and the observation that $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is a Fréchet space by Proposition 3.7 (b), which coincides with $(X, \gamma)'_b$ by Proposition 3.17 (a) since $\gamma = \widetilde{\gamma}$ and (X, γ) satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\gamma)$ by Proposition 3.16 and Remark 3.2 (c).

(c)⇒(f) Due to the equivalence (a)⇔(e) the Fréchet space $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is distinguished by [26, Theorem 8.3.44, p. 261] since $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)'_b$ is topologically isomorphic to the bornological space (X, τ).

(f)⇒(a) By [1, 1. Theorem (Mujica), p. 115] the map

$$\widetilde{\mathcal{I}}\colon (X, \tau) \to (X'_{\mathcal{B},\widetilde{\tau}}, \beta)'_i,\ x \longmapsto [x' \mapsto x'(x)],$$

is a topological isomorphism where $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)'_i$ is the inductive dual. Since $E := (X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is a Fréchet space, thus barrelled, every null sequence in E'_b is equicontinuous by [15, 11.1.1 Proposition, p. 220]. Thus it follows from [26, Observation 8.3.40 (b), p. 260] that $E'_i = E'_b$ because E'_b is bornological by [26, Theorem 8.3.44, p. 261] as E is a distinguished Fréchet space. □

We note that the space $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is distinguished by [26, Proposition 8.3.45 (iii), p. 262] if it is a quasi-normable Fréchet space. Sufficient conditions that $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is a quasi-normable Fréchet space are given in [1, 4. Remark, p. 117]. The following example is a slight generalisation of [1, 3. Examples B, p. 125–126] where holomorphic functions are considered.

Example 3.25 Let $\Omega \subset \mathbb{R}^d$ be open, $P(\partial)$ a hypoelliptic linear partial differential operator on $\mathcal{C}^\infty(\Omega)$ and $\mathcal{V} := (v_n)_{n\in\mathbb{N}}$ a *drecreasing* family, i.e. $v_{n+1} \leq v_n$ for all $n \in \mathbb{N}$, of continuous functions $v_n\colon \Omega \to (0, \infty)$. We define the inductive limit

$$\mathcal{V}\mathcal{C}_P(\Omega) := \varinjlim_{n\in\mathbb{N}} \mathcal{C}_P v_n(\Omega)$$

of the Banach spaces $(\mathcal{C}_P v_n(\Omega), \|\cdot\|_{v_n})$, and equip $\mathcal{VC}_P(\Omega)$ with its locally convex inductive limit topology $\gamma\tau$. The space $(\mathcal{VC}_P(\Omega), \gamma\tau)$ is a strict inductive limit and thus $\gamma\tau$ is Hausdorff by [15, 4.6.1 Theorem, p. 84] and $(\mathcal{VC}_P(\Omega), \gamma\tau)$ complete by [15, 4.6.4 Theorem, p. 86]. Further, the inductive limit is a (ultra)bornological DF-space by [21, Proposition 25.16, p. 301] and $\tau_{\text{co}} := \tau_{\text{co}|\mathcal{VC}_P(\Omega)} \leq \gamma\tau$. The $\|\cdot\|_{v_n}$-closed unit balls $B_{\|\cdot\|_{v_n}}$ of $\mathcal{C}_P v_n(\Omega)$ are τ_{co}-closed. It follows that they are also τ_{co}-compact for all $n \in \mathbb{N}$ as $(\mathcal{C}_P(\Omega), \tau_{\text{co}})$ is a Fréchet–Schwartz space. Since $\tau_{\text{co}} \leq \gamma\tau$, they are $\gamma\tau$-closed as well and thus they form a basis of absolutely convex $\gamma\tau$-bounded sets by [21, Proposition 25.16, p. 301]. Therefore $(\mathcal{VC}_P(\Omega), \gamma\tau)$ satisfies (BBC) for τ_{co}. If $\mathcal{V}$ is *regularly decreasing*, i.e. for every $n \in \mathbb{N}$ there is $m \geq n$ such that for every $U \subset \Omega$ with $\inf_{x\in U} v_m(x)/v_n(x) > 0$ we also have $\inf_{x\in U} v_k(x)/v_n(x) > 0$ for all $k \geq m+1$, then the Fréchet space $(\mathcal{VC}_P(\Omega)'_{\mathcal{B},\tau_{\text{co}}}, \beta)$ is quasi-normable by [1, p. 125–126] and so distinguished.

Corollary 3.26 *Let (X, τ) be a bornological locally convex Hausdorff space. Then the following assertions are equivalent.*

(a) (X, τ) has a Banach predual.
(b) (X, τ) has a semi-Montel prebidual (Y, φ) such that Y'_b is completely normable.
(c) (X, τ) is a completely normable space satisfying (BBC) *and* (CNC) *for some $\widetilde{\tau}$ and (X, γ) satisfies* (B τ B γ).
(d) (X, τ) is a completely normable space satisfying (BBCl) *for some $\widetilde{\tau} \leq \tau$ and (X, γ) is semi-reflexive.*
(e) (X, τ) is a completely normable space satisfying (BBC) *for some $\widetilde{\tau}$.*
(f) There is a norm $|||\cdot|||$ on X which induces τ such that $B_{|||\cdot|||}$ $\widetilde{\tau}$-compact for some $\widetilde{\tau}$.

Proof (b)⇒(a), (c)⇒(d), (c)⇒(e) These implications follow from the proof of the corresponding implications of Corollary 3.24.

(a)⇒(c) This implication follows from the proof of the implication (a)⇒(c) of Corollary 3.24 and the observation that the strong dual of a Banach predual (Y, φ) of (X, τ) is completely normable.

(c)⇒(b), (d)⇒(a), (e)⇒(a) These implications follow from the proof of the corresponding implications of Corollary 3.22 and the observation that $(X'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is completely normable by Proposition 3.7 (c), which coincides with $(X, \gamma)'_b$.

(e)⇔(f) This equivalence follows from Proposition 3.4. □

4 Linearisation

In this section we study necessary and sufficient conditions for the existence of a strong linearisation of a bornological function space. If $X = \mathcal{F}(\Omega)$ is a space of $\mathbb{K}$-valued functions on a non-empty set Ω that satisfies the assumptions of Theorem 3.8 or Corollary 3.20 and we choose $Y := \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ and $T := \mathcal{I}$, then we only need one additional ingredient to obtain a strong linearisation of $\mathcal{F}(\Omega)$ from the tuple $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$, namely a suitable map $\delta\colon \Omega \to \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ which fulfils $\mathcal{I}(f) \circ \delta = f$ for every $f \in \mathcal{F}(\Omega)$. In order to fulfil $\mathcal{I}(f) \circ \delta = f$ for every $f \in \mathcal{F}(\Omega)$, i.e.

$$f(x) = (\mathcal{I}(f) \circ \delta)(x) = \mathcal{I}(f)(\delta(x)) = \delta(x)(f), \quad x \in \Omega,$$

for every $f \in \mathcal{F}(\Omega)$, the map $\delta(x)$ has to be the point evaluation functional δ_x given by $\delta_x(f) := f(x)$ for every $x \in \Omega$ and $f \in \mathcal{F}(\Omega)$. Thus we obtain a strong linearisation of $\mathcal{F}(\Omega)$ if $\delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ for every $x \in \Omega$ and arrive at the following results.

Corollary 4.1 *Let $(\mathcal{F}(\Omega),\tau)$ be a bornological locally convex Hausdorff space of $\mathbb{K}$-valued functions on a non-empty set Ω satisfying* (BBC) *and* (CNC) *for some $\widetilde{\tau}$ and $\mathcal{B}$ the family of τ-bounded sets. If $\Delta(x) := \delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ for all $x\in\Omega$, then $(\Delta, \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$ is a strong complete barrelled linearisation of $\mathcal{F}(\Omega)$.*

Corollary 4.2 *Let $(\mathcal{F}(\Omega),\tau)$ be a bornological locally convex Hausdorff space of $\mathbb{K}$-valued functions on a non-empty set Ω satisfying* (BBCl) *and* (CNC) *for some $\widetilde{\tau}$ and $(\mathcal{F}(\Omega),\widetilde{\gamma})$ a semi-reflexive space satisfying* (B τ B $\widetilde{\gamma}$). *If $\Delta(x) := \delta_x \in (\mathcal{F}(\Omega),\widetilde{\gamma})' = \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ for all $x\in\Omega$, then $(\Delta, (\mathcal{F}(\Omega),\widetilde{\gamma})', \mathcal{I})$ is a strong complete barrelled linearisation of $\mathcal{F}(\Omega)$.*

Remark 4.3 Let $(\mathcal{F}(\Omega),\tau)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a non-empty set Ω, $\mathcal{B}$ the family of τ-bounded sets and $\widetilde{\tau}$ a locally convex Hausdorff topology on $\mathcal{F}(\Omega)$. If $\tau_{\mathrm{p}} \leq \widetilde{\tau}$, then $\delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ for all $x\in\Omega$. Indeed, for every $x\in\Omega$ we have $\delta_x \in (\mathcal{F}(\Omega),\tau_{\mathrm{p}})'$ by definition and thus $\delta_x \in (\mathcal{F}(\Omega),\widetilde{\tau})' \subset \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ since $\tau_{\mathrm{p}} \leq \widetilde{\tau}$.

Corollary 4.1 and Remark 4.3 also give us a simple sufficient criterion when a bornological space $(\mathcal{F}(\Omega),\tau)$ of continuous functions has a separable predual, which generalises [23, 2.2 Remark, p. 870].

Proposition 4.4 *Let $(\mathcal{F}(\Omega),\tau)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued continuous functions on a non-empty separable topological Hausdorff space Ω satisfying* (BBC) *and* (CNC) *for some $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ and $\mathcal{B}$ the family of τ-bounded sets. Then $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$ is a complete barrelled predual of $(\mathcal{F}(\Omega),\tau)$ and $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is separable.*

Proof $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$ is a complete barrelled predual of $(\mathcal{F}(\Omega),\tau)$ by Corollary 4.1 and Remark 4.3. By the bipolar theorem the span of $\{\delta_x \mid x\in D\}$ is β-dense in $\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ for any dense subspace D of Ω since $\mathcal{F}(\Omega)$ is a space of continuous functions. This implies our statement on separability. □

Next, we show that our sufficient conditions for the existence of a strong complete barrelled linearisation are also necessary.

Theorem 4.5 *Let $(\mathcal{F}(\Omega),\tau)$ be a bornological locally convex Hausdorff space of $\mathbb{K}$-valued functions on a non-empty set Ω and $\mathcal{B}$ the family of τ-bounded sets. Then the following assertions are equivalent.*

(a) $(\mathcal{F}(\Omega),\tau)$ *admits a strong complete barrelled linearisation.*
(b) $(\mathcal{F}(\Omega),\tau)$ *has a semi-Montel prebidual (Y,φ) such that Y'_b is complete and for every $x\in\Omega$ there is a unique $y'_x\in Y'$ such that $\delta_x = \varphi(\cdot)(y'_x)$.*
(c) $(\mathcal{F}(\Omega),\tau)$ *satisfies* (BBC) *and* (CNC) *for some $\widetilde{\tau}$ such that $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ and $(\mathcal{F}(\Omega),\widetilde{\gamma})$ satisfies* (B τ B $\widetilde{\gamma}$).
(d) $(\mathcal{F}(\Omega),\tau)$ *satisfies* (BBCl) *and* (CNC) *for some $\widetilde{\tau}$ such that $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ and $(\mathcal{F}(\Omega),\widetilde{\gamma})$ is semi-reflexive and satisfies* (B τ B $\widetilde{\gamma}$).
(e) $(\mathcal{F}(\Omega),\tau)$ *satisfies* (BBC) *and* (CNC) *for some $\widetilde{\tau}$ such that $\tau_{\mathrm{p}} \leq \widetilde{\tau}$.*
(f) $(\mathcal{F}(\Omega),\tau)$ *satisfies* (BBC) *and* (CNC) *for some $\widetilde{\tau}$ such that $\delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ for all $x\in\Omega$.*

Proof (b)⇒(a) By the proof of Corollary 3.22 the tuple (Y'_b,φ) is a complete barrelled predual of $(\mathcal{F}(\Omega),\tau)$. We set $\delta\colon \Omega\to Y'$, $\delta(x) := y'_x$. Then we have

$$(\varphi(f)\circ\delta)(x) = \varphi(f)(y'_x) = \delta_x(f) = f(x)$$

for all $f\in\mathcal{F}(\Omega)$ and $x\in\Omega$. Hence (δ, Y'_b, φ) is a strong complete barrelled linearisation of $\mathcal{F}(\Omega)$.

(a)⇒(c) Since $(\mathcal{F}(\Omega), \tau)$ admits a strong complete barrelled linearisation, there are a complete barrelled locally convex Hausdorff space Y, a map $\delta\colon \Omega \to Y$ and a topological isomorphism $T\colon (\mathcal{F}(\Omega), \tau) \to Y_b'$ such that $T(f) \circ \delta = f$ for all $f \in \mathcal{F}(\Omega)$. Then $(\mathcal{F}(\Omega), \tau)$ satisfies (BBC) and (CNC) for $\widetilde{\tau} := \sigma_T(\mathcal{F}(\Omega), Y)$ and $(\mathcal{F}(\Omega), \widetilde{\gamma})$ satisfies $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$ by Proposition 3.21 (a). From $T(f)(\delta(x)) = f(x)$ for all $f \in \mathcal{F}(\Omega)$ and $x \in \Omega$, we deduce that τ_{p} and $\sigma_T(\mathcal{F}(\Omega), Y_0)$ coincide on $\mathcal{F}(\Omega)$ where Y_0 denotes the span of $\{\delta(x) \mid x \in \Omega\}$ which is dense in Y by Proposition 2.7, and $\sigma_T(\mathcal{F}(\Omega), Y_0)$ is defined by the system of seminorms

$$p_N(f) := \sup_{y \in N} |T(f)(y)|, \quad f \in \mathcal{F}(\Omega),$$

for finite sets $N \subset Y_0$. Hence we have $\tau_{\mathrm{p}} = \sigma_T(\mathcal{F}(\Omega), Y_0) \leq \sigma_T(\mathcal{F}(\Omega), Y) = \widetilde{\tau}$.

(c)⇒(b) Due to Corollary 3.14 $(\mathcal{F}(\Omega), \widetilde{\gamma})$ is a semi-Montel space and thus semi-reflexive. It follows from Corollary 4.2 and Remark 4.3 with $Y := (\mathcal{F}(\Omega), \widetilde{\gamma})$ and $\varphi := \mathcal{I}$ that (Y, φ) is a semi-Montel prebidual of $(\mathcal{F}(\Omega), \tau)$ such that Y_b' is a complete barrelled space and $\delta_x = \varphi(\cdot)(\delta_x)$ with $\delta_x \in Y'$. Suppose that for $x \in \Omega$ there is another $y_x' \in Y'$ such that $\delta_x = \varphi(\cdot)(y_x')$. This implies that $\Phi_\varphi(y_x') = \Phi_\varphi(\delta_x)$ for the map $\Phi_\varphi\colon Y_b' \to (\mathcal{F}(\Omega), \tau)_b'$ from Proposition 2.2 and thus $y_x' = \delta_x$ by Proposition 2.2.

(c)⇒(d) This implication follows from the proof of Corollary 3.22.

(d)⇒(a) This implication follows from Corollary 4.2 and Remark 4.3 with $Y := (\mathcal{F}(\Omega), \widetilde{\gamma})$ and $\varphi := \mathcal{I}$.

(c)⇒(e) This implication is obvious.

(e)⇒(f) We only need to show that $\delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ for all $x \in \Omega$, which is a consequence of Remark 4.3.

(f)⇒(a) This implication follows from Corollary 4.1. □

Corollary 4.6 *Let $(\mathcal{F}(\Omega), \tau)$ be a bornological locally convex Hausdorff space of $\mathbb{K}$-valued functions on a non-empty set Ω and $\mathcal{B}$ the family of τ-bounded sets. Then the following assertions are equivalent.*

(a) $(\mathcal{F}(\Omega), \tau)$ *admits a strong complete barrelled DF-linearisation.*
(b) $(\mathcal{F}(\Omega), \tau)$ *has a semi-Montel prebidual* (Y, φ) *such that* Y_b' *is a complete DF-space and for every* $x \in \Omega$ *there is a unique* $y_x' \in Y'$ *such that* $\delta_x = \varphi(\cdot)(y_x')$.
(c) $(\mathcal{F}(\Omega), \tau)$ *is a Fréchet space satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ *and* $(\mathcal{F}(\Omega), \widetilde{\gamma})$ *satisfies* $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$.
(d) $(\mathcal{F}(\Omega), \tau)$ *is a Fréchet space satisfying* (BBCl) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ *and* $(\mathcal{F}(\Omega), \widetilde{\gamma})$ *is semi-reflexive and satisfies* $(\mathrm{B}\,\tau\,\mathrm{B}\,\widetilde{\gamma})$.
(e) $(\mathcal{F}(\Omega), \tau)$ *is a Fréchet space satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$.
(f) $(\mathcal{F}(\Omega), \tau)$ *is a Fréchet space satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ *for all* $x \in \Omega$.

Proof This statement follows from the proofs of Theorem 4.5 and Corollary 3.23. □

Corollary 4.7 *Let $(\mathcal{F}(\Omega), \tau)$ be a bornological locally convex Hausdorff space of $\mathbb{K}$-valued functions on a non-empty set Ω and $\mathcal{B}$ the family of τ-bounded sets. Then the following assertions are equivalent.*

(a) $(\mathcal{F}(\Omega), \tau)$ *admits a strong Fréchet linearisation.*
(b) $(\mathcal{F}(\Omega), \tau)$ *has a semi-Montel prebidual* (Y, φ) *such that* Y_b' *is a Fréchet space and for every* $x \in \Omega$ *there is a unique* $y_x' \in Y'$ *such that* $\delta_x = \varphi(\cdot)(y_x')$.
(c) $(\mathcal{F}(\Omega), \tau)$ *is a complete DF-space satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ *and* $(\mathcal{F}(\Omega), \gamma)$ *satisfies* $(\mathrm{B}\,\tau\,\mathrm{B}\,\gamma)$.

(d) $(\mathcal{F}(\Omega), \tau)$ *is a complete DF-space satisfying* (BBCl) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau} \leq \tau$ *and* $(\mathcal{F}(\Omega), \gamma)$ *is semi-reflexive.*
(e) $(\mathcal{F}(\Omega), \tau)$ *is a complete DF-space satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$.
(f) $(\mathcal{F}(\Omega), \tau)$ *is a complete DF-space satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ *for all* $x \in \Omega$.
(g) $(\mathcal{F}(\Omega), \tau)$ *is a complete DF-space satisfying* (BBC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ *and the Fréchet space* $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \beta)$ *is distinguished.*
(h) $(\mathcal{F}(\Omega), \tau)$ *is a complete DF-space satisfying* (BBC) *for* τ_{p} *and the Fréchet space* $(\mathcal{F}(\Omega)'_{\mathcal{B},\tau_{\mathrm{p}}}, \beta)$ *is distinguished.*

Proof The first seven equivalences follow from the proofs of Theorem 4.5 and Corollary 3.24 where one uses for the implication (g)⇒(a) in addition Remark 4.3.

(g)⇒(h) If $(\mathcal{F}(\Omega), \tau)$ satisfies (BBC) for some $\widetilde{\tau}$ such that $\tau_{\mathrm{p}} \leq \widetilde{\tau}$, then it satisfies (BBC) for τ_{p} by Remark 3.2 (d). Further, we have $\mathcal{F}(\Omega)'_{\mathcal{B},\tau_{\mathrm{p}}} = \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ and $\beta_{\mathcal{B},\tau_{\mathrm{p}}} = \beta_{\mathcal{B},\widetilde{\tau}} = \beta$ by Remark 3.6, implying that $(\mathcal{F}(\Omega)'_{\mathcal{B},\tau_{\mathrm{p}}}, \beta)$ is distinguished.

(h)⇒(g) This implication is obvious. □

Corollary 4.8 *Let* $(\mathcal{F}(\Omega), \tau)$ *be a bornological locally convex Hausdorff space of* $\mathbb{K}$*-valued functions on a non-empty set* Ω *and* $\mathcal{B}$ *the family of* τ*-bounded sets. Then the following assertions are equivalent.*

(a) $(\mathcal{F}(\Omega), \tau)$ *admits a strong Banach linearisation.*
(b) $(\mathcal{F}(\Omega), \tau)$ *has a semi-Montel prebidual* (Y, φ) *such that* Y'_b *is completely normable and for every* $x \in \Omega$ *there is a unique* $y'_x \in Y'$ *such that* $\delta_x = \varphi(\cdot)(y'_x)$.
(c) $(\mathcal{F}(\Omega), \tau)$ *is a completely normable space satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ *and* $(\mathcal{F}(\Omega), \gamma)$ *satisfies* (B τ B γ).
(d) $(\mathcal{F}(\Omega), \tau)$ *is a completely normable space satisfying* (BBCl) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau} \leq \tau$ *and* $(\mathcal{F}(\Omega), \gamma)$ *is semi-reflexive.*
(e) $(\mathcal{F}(\Omega), \tau)$ *is a completely normable space satisfying* (BBC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$.
(f) *There is a norm* $|||\cdot|||$ *on* $\mathcal{F}(\Omega)$ *which induces* τ *such that* $B_{|||\cdot|||}$ *is* $\widetilde{\tau}$*-compact for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$.
(g) *There is a norm* $|||\cdot|||$ *on* $\mathcal{F}(\Omega)$ *which induces* τ *such that* $B_{|||\cdot|||}$ *is* $\widetilde{\tau}$*-compact for some* $\widetilde{\tau}$ *such that* $\delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ *for all* $x \in \Omega$.
(h) $(\mathcal{F}(\Omega), \tau)$ *is a completely normable space satisfying* (BBC) *for* τ_{p}.
(i) *There is a norm* $|||\cdot|||$ *on* $\mathcal{F}(\Omega)$ *which induces* τ *such that* $B_{|||\cdot|||}$ *is* τ_{p}*-compact.*

Proof The first seven equivalences follow from the proofs of Theorem 4.5 and Corollary 3.26.

(e)⇒(h) This implication follows from Remark 3.2 (d) with $\widetilde{\tau}_0 := \tau_{\mathrm{p}}$.

(h)⇒(e) This implication is obvious.

(h)⇔(i) This equivalence follows from Proposition 3.4 with $\widetilde{\tau} := \tau_{\mathrm{p}}$. □

We close this section with a characterisation of continuous strong linearisations. We call a topological space Ω a $gk_{\mathbb{R}}$*-space* if for any completely regular space Y and any map $f: \Omega \to Y$, whose restriction to each compact $K \subset \Omega$ is continuous, the map is already continuous on Ω. If a $gk_{\mathbb{R}}$-space Ω is also completely regular, then it is called a $k_{\mathbb{R}}$*-space* (see [5, (2.3.7) Proposition, p. 22]). Examples of Hausdorff $gk_{\mathbb{R}}$-spaces are Hausdorff k-spaces by [9, 3.3.21 Theorem, p. 152]. Examples of Hausdorff $k_{\mathbb{R}}$-spaces are metrisable spaces by [13, Proposition 11.5, p. 181] and [9, 3.3.20 Theorem, p. 152], locally compact Hausdorff spaces and strong duals of Fréchet–Montel spaces (*DFM-spaces*) by [10, Proposition 3.27,

p. 95] and [19, 4.11 Theorem (5), p. 39]. The underlying idea of the proof of our next result comes from [14, Theorem 2.1, p. 187].

Proposition 4.9 *Let $(\mathcal{F}(\Omega),\tau)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued continuous functions on a non-empty Hausdorff $gk_{\mathbb{R}}$-space Ω and $\mathcal{B}$ the family of τ-bounded sets. If $(\mathcal{F}(\Omega),\tau)$ satisfies* (BBC) *for some $\widetilde{\tau}$ such that $\tau_{\mathrm{co}}\leq\widetilde{\tau}$, then the map*

$$\Delta\colon \Omega\to(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}},\beta),\ \ \Delta(x):=\delta_x,$$

is continuous.

Proof First, we note that the map Δ is well-defined by Remark 4.3 because $\tau_{\mathrm{p}}\leq\tau_{\mathrm{co}}\leq\widetilde{\tau}$. Since Ω is a $gk_{\mathbb{R}}$-space and the locally convex Hausdorff space $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}},\beta)$ completely regular by [10, Proposition 3.27, p. 95], we only need to show that the restricted map $\Delta_{|K}$ is continuous for every compact set $K\subset\Omega$. Let $B\in\mathcal{B}$ be absolutely convex and $\widetilde{\tau}$-compact and $K\subset\Omega$ a compact set. Due to Remark 3.2 (d) B is also τ_{co}-compact. The restriction $(\mathcal{F}(\Omega),\tau_{\mathrm{co}})\to(\mathcal{C}(K),\|\cdot\|_K)$, $f\mapsto f_{|K}$, is continuous where $\|g\|_K:=\sup_{x\in K}|g(x)|$ for all $g\in\mathcal{C}(K)$. This implies that $B_{|K}:=\{f_{|K}\mid f\in B\}$ is $\|\cdot\|_K$-compact. Due to [24, Theorem 47.1 (Ascoli's theorem) (b), p. 290] we obtain that $B_{|K}$ is equicontinuous. Hence for every $\varepsilon>0$ and $x\in K$ there is a neighbourhood $U(x)\subset K$ of x such that for all $y\in U(x)$ we have

$$\sup_{f\in B}|f(x)-f(y)|\leq\varepsilon,$$

which implies

$$\sup_{f\in B}|\Delta(x)(f)-\Delta(y)(f)|=\sup_{f\in B}|f(x)-f(y)|\leq\varepsilon.$$

Therefore $\Delta_{|K}$ is continuous, which closes the proof. □

Theorem 4.10 *Let $(\mathcal{F}(\Omega),\tau)$ be a bornological locally convex Hausdorff space of $\mathbb{K}$-valued continuous functions on a non-empty topological Hausdorff space Ω and $\mathcal{B}$ the family of τ-bounded sets. Consider the following assertions.*

(a) *$(\mathcal{F}(\Omega),\tau)$ admits a continuous strong complete barrelled linearisation.*
(b) *$(\mathcal{F}(\Omega),\tau)$ satisfies* (BBC) *and* (CNC) *for some $\widetilde{\tau}$ such that $\tau_{\mathrm{p}}\leq\widetilde{\tau}$, and every $B\in\mathcal{B}$ is equicontinuous.*
(c) *$(\mathcal{F}(\Omega),\tau)$ satisfies* (BBC) *and* (CNC) *for some $\widetilde{\tau}$ such that*

$$\Delta\colon \Omega\to(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}},\beta),\ \ \Delta(x):=\delta_x,$$

is a well-defined continuous map.
(d) *$(\mathcal{F}(\Omega),\tau)$ satisfies* (BBC) *and* (CNC) *for some $\widetilde{\tau}$ such that $\tau_{\mathrm{co}}\leq\widetilde{\tau}$.*

Then it holds (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). If Ω is a $gk_{\mathbb{R}}$-space, then it holds (d) $\Rightarrow$ (c).

Proof (a)⇒(b) Due to Theorem 4.5 we only need to show that every $B\in\mathcal{B}$ is equicontinuous. Since $(\mathcal{F}(\Omega),\tau)$ admits a continuous strong complete barrelled linearisation, there are a complete barrelled locally convex Hausdorff space Y, a continuous map $\delta\colon\Omega\to Y$ and a topological isomorphism $T\colon(\mathcal{F}(\Omega),\tau)\to Y_b'$ such that $T(f)\circ\delta=f$ for all $f\in\mathcal{F}(\Omega)$. For every $B\in\mathcal{B}$ we have

$$\sup_{f\in B}|f(x)-f(y)|=\sup_{f\in B}|T(f)(\delta(x))-T(f)(\delta(y))|=\sup_{f\in B}|\Phi_T(\delta(x))(f)-\Phi_T(\delta(y))(f)|$$

for all $x, y \in \Omega$ with Φ_T from Proposition 2.2 for $X := \mathcal{F}(\Omega)$. From the continuity of δ and Proposition 2.2 we deduce that every $B \in \mathcal{B}$ is equicontinuous.

(b)⇒(c) Due to Theorem 4.5 we only need to show that Δ is continuous. This directly follows from the observation that

$$\sup_{f\in B} |\Delta(x)(f) - \Delta(y)(f)| = \sup_{f\in B} |f(x) - f(y)|.$$

for all $B \in \mathcal{B}$ and $x, y \in \Omega$ and the equicontinuity of every $B \in \mathcal{B}$.

(c)⇒(a) Due to Corollary 4.1 $(\Delta, \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$ is a strong complete barrelled linearisation of $\mathcal{F}(\Omega)$. Since Δ is continuous, this triple is also a continuous linearisation.

(d)⇒(c) if Ω a $gk_{\mathbb{R}}$-space. This implication follows from Corollary 4.1 and Proposition 4.9. □

Using Corollary 4.6, we may add further statements to Theorem 4.10 that are equivalent to statement (a). In the same way we may prove a corresponding characterisation of continuous strong complete barrelled DF-linearisation by using Corollary 4.6, of continuous strong Fréchet linearisations by using Corollary 4.7 and of continuous strong Banach linearisations by using Corollary 4.8. In the case of continuous strong Banach linearisations we get [14, Theorem 2.2, Corollary 2.3, p. 188–189] back with an improvement of [14, Corollary 2.3, p. 189] from Hausdorff k-spaces $U = \Omega$ to Hausdorff $gk_{\mathbb{R}}$-spaces.

Example 4.11 Let $\Omega \subset \mathbb{R}^d$ be open and $P(\partial)$ a hypoelliptic linear partial differential operator on $\mathcal{C}^\infty(\Omega)$.

(i) Let $\mathcal{V}$ be a point-detecting directed family of continuous weights. If the space $(\mathcal{C}_P\mathcal{V}(\Omega), \tau_{\mathcal{V}})$ is bornological, then $(\Delta, \mathcal{C}_P\mathcal{V}(\Omega)'_{\mathcal{B},\tau_{\mathrm{co}}}, \mathcal{I})$ is a continuous strong complete barrelled linearisation of $\mathcal{C}_P\mathcal{V}(\Omega)$ by Example 3.3 and the proof of the implication (d)⇒(c) of Theorem 4.10 since Ω is metrisable and thus a Hausdorff $k_{\mathbb{R}}$-space. If $\mathcal{V}$ is countable and increasing, then $(\Delta, \mathcal{C}_P\mathcal{V}(\Omega)'_{\mathcal{B},\tau_{\mathrm{co}}}, \mathcal{I})$ is a continuous strong complete barrelled DF-linearisation of $\mathcal{C}_P\mathcal{V}(\Omega)$, and if $\mathcal{V} = \{v\}$, then $(\Delta, \mathcal{C}_P v(\Omega)'_{\mathcal{B},\tau_{\mathrm{co}}}, \mathcal{I})$ is a continuous strong Banach linearisation of $\mathcal{C}_P v(\Omega)$.

(ii) Let $\mathcal{V} := (v_n)_{n\in\mathbb{N}}$ be a drecreasing, regularly decreasing family of continuous functions $v_n \colon \Omega \to (0, \infty)$. Then $(\Delta, \mathcal{V}\mathcal{C}_P(\Omega)'_{\mathcal{B},\tau_{\mathrm{co}}}, \mathcal{I})$ is a continuous strong Fréchet linearisation of $\mathcal{V}\mathcal{C}_P(\Omega)$ by Example 3.25, the proof of the implication (f)⇒(a) of Corollary 3.24 and Proposition 4.9.

Declarations

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