A Uni-directional Model for Narrow and Broad Pulse Propagation in Second Order Nonlinear Media

E. van Groesen\textsuperscript{1}, E. Cahyono and A. Suryanto

Applied Analysis & Mathematical Physics

MESA+ Research Institute, University of Twente, The Netherlands

\textsuperscript{1}email: Groesen@math.utwente.nl

1. Introduction

The validity of the NLS-equation to describe the non-linear deformation of pulses is restricted to broad pulses. Here (see 2) we present for quadratic non-linear material a uni-directional model equation that describes both broad and narrow pulses.

2. Uni-directionalisation of the Maxwell equations

The Maxwell equations for pulses in one spatial dimension for TE-modes is described by the bi-directional wave equation for the electric field component in terms of the dielectric displacement like

$$\partial_t E = \mu_0 \partial_x D$$

- In vacuum the splitting into waves running to the right and to the left is exact:

$$-\frac{1}{c^2} \partial_x \partial_t E = 0$$

for right travelling waves:

$$\partial_x E - \frac{1}{c^2} \partial_t E = 0.$$  

• In a dispersive ($\chi_d$), quadratic nonlinear ($\chi_2$), nonmagnetic medium

$$D = \varepsilon_0 (1 + \chi_d) E + \varepsilon_2 \chi_2 E^2$$

the splitting leads to an equation for right travelling waves (cf [1]):

$$\partial_t E + \frac{1}{c^2} \partial_x (RE + \chi_2 E^2) = 0,$$  

with the operator $R$ determined to satisfy the dispersive properties

$$k = K(\omega) = \frac{\omega}{c R(\omega)}; \ R^2 = 1 + \chi_d(\omega).$$

This equation is valid for broad and narrow pulses since the correct dispersion is modelled; this is a KdV-type of equation, known in surface water waves (1895, Korteweg – de Vries).

3. Double-pumped pulses

In a linear medium the field for a pulse of amplitude $q$ and frequencies $\omega \pm \Delta \omega$ is a modulation of a carrier wave:

$$E_{\text{envelope}} = 4q \cos(\Delta \omega K(\omega)z - t) \cos(kz - \sigma t)$$

Non-linearity induces deformations of the envelope; but the size of the deformations depends just as well on the frequency difference; the determining parameter is the quotient $q / \Delta \omega$.

The following plots illustrate the deformations in the envelope of the signal at different positions.

Using the uni-directional equation we study a double pumped signal. The size of the deformations of the envelope (see 3) is not only determined by the amplitude, but just as well by the frequency difference. We present an analytic solution (simplified in 4) to explain this as a dominating third order contribution.

4. Explanation of the envelope deformation

Looking for a solution of $\psi$ as series expansion in the amplitude $q$:

$$E = E^{(1)} + E^{(2)} + E^{(3)} + \cdots,$$  

this expansion turns out to be not uniform in $\Delta \omega$.

Actually, it holds that $E^{(n)} \sim q^n / (\Delta \omega)^n$, and hence the expansion involves two parameters and the ordering becomes mixed up if $q / \Delta \omega > 1$.

Some details of the solution (see the schematic plot of the mode generation from the quadratic nonlinearity):

• the first order term is a modification of the linear solution, with a correction of the wave number (non-linear dispersion relation) to prevent resonances in the third order:

$$E^{(1)} = 4q \cos(\Delta \omega (K(\omega)z - t)) \cos(kz - \sigma t + k_{\text{new}} z)$$

• the second order results from interactions of the basic frequencies;

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Example of the second order term.}
\end{figure}

• the third order term contains side-band contributions from interactions of the 1st and 2nd order mode; this side-band contribution reads

$$E_{\text{side}}^{(3)} = q^3 \Delta \omega K(\omega) z \cos(3\Delta \omega (K(\omega)z - t)) \cos(kz - \sigma t),$$

with the coefficients $\beta_1 = O(1)$ and $\delta = K''''(\omega)$.

Observe the effects 1) the frequency difference and 2) the asymmetrization if $\sigma = 0$.

5. Conclusion and remarks about NLS

• The third order contribution will dominate the second order terms for small frequency difference; this leads to deformations of the envelope: amplitude increase and characteristic secondary modulations of a third of the original modulation length.

• Equation $\psi$ describes the complete uni-directional electric field. For broad pulses it is possible to derive an NLS-amplitude equation from $\psi$.

Using this NLS, formally the double pumped pulse can also be calculated, provided that the frequency difference is small (but not too small to prevent a mix-up of orders in a series expansion as above). The NLS-solution is comparable, but less accurate, than the solution described in section 4.

In fact: the NLS solution will not show the asymmetrisation from the different contributions of the two side bands (since $\delta = 0$).


6. References