IMA Journal of Numerical Analysis (2024) **00**, 1–50
https://doi.org/10.1093/imanum/drae008

Port-Hamiltonian discontinuous Galerkin finite element methods

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[Received on 30 January 2023; revised on 25 October 2023]

A port-Hamiltonian (pH) system formulation is a geometrical notion used to formulate conservation laws for various physical systems. The distributed parameter port-Hamiltonian formulation models infinite dimensional Hamiltonian dynamical systems that have a nonzero energy flow through the boundaries. In this paper, we propose a novel framework for discontinuous Galerkin (DG) discretizations of pH-systems. Linking DG methods with pH-systems gives rise to compatible structure preserving semidiscrete finite element discretizations along with flexibility in terms of geometry and function spaces of the variables involved. Moreover, the port-Hamiltonian formulation makes boundary ports explicit, which makes the choice of structure and power preserving numerical fluxes easier. We state the Discontinuous Finite Element Stokes–Dirac structure with a power preserving coupling between elements, which provides the mathematical framework for a large class of pH discontinuous Galerkin discretizations. We also provide an *a priori* error analysis for the port-Hamiltonian discontinuous Galerkin Finite Element Method (pH-DGFEM). The port-Hamiltonian discontinuous Galerkin finite element method is demonstrated for the scalar wave equation showing optimal rates of convergence.

Keywords: discontinuous Galerkin FEM; port-Hamiltonian systems.

1. Introduction

This paper defines a novel elementwise discontinuous finite element Stokes–Dirac structure, which provides the mathematical framework for elementwise discontinuous Galerkin (DG) discretizations of port-Hamiltonian (pH) systems. The pH-system formulation (van der Schaft & Maschke, 2002; Schöberl & Schlacher, 2011; Jacob & Zwart, 2012; Rashad *et al.*, 2020) is an approach for modeling and control of various kinds of physical systems. Infinite dimensional pH-systems with nonzero energy flow through it (spatial) boundaries are called distributed parameter pH-systems (van der Schaft & Maschke, 2002; Jacob & Zwart, 2012; van der Schaft *et al.*, 2014). These distributed parameter pH-systems comprise of an interconnection structure called Stokes–Dirac structure and an energy functional called Hamiltonian. The Stokes–Dirac structure gives us an effective way to express the behaviour of a system through pairs of input–output variables (whose product gives the power) at the boundary. This structural property of pH-systems makes them very suitable for the modeling and analysis of interconnected multi-physics systems. Due to the varied complexities of these multi-physics systems (inter-domain coupling, nonlinearities, etc.), the interest in numerical discretization techniques for pH-systems is increasing. In particular, numerical techniques that can preserve the mathematical structure (power balance) of the system upon discretization are important.

Various approaches have been followed to obtain structure preserving discretizations of port-Hamiltonian systems. Finite difference and model order reduction methods have been proposed by Lopezlena *et al.* (2003) and Trenchant *et al.* (2018). Talasila *et al.* (2002); Golo *et al.* (2004) proposed a method that is effective in preserving the structure of a pH-system on discretization, but this method has restrictive compatibility conditions and becomes complicated for higher dimensions.

Based on Discrete Exterior Calculus, Hirani (2003), Seslija *et al.* (2012), and using Finite Element Exterior Calculus (Arnold *et al.*, 2006, 2010), Brugnoli *et al.* (2022) proposed discrete formulations of pH conservation laws, along with a consistent approximation of the closure equations, which result in compatible discretizations. In Cardoso-Ribeiro *et al.* (2021), Cardoso-Ribeiro *et al.* have proposed the partitioned Finite Element Method (pFEM) for pH-systems with 2 conservation laws, e.g., shallow water equations, acoustic wave equations. The pFEM for structure preserving discretizations of pH-systems solves one conservation law in weak form and the other one in strong form. Serhani *et al.* (2019) have extended this pFEM to damped pH-systems. Kotyczka *et al.* (2018); Kotyczka (2019) proposed a mixed finite element discretization based on a general pH weak formulation. In this approach, the weak form of the Dirac structure on each element is defined as the basis for a mixed-Galerkin pH formulation. Using a power-preserving interconnection, a structure preserving discretization of the entire system is obtained.

The consistent approximation of the closure equation in the Discrete Exterior Calculus approach and the weak form in the partitioned-FEM and mixed-FEM methods provide more flexibility in terms of consistency and stability. However, these methods are restricted in terms of mesh geometry and the flexibility in the choice of function spaces chosen for the variables in the conservation laws. In this paper, using the concept of an elementwise Dirac structure and power preserving interconnection, we move a step forward by proposing a discontinuous Galerkin (DG) method for structure preserving discretizations of pH-systems. The flexibility in terms of mesh geometry (e.g., hanging nodes) and function spaces allowing hp-adaptive methods, which DG methods (Ern & Guermond, 2004; Hesthaven & Warburton, 2007; Riviere, 2008; Di Pietro & Ern, 2011) provide, makes them a rich environment for the discretization of various physical systems. DG methods use the basic idea of deriving an element by element weak formulation and then through appropriate numerical fluxes at the shared boundaries of the elements a stable discretization is ensured for the whole system. DG methods result in elementwise conservative numerical discretizations (Di Pietro & Ern, 2011), which often can also preserve the discrete energy when the proper numerical fluxes are chosen such as in the Local Discontinuous Galerkin (LDG) method. Due to the nonconforming nature of DG discretizations it is, however, nontrivial to preserve the Hamiltonian structure of the PDEs (if the PDEs possess one) (Castillo & Gómez, 2020a,b; Celledoni & Jackaman, 2021) especially on locally refined meshes. Hence, linking DG methods with the concept of pH-systems has a twofold advantage. On one hand the pH formulation makes the boundary port-variables explicit, so choosing a numerical flux that preserves the mathematical structure for DG discretizations becomes easier. On the other hand DG discretizations provide a flexible (in terms of mesh geometry and parallel computing) structure preserving discretization of pH-systems.

Since the power preserving interconnection of two Dirac structures again gives a Dirac structure (Cervera *et al.*, 2007; Jacob & Zwart, 2012), the elementwise DG formulation is extended to the whole manifold using a power preserving interconnection structure. We also provide an *a priori* error analysis of the port-Hamiltonian DG discretization.

The paper is organized as follows. In order to define the mathematical framework and notation we start with a brief introduction to differential forms in Section 2. In Section 3 we discuss the main aspects of distributed parameter pH-systems, including the definition of the Stokes–Dirac structure and the Hamiltonian, and then relate the port variables in the Stokes–Dirac structure to the Hamiltonian of the system. With this background we discuss in Section 4 the key results of this paper. We first define the function spaces for the port-variables of the Dirac structure. Next, we state the duality relations on the elements and their boundaries. Finally, at the end of this section, we state the definition of a generalized Stokes–Dirac structure (discontinuous finite element Stokes–Dirac structure) along with a definition of the interconnection Dirac structure (structure representing the connection between discontinuous finite elements). We state the power preserving coupling between finite elements in Section 5. In Section 6 we present the DG formulation of the discontinuous finite element Stokes–Dirac structure, together with the chosen numerical fluxes. In Section 7 we analyse the stability and in Section 8 we give an *a priori* error analysis of the DG discretization suitable for a large class of pH-systems. To support the theory we show some numerical results for the port-Hamiltonian discontinuous Galerkin finite element discretization of the scalar wave equation in Section 9.

2. Summary of some properties of differential forms

In order to express the geometrical structure of port-Hamiltonian systems we will use the language of differential forms. To make the paper self contained we will summarize the main properties of differential forms and their function spaces used in this paper. For more details, we refer to Flanders (1963); Frankel (2011); Abraham *et al.* (2012); Arnold (2013).

2.1 *Smooth differential forms*

Let Ω be an open, bounded, connected and oriented n-dimensional manifold in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and let $\Lambda^k(\Omega)$ be the space of smooth k-forms on Ω. The space of k-forms $\Lambda^k(\Omega)$ and the space of $(n-k)$-forms $\Lambda^{n-k}(\Omega)$ have equal dimension and therefore are isomorphic. There exists a duality product between a k-form λ and an $(n-k)$-form μ,

$$\langle \lambda | \mu \rangle_\Omega = \int_\Omega \lambda \wedge \mu, \tag{2.1}$$

where $\wedge$ is the (usual) wedge product of differential forms. The continuous extension of a smooth k-form $\omega \in \Lambda^k(\Omega)$ to the boundary $\partial\Omega$ is done through the trace operator $\mathrm{tr}(\omega) \in \Lambda^k(\partial\Omega)$. For $\omega \in \Lambda^{n-1}(\Omega)$, $\mathrm{tr}(\omega) \in \Lambda^{n-1}(\partial\Omega)$, the generalized Stokes theorem is given by

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \mathrm{tr}(\omega), \tag{2.2}$$

where d is the exterior derivative operator for differential forms. Using the duality product (2.1), the generalized Stokes theorem (2.2) and Leibniz rule (Frankel, 2011; Abraham *et al.*, 2012; Marsden & Ratiu, 2013), we state the integration by parts rule for smooth differential forms $\lambda \in \Lambda^k(\Omega)$ and $\mu \in \Lambda^{n-k-1}(\Omega)$,

$$\langle \mathrm{d}\lambda \mid \mu \rangle_\Omega = \langle \lambda | \, \mu \,\rangle_{\partial\Omega} + (-1)^{k-1} \langle \lambda \mid \mathrm{d}\mu \rangle_\Omega, \tag{2.3}$$

where with an abuse of notation we write, $\langle \lambda \mid \mu \rangle_{\partial\Omega} = \int_{\partial\Omega} \mathrm{tr}(\lambda) \wedge \mathrm{tr}(\mu)$ (Arnold *et al.*, 2010).

2.2 *Hodge duality*

The inner product g between two k-forms $\alpha^k, \beta^k \in \Lambda^k(\Omega)$ is written as

$$g(\alpha^k, \beta^k) = \int_\Omega \alpha^k(\mu)\beta^k(\mu)\,\mathrm{d}\mu, \tag{2.4}$$

where $\mathrm{d}\mu$ is the Lebesque measure on Ω and $\alpha^k(\mu), \beta^k(\mu) \in C^\infty(\Omega)$ are the coefficient functions of the smooth differential forms $\alpha^k, \beta^k \in \Lambda^k(\Omega)$. With this inner product we can construct an isomorphism between the spaces $\Lambda^k(\Omega)$ and $\Lambda^{n-k}(\Omega)$, which is denoted as $* : \Lambda^k \to \Lambda^{n-k}$ and called the Hodge star operator (Flanders, 1963; Frankel, 2011; Abraham *et al.*, 2012).

DEFINITION 2.1 Let Ω be an oriented manifold with $\dim(\Omega) = n$ and let $\Lambda^k(\Omega)$ be the space of differential k-forms. Let the inner product on the space of k forms be given by $g : \Lambda^k(\Omega) \times \Lambda^k(\Omega) \to \mathbb{R}$. Then the Hodge star operator $* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ is the linear mapping defined through

$$\int_\Omega \lambda \wedge *\omega = g(\omega, \lambda), \qquad \lambda,\ \omega \in \Lambda^k(\Omega). \tag{2.5}$$

If the Hodge star operator is applied twice on a k-form $\alpha^k \in \Lambda^k(\Omega)$ (Section 14.1, Frankel, 2011), we have

$$* * \alpha^k = (-1)^{k(n-k)}\alpha^k. \tag{2.6}$$

2.3 *Function spaces for differential forms*

We will now extend the above properties of differential forms to weaker smoothness conditions. For more information we refer to Section 4 of Arnold *et al.* (2010) or Section 2 of Arnold *et al.* (2006). The function spaces $H^s\Lambda^k(\Omega)$, with $s \in \mathbb{R}, s \geq 0$ and $L^p\Lambda^k(\Omega)$ contain differential k-forms whose coefficient functions belong to the standard Sobolev space $H^s(\Omega)$ and the Lebesgue space $L^p(\Omega)$, respectively. The weak exterior derivative for $\lambda \in L^2\Lambda^k(\Omega)$ is defined via the integration by parts formula (2.3). Namely, for infinitely differentiable differential forms with compact support $\mu \in C_c^\infty\Lambda^{n-k-1}(\Omega)$, $k \leq n-1$, $\mathrm{d}\lambda$ is defined to satisfy

$$\langle \mathrm{d}\lambda | \mu \rangle_\Omega := -(-1)^k \langle \lambda | \,\mathrm{d}\mu \rangle_\Omega, \tag{2.7}$$

where we use the symbol d for the weak exterior derivative as well. This weak exterior derivative operator allows us to apply the exterior derivative to differential forms whose coefficients are not differentiable in the classical sense. In analogy with the definition of Sobolev spaces (Section 5.2.2, Evans, 2010), $H\Lambda^k(\Omega)$ is defined to be the space of differential forms in $L^2\Lambda^k(\Omega)$ with a weak exterior derivative in $L^2\Lambda^{k+1}(\Omega)$ (Arnold *et al.*, 2006, page 17),

$$H\Lambda^k(\Omega) := \{\lambda^k \in L^2\Lambda^k(\Omega) \mid \mathrm{d}\lambda^k \in L^2\Lambda^{k+1}(\Omega)\}. \tag{2.8}$$

We also extend the inner product g defined in (2.4) to $L^2\Lambda^k(\Omega)$ for two k-forms α^k and β^k, Section 4, Arnold *et al.* (2006)

$$\langle \alpha^k, \beta^k \rangle_{L^2\Lambda^k(\Omega)} = g(\alpha^k, \beta^k) = \int_\Omega \alpha^k(\mu)\beta^k(\mu)\,\mathrm{d}\mu. \tag{2.9}$$

Using (2.9) and extending Definition 2.1 to $L^2\Lambda^k(\Omega)$, Section 4 (Arnold *et al.*, 2006), we obtain

$$\int_\Omega \lambda \wedge *\omega = g(\omega, \lambda), \qquad \lambda,\ \omega \in L^2\Lambda^k(\Omega). \tag{2.10}$$

Combining (2.9) and (2.10), we finally obtain

$$\langle \alpha^k, \beta^k \rangle_{L^2\Lambda^k(\Omega)} = g(\alpha^k, \beta^k) = \int_\Omega \alpha^k \wedge *\beta^k. \tag{2.11}$$

This also gives the Hilbert space $H\Lambda^k(\Omega)$ under the inner product, Section 4.2 (Arnold *et al.*, 2006)

$$\langle \alpha, \beta \rangle_{H\Lambda^k(\Omega)} = \langle \alpha, \beta \rangle_{L^2\Lambda^k(\Omega)} + \langle \mathrm{d}\alpha, \mathrm{d}\beta \rangle_{L^2\Lambda^{k+1}(\Omega)}. \tag{2.12}$$

The space $H\Lambda^0(\Omega)$ coincides with $H^1\Lambda^0(\Omega)$ (or simply $H^1(\Omega)$), while the space $H\Lambda^n(\Omega)$ coincides with $L^2\Lambda^n(\Omega)$. Since we deal with boundary controlled systems we need to extend the differential forms to domains with a boundary. Using the theory of trace operators in Sobolev spaces, we find that tr : $\Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$ extends by continuity to a mapping of $H^1\Lambda^k(\Omega)$ onto the space $H^{1/2}\Lambda^k(\partial\Omega)$. Next, consider the space $H\Lambda^k(\Omega)$. The trace of $\lambda \in H\Lambda^k(\Omega)$ is a bounded operator on $H\Lambda^k(\Omega)$ with values in $H^{-1/2}\Lambda^k(\partial\Omega)$. Given $\rho \in H^{1/2}\Lambda^k(\partial\Omega)$ the Hodge star of any k-form with respect to the boundary is given by $\bar{*}\rho \in H^{1/2}\Lambda^{n-k-1}(\partial\Omega)$. We can then state the integration by parts rule for $\lambda \in H\Lambda^k(\Omega)$, $\mu \in H^1\Lambda^{n-k-1}(\Omega)$ with $\mathrm{tr}\mu = \bar{*}\rho$ as

$$\langle \mathrm{d}\lambda \,|\, \mu \rangle_\Omega = \langle \lambda \,|\, \mu \rangle_{\partial\Omega} - (-1)^k \langle \lambda \,|\, \mathrm{d}\mu \rangle_\Omega, \tag{2.13}$$

where the pairing at the boundary is interpreted as the duality pairing between $H^{-1/2}\Lambda^k(\partial\Omega)$ and $H^{1/2}\Lambda^{n-k-1}(\partial\Omega)$ (Arnold *et al.*, 2006, page 19). Introducing the codifferential operator $\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$, which satisfies the relation $*\delta\lambda = (-1)^k d(*\lambda)$, $\lambda \in \Lambda^k(\Omega)$ (Arnold *et al.*, 2006, Eq. 2.7), we can convert the duality product into an inner product and by introducing $\mu = *\omega$ rewrite (2.13) as

$$(\mathrm{d}\lambda, \omega)_{L^2\Lambda^{k+1}(\Omega)} = \int_{\partial\Omega} \mathrm{tr}(\lambda) \wedge \mathrm{tr}(*\omega) + (\lambda, \delta\omega)_{L^2\Lambda^k(\Omega)}, \tag{2.14}$$

with $\lambda \in H\Lambda^k(\Omega)$ and $\omega \in H^1\Lambda^{k+1}(\Omega)$. Equation (2.14) directly shows that for compactly supported functions the weak exterior derivative in (2.7) is the adjoint of the codifferential.

3. Distributed port-Hamiltonian systems

3.1 *Dirac structure*

The concept of a Dirac structure, as introduced in Courant & Weinstein (1988); Courant (1990), is the geometrical notion formalizing general power-conserving interconnections, thereby allowing the Hamiltonian formulation of various kinds of combined systems. The definition of a Dirac structure is as follows. Let $\mathscr{F}$ and $\mathscr{E}$ be linear spaces, with a bilinear operation: $\mathscr{F} \times \mathscr{E} \to \mathbb{R}$. The bilinear product is denoted as $\langle e \,|\, f \rangle, f \in \mathscr{F}, e \in \mathscr{E}$. We call this bilinear pairing a nondegenerate duality pairing if $\langle e \,|\, f \rangle = 0$

for all $f \in \mathscr{F}$ implies $e = 0$ and if $\langle e \,|\, f \rangle = 0$ for all $e \in \mathscr{E}$ implies $f = 0$. By symmetrizing the pairing we get a symmetric bilinear pairing $\langle\langle\ \rangle\rangle$ on $\mathscr{F} \times \mathscr{E}$, with values in $\mathbb{R}$ given by

$$\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle = \langle e_1 \,|\, f_2 \rangle + \langle e_2 \,|\, f_1 \rangle, \qquad (f_i, e_i) \in \mathscr{F} \times \mathscr{E}. \tag{3.1}$$

A Dirac structure is a linear subspace $\mathscr{D} \subset \mathscr{F} \times \mathscr{E}$ such that $\mathscr{D} = \mathscr{D}^{\perp}$ with $\perp$ denoting the orthogonal complement with respect to the bilinear pairing $\langle\langle\ \rangle\rangle$ (van der Schaft *et al.*, 2014). From the above definition it follows that for any (f, e) in the Dirac structure $\mathscr{D}$, we have

$$0 = \langle\langle (f, e), (f, e) \rangle\rangle = 2\langle e | f \rangle. \tag{3.2}$$

Thus, if (f, e) is a pair of power variables, where $\langle e \,|\, f \rangle$ equals the power, then the condition $(f, e) \in \mathscr{D}$ implies power conservation $\langle e \,|\, f \rangle = 0$.

3.2 *The Stokes–Dirac structure*

The key concept in the treatment of the underlying geometric framework of a distributed-parameter port-Hamiltonian system with nonzero energy flow through the boundary is the introduction of a special type of Dirac structure. This structure is introduced by selecting suitable spaces of differential forms on the spatial domain and its boundary, making use of Stokes theorem.

We now define the linear space over the n-dimensional oriented manifold Ω

$$\mathscr{F}_{p,q} = L^2 \Lambda^p(\Omega) \times L^2 \Lambda^q(\Omega) \times H^{1/2} \Lambda^{n-p}(\partial\Omega), \tag{3.3}$$

for any pair p, q of non-negative integers satisfying

$$p + q = n + 1, \tag{3.4}$$

and correspondingly we define

$$\mathscr{E}_{p,q} = H^1 \Lambda^{n-p}(\Omega) \times H\Lambda^{n-q}(\Omega) \times H^{-1/2} \Lambda^{n-q}(\partial\Omega). \tag{3.5}$$

By Sections 2.1, 3.1 and (3.1) we have the symmetric nondegenerate bilinear pairing (using L^2 as the pivot space in the duality pairing)

$$\left\langle\left\langle \left(f_1^p, f_1^q, f_1^b, e_1^p, e_1^q, e_1^b\right), \left(f_2^p, f_2^q, f_2^b, e_2^p, e_2^q, e_2^b\right) \right\rangle\right\rangle$$
$$= \int_{\Omega} \left[e_1^p \wedge f_2^p + e_1^q \wedge f_2^q + e_2^p \wedge f_1^p + e_2^q \wedge f_1^q \right] + \int_{\partial\Omega} \left[e_1^b \wedge f_2^b + e_2^b \wedge f_1^b \right] \tag{3.6}$$

for $f_1^p, f_2^p \in L^2 \Lambda^p(\Omega)$, $f_1^q, f_2^q \in L^2 \Lambda^q(\Omega)$, $f_1^b, f_2^b \in H^{1/2} \Lambda^{n-p}(\partial\Omega)$ and $e_1^p, e_2^p \in H^1 \Lambda^{n-p}(\Omega)$, $e_1^q, e_2^q \in H\Lambda^{n-q}(\Omega)$, $e_1^b, e_2^b \in H^{-1/2} \Lambda^{n-q}(\partial\Omega)$. Note, using $H^1 \Lambda^{n-p}(\Omega) \subset L^2 \Lambda^{n-p}(\Omega)$ and $H\Lambda^{n-q}(\Omega) \subset L^2 \Lambda^{n-q}(\Omega)$ we have on Ω duality between $L^2 \Lambda^p(\Omega)$ and $L^2 \Lambda^{n-p}(\Omega)$, and between $L^2 \Lambda^q(\Omega)$ and $L^2 \Lambda^{n-q}(\Omega)$. At the boundary $\partial\Omega$ we have duality between $H^{1/2} \Lambda^{n-p}(\Omega)$ and $H^{-1/2} \Lambda^{n-q}(\Omega)$ using the relation $p + q = n + 1$. We consider in (3.6) Hodge duality on Ω and $\partial\Omega$. The analytic duality at $\partial\Omega$ is imposed since the trace of $H^1 \Lambda^{n-p}(\Omega)$ is $H^{1/2} \Lambda^{n-p}(\partial\Omega)$, which requires in the Dirac structure that $f^b \in H^{1/2}\Lambda^{n-p}(\partial\Omega)$. Similarly, the trace space of $H\Lambda^{n-q}(\Omega)$ is $H^{-1/2}\Lambda^{n-q}(\partial\Omega)$, which requires $e^b \in H^{-1/2}\Lambda^{n-q}(\partial\Omega)$. Analytic duality for the spaces $H^1\Lambda^{n-p}(\Omega) \times H\Lambda^{n-q}(\Omega)$ is not imposed and is not needed in this paper. We define the linear subspace $\mathscr{D}$ of $\mathscr{F}_{p,q} \times \mathscr{E}_{p,q}$ as

$$\mathscr{D} = \left\{ (f^p, f^q, f^b, e^p, e^q, e^b) \in \mathscr{F}_{p,q} \times \mathscr{E}_{p,q} \middle| \begin{bmatrix} f^p \\ f^q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{r_1} d \\ d & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} \mathrm{tr}(e^p) \\ \mathrm{tr}(e^q) \end{bmatrix} \right\}, \tag{3.7}$$

where d is the exterior derivative operator and r_1 is defined as

$$r_1 = pq + 1. \tag{3.8}$$

For smooth differential forms, it is proven in van der Schaft & Maschke (2002) that $\mathscr{D}$ is a Dirac structure. In Kumar *et al.* (in press) we will prove this also holds for the Sobolev spaces $\mathscr{F}_{p,q}$ and $\mathscr{E}_{p,q}$.

3.3 *Distributed-parameter port-Hamiltonian systems*

We consider the Hamiltonian density function

$$\mathscr{H} : L^2\Lambda^p(\Omega) \times L^2\Lambda^q(\Omega) \times \Omega \to L^2\Lambda^n(\Omega),$$

which results in the total Hamiltonian

$$H(\alpha^p, \alpha^q) = \int_\Omega \mathscr{H}(\alpha^p, \alpha^q, z) \in \mathbb{R}.$$

Following van der Schaft & Maschke (2002), if $\alpha^p, \partial\alpha^p \in L^2\Lambda^p(\Omega)$ and $\alpha^q, \partial\alpha^q \in L^2\Lambda^q(\Omega)$, then under weak smoothness conditions, we have

$$H(\alpha^p + \partial\alpha^p, \alpha^q + \partial\alpha^q) = \int_\Omega \mathscr{H}(\alpha^p, \alpha^q, z) + \left[\delta_p H \wedge \partial\alpha^p + \delta_q H \wedge \partial\alpha^q\right]$$
$$+ \text{higher order terms in } \partial\alpha^p, \partial\alpha^q, \tag{3.9}$$

for certain differential forms

$$\begin{aligned} \beta^p &= \delta_p H \in L^2\Lambda^{n-p}(\Omega), \\ \beta^q &= \delta_q H \in L^2\Lambda^{n-q}(\Omega), \end{aligned} \tag{3.10}$$

with $(\delta_p H, \delta_q H)$ the partial functional derivatives of H (Marsden & Ratiu, 2013). If we consider the time dependent function $(\alpha^p(t), \alpha^q(t)) \in L^2\Lambda^p(\Omega) \times L^2\Lambda^q(\Omega)$, where we assume that the coefficients are smooth functions of time, and the Hamiltonian $H(t) := H(\alpha^p(t), \alpha^q(t))$ is evaluated along this trajectory, then by the chain rule

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\Omega} \left[\beta^p \wedge \frac{\partial \alpha^p}{\partial t} + \beta^q \wedge \frac{\partial \alpha^q}{\partial t} \right]. \tag{3.11}$$

The differential forms $(\frac{\partial \alpha^p}{\partial t}, \frac{\partial \alpha^q}{\partial t})$ are the generalized velocities of the energy variables α^p, α^q.

DEFINITION 3.1 A **distributed-parameter port-Hamiltonian system** on an n-dimensional oriented manifold Ω, with state space $\mathscr{F}_{p,q} \times \mathscr{E}_{p,q}$ (with $p + q = n + 1$), Stokes–Dirac structure $\mathscr{D}$ stated in (3.7) and Hamiltonian H, is given by the conservation laws

$$\begin{aligned} \begin{bmatrix} -\frac{\partial \alpha^p}{\partial t} \\ -\frac{\partial \alpha^q}{\partial t} \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^{r_1} d \\ d & 0 \end{bmatrix} \begin{bmatrix} \beta^p \\ \beta^q \end{bmatrix}, \\ \begin{bmatrix} f^b \\ e^b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} \mathrm{tr}(\beta^p) \\ \mathrm{tr}(\beta^q) \end{bmatrix}, \end{aligned} \tag{3.12}$$

where $r_1 = pq + 1$ and β^p, β^q as in (3.10).

Comparing (3.7) and (3.12), we see that in (3.7) the relation

$$\begin{aligned} f^p &= -\frac{\partial \alpha^p}{\partial t}, \qquad e^p = \beta^p, \\ f^q &= -\frac{\partial \alpha^q}{\partial t}, \qquad e^q = \beta^q, \end{aligned} \tag{3.13}$$

has been substituted. By the power conservation property (3.2) of any Dirac structure it follows that for any $(f^p, f^q, f^b, e^p, e^q, e^b) \in \mathscr{F}_{p,q} \times \mathscr{E}_{p,q}$, in the Stokes–Dirac structure (3.7), there holds

$$\int_{\Omega} [e^p \wedge f^p + e^q \wedge f^q] + \int_{\partial\Omega} e^b \wedge f^b = 0. \tag{3.14}$$

Using (3.11), (3.13) and (3.14), we obtain

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial\Omega} e^b \wedge f^b,$$

which states that the increase in internal energy in the domain Ω is equal to the power supplied to the system through the boundary $\partial\Omega$.

4. Discontinuous finite element port-Hamiltonian system

In this section we extend the definition of a Stokes–Dirac structure to discontinuous finite element spaces. We start the discussion with the tessellation of an n-dimensional oriented manifold and later discuss the function spaces and duality relations between the port-variables. This provides us with the basic constituents to define a Stokes–Dirac structure for the discontinuous finite element setting, which we call generalized Stokes–Dirac structure.

4.1 *Tessellation*

A finite element can be defined as the triplet $(K, \mathscr{P}, \mathscr{N})$, where K is the element domain, $\mathscr{P}$ represents the basis functions for the (energy and co-energy) variables and $\mathscr{N}$ the nodal variables.

We start by introducing a tessellation $\mathscr{T}_h = \{K\}$ of an orientable manifold Ω with Lipschitz continuous boundary $\partial\Omega$ into shape-regular nonoverlapping simplicial elements (e.g., intervals, triangles, tetrahedra, etc.). Each $K \in \mathscr{T}_h$ is a bounded open set with a nonempty interior and piecewise smooth boundary, which inherits its orientation from Ω. We denote the set of interior faces in the tessellation as $\mathscr{F}_i$. Thus, $\Gamma_i \in \mathscr{F}_i$ if and only if $\exists K_L, K_R \in \mathscr{T}_h$ with $K_L \neq K_R$ such that $\Gamma_i = \bar{K}_L \cap \bar{K}_R \neq \emptyset$. Likewise, we denote the set of exterior faces in the tessellation with $\mathscr{F}_o$. Thus, $\Gamma_o \in \mathscr{F}_o$ if and only if $\exists K \in \mathscr{T}_h$ such that $\Gamma_0 = \bar{K} \cap \partial\Omega \neq \emptyset$. The set of all faces is $\mathscr{F}_h = \mathscr{F}_i \cup \mathscr{F}_o$. Let $\Delta(K)$ be the set of all subelements of element K, e.g., the element, its faces, edges and vertices. By the definition of an orientable manifold it follows that each element $K \in \mathscr{T}_h$ inherits the orientation of the manifold Ω such that all elements have a positive Jacobian. Following this orientation all internal faces connect to a right and a left element, whose orientation is in the opposite direction when viewed locally from the face. As we focus on simplices in this paper, we will refer to the elements $K \in \mathscr{T}_h$ as simplices and $\Delta(K)$ to be the set of all subsimplices of the simplex K. Note that the set of all subsimplices of simplex K, $\Delta(K)$, also includes the simplex K.

4.2 *Function spaces for port variables*

With the n-dimensional oriented manifold Ω discretized into simplices $K \in \mathscr{T}_h$, we can extend the function spaces (3.3) and (3.5) to broken function spaces for the port variables to be used in a discontinuous Galerkin discretization,

$$\begin{aligned} \mathscr{F}_h &= L^2\Lambda^p(\Omega) \times L^2\Lambda^q(\Omega) \times H^{1/2}\Lambda^{n-p}(\mathscr{F}_h), \\ \mathscr{E}_h &= H^1\Lambda^{n-p}(\mathscr{T}_h) \times H\Lambda^{n-q}(\mathscr{T}_h) \times H^{-1/2}\Lambda^{n-q}(\mathscr{F}_h) \end{aligned} \tag{4.1}$$

with the broken Sobolev spaces of differential forms

$$\begin{aligned} H^1\Lambda^k(\mathscr{T}_h) &:= \left\{\lambda_h^k \in L^2\Lambda^k(\Omega) \,\middle|\, \lambda_h^k\Big|_K \in H^1\Lambda^k(K) \quad \forall K \in \mathscr{T}_h\right\}, \\ H\Lambda^k(\mathscr{T}_h) &:= \left\{\lambda_h^k \in L^2\Lambda^k(\Omega) \,\middle|\, \lambda_h^k\Big|_K \in H\Lambda^k(K) \quad \forall K \in \mathscr{T}_h\right\}, \\ H^{1/2}\Lambda^k(\mathscr{F}_h) &:= \left\{\lambda_h^k \in L^2\Lambda^k(\mathscr{F}_h) \,\middle|\, \lambda_h^k\Big|_F \in H^{1/2}\Lambda^k(F) \quad \forall F \in \mathscr{F}_h\right\}, \\ H^{-1/2}\Lambda^k(\mathscr{F}_h) &:= \left\{\lambda_h^k \in L^2\Lambda^k(\mathscr{F}_h) \,\middle|\, \lambda_h^k\Big|_F \in H^{-1/2}\Lambda^k(F) \quad \forall F \in \mathscr{F}_h\right\}. \end{aligned} \tag{4.2}$$

Before defining the discontinuous finite element function spaces for the port variables, we will briefly describe the polynomial and broken polynomial spaces for differential forms. Let $\mathscr{P}_r(\mathbb{R}^n)$ and $\mathscr{H}_r(\mathbb{R}^n)$ denote, respectively, the spaces of polynomials in n variables of degree at most r and homogeneous polynomials of degree r. For more in depth understanding of the degree and basis of these polynomial spaces we refer to Sections 3 and 4 in Arnold *et al.* (2006). The space of all polynomials is $\mathscr{P}(\mathbb{R}^n) = \oplus_{r=0}^{\infty}\mathscr{H}_r(\mathbb{R}^n)$. The space of polynomial differential forms $\mathscr{P}_r\Lambda^k(\mathbb{R}^n)$ is defined to be the space of differential forms whose coefficient functions are from $\mathscr{P}_r(\mathbb{R}^n)$. Similarly, the space of homogeneous polynomial differential forms $\mathscr{H}_r\Lambda^k(\mathbb{R}^n)$ is the space of differential forms whose coefficient functions are from $\mathscr{H}_r(\mathbb{R}^n)$ (Arnold *et al.*, 2006; Arnold, 2013). For $r < 0$ the spaces of polynomial differential forms $\mathscr{P}_r\Lambda^k(\mathbb{R}^n)$ and $\mathscr{H}_r\Lambda^k(\mathbb{R}^n)$ are the zero space. For each polynomial degree $r \geq 0$, a homogeneous polynomial subcomplex of the de Rham complex (Arnold *et al.*, 2006, page 29) is

$$0 \longrightarrow \mathscr{H}_r\Lambda^0 \xrightarrow{d} \mathscr{H}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{H}_{r-n}\Lambda^n \longrightarrow 0, \tag{4.3}$$

which is exact, where the cohomology vanishes for $r < 0$ and also for $r = 0$ except for the lowest degree where the cohomology space is $\mathbb{R}$. Taking the direct sum over all polynomial with degree up to r of the homogeneous polynomial de Rham complex (4.3) gives the polynomial de Rham complex

$$\mathbb{R} \hookrightarrow \mathscr{P}_r\Lambda^0 \xrightarrow{d} \mathscr{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{P}_{r-n}\Lambda^n \longrightarrow 0, \tag{4.4}$$

which is exact for $r \geq 0$ (Arnold *et al.*, 2006, 2010). Using the polynomial differential form space $\mathscr{P}_r\Lambda^k(\mathbb{R}^n)$, the homogeneous polynomial differential form space $\mathscr{H}_r\Lambda^k(\mathbb{R}^n)$ and the Koszul differential operator κ (Arnold *et al.*, 2006; Arnold, 2013; Loday, 2013), we can define a third kind of polynomial de Rham complex as

$$\mathbb{R} \hookrightarrow \mathscr{P}_r^-\Lambda^0 \xrightarrow{d} \mathscr{P}_r^-\Lambda^1 \xrightarrow{d} \ldots. \xrightarrow{d} \mathscr{P}_r^-\Lambda^n \longrightarrow 0, \tag{4.5}$$

which is exact for $r > 0$ and where $\mathscr{P}_r^-\Lambda^k = \mathscr{P}_{r-1}\Lambda^k + \kappa\mathscr{H}_{r-1}\Lambda^{k+1}$ (Arnold *et al.*, 2006, 2010). For these polynomial differential form spaces we define the broken-polynomial differential form spaces as

$$\begin{aligned}
\mathscr{H}_r\Lambda^k(\mathscr{T}_h) &:= \left\{\lambda_h^k \in L^2\Lambda^k(\Omega) \,\middle|\, \lambda_h^k\big|_K \in \mathscr{H}_r\Lambda^k(K)\ \forall K \in \mathscr{T}_h\right\},\\
\mathscr{P}_r\Lambda^k\mathscr{T}_h) &:= \left\{\lambda_h^k \in L^2\Lambda^k(\Omega) \,\middle|\, \lambda_h^k\big|_K \in \mathscr{P}_r\Lambda^k(K)\ \forall K \in \mathscr{T}_h\right\},\\
\mathscr{P}_r^-\Lambda^k(\mathscr{T}_h) &:= \left\{\lambda_h^k \in L^2\Lambda^k(\Omega) \,\middle|\, \lambda_h^k\big|_K \in \mathscr{P}_r^-\Lambda^k(K)\ \forall K \in \mathscr{T}_h\right\},
\end{aligned} \tag{4.6}$$

with $\mathscr{H}_r\Lambda^k(K) = \mathscr{H}_r\Lambda^k(\mathbb{R}^n)|_K$, the restriction of $\mathscr{H}_r\Lambda^k(\mathbb{R}^n)$ to element $K \in \mathscr{T}_h$ and similar expressions for $\mathscr{P}_r\Lambda^k(K)$ and $\mathscr{P}_r^-\Lambda^k(K)$. For any subsimplex f of K we restrict the broken polynomial differential form spaces to f as

$$\begin{aligned}
\mathscr{H}_r\Lambda^k(f) &= \mathrm{tr}_{K,f}\mathscr{H}_r\Lambda^k(K),\\
\mathscr{P}_r\Lambda^k(f) &= \mathrm{tr}_{K,f}\mathscr{P}_r\Lambda^k(K),\\
\mathscr{P}_r^-\Lambda^k(f) &= \mathrm{tr}_{K,f}\mathscr{P}_r^-\Lambda^k(K),
\end{aligned} \tag{4.7}$$

where the notation $\mathrm{tr}_{K,f}$ represents the tangential trace at the subsimplex $f \in \Delta(K)$ of the polynomial differential forms defined on the simplex K. The restriction (4.7) is only valid if $k \leq \dim(f) \leq n$, because a k-form cannot be defined for subsimplices with $\dim(f) < k$. The restriction of the spaces on element K to the subsimplices $f \in \Delta(K)$ can be used to form the following de Rham complexes on the subsimplices

f of simplex K

$$0 \longrightarrow \mathscr{P}_r\Lambda^0(f) \xrightarrow{d} \mathscr{P}_{r-1}\Lambda^1(f) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{P}_{r-\dim(f)}\Lambda^{\dim(f)}(f) \longrightarrow 0, \; r \geq 0, \tag{4.8}$$

and

$$0 \longrightarrow \mathscr{P}_r^-\Lambda^0(f) \xrightarrow{d} \mathscr{P}_r^-\Lambda^1(f) \xrightarrow{d} .... \xrightarrow{d} \mathscr{P}_r^-\Lambda^{\dim(f)}(f) \longrightarrow 0, \; r > 0. \tag{4.9}$$

Consider a simplex $K \in \mathscr{T}_h$ and let $f \in \Delta(K)$. The dual of the polynomial differential form space $\mathscr{P}_r\Lambda^k(K)$ for $0 \leq k \leq n, r \geq 1$ is denoted by $\mathscr{P}_r\Lambda^k(K)^*$, and can be understood in the following manner. For $f \in \Delta(K)$ with $k \leq \dim(f) \leq \min(n, k+r-1)$, let η_f be an element of $\mathscr{P}^-_{r+k-\dim(f)}\Lambda^{\dim(f)-k}(f)$. Then the following defines a linear functional on $\mathscr{P}_r\Lambda^k(K)$

$$\phi(\omega) = \int_f \mathrm{tr}_{K,f}\omega \wedge \eta_f, \qquad \omega \in \mathscr{P}_r\Lambda^k(K). \tag{4.10}$$

Hence, for every f and η_f satisfying these conditions, we can identify a dual element of $\mathscr{P}_r\Lambda^k(K)$. From Theorems 4.8 and 4.10 (Arnold *et al.*, 2006), it follows that this forms the basis to represent all dual elements of $\mathscr{P}_r\Lambda^k(K)^*$ in a unique way, i.e.,

$$\mathscr{P}_r\Lambda^k(K)^* \cong \bigoplus_{f\in\Delta(K),\ \dim(f)\in[k,\ k+r-1]} \mathscr{P}^-_{r+k-\dim(f)}\Lambda^{\dim(f)-k}(f), \tag{4.11}$$

with $0 \leq k \leq n, \; r \geq 1$. So, any element $\mu_h^k \in \mathscr{P}_r\Lambda^k(K)^*$ can be written as a vector comprising of elements $\mu^k_{h,f} \in \mathscr{P}^-_{r+k-\dim(f)}\Lambda^{\dim(f)-k}(f)$ defined on the subelements $f \in \Delta(K)$, i.e.,

$$\mu_h^k = \left(\mu^k_{h,f_1}, \cdots, \mu^k_{h,f_m}\right), \quad f_1, \cdots, f_m \in \Delta(K). \tag{4.12}$$

Using the representation (4.12), for $\lambda_h^k \in \mathscr{P}_r\Lambda^k(K)$, $\mu_h^k \in \mathscr{P}_r\Lambda^k(K)^*$ and $\mu^k_{h,f} \in \mathscr{P}^-_{r+k-\dim(f)}\Lambda^{\dim(f)-k}(f)$, $0 \leq k \leq n, \; r \geq 1$, we define the duality product

$$\left\langle \lambda_h^k \mid \mu_h^k \right\rangle_K = \sum_{f\in\Delta(K),\ \dim(f)\in[k,\ k+r-1]} \int_f \mathrm{tr}_{K,f}\lambda_h^k \wedge \mu^k_{h,f}. \tag{4.13}$$

The dual of the polynomial differential form space $\mathscr{P}_r^-\Lambda^k(K)$ for $0 \leq k \leq n, r \geq 1$ is denoted by $\mathscr{P}_r^-\Lambda^k(K)^*$, and is constructed in a similar manner. For each $f \in \Delta(K)$ with $k \leq \dim(f) \leq \min(n, k + r-1)$, let η_f be an element of $\mathscr{P}_{r+k-\dim(f)-1}\Lambda^{\dim(f)-k}(f)$. Then the following defines a linear functional on $\mathscr{P}_r^-\Lambda^k(K)$

$$\phi(\omega) = \int_f \mathrm{tr}_{K,f}\omega \wedge \eta_f, \qquad \omega \in \mathscr{P}_r^-\Lambda^k(K). \tag{4.14}$$

Hence, for every f and η_f satisfying these conditions, we can identify a dual element of $\mathscr{P}_r^- \Lambda^k(K)$. From Theorem 4.14 (Arnold *et al.*, 2006), it follows that this forms the basis to represent all dual elements of $\mathscr{P}_r^- \Lambda^k(K)^*$ in a unique way,

$$\mathscr{P}_r^- \Lambda^k(K)^* \cong \bigoplus_{f\in\Delta(K),\dim(f)\in[k,\, k+r-1]} \mathscr{P}_{r+k-\dim(f)-1}\Lambda^{\dim(f)-k}(f). \tag{4.15}$$

We use this identification for the dual and the duality product on $\mathscr{P}_r^- \Lambda^k(K)$ for the port-Hamiltonian variables λ_h^k and μ_h^k. So similar to (4.13), for $\lambda_h^k \in \mathscr{P}_r^- \Lambda^k(K)$, $\mu_h^k \in \mathscr{P}_r^- \Lambda^k(K)^*$ and $\mu_{h,f}^k \in \mathscr{P}_{r+k-\dim(f)-1}\Lambda^{\dim(f)-k}(f)$, $0 \leq k \leq n,\ r \geq 1$, we have the duality product

$$\left\langle \lambda_h^k \mid \mu_h^k \right\rangle_K = \sum_{f\in\Delta(K),\dim(f)\in[k,\, k+r-1]} \int_f \mathrm{tr}_{K,f}\lambda_h^k \wedge \mu_{h,f}^k, \tag{4.16}$$

where $\mu_h^k = (\mu_{h,f_1}^k, \cdots, \mu_{h,f_m}^k), f_1, \cdots, f_m \in \Delta(K)$, as was explained in (4.12).

For $p + q = n + 1$ we now consider two cases for the pairs of port variables.

Case 1:

$$E_q(K) := \mathscr{P}_{r+1}\Lambda^{n-q}(K),\ F_p(K) := \mathscr{P}_r\Lambda^p(K). \tag{4.17}$$

These two spaces of polynomial differential forms are related via the exterior derivative operation

$$d_{pq} : E_q(K) \to F_p(K), \tag{4.18}$$

where d_{pq} is the (usual) exterior derivative for differential forms, see also (4.8). The dual of the differential form space F_p is denoted by E_p and the dual of the differential form space E_q is denoted by F_q. Using (4.11), the spaces E_p and F_q for Case 1 become

$$\begin{aligned} E_p(K) &:= \bigoplus_{f\in\Delta(K),\dim(f)\in[p,\, p+r-1]} \mathscr{P}^-_{r+p-\dim(f)}\Lambda^{\dim(f)-p}(f) \\ &=: \bigoplus_{f\in\Delta(K),\dim(f)\in[p,\, p+r-1]} E_p(f), \\ F_q(K) &:= \bigoplus_{f\in\Delta(K),\dim(f)\in[n-q,\, n-q+r]} \mathscr{P}^-_{r+1+n-q-\dim(f)}\Lambda^{\dim(f)-n+q}(f) \\ &=: \bigoplus_{f\in\Delta(K),\dim(f)\in[n-q,\, n-q+r]} F_q(f) \end{aligned} \tag{4.19}$$

with $0 \leq k \leq n,\ r \geq 1$.

We have a relationship between E_p and F_q via the exterior derivative operation, that is,

$$d_{qp} : E_p \to F_q, \tag{4.20}$$

where d_{qp} is the (usual) exterior derivative for differential forms applied on the subsimplices f of simplex K, see also (4.9).

Case 2:

$$E_q(K) := \mathscr{P}_r^- \Lambda^{n-q}(K),\ F_p(K) := \mathscr{P}_r^- \Lambda^p(K). \tag{4.21}$$

These two spaces of polynomial differential forms are related via the exterior derivative operation

$$d_{pq} : E_q(K) \to F_p(K), \tag{4.22}$$

where d_{pq} is the (usual) exterior derivative for differential forms, see also (4.9). The dual of the differential form space F_p is denoted by E_p and the dual of the differential form space E_q is denoted by F_q. Using (4.15), the spaces E_p and F_q become

$$\begin{aligned}
E_p(K) &:= \bigoplus_{f\in\Delta(K),\dim(f)\in[p,\, p+r-1]} \mathscr{P}_{r+p-\dim(f)-1}\Lambda^{\dim(f)-p}(f)\\
&=: \bigoplus_{f\in\Delta(K),\dim(f)\in[p,\, p+r-1]} E_p(f),\\
F_q(K) &:= \bigoplus_{f\in\Delta(K),\dim(f)\in[n-q,\, n-q+r-1]} \mathscr{P}_{r+n-q-\dim(f)-1}\Lambda^{\dim(f)-n+q}(f)\\
&=: \bigoplus_{f\in\Delta(K),\dim(f)\in[n-q,\, n-q+r-1]} F_q(f).
\end{aligned} \tag{4.23}$$

We have a relationship between E_p and F_q via the exterior derivative operation, that is,

$$d_{qp} : E_p \to F_q, \tag{4.24}$$

where d_{qp} is the (usual) exterior derivative for differential forms applied on the subsimplices f of simplex K, see also (4.8).

4.3 *Duality product on element K for discrete dual port-variable pairs*

The port-variables have a well-defined nondegenerate duality product (2.1). We extend the duality product now to the discrete dual port-variable pairs. On each subsimplex $f \in \Delta(K)$, we have a duality product between $e^p_{h,f} \in E_p(f)$ and $\mathrm{tr}_{K,f} f^p_h \in \mathrm{tr}_{K,f} F_p(K)$. Summing over all $f \in \Delta(K)$ and using the fact that $\{E_p(f), \mathrm{tr}_{K,f} F_p(K)\}$ are dual pairs, we have

$$\langle e^p_h \mid f^p_h \rangle_K = (-1)^{p(n-p)} \langle f^p_h \mid e^p_h \rangle = (-1)^{p(n-p)} \sum_{f\in\Delta(K)} \int_f \mathrm{tr}_{K,f} f^p_h \wedge e^p_{h,f}, \tag{4.25}$$

where $e^p_h = (e^p_{h,f_1}, \cdots, e^p_{h,f_m}), f_1, \cdots, f_m \in \Delta(K)$, as explained in (4.12).

Using Theorem 4.8 in Arnold *et al.* (2006), we can prove that if

$$\langle e^p_h \mid f^p_h \rangle_K = (-1)^{p(n-p)} \sum_{f\in\Delta(K)} \int_f \mathrm{tr}_{K,f} f^p_h \wedge e^p_{h,f} = 0 \quad \forall e^p_{h,f} \in E_p(f), \tag{4.26}$$

then $f^p_h = 0$.

Similarly, on each subsimplex $f \in \varDelta(K)$ we have a duality product between $\mathrm{tr}_{K,f} e_h^q \in \mathrm{tr}_{K,f} E_q(K)$ and $f_{h,f}^q \in F_q(f)$. Summing over all $f \in \varDelta(K)$ and using the fact that $\{\mathrm{tr}_{K,f} E_q(K), F_q(f)\}$ are dual pairs we have that if

$$\langle e_h^q \mid f_h^q \rangle_K = \sum_{f\in\varDelta(K)} \int_f \mathrm{tr}_{K,f} e_h^q \wedge f_{h,f}^q = 0 \qquad \forall f_{h,f}^q \in F_q(f), \tag{4.27}$$

where $f_h^q = (f_{h,f_1}^q, \cdots, f_{h,f_m}^q), f_1, \cdots, f_m \in \varDelta(K)$, as explained in (4.12), then $e_h^q = 0$, see Theorem 4.12 in Arnold *et al.* (2006). Hence, on element K there exists a nondegenerate duality product between $E_p(K)$ and $F_p(K)$ and a nondegenerate duality product between $E_q(K)$ and $F_q(K)$, given by (4.13) and (4.16).

4.4 *Duality product on subelements of element K for discrete dual port-variable pairs*

The trace operator $\mathrm{tr}_{f,\partial f}$ restricts the spaces defined on the subsimplices $f \in \varDelta(K)$ to the boundaries ∂f. For example, A $(k-1)$-form, $\lambda_{h,f}$ defined on a subsimplex $f \in \varDelta(K)$ with $k \leq \dim(f) \leq n$, is restricted on the boundary ∂f as

$$\lambda_{h,f}^k(\partial f) = \mathrm{tr}_{f,\partial f} \lambda_{h,f}^k. \tag{4.28}$$

We stated the duality product over the element K as the sum of duality products over the subsimplices $f \in \varDelta(K)$. Similarly, we state the duality product over the boundaries of K as the sum of duality products over the boundaries ∂f of $f \in \varDelta(K)$, which are denoted as $\partial\varDelta(K)$. Considering the theory of port-Hamiltonian systems, at the boundaries $\partial\varOmega$ of the oriented manifold $\varOmega$ we have using the Dirac structure (3.7) duality between the trace of efforts at the boundaries. Hence, at the boundaries ∂f of each $f \in \varDelta(K)$ we have to express the duality between $E_p(\partial\varDelta(K))$ and $E_q(\partial\varDelta(K))$. Since the spaces $E_p(\partial\varDelta(K))$ and $E_q(\partial\varDelta(K))$ will in general not have the same dimension we need to do this in two steps. First, we define a bilinear product between $E_p(\partial\varDelta(K))$ and $E_q(\partial\varDelta(K))$. Next, we will introduce a conversion operator Q that accounts for the differences in the dimension of $E_p(\partial\varDelta(K))$ and $E_q(\partial\varDelta(K))$. Using (4.28) and Theorems 4.15 and 4.21 in Arnold *et al.* (2006), these two spaces at the boundary ∂f of $f \in \varDelta(K)$ are,

$$\begin{aligned} E_p(\partial\varDelta(K)) &= \bigoplus_{f\in\varDelta(K),\dim(f)\in[p,r+p-1]} \mathrm{tr}_{f,\partial f} E_p(f) \\ E_q(\partial\varDelta(K)) &= \bigoplus_{f\in\varDelta(K),\dim(f)\in[p,r+p-1]} \mathrm{tr}_{K,\partial f} E_q(K). \end{aligned} \tag{4.29}$$

Using the relation $p + q = n + 1$, these spaces for Case 1 and Case 2 then become
Case 1:

$$\begin{aligned} E_p(\partial\varDelta(K)) &= \bigoplus_{f\in\varDelta(K),\dim(f)\in[p,r+p-1]} \mathrm{tr}_{f,\partial f} \mathscr{P}_{r+p-\dim(f)}^- \varLambda^{\dim(f)-p}(f), \\ E_q(\partial\varDelta(K)) &= \bigoplus_{f\in\varDelta(K),\dim(f)\in[p,r+p-1]} \mathrm{tr}_{K,\partial f} \mathscr{P}_{r+1} \varLambda^{n-q}(K). \end{aligned} \tag{4.30}$$

Case 2:

$$\begin{aligned} E_p(\partial\varDelta(K)) &= \bigoplus_{f\in\varDelta(K),\dim(f)\in[p,r+p-1]} \mathrm{tr}_{f,\partial f} \mathscr{P}_{r+p-\dim(f)-1} \varLambda^{\dim(f)-p}(f), \\ E_q(\partial\varDelta(K)) &= \bigoplus_{f\in\varDelta(K),\dim(f)\in[p,r+p-1]} \mathrm{tr}_{K,\partial f} \mathscr{P}_r^- \varLambda^{n-q}(K). \end{aligned} \tag{4.31}$$

The bilinear product between $u_h^p \in E_p(\partial\Delta(K)), u_h^q \in E_q(\partial\Delta(K))$ is stated as

$$\begin{aligned}\langle u_h^q \mid u_h^p \rangle_{\partial K} &= \langle \mathrm{tr} e_h^q \mid \mathrm{tr} e_h^p \rangle_{\partial K} \\ &= \sum_{f\in\Delta(K),\dim(f)\in[p,r+p-1]} \int_{\partial f} \mathrm{tr}_{K,\partial f} e_h^q \wedge \mathrm{tr}_{f,\partial f} e_{h,f}^p,\end{aligned} \tag{4.32}$$

with $e_h^q \in E_p(K), e_h^p \in E_p(K), e_{h,f}^p \in E_p(f)$ and e_h^p can be represented as $e_h^p = (e_{h,f_1}^p, \cdots, e_{h,f_m}^p)$, $f_1, \cdots, f_m \in \Delta(K)$ as explained in (4.12).

In general, the function spaces $E_p(\partial\Delta(K))$ and $E_q(\partial\Delta(K))$ do not have the same dimensions. This poses the crucial problem for the bilinear product (4.32) to be nondegenerate. Hence, in general (4.32) is a bilinear product, but not a duality product. To convert this bilinear product into a duality product we will use a linear (surjective) operator Q. We will discuss the general construction of the operator Q in the next subsection. Note that from now on with an abuse of notation we will write $\partial\Delta(K)$ as ∂K.

4.4.1 *The conversion operator Q.* Let us take two finite-dimensional linear spaces $X(\partial K)$ and $Y(\partial K)$ such that $\dim(X(\partial K)) \geq \dim(Y(\partial K))$. Suppose that, as in (4.32), we have the bilinear product between $X(\partial K)$ and $Y(\partial K)$, $\langle x \mid y\rangle_{\partial K}$, $x \in X(\partial K)$ and $y \in Y(\partial K)$. For fixed $x \in X(\partial K)$, $q_x(y) = \langle x \mid y\rangle_{\partial K}$, is a linear map from $Y(\partial K)$ to $\mathbb{R}$. Thus, there exists a unique $y^* \in Y^*(\partial K)$, ($Y^*(\partial K)$ is the dual function space to $Y(\partial K)$) such that $q_x(y) = \langle y^* \mid y\rangle_{Y(\partial K)^*\times Y(\partial K)}$. So, to $x \in X(\partial K)$ we have uniquely assigned a $y^* \in Y^*(\partial K)$. This we can do for all $x \in X(\partial K)$, and we define the operator Q,

$$Q : X(\partial K) \to Y(\partial K)^*, \tag{4.33}$$

such that $Q(x) = y^*$, and thus

$$q_x(y) = \langle x \mid y\rangle_{\partial K} = \langle Q(x) \mid y\rangle_{Y(\partial K)^*\times Y(\partial K)}. \tag{4.34}$$

Since, y^* is unique the operator Q is well-defined. Also, for all $y \in Y(\partial K)$

$$\begin{aligned} q_{\alpha x+\beta\tilde{x}}(y) &= \langle \alpha x + \beta\tilde{x} \mid y\rangle_{\partial K} \\ &= \alpha\langle x \mid y\rangle_{\partial K} + \beta\langle \tilde{x} \mid y\rangle_{\partial K} \\ &= \alpha\langle Q(x) \mid y\rangle_{Y(\partial K)^*\times Y(\partial K)} + \beta\langle Q(\tilde{x}) \mid y\rangle_{Y(\partial K)^*\times Y(\partial K)}. \end{aligned} \tag{4.35}$$

On the other hand,

$$q_{\alpha x+\beta\tilde{x}}(y) = \langle Q(\alpha x + \beta\tilde{x}) \mid y\rangle_{Y(\partial K)^*\times Y(\partial K)}. \tag{4.36}$$

Thus, Q is a linear operator.

DEFINITION 4.1 The bilinear product $\langle x \mid y\rangle_{\partial K}$ is defined to be half-degenerate if for any $y \in Y(\partial K)$ the assertion: $\langle x \mid y\rangle_{\partial K} = 0\ \forall x \in X(\partial K)$ implies that $y = 0$.

LEMMA 4.2 The operator Q of (4.34) is surjective if and only if the bilinear product $\langle x \mid y\rangle_{\partial K}$ is half-degenerate.

Proof. Assume that Q is surjective and let $y_0 \in Y(\partial K)$ be such that $\langle x \mid y_0 \rangle_{\partial K} = 0$, $\forall x \in X(\partial K)$. This gives that $\langle y^* \mid y_0 \rangle_{Y(\partial K)^* \times Y(\partial K)} = 0$, $\forall y^* \in Y(\partial K)^*$, which implies $y_0 = 0$. Now we prove the other implication. Assume that Q is not surjective. Since the range of Q is a linear subspace of $Y(\partial K)^*$ and $Y(\partial K)$ has the dual $Y(\partial K)^*$, there exists a nonzero $y_0 \in Y(\partial K)$ such that $\langle Q(x) \mid y_0 \rangle_{Y(\partial K)^* \times Y(\partial K)} = 0$, $\forall x \in X(\partial K)$. By (4.34) we see that $\langle x \mid y_0 \rangle_{\partial K} = 0, \forall x \in X(\partial K)$ for a nonzero y_0, providing the contradiction. □

As the operator Q converts the bilinear product on $X(\partial K) \times Y(\partial K)$ into a duality product on $Y(\partial K)^* \times Y(\partial K)$, we can define dual operators with respect to the bilinear form $\langle \cdot \mid \cdot \rangle_{\partial K}$. Let S be a linear operator that maps the space $X(K)$ to the space $X(\partial K)$, i.e., $S : X(K) \to X(\partial K)$. So for $x = S(x_h)$, with $x_h \in X(K)$, and $y \in Y(\partial K)$, we have

$$\langle x \mid y \rangle_{\partial K} = \langle S(x_h) \mid y \rangle_{\partial K}. \tag{4.37}$$

Using the operator Q, (4.34),

$$\begin{aligned} \langle S(x_h) \mid y \rangle_{\partial K} &= \langle QS(x_h) \mid y \rangle_{Y(\partial K)^* \times Y(\partial K)} \\ &= \left\langle x_h \mid (QS)^*(y) \right\rangle_{X(K) \times X(K)^*}. \end{aligned} \tag{4.38}$$

Here, $(QS)^* : Y(\partial K) \to X(K)^*$ denotes the (standard) dual of QS. Similarly, let T be an operator that maps the space $Y(K)$ to the space $Y(\partial K)$, i.e., $T : Y(K) \to Y(\partial K)$. So for $y = T(y_h)$, with $y_h \in Y(K)$, and $x \in X(\partial K)$, we have

$$\langle x \mid y \rangle_{\partial K} = \langle x \mid T(y_h) \rangle_{\partial K}. \tag{4.39}$$

Using the operator Q, (4.34),

$$\begin{aligned} \langle x \mid T(y_h) \rangle_{\partial K} &= \langle Q(x) \mid T(y_h) \rangle_{Y(\partial K)^* \times Y(\partial K)} \\ &= \langle T^* Q(x) \mid y_h \rangle_{Y(K)^* \times Y(K)}. \end{aligned} \tag{4.40}$$

Here, $(T^* Q) : X(\partial K) \to Y(K)^*$.

The operator Q to define a nondegenerate duality product at the boundary ∂K of K is only used in the Dirac structure for theoretical reasons. In the implementation it is, however, sufficient to use the degenerate version of the duality product for the duality pairing.

4.4.2 *Duality on the boundaries of K.* We assume that $\dim(E_q(\partial K)) \geq \dim(E_p(\partial K))$. Note that this is just an assumption it can very well happen that $\dim(E_q(\partial K)) \leq \dim(E_p(\partial K))$, but the analysis is similar for this case.

On $E_q(\partial K) \times E_p(\partial K)$ we have the bilinear product (4.32). Let $Q : E_q(\partial K) \to E_p(\partial K)^*$ be the mapping associated to this product, see (4.33) and (4.34).

The trace operator tr_p is the map $\mathrm{tr}_p : E_p(K) \to E_p(\partial K)$ and the trace operator tr_q is the map $\mathrm{tr}_q : E_q(K) \to E_q(\partial K)$. For port variables $e_h^p \in E_p(K)$ and $e_h^q \in E_q(K)$, we have

$$\begin{aligned} \left\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p e_h^p \right\rangle_{\partial K} &= \left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\ &= \left\langle \mathrm{tr}_p^* \left(Q\left(\mathrm{tr}_q e_h^q\right)\right) \mid e_h^p \right\rangle_{E_p(K)^* \times E_p(K)}. \end{aligned} \tag{4.41}$$

Here, $(\mathrm{tr}_p^* \circ Q) : E_q(\partial K) \to E_p(K)^*$.

Similarly,

$$\left\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p e_h^p \right\rangle_{\partial K} = \left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)}$$
$$= \left\langle e_h^q \mid (Q\mathrm{tr}_q)^* \left(\mathrm{tr}_p e_h^p\right)\right\rangle_{E_q(K)\times E_q(K)^*}. \tag{4.42}$$

Here, $(Q \circ \mathrm{tr}_q)^* : E_p(\partial K) \to E_q(K)^*$.

LEMMA 4.3 For $\mathrm{tr}_p e_h^p \in E_p(\partial K)$ and $\mathrm{tr}_q e_h^q \in E_q(\partial K)$, the bilinear product $\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p e_h^p\rangle_{\partial K}$ is half-degenerate.

Proof. Using the relation $\dim(E_q(\partial K)) \geq \dim(E_p(\partial K))$ and the fact that the wedge product on $\partial f \subseteq \partial K$ is nondegenerated, we find with (4.32) that if $\mathrm{tr}_p e_h^p \in E_p(\partial K)$ is such that $\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p e_h^p\rangle_{\partial K} = 0$ for all $\mathrm{tr}_q e_h^q \in E_q(\partial K)$, then $\mathrm{tr}_p e_h^p = 0$. Hence, $\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p e_h^p\rangle_{\partial K}$ is half-degenerate. □

Lemmas 4.2 and 4.3 imply that the operator Q is surjective.

LEMMA 4.4 For $e_h^p, v_h^p \in E_p(K)$, $v_{h,f}^p \in E_p(f)$ and $e_h^q, v_h^q \in E_q(K)$ with $p + q = n + 1$, the integration by parts rule (2.13) for an element K gives

$$\left\langle \mathrm{d}_{pq} e_h^q \mid v_h^p \right\rangle_K = (-1)^p \left\langle e_h^q \mid \mathrm{d}_{qp} v_h^p \right\rangle_K + \left\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p v_h^p \right\rangle_{\partial K}. \tag{4.43}$$

Proof. For $e_h^q \in E_q(K), v_h^p \in E_p(K)$, using the exterior derivative operator $d_{pq}(e_h^q(K)) \in F_p(K)$ as the exterior derivative operator operator d_{pq} is defined as the map $d_{pq} : E_q(K) \to F_p(K)$. Using the duality product between $E_p(K)$ and $F_p(K)$, (4.13) or (4.16), we write

$$\left\langle \mathrm{d}_{pq} e_h^q \mid v_h^p \right\rangle_K = \sum_{f \in \Delta(K), p \leq \dim(f) \leq r+p-1} \int_f \mathrm{tr}_{K,f}\left(\mathrm{d}e_h^q\right) \wedge v_{h,f}^p, \tag{4.44}$$

where v_h^p can be represented as $v_h^p = (v_{h,f_1}^p, \cdots, v_{h,f_m}^p), f_1, \cdots, f_m \in \Delta(K)$, as explained in (4.12). As the exterior derivative operator d and the trace operator tr are commutative, we can simplify the above as

$$\left\langle \mathrm{d}_{pq} e_h^q \mid v_h^p \right\rangle_K = \sum_{f \in \Delta(K), p \leq \dim(f) \leq r+p-1} \int_f \mathrm{d}(\mathrm{tr}_{K,f} e_h^q) \wedge v_{h,f}^p. \tag{4.45}$$

Using (2.13) and $E_q(K) \subset \Lambda^{n-k}(K)$ in the first step and then (4.13), (4.32), we obtain

$$\left\langle \mathrm{d}_{pq} e_h^q \mid v_h^p \right\rangle_K = \sum_{f \in \Delta(K), p \leq \dim(f) \leq r+p-1} \left((-1)^p \left(\int_f \mathrm{tr}_{K,f} e_h^q \wedge \mathrm{d}v_{h,f}^p \right) \right.$$
$$\left. + \int_{\partial f} \mathrm{tr}_{K,\partial f} e_h^q \wedge \mathrm{tr}_{f,\partial f} v_{h,f}^p \right)$$
$$= (-1)^p \left\langle e_h^q \mid d_{qp} v_h^p \right\rangle_K + \left\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p v_h^p \right\rangle_{\partial K}. \tag{4.46}$$

□

Using the duality pairings (4.13), (4.16) and (4.32), we define a symmetric bilinear product for discrete port variables.

DEFINITION 4.5 (Extended bilinear form). Let K be an element of $\mathscr{T}_h$. For $\{f_h^p, f_h'^p\} \in F_p(K)$, $\{f_h^q, f_h'^q\} \in F_q(K)$, $\{e_h^p, e_h'^p\} \in E_p(K)$, $\{e_h^q, e_h'^q\} \in E_q(K)$, the symmetric extended bilinear form is defined as

$$\begin{aligned} &\left\langle\!\left\langle \left(f_h^p, f_h^q, e_h^p, e_h^q\right), \left(f_h'^p, f_h'^q, e_h'^p, e_h'^q\right) \right\rangle\!\right\rangle_h \\ &= \left\langle e_h'^p \mid f_h^p \right\rangle_K + \left\langle e_h^p \mid f_h'^p \right\rangle_K + \left\langle e_h'^q \mid f_h^q \right\rangle_K + \left\langle e_h^q \mid f_h'^q \right\rangle_K \\ &\quad + \left\langle \mathrm{tr} e_h'^q \mid \mathrm{tr} e_h^p \right\rangle_{\partial K} + \left\langle \mathrm{tr} e_h^q \mid \mathrm{tr} e_h'^p \right\rangle_{\partial K} + \left\langle \mathrm{tr} e_h'^p \mid \mathrm{tr} e_h^q \right\rangle_{\partial K} + \left\langle \mathrm{tr} e_h^p \mid \mathrm{tr} e_h'^q \right\rangle_{\partial K}. \end{aligned} \tag{4.47}$$

4.5 *Discontinuous Finite Element Dirac structure*

In the discontinuous finite element discretization we consider elements to be independent from each other. The interaction occurs only through common boundaries. We define now a Dirac structure, called generalized Stokes–Dirac structure that is suitable for a discontinuous finite element discretization. We also state the interconnection between adjacent elements in the spatial domain using a Dirac structure at the element boundaries, called interconnection Dirac structure. This will provide the mathematical framework for defining port-Hamiltonian discontinuous Galerkin discretizations.

4.5.1 *The generalized Stokes–Dirac structure.* The polynomial differential form spaces on which the port-variables are projected are denoted as

$$\begin{aligned} F_{p,q} &:= F_p(K) \times F_q(K) \times E_p(\partial K) \times E_q(\partial K), \\ E_{p,q} &:= E_p(K) \times E_q(K) \times E_q(\partial K) \times E_p(\partial K). \end{aligned} \tag{4.48}$$

For port variables $e_h^p \in E_p(K), e_h^q \in E_q(K), f_h^p \in F_p(K), f_h^q \in F_q(K)$ and the input and output port boundary pairs $\{y_h^p, u_h^p\} \in E_p(\partial K), \{y_h^q, u_h^q\} \in E_q(\partial K)$, we have the following bilinear product:

$$\langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \langle u_h^q \mid y_h^p \rangle_{\partial K} + \langle y_h^q \mid u_h^p \rangle_{\partial K}. \tag{4.49}$$

However since, $\dim(E_p(\partial K)) \neq \dim(E_q(\partial K))$, the bilinear product (4.49) is degenerated. As explained in subsection 4.3.4 we use the linear (surjective) operator Q, (4.34) to transform this bilinear product into a nondegenerate duality product. Recall that we assume $\dim(E_q(\partial K)) \geq \dim(E_p(\partial K))$. Using the abstract operator Q, (4.34), we can map the boundary port variable space $E_q(\partial K)$ to the dual of $E_p(\partial K)$, while maintaining the bilinear product. That is, for each $u_h^q \in E_q(\partial K)$, we have an element $u_h^{p*} \in E_p(\partial K)^*$, such that $Q(u_h^q) = u_h^{p*}$ and $\langle u_h^q \mid y_h^p \rangle_{\partial K} = \langle Q(u_h^q) \mid y_h^p \rangle_{E_p(\partial K)^* \times E_p(\partial K)}$. Using this we can rewrite the bilinear product (4.49) as

$$\begin{aligned} &\langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \langle u_h^q \mid y_h^p \rangle_{\partial K} + \langle y_h^q \mid u_h^p \rangle_{\partial K} \\ &= \langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \langle Q\left(u_h^q\right) \mid y_h^p \rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\ &\quad + \langle Q\left(y_h^q\right) \mid u_h^p \rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\ &= \langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\ &\quad + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)}. \end{aligned} \tag{4.50}$$

We replace the polynomial differential form spaces, (4.48), with the (abstract) spaces

$$\begin{aligned} F'_{p,q} &:= F_p(K) \times F_q(K) \times E_p(\partial K) \times E_p(\partial K)^*, \\ E'_{p,q} &:= E_p(K) \times E_q(K) \times E_p(\partial K)^* \times E_p(\partial K). \end{aligned} \tag{4.51}$$

Definition 4.6 For the port variables $e_h^p \in E_p(K), e_h^q \in E_q(K), f_h^p \in F_p(K), f_h^q \in F_q(K)$, the input and output port boundary pairs $\{y_h^p, u_h^p\} \in E_p(\partial K)$, $\{y_h^{p*}, u_h^{p*}\} \in E_p(\partial K)^*$, we have the following duality product on $F'_{p,q} \times E'_{p,q}$

$$\langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)}. \tag{4.52}$$

Using the duality product (4.52) as the power we can now define the generalized Stokes–Dirac structure.

Definition 4.7 (Generalized Stokes–Dirac Structure). For $K \in \mathscr{T}_h$, we define the subspace D_h of $F'_{p,q} \times E'_{p,q}$ as

$$\begin{aligned} D_h = \Bigg\{ & \left(f_h^p, f_h^q, y_h^p, y_h^{p*}, e_h^p, e_h^q, u_h^{p*}, u_h^p\right) \in F'_{p,q} \times E'_{p,q} \mid \\ \begin{bmatrix} f_h^p \\ f_h^q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^{r_1} d_{pq} + (-1)^{r_1+q} \frac{1}{2} \mathrm{tr}_p^* Q(\mathrm{tr}_q) \\ d_{qp} + (-1)^p \frac{1}{2} (Q \mathrm{tr}_q)^* \mathrm{tr}_p & 0 \end{bmatrix} \begin{bmatrix} e_h^p \\ e_h^q \end{bmatrix} \\ &+ \begin{bmatrix} 0 & (-1)^{r_1+q} \mathrm{tr}_p^* \\ (-1)^p (Q \mathrm{tr}_q)^* & 0 \end{bmatrix} \begin{bmatrix} u_h^p \\ u_h^{p*} \end{bmatrix}, \\ \begin{bmatrix} y_h^{p*} \\ y_h^p \end{bmatrix} &= \begin{bmatrix} 0 & -Q(\mathrm{tr}_q) \\ -(-1)^p \mathrm{tr}_p & 0 \end{bmatrix} \begin{bmatrix} e_h^p \\ e_h^q \end{bmatrix} \Bigg\}, \end{aligned} \tag{4.53}$$

where $p+q = n+1, r_1 = pq+1, d_{pq}$ and d_{qp} as defined in (4.18) and (4.20), the trace operation defined as in (4.28) and the operator Q defined as in (4.33), (4.34).

We now prove that the generalized Stokes–Dirac structure given in the definition above is indeed a Dirac structure. For that we use the following lemma. Although the statement in the following lemma is well known, it is difficult to find the proof, so we state the proof as well.

Lemma 4.8 Consider the finite dimensional linear spaces F_h and E_h, having a nondegenerate bilinear product defined on $F_h \times E_h$. A subspace $D_h \in F_h \times E_h$ is a Dirac structure if and only if the power is conserved and $\dim(D_h) = \dim(F_h)$.

Proof. Given, $\langle e \mid f \rangle = 0, \forall (f, e) \in D_h$ and $\dim(D_h) = \dim(F_h)$, we have to prove that D_h is a Dirac structure, i.e., $D_h = (D_h)^\perp$. And, conversely if D_h is a Dirac structure, i.e., $D_h = D_h{}^\perp$, we have to prove that $\dim(D_h) = \dim(F_h)$. Suppose $\forall (f, e) \in D_h$, we have

$$\langle\langle (f, e), (f, e) \rangle\rangle_h = \langle e \mid f \rangle + \langle e \mid f \rangle = 2\langle e \mid f \rangle = 0, \tag{4.54}$$

then using (3.2) $D_h \subseteq (D_h)^\perp$. Let n be the dimension of F_h and let $\{[f_1, e_1]^T, ..., [f_n, e_n]^T\}$ be a basis of D_h. For $[\tilde{f}, \tilde{e}]^T \in D_h^\perp$ it implies

$$\langle\langle (f, e), (\tilde{f}, \tilde{e})\rangle\rangle_h = \langle e_k \mid \tilde{f}\rangle + \langle \tilde{e} \mid f_k\rangle = 0, \qquad k = 1, \cdots, n. \tag{4.55}$$

This gives n independent linear equations to be solved in a $2n$ dimensional space. This implies that the solution set is of $2n - n = n$ dimensions. So, $\dim(D_h^\perp) = n$. Since, $\dim(D_h) = \dim(E_h) = \dim(F_h) = n$ and $D_h \subseteq (D_h)^\perp$, we have $D_h = (D_h)^\perp$. Conversely, if D_h is a Dirac structure, then $D_h = (D_h)^\perp$, and (3.2) implies in particular that

$$\langle e \mid f\rangle = 0, \quad \forall (f, e) \in D_h. \tag{4.56}$$

Let $\{[f_1, e_1]^T, ..., [f_m, e_m]^T\}$ be a basis of D_h, for $[\tilde{f}, \tilde{e}]^T \in (D_h)^\perp$ it implies

$$\langle e_k \mid \tilde{f}\rangle + \langle \tilde{e} \mid f_k\rangle = 0, \qquad k = 1, \cdots, m. \tag{4.57}$$

This gives m independent linear equations to be solved in a $2n$ dimensional space. This implies that the solution set is of $2n - m$ dimensions. So, $\dim(D^\perp) = 2n - m$. Since, $D_h = (D_h)^\perp$, we have $2n - m = n$, or $m = n$. This proves that $\dim(D_h)^\perp = \dim(D_h) = \dim(F_h) = n$. □

Lemma 4.8 states the conditions for a linear subspace to be a Dirac structure. Using this lemma we prove now that the subspace (4.53) is a Dirac structure. Along with Lemma 4.8, we state two more lemmas, which will be used to prove that the structure (4.53) is a Dirac structure.

LEMMA 4.9 For port variables $e_h^p \in E_p(K), e_h^q \in E_q(K), f_h^p \in F_p(K), f_h^q \in F_q(K)$, input and output port boundary pairs $\{y_h^p, u_h^p\} \in E_p(\partial K)$, $\{y_h^{p*}, u_h^{p*}\} \in E_p(\partial K)^*$ the following holds. If

$$\begin{aligned} f_h^p &= \left((-1)^{r_1} d_{pq} + (-1)^{r_1+q}\frac{1}{2}\mathrm{tr}_p^* Q(\mathrm{tr}_q)\right) e_h^q + (-1)^{r_1+q}\mathrm{tr}_p^* u_h^{p*}, \\ f_h^q &= \left(d_{qp} + (-1)^p \frac{1}{2}(Q\mathrm{tr}_q)^*\mathrm{tr}_p\right) e_h^p + (-1)^p (Q\mathrm{tr}_q)^* u_h^p, \end{aligned} \tag{4.58}$$

then for $v_h^p \in E_p(K)$, $v_h^q \in E_q(K)$,

$$\begin{aligned} \langle v_h^p \mid f_h^p\rangle_K &= -\left\langle e_h^q \mid d_{qp} v_h^p\right\rangle_K + (-1)^{p+1}\frac{1}{2}\left\langle Q(\mathrm{tr}_q e_h^q) \mid \mathrm{tr}_p v_h^p\right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\ &\quad + (-1)^p \left\langle u_h^{p*} \mid \mathrm{tr}_p v_h^p\right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\ \langle v_h^q \mid f_h^q\rangle_K &= (-1)^p \left\langle d_{pq} v_h^q \mid e_h^p\right\rangle_K + (-1)^{p+1}\frac{1}{2}\left\langle Q\left(\mathrm{tr}_q v_h^q\right) \mid \mathrm{tr}_p e_h^p\right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\ &\quad + (-1)^p \left\langle Q\mathrm{tr}_q\left(v_h^q\right) \mid u_h^p\right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \end{aligned} \tag{4.59}$$

with $p + q = n + 1$ and $r_1 = pq + 1$.

Proof. For the test differential forms $\nu_h^p \in E_p(K)$ and $\nu_h^q \in E_q(K)$, we have

$$\begin{aligned}\langle \nu_h^p \mid f_h^p\rangle_K &= \left\langle \nu_h^p \mid \left((-1)^{r_1} d_{pq} + (-1)^{r_1+q}\frac{1}{2}\mathrm{tr}_p^* Q(\mathrm{tr}_q)\right) e_h^q\right\rangle_K + (-1)^{r_1+q}\left\langle \nu_h^p \mid \mathrm{tr}_p^* u_h^{p*}\right\rangle_K \\ \langle \nu_h^q \mid f_h^q\rangle_K &= \left\langle \nu_h^q \mid \left(d_{qp} + (-1)^p\frac{1}{2}(Q\mathrm{tr}_q)^*\mathrm{tr}_p\right) e_h^p\right\rangle_K + (-1)^p\left\langle \nu_h^q \mid (Q\mathrm{tr}_q)^* u_h^p\right\rangle_K .\end{aligned} \tag{4.60}$$

Using (4.41) and (4.42), (4.60) can be written as

$$\begin{aligned}\langle \nu_h^p \mid f_h^p\rangle_K &= (-1)^{r_1}\left\langle \nu_h^p \mid d_{pq}e_h^q\right\rangle_K + (-1)^{r_1+q}\frac{1}{2}\left\langle \mathrm{tr}_p \nu_h^p \mid Q\left(\mathrm{tr}_q e_h^q\right)\right\rangle_{E_p(\partial K)\times E_p(\partial K)^*} \\ &\quad + (-1)^{r_1+q}\left\langle \mathrm{tr}_p \nu_h^p \mid u_h^{p*}\right\rangle_{E_p(\partial K)\times E_p(\partial K)^*} \\ \langle \nu_h^q \mid f_h^q\rangle_K &= \left\langle \nu_h^q \mid d_{qp}e_h^p\right\rangle_K + (-1)^p\frac{1}{2}\left\langle Q\left(\mathrm{tr}_q \nu_h^q\right) \mid \mathrm{tr}_p e_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} \\ &\quad + (-1)^p\left\langle Q\mathrm{tr}_q\left(\nu_h^q\right) \mid u_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} .\end{aligned} \tag{4.61}$$

Using the relation $p+q=n+1$, we rewrite (4.61) as

$$\begin{aligned}\langle \nu_h^p \mid f_h^p\rangle_K &= (-1)^{p+1}\left\langle d_{pq}e_h^q \mid \nu_h^p\right\rangle_K + (-1)^p\frac{1}{2}\left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p \nu_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} \\ &\quad + (-1)^p\left\langle u_h^{p*} \mid \mathrm{tr}_p \nu_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} \\ \langle \nu_h^q \mid f_h^q\rangle_K &= \left\langle \nu_h^q \mid d_{qp}e_h^p\right\rangle_K + (-1)^p\frac{1}{2}\left\langle Q\left(\mathrm{tr}_q \nu_h^q\right) \mid \mathrm{tr}_p e_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} \\ &\quad + (-1)^p\left\langle Q\mathrm{tr}_q\left(\nu_h^q\right) \mid u_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} .\end{aligned} \tag{4.62}$$

Using integration by parts rule, (4.43) and the relation $p+q=n+1$,

$$\begin{aligned}\langle \nu_h^p \mid f_h^p\rangle_K &= -\left\langle e_h^q \mid d_{qp}\nu_h^p\right\rangle_K + (-1)^{p+1}\left\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p \nu_h^p\right\rangle_{\partial K} \\ &\quad + (-1)^p\frac{1}{2}\left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p \nu_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} \\ &\quad + (-1)^p\left\langle u_h^{p*} \mid \mathrm{tr}_p \nu_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} \\ \langle \nu_h^q \mid f_h^q\rangle_K &= (-1)^p\left\langle d_{pq}\nu_h^q \mid e_h^p\right\rangle_K + (-1)^{p+1}\left\langle \mathrm{tr}_q \nu_h^q \mid \mathrm{tr}_p e_h^p\right\rangle_{\partial K} \\ &\quad + (-1)^p\frac{1}{2}\left\langle Q\left(\mathrm{tr}_q \nu_h^q\right) \mid \mathrm{tr}_p e_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} \\ &\quad + (-1)^p\left\langle Q\mathrm{tr}_q\left(\nu_h^q\right) \mid u_h^p\right\rangle_{E_p(\partial K)^*\times E_p(\partial K)} .\end{aligned}$$

Using (4.34), we get (4.59). □

THEOREM 4.10 The structure (4.53) is a Dirac structure, which implies that

$$\langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} = 0. \quad (4.63)$$

Proof. The power for the Dirac structure (4.53) is

$$\langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)}. \quad (4.64)$$

Using Lemma 4.9 and replacing $v_h^p \in E_p, v_h^q \in E_q$ with $e_h^p \in E_p, e_h^q \in E_q$, we get

$$\begin{aligned}
&\langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&= -\left\langle e_h^q \mid d_{qp} e_h^p \right\rangle_K + (-1)^{p+1} \frac{1}{2} \left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + (-1)^p \left\langle u_h^{p*} \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + (-1)^p \left\langle d_{pq} e_h^q \mid e_h^p \right\rangle_K + (-1)^{p+1} \frac{1}{2} \left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + (-1)^p \left\langle Q \mathrm{tr}_q \left(e_h^q\right) \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)}.
\end{aligned} \quad (4.65)$$

Using (4.43), we can further simplify

$$\begin{aligned}
&-\left\langle e_h^q \mid d_{qp} e_h^p \right\rangle_K + (-1)^{p+1} \frac{1}{2} \left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + (-1)^p \left\langle u_h^{p*} \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + (-1)^p \left\langle d_{pq} e_h^q \mid e_h^p \right\rangle_K + (-1)^{p+1} \frac{1}{2} \left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + (-1)^p \left\langle Q \mathrm{tr}_q \left(e_h^q\right) \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + \langle y_h^{p*} \mid u_h^p \rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \langle u_h^{p*} \mid y_h^p \rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&= (-1)^p \left\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p e_h^p \right\rangle_{\partial K} + (-1)^{p+1} \left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + (-1)^p \left\langle u_h^{p*} \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + (-1)^p \left\langle Q \mathrm{tr}_q \left(e_h^q\right) \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)}.
\end{aligned}$$

Using (4.34), we can further simplify as follows

$$\begin{aligned}
&(-1)^p \left\langle \mathrm{tr}_q e_h^q \mid \mathrm{tr}_p e_h^p \right\rangle_{\partial K} + (-1)^{p+1} \left\langle Q\left(\mathrm{tr}_q e_h^q\right) \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + (-1)^p \left\langle u_h^{p*} \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + (-1)^p \left\langle Q\mathrm{tr}_q\left(e_h^q\right) \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&= (-1)^p \left\langle u_h^{p*} \mid \mathrm{tr}_p e_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + (-1)^p \left\langle Q\mathrm{tr}_q\left(e_h^q\right) \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)}.
\end{aligned} \tag{4.66}$$

Using the relations $y_h^{p*} = -Q(\mathrm{tr}_q(e_h^q)), y_h^p = -(-1)^p \mathrm{tr}_p e_h^p$ from (4.53) in (4.66), we obtain

$$\begin{aligned}
&- \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} - \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} = 0.
\end{aligned}$$

Furthermore, because of the duality $\dim(E'_{p,q}) = \dim(F'_{p,q}) = \dim(D_h)$. Using Lemma 4.8, it follows that (4.53) is a Dirac structure. □

Thus, for the Stokes–Dirac structure (4.53), we obtain

$$\langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} = 0. \tag{4.67}$$

Hence power is conserved.

Along with the elementwise Stokes–Dirac structure of Definition 4.7, we need a power preserving interconnection structure at the boundaries of the elements so that we can connect all elements in the discretized manifold. We use the power preserving property of a Dirac structure and define a Dirac structure at the faces of the elements. We call such a Dirac structure an interconnection Dirac Structure.

4.5.2 *The interconnection Dirac structure.* Let $\{K_1, K_2\} \in \mathscr{T}_h$ be two arbitrary elements sharing a common boundary $\partial K_1 \cap \partial K_2$. Using (4.29), we have

$$\begin{aligned}
E_p(\partial K_1) = \Big\{ &\left(\mathrm{tr}_{f_1,\partial f_1} e_{h,f_1}^p, \mathrm{tr}_{f_2,\partial f_2} e_{h,f_2}^p, \cdots, \mathrm{tr}_{f_m,\partial f_m} e_{h,f_m}^p\right) \mid f_i \in \Delta(K_1), \\
&\dim(f_i) \in [p, r+p-1], e_{h,f_i}^p \in E_p(f) \Big\} \\
E_p(\partial K_2) = \Big\{ &\left(\mathrm{tr}_{f_1,\partial f_1} e_{h,f_1}^p, \mathrm{tr}_{f_2,\partial f_2} e_{h,f_2}^p, \cdots, \mathrm{tr}_{f_m,\partial f_m} e_{h,f_m}^p\right) \mid f_i \in \Delta(K_2), \\
&\dim(f_i) \in [p, r+p-1], e_{h,f_i}^p \in E_p(f) \Big\},
\end{aligned} \tag{4.68}$$

with m being the number of subsimplices in $\partial\varDelta(K)$. As only the common boundaries between two elements will be connected through the interconnection Dirac Structure we split the function spaces $E_p(\partial K_1)$ and $E_p(\partial K_2)$ as

$$\begin{aligned} E_p(\partial K_1) &= E_p(\varDelta_{1\cap 2}) \oplus E_p(\varDelta_{1\neg 2}), \\ E_p(\partial K_2) &= E_p(\varDelta_{1\cap 2}) \oplus E_p(\varDelta_{2\neg 1}), \end{aligned} \tag{4.69}$$

where $\varDelta_{1\cap 2} := \partial K_1 \cap \partial K_2$, $\varDelta_{1\neg 2} = \partial K_1 - (\partial K_1 \cap \partial K_2)$ and $\varDelta_{2\neg 1} = \partial K_2 - (\partial K_1 \cap \partial K_2)$. We can then define

$$Z := E_p(\varDelta_{1\cap 2}) = \left(\left(w^p_{h,L} \in E_p(\partial K_1)|_f, w^p_{h,R} \in E_p(\partial K_2)|_f\right) \mid f \in \varDelta_{1\cap 2}\right), \tag{4.70}$$

and its dual function space as

$$Z^* := E_p(\varDelta_{1\cap 2})^* = \left(\left(w^{p*}_{h,L} \in E_p(\partial K_1)^*|_f, w^{p*}_{h,R} \in E_p(\partial K_2)^*|_f\right) \mid f \in \varDelta_{1\cap 2}\right). \tag{4.71}$$

We choose the polynomial differential form space for the *interconnection* Dirac structure to be

$$Z_T := Z \times Z^*. \tag{4.72}$$

The dual of Z_T, $Z^*_T = Z^* \times Z$. For, $((w^p_{h,L}, w^p_{h,R}), (w^{p*}_{h,L}, w^{p*}_{h,R})) \in Z_T$ and $((y^{p*}_{h,L}, y^{p*}_{h,R}), (y^p_{h,L}, y^p_{h,R})) \in Z^*_T$, the duality product on the space $Z_T \times Z^*_T$ becomes

$$\left\langle y^{p*}_{h,L} \mid w^p_{h,L}\right\rangle_{Z^*\times Z} + \left\langle w^{p*}_{h,L} \mid y^p_{h,L}\right\rangle_{Z^*\times Z} + \left\langle y^{p*}_{h,R} \mid w^p_{h,R}\right\rangle_{Z^*\times Z} + \left\langle w^{p*}_{h,R} \mid y^p_{h,R}\right\rangle_{Z^*\times Z}. \tag{4.73}$$

Based on the duality product (4.73) we can now state the interconnection Dirac structure.

DEFINITION 4.11 (Interconnection Dirac Structure). For the boundary port variables, $((w^p_{h,L}, w^p_{h,R}), (w^{p*}_{h,L}, w^{p*}_{h,R})) \in Z_T$ and $((y^{p*}_{h,L}, y^{p*}_{h,R}), (y^p_{h,L}, y^p_{h,R})) \in Z^*_T$,

$$D_c = \left\{ \left(w^p_{h,L}, w^p_{h,R}, w^{p*}_{h,L}, w^{p*}_{h,R}, y^{p*}_{h,L}, y^{p*}_{h,R}, y^p_{h,L}, y^p_{h,R}\right) \in (Z_T \times Z^*_T) \mid \right.$$
$$\left. \begin{bmatrix} w^p_{h,L} \\ w^{p*}_{h,L} \\ w^p_{h,R} \\ w^{p*}_{h,R} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}+\theta & 0 & -\theta \\ \frac{1}{2}-\theta & 0 & \theta-1 & 0 \\ 0 & 1-\theta & 0 & \theta-\frac{1}{2} \\ \theta & 0 & \frac{1}{2}-\theta & 0 \end{bmatrix} \begin{bmatrix} y^{p*}_{h,L} \\ y^p_{h,L} \\ y^{p*}_{h,R} \\ y^p_{h,R} \end{bmatrix} \right\}, \tag{4.74}$$

where $\theta \in [0, 1]$.

THEOREM 4.12 The interconnection structure (4.74) is a Dirac structure.

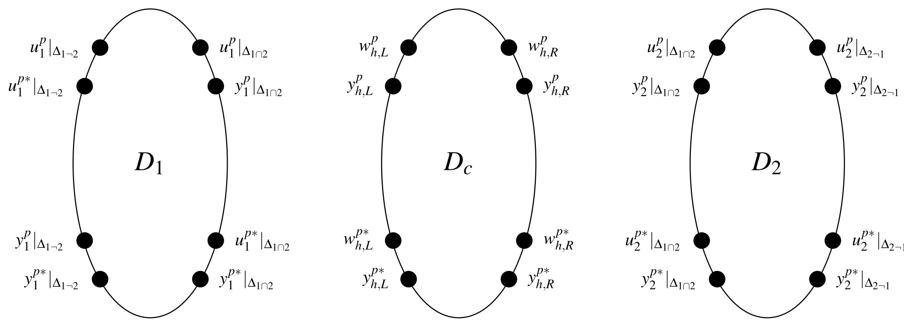

FIG. 1. Coupling of two Dirac structures via the interconnection Dirac structure D_c.

Proof. Using Lemma 4.8 we will be able to prove that (4.74) is a Dirac structure if the power of (4.74) is zero and the dimensions of the spaces are equal. The power of the Dirac structure, (4.74) is given by

$$
\begin{aligned}
&\left\langle y^{p*}_{h,L} \mid w^{p}_{h,L}\right\rangle_{Z^*\times Z} + \left\langle w^{p*}_{h,L} \mid y^{p}_{h,L}\right\rangle_{Z^*\times Z} + \left\langle y^{p*}_{h,R} \mid w^{p}_{h,R}\right\rangle_{Z^*\times Z} \\
&\quad + \left\langle w^{p*}_{h,R} \mid y^{p}_{h,R}\right\rangle_{Z^*\times Z} = \left(-\frac{1}{2}+\theta\right)\left\langle y^{p*}_{h,L} \mid y^{p}_{h,L}\right\rangle_{Z^*\times Z} - \theta\left\langle y^{p*}_{h,L} \mid y^{p}_{h,R}\right\rangle_{Z^*\times Z} \\
&\quad + \left(\frac{1}{2}-\theta\right)\left\langle y^{p*}_{h,L} \mid y^{p}_{h,L}\right\rangle_{Z^*\times Z} + (\theta-1)\left\langle y^{p*}_{h,R} \mid y^{p}_{h,L}\right\rangle_{Z^*\times Z} \\
&\quad + (1-\theta)\left\langle y^{p*}_{h,R} \mid y^{p}_{h,L}\right\rangle_{Z^*\times Z} + \left(\theta-\frac{1}{2}\right)\left\langle y^{p*}_{h,R} \mid y^{p}_{h,R}\right\rangle_{Z^*\times Z} \\
&\quad + \theta\left\langle y^{p*}_{h,L} \mid y^{p}_{h,R}\right\rangle_{Z^*\times Z} + \left(\frac{1}{2}-\theta\right)\left\langle y^{p*}_{h,R} \mid y^{p}_{h,R}\right\rangle_{Z^*\times Z} \\
&= 0. \qquad (4.75)
\end{aligned}
$$

Moreover, the dual of the function space Z_T, $Z_T{}^*$ is the same function space Z_T so $\dim(Z_T) = \dim(Z_T{}^*) = \dim(D_c)$. Hence, (4.74) is a Stokes–Dirac structure. □

Any skew symmetric matrix represents a power conserving interconnection, but the matrix (4.74) is chosen such that a power conserving numerical flux is obtained in the DG discretization. It is not necessary to ensure that w^p_h, y^{p*}_h and w^{p*}_h, y^p_h are isomorphic. If we ensure a connection between w^p_h and y^{p*}, and analogously w^{p*} and y^p_h so that they are isomorphic at the discrete level we will obtain a different connection structure and hence a different form of discontinuous Galerkin discretization.

5. Choice of interconnection variable

In this section, we prove that upon identification of the correct interconnection port variables, the coupling of discrete Dirac structures results again in a discrete Dirac structure. The interconnection is diagrammatically shown in Fig. 1.

Let us consider two simplices $K_1, K_2 \in \mathscr{T}_h$ with a common boundary. The conservation laws on K_1 are described by the port variables $(f_1^p, f_1^q, y_1^p, y_1^{p*}, e_1^p, e_1^q, u_1^{p*}, u_1^p) \in D_1 \subset F'_{p,q}(K_1) \times E'_{p,q}(K_1)$ and the conservation laws on K_2 by the port variables $(f_2^p, f_2^q, y_2^p, y_2^{p*}, e_2^p, e_2^q, u_2^{p*}, u_2^p) \in D_2 \subset F'_{p,q}(K_2) \times E'_{p,q}(K_2)$, respectively. We connect these two simplices through their common boundary $\Delta_{1\cap 2}$. For simplicity of notation we use

$$Z := E_p(\Delta_{1\cap 2}), \qquad Z_1 := E_p(\Delta_{1\neg 2}), \qquad Z_2 := E_p(\Delta_{2\neg 1}). \tag{5.1}$$

For $F'_{p,q}(K_1) := F_p(K_1) \times F_q(K_1) \times E_p(\partial K_1) \times E_p(\partial K_1)^*$ and $E'_{p,q}(K_1) := E_p(K_1) \times E_q(K_1) \times E_p(\partial K_1)^* \times E_p(\partial K_1)$ the generalized Stokes–Dirac structure D_1 is defined as in Definition 4.7 and the interconnection Dirac structure D_c is defined as in Definition 4.11. Interconnecting D_1 and D_c, we define the structure $D_1 \circ D_c$ on $F'_{p,q}(K_1) \times E'_{p,q}(K_1)$,

$$\begin{aligned}
D_1 \circ D_c = \Bigg\{ \Bigg(& f_1^p, f_1^q, \left(y_1^p|_{\Delta_{1\neg 2}},\, y_{h,R}^p\right)^T, \left(y_1^{p*}|_{\Delta_{1\neg 2}},\, y_{h,R}^{p*}\right)^T, e_1^p, e_1^q, \\
& \left(u_1^{p*}|_{\Delta_{1\neg 2}},\, -w_{h,R}^{p*}\right)^T, \left(u_1^p|_{\Delta_{1\neg 2}},\, -w_{h,R}^p\right)^T \Bigg) \in F'_{p,q}(K_1) \times E'_{p,q}(K_1) \mid \\
& \exists\, u_1^p|_{\Delta_{1\cap 2}}, u_1^{p*}|_{\Delta_{1\cap 2}}, y_1^p|_{\Delta_{1\cap 2}}, y_1^{p*}|_{\Delta_{1\cap 2}} \text{ s.t. with} \\
& y_1^{p*} = \left(y_1^{p*}|_{\Delta_{1\neg 2}},\, y_1^{p*}|_{\Delta_{1\cap 2}}\right)^T, y_1^p = \left(y_1^p|_{\Delta_{1\neg 2}},\, y_1^p|_{\Delta_{1\cap 2}}\right)^T, \\
& u_1^{p*} = \left(u_1^{p*}|_{\Delta_{1\neg 2}},\, u_1^{p*}|_{\Delta_{1\cap 2}}\right)^T, u_1^p = \left(u_1^p|_{\Delta_{1\neg 2}},\, u_1^p|_{\Delta_{1\cap 2}}\right)^T, \\
& \text{holds } \left(f_1^p, f_1^q, y_1^p, y_1^{p*}, e_1^p, e_1^q, u_1^{p*}, u_1^p\right) \in D_1 \text{ and } \Big(u_1^p|_{\Delta_{1\cap 2}}, w_{h,R}^p, \\
& u_1^{p*}|_{\Delta_{1\cap 2}}, w_{h,R}^{p*}, y_1^{p*}|_{\Delta_{1\cap 2}}, y_{h,R}^{p*}, y_1^p|_{\Delta_{1\cap 2}}, y_{h,R}^p\Big) \in D_c \Bigg\}.
\end{aligned} \tag{5.2}$$

Since each subdomain has an orientation induced by the orientation of Ω, the minus sign in front of $w_{h,R}^{p*}$ and $w_{h,R}^p$ is due to the fact that at the connection of two elements at a shared edge or face the outer normal of the connected elements have opposite signs. So when connecting D_1 and D_c, we basically take in Fig. 1 $u_1^{p*}|_{\Delta_{1\cap 2}} = w_{h,L}^{p*}$, $u_1^p|_{\Delta_{1\cap 2}} = w_{h,L}^p$, $y_1^{p*}|_{\Delta_{1\cap 2}} = y_{h,L}^{p*}$ and $y_1^p|_{\Delta_{1\cap 2}} = y_{h,L}^p$. The power of $D_1 \circ D_c$ is given by

$$\begin{aligned}
& \langle e_1^p \mid f_1^p \rangle_{K_1} + \langle e_1^q \mid f_1^q \rangle_{K_1} + \left\langle y_1^{p*} \mid u_1^p \right\rangle_{Z_1^* \times Z_1} + \left\langle u_1^{p*} \mid y_1^p \right\rangle_{Z_1^* \times Z_1} \\
& - \left\langle y_{h,R}^{p*} \mid w_{h,R}^p \right\rangle_{Z^* \times Z} - \left\langle w_{h,R}^{p*} \mid y_{h,R}^p \right\rangle_{Z^* \times Z}.
\end{aligned} \tag{5.3}$$

LEMMA 5.1 The structure $D_1 \circ D_c$, (5.2) with power given by (5.3) is a Dirac structure.

Proof. Using Theorem 4.10 we can state

$$\begin{aligned}
& \langle e_1^p \mid f_1^p \rangle_{K_1} + \langle e_1^q \mid f_1^q \rangle_{K_1} + \left\langle y_1^{p*} \mid u_1^p \right\rangle_{Z_1^* \times Z_1} + \left\langle u_1^{p*} \mid y_1^p \right\rangle_{Z_1^* \times Z_1} \\
& + \left\langle y_1^{p*} \mid u_1^p \right\rangle_{Z^* \times Z} + \left\langle u_1^{p*} \mid y_1^p \right\rangle_{Z^* \times Z} = 0.
\end{aligned} \tag{5.4}$$

We can rewrite (5.4) as

$$\langle e_1^p \mid f_1^p\rangle_{K_1} + \langle e_1^q \mid f_1^q\rangle_{K_1} + \left\langle y_1^{p*} \mid u_1^p\right\rangle_{Z_1^*\times Z_1} + \left\langle u_1^{p*} \mid y_1^p\right\rangle_{Z_1^*\times Z_1}$$
$$+ \left\langle y_{h,L}^{p*} \mid w_{h,L}^p\right\rangle_{Z^*\times Z} + \left\langle w_{h,L}^{p*} \mid y_{h,L}^p\right\rangle_{Z^*\times Z} = 0. \tag{5.5}$$

Using the power preserving property of interconnection Dirac structure D_c, Theorem 4.12, we have

$$\left\langle y_{h,L}^{p*} \mid w_{h,L}^p\right\rangle_{Z^*\times Z} + \left\langle w_{h,L}^{p*} \mid y_{h,L}^p\right\rangle_{Z^*\times Z} = -\left\langle y_{h,R}^{p*} \mid w_{h,R}^p\right\rangle_{Z^*\times Z} - \left\langle w_{h,R}^{p*} \mid y_{h,R}^p\right\rangle_{Z^*\times Z}.$$

Thus, (5.5) finally becomes

$$\langle e_1^p \mid f_1^p\rangle_{K_1} + \langle e_1^q \mid f_1^q\rangle_{K_1} + \left\langle y_1^{p*} \mid u_1^p\right\rangle_{Z_1^*\times Z_1} + \left\langle u_1^{p*} \mid y_1^p\right\rangle_{Z_1^*\times Z_1}$$
$$- \left\langle y_{h,R}^{p*} \mid w_{h,R}^p\right\rangle_{Z^*\times Z} - \left\langle w_{h,R}^{p*} \mid y_{h,R}^p\right\rangle_{Z^*\times Z} = 0. \tag{5.6}$$

Hence, $D_1 \circ D_c$ is a Dirac structure. □

Similarly, the generalized Stokes–Dirac structure D_2 can be defined as in Definition 4.7 on $F'_{p,q}(K_2) \times E'_{p,q}(K_2)$. Interconnecting $D_1 \circ D_c$ and D_2, we define the structure $D_1 \circ D_c \circ D_2$ on $F'_{p,q}(K_1 \cup K_2) \times E'_{p,q}(K_1 \cup K_2)$

$$\begin{aligned} D_1 \circ D_c \circ D_2 =\Big\{ \Big(& (f_1^p, f_2^p)^T, (f_1^q, f_2^q)^T, (y_1^p|_{\Delta_{1\neg 2}}, y_2^p|_{\Delta_{2\neg 1}})^T, \left(y_1^{p*}|_{\Delta_{1\neg 2}}, y_2^{p*}|_{\Delta_{2\neg 1}}\right)^T, \\ & (e_1^p, e_2^p)^T, (e_1^q, e_2^q)^T, \left(u_1^{p*}|_{\Delta_{1\neg 2}}, u_2^{p*}|_{\Delta_{2\neg 1}}\right)^T, \\ & (u_1^p|_{\Delta_{1\neg 2}}, u_2^p|_{\Delta_{2\neg 1}})^T \Big) \in F'_{p,q}(K_1 \cup K_2) \times E'_{p,q}(K_1 \cup K_2) \mid \\ & \exists\, u_2^p|_{\Delta_{1\cap 2}}, u_2^{p*}|_{\Delta_{1\cap 2}}, y_2^p|_{\Delta_{1\cap 2}}, y_2^{p*}|_{\Delta_{1\cap 2}} \text{ s.t. with} \\ & y_2^{p*} = \left(y_2^{p*}|_{\Delta_{2\neg 1}}, y_2^{p*}|_{\Delta_{1\cap 2}}\right)^T, y_2^p = \left(y_2^p|_{\Delta_{2\neg 1}}, y_2^p|_{\Delta_{1\cap 2}}\right)^T, \\ & u_2^{p*} = \left(u_2^{p*}|_{\Delta_{2\neg 1}}, u_2^{p*}|_{\Delta_{1\cap 2}}\right)^T, u_2^p = \left(u_2^p|_{\Delta_{2\neg 1}}, u_2^p|_{\Delta_{1\cap 2}}\right)^T, \\ & \text{holds } \Big(f_1^p, f_1^q, (y_1^p|_{\Delta_{1\neg 2}}, y_2^p|_{\Delta_{1\cap 2}})^T, (y_1^{p*}|_{\Delta_{1\neg 2}}, y_2^{p*}|_{\Delta_{1\cap 2}})^T, e_1^p, e_1^q, \\ & (u_1^{p*}|_{\Delta_{1\neg 2}}, u_2^{p*}|_{\Delta_{1\cap 2}})^T, (u_1^p|_{\Delta_{1\neg 2}}, u_2^p|_{\Delta_{1\cap 2}})^T\Big) \in D_1 \circ D_c \text{ and } (f_2^p, f_2^q, \\ & y_2^p, y_2^{p*}, e_2^p, e_2^q, u_2^{p*}, u_2^p) \in D_2 \Big\}. \end{aligned} \tag{5.7}$$

So when connecting $D_1 \circ D_c$ and D_2, we basically take in Fig. 1 $u_2^{p*}|_{\Delta_{1\cap 2}} = w_{h,R}^{p*}$, $u_2^p|_{\Delta_{1\cap 2}} = w_{h,R}^p$, $y_2^{p*}|_{\Delta_{1\cap 2}} = y_{h,R}^{p*}$ and $y_2^p|_{\Delta_{1\cap 2}} = y_{h,R}^p$. The power of $D_1 \circ D_c \circ D_2$ is given by

$$\langle e_1^p \mid f_1^p\rangle_{K_1} + \langle e_1^q \mid f_1^q\rangle_{K_1} + \langle e_2^p \mid f_2^p\rangle_{K_2} + \langle e_2^q \mid f_2^q\rangle_{K_2} + \left\langle y_1^{p*} \mid u_1^p\right\rangle_{Z_1^*\times Z_1}$$
$$+ \left\langle u_1^{p*} \mid y_1^p\right\rangle_{Z_1^*\times Z_1} + \left\langle y_2^{p*} \mid u_2^p\right\rangle_{Z_2^*\times Z_2} + \left\langle u_2^{p*} \mid y_2^p\right\rangle_{Z_2^*\times Z_2}. \tag{5.8}$$

LEMMA 5.2 The structure $D_1 \circ D_c \circ D_2$, (5.7) with power given by (5.8) is a Dirac structure.

Proof. The proof is similar to the proof of Lemma 5.1. □

6. Galerkin formulation of the discontinuous Finite element port-Hamiltonian system

After defining the Stokes–Dirac structure for individual elements (4.53) and connecting them via the interconnection Dirac structure (4.74) the next step is to formulate the discontinuous Galerkin formulation for the Stokes–Dirac structure on a single element $K \in \mathcal{T}_h$ and then to generalize it to the whole discretized manifold Ω by connections through the interconnection Dirac structure.

6.1 *DG formulation in port variables*

For element $K \in \mathcal{T}_h$, port variables $e_h^p \in E_p(K)$, $e_h^q \in E_q(K)$, $f_h^p \in F_p(K)$, $f_h^q \in F_q(K)$, and external input boundary port variables $u_h^p \in E_p(\partial K), u_h^{p*} \in E_p(\partial K)^*$, the conservation laws stated through the Stokes–Dirac structure (4.53) are

$$f_h^p = (-1)^{r_1} d_{pq} e_h^q + (-1)^{r_1+q} \frac{1}{2} \mathrm{tr}_p^* Q \left(\mathrm{tr}_q e_h^q\right) + (-1)^{r_1+q} \mathrm{tr}_p^* u_h^{p*}, \tag{6.1a}$$

$$f_h^q = d_{qp} e_h^p + (-1)^p \frac{1}{2} (Q \mathrm{tr}_q)^* \left(\mathrm{tr}_p e_h^p\right) + (-1)^p (Q\mathrm{tr})^* u_h^p. \tag{6.1b}$$

Note that $p + q = n + 1$ and $r_1 = pq + 1$. Consider two elements, $K_1, K_2 \in \mathcal{T}_h$ connected through the common boundary $\partial K_1 \cap \partial K_2$. We split the boundaries between the two elements as in (4.69), (4.70) and (4.71). Using Lemma 4.9, (5.1) and the linearity of duality products, for $v_h^p \in E_p(K_1), v_h^q \in E_q(K_1)$, we get

$$\begin{aligned} \langle v_h^p \mid f_h^p \rangle_{K_1} &= -\left\langle e_h^q \mid d_{qp} v_h^p \right\rangle_{K_1} + (-1)^{p+1} \left\langle \frac{1}{2} Q \left(\mathrm{tr}_q e_h^q\right) - u_h^{p*} \mid \mathrm{tr}_p v_h^p \right\rangle_{Z_1^* \times Z_1} \\ &\quad + (-1)^{p+1} \left\langle \frac{1}{2} Q \left(\mathrm{tr}_q e_h^q\right) - u_h^{p*} \mid \mathrm{tr}_p v_h^p \right\rangle_{Z^* \times Z}, \\ \langle v_h^q \mid f_h^q \rangle_{K_1} &= (-1)^p \left\langle d_{pq} v_h^q \mid e_h^p \right\rangle_{K_1} + (-1)^{p+1} \left\langle Q \left(\mathrm{tr}_q v_h^q\right) \mid \frac{1}{2} \mathrm{tr}_p e_h^p - u_h^p \right\rangle_{Z_1^* \times Z_1} \\ &\quad + (-1)^{p+1} \left\langle Q \left(\mathrm{tr}_q v_h^q\right) \mid \frac{1}{2} \mathrm{tr}_p e_h^p - u_h^p \right\rangle_{Z^* \times Z}, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} \left\langle \tilde{v}_h^p \mid \tilde{f}_h^p \right\rangle_{K_2} &= -\left\langle \tilde{e}_h^q \mid d_{qp} \tilde{v}_h^p \right\rangle_{K_2} + (-1)^{p+1} \left\langle \frac{1}{2} Q \left(\mathrm{tr}_q \tilde{e}_h^q\right) - \tilde{u}_h^{p*} \mid \mathrm{tr}_p \tilde{v}_h^p \right\rangle_{Z_2^* \times Z_2} \\ &\quad + (-1)^{p+1} \left\langle \frac{1}{2} Q \left(\mathrm{tr}_q \tilde{e}_h^q\right) - \tilde{u}_h^{p*} \mid \mathrm{tr}_p \tilde{v}_h^p \right\rangle_{Z^* \times Z}, \\ \left\langle \tilde{v}_h^q \mid \tilde{f}_h^q \right\rangle_{K_2} &= (-1)^p \left\langle d_{pq} \tilde{v}_h^q \mid \tilde{e}_h^p \right\rangle_{K_2} + (-1)^{p+1} \left\langle Q \left(\mathrm{tr}_q \tilde{v}_h^q\right) \mid \frac{1}{2} \mathrm{tr}_p \tilde{e}_h^p - \tilde{u}_h^p \right\rangle_{Z_2^* \times Z_2} \\ &\quad + (-1)^{p+1} \left\langle Q \left(\mathrm{tr}_q \tilde{v}_h^q\right) \mid \frac{1}{2} \mathrm{tr}_p \tilde{e}_h^p - \tilde{u}_h^p \right\rangle_{Z^* \times Z}. \end{aligned} \tag{6.3}$$

With the same argument given in Section 5 and using Fig. 1 as a reference we choose the port variables defined on the interconnection to be

$$
\begin{aligned}
&w^p_{h,L} = u^p_h|_{\Delta_{1\cap 2}},\ w^{p*}_{h,L} = u^{p^*}_h|_{\Delta_{1\cap 2}},\ w^p_{h,R} = -\tilde{u}^p_h|_{\Delta_{1\cap 2}},\ w^{p*}_{h,R} = -\tilde{u}^{p^*}_h|_{\Delta_{1\cap 2}},\\
&y^p_{h,L} = \mathrm{tr}_p e^p_h|_{\Delta_{1\cap 2}},\ y^{p*}_{h,L} = Q\left(\mathrm{tr}_q e^q_h\right)|_{\Delta_{1\cap 2}},\ y^p_{h,R} = \mathrm{tr}_p \tilde{e}^p_h|_{\Delta_{1\cap 2}},\\
&y^{p*}_{h,R} = Q\left(\mathrm{tr}_q \tilde{e}^q_h\right)|_{\Delta_{1\cap 2}}.
\end{aligned}
\tag{6.4}
$$

The minus sign in front of $w^{p*}_{h,R}$ and $w^p_{h,R}$ is due to the fact that at the connection of two elements at a shared edge or face the outer normal changes sign. Using (4.74) we then obtain

$$
\begin{aligned}
u^p_h|_{\Delta_{1\cap 2}} &= \left(\theta - \frac{1}{2}\right)\mathrm{tr}_p e^p_h|_{\Delta_{1\cap 2}} - \theta\,\mathrm{tr}_p \tilde{e}^p_h|_{\Delta_{1\cap 2}},\\
u^{p^*}_h|_{\Delta_{1\cap 2}} &= \left(\frac{1}{2} - \theta\right) Q\left(\mathrm{tr}_q e^q_h|_{\Delta_{1\cap 2}}\right) + (\theta - 1) Q\left(\mathrm{tr}_q \tilde{e}^q_h|_{\Delta_{1\cap 2}}\right),\\
\tilde{u}^p_h|_{\Delta_{1\cap 2}} &= -(1-\theta)\mathrm{tr}_p e^p_h|_{\Delta_{1\cap 2}} - \left(\theta - \frac{1}{2}\right)\mathrm{tr}_p \tilde{e}^p_h|_{\Delta_{1\cap 2}},\\
\tilde{u}^{p^*}_h|_{\Delta_{1\cap 2}} &= -\theta Q\left(\mathrm{tr}_q e^q_h|_{\Delta_{1\cap 2}}\right) - \left(\frac{1}{2} - \theta\right) Q\left(\mathrm{tr}_q \tilde{e}^q_h|_{\Delta_{1\cap 2}}\right).
\end{aligned}
\tag{6.5}
$$

Substituting the values for the input ports as given in (6.5), we can simplify (6.2) and (6.3) as

$$
\begin{aligned}
\langle v^p_h \mid f^p_h\rangle_{K_1} &= -\left\langle e^q_h \mid d_{qp} v^p_h\right\rangle_{K_1} + (-1)^{p+1}\left\langle \frac{1}{2} Q\left(\mathrm{tr}_q e^q_h\right) - u^{p^*}_h \mid \mathrm{tr}_p v^p_h\right\rangle_{Z_1^*\times Z_1}\\
&\quad + (-1)^{p+1}\left\langle \theta Q\left(\mathrm{tr}_q e^q_h\right) + (1-\theta) Q\left(\mathrm{tr}_q \tilde{e}^q_h\right) \mid \mathrm{tr}_p v^p_h\right\rangle_{Z^*\times Z},\\
\langle v^q_h \mid f^q_h\rangle_{K_1} &= (-1)^p \left\langle d_{pq} v^q_h \mid e^p_h\right\rangle_{K_1} + (-1)^{p+1}\left\langle Q\left(\mathrm{tr}_q v^q_h\right) \mid \frac{1}{2}\mathrm{tr}_p e^p_h - u^p_h\right\rangle_{Z_1^*\times Z_1}\\
&\quad + (-1)^{p+1}\left\langle Q\left(\mathrm{tr}_q v^q_h\right) \mid (1-\theta)\mathrm{tr}_p e^p_h + \theta\,\mathrm{tr}_p \tilde{e}^p_h\right\rangle_{Z^*\times Z},
\end{aligned}
\tag{6.6}
$$

and

$$
\begin{aligned}
\left\langle \tilde{v}^p_h \mid \tilde{f}^p_h\right\rangle_{K_2} &= -\left\langle \tilde{e}^q_h \mid d_{qp} \tilde{v}^p_h\right\rangle_{K_2} + (-1)^{p+1}\left\langle \frac{1}{2} Q\left(\mathrm{tr}_q \tilde{e}^q_h\right) - \tilde{u}^{p^*}_h \mid \mathrm{tr}_p \tilde{v}^p_h\right\rangle_{Z_2^*\times Z_2}\\
&\quad + (-1)^{p+1}\left\langle \theta Q\left(\mathrm{tr}_q e^q_h\right) + (1-\theta) Q\left(\mathrm{tr}_q \tilde{e}^q_h\right) \mid \mathrm{tr}_p \tilde{v}^p_h\right\rangle_{Z^*\times Z},\\
\left\langle \tilde{v}^q_h \mid \tilde{f}^q_h\right\rangle_{K_2} &= (-1)^p \left\langle d_{pq} \tilde{v}^q_h \mid \tilde{e}^p_h\right\rangle_{K_2} + (-1)^{p+1}\left\langle Q\left(\mathrm{tr}_q \tilde{v}^q_h\right) \mid \frac{1}{2}\mathrm{tr}_p \tilde{e}^p_h - \tilde{u}^p_h\right\rangle_{Z_2^*\times Z_2}\\
&\quad + (-1)^{p+1}\left\langle Q\left(\mathrm{tr}_q \tilde{v}^q_h\right) \mid (1-\theta)\mathrm{tr}_p e^p_h + \theta\,\mathrm{tr}_p \tilde{e}^p_h\right\rangle_{Z^*\times Z}.
\end{aligned}
\tag{6.7}
$$

As the operator $Q : E_q(\partial K) \to E_p(\partial K)^*$ is surjective, using (4.33), (4.34), there exists a $u^q_h \in E_q(\partial K)$ such that $u^{p^*}_h = Q(u^q_h)$. But, we have not defined the exact definition of the operator Q that can be used for numerical computation so we need to remove Q from our DG formulation. Using $\langle \cdot \mid \cdot\rangle_{E_p(\partial K)^*\times E_p(\partial K)} =$

$\langle \cdot \mid \cdot \rangle_{\partial K}$ in (6.6) and (6.7) and then adding results for the two element system gives

$$
\begin{aligned}
\sum_{K\in\{K_1,K_2\}} \langle v_h^p \mid f_h^p \rangle_K &= \sum_{K\in\{K_1,K_2\}} - \langle e_h^q \mid d_{qp} v_h^p \rangle_K \\
&+ (-1)^{p+1} \left\langle \frac{1}{2}\mathrm{tr}_q e_h^q - u_h^q \mid \mathrm{tr}_p v_h^p \right\rangle_{\Delta_{1\neg 2}} + (-1)^{p+1} \left\langle \frac{1}{2}\mathrm{tr}_q \tilde{e}_h^q - \tilde{u}_h^q \mid \mathrm{tr}_p \tilde{v}_h^p \right\rangle_{\Delta_{2\neg 1}} \\
&+ (-1)^{p+1} \left\langle \theta \mathrm{tr}_q e_h^q + (1-\theta)\mathrm{tr}_q \tilde{e}_h^q \mid \mathrm{tr}_p v_h^p + \mathrm{tr}_p \tilde{v}_h^p \right\rangle_{\Delta_{1\cap 2}}, \\
\sum_{K\in\{K_1,K_2\}} \left\langle v_h^q \mid f_h^q \right\rangle_K &= \sum_{K\in\{K_1,K_2\}} (-1)^p \left\langle d_{pq} v_h^q \mid e_h^p \right\rangle_K \\
&+ (-1)^{p+1} \left\langle \mathrm{tr}_q v_h^q \mid \frac{1}{2}\mathrm{tr}_p e_h^p - u_h^p \right\rangle_{\Delta_{1\neg 2}} + (-1)^{p+1} \left\langle \mathrm{tr}_q \tilde{v}_h^q \mid \frac{1}{2}\mathrm{tr}_p \tilde{e}_h^p - \tilde{u}_h^p \right\rangle_{\Delta_{2\neg 1}} \\
&+ (-1)^{p+1} \left\langle \mathrm{tr}_q v_h^q + \mathrm{tr}_q \tilde{v}_h^q \mid (1-\theta)\mathrm{tr}_p e_h^p + \theta \mathrm{tr}_p \tilde{e}_h^p \right\rangle_{\Delta_{1\cap 2}}.
\end{aligned}
\tag{6.8}
$$

Equation (6.8) can be generalized to any two elements K_i and K_j, connected through a common internal face, $\partial K_i \cap \partial K_j$. For the tessellation $\mathscr{T}_h$ of the manifold Ω, we denote the variables $\mathrm{tr}_q v_h^q(K_1)|_{\partial K_1 \cap \partial K_2}$ as $\mathrm{tr}_q v_h^q|_L$ (contribution from the left element) and $\mathrm{tr}_q v_h^q(K_2)|_{\partial K_1 \cap \partial K_2}$ as $\mathrm{tr}_q v_h^q|_R$ (contribution from the right element). We can then add all elements K in the discretized manifold $\mathscr{T}_h$. Let $\mathscr{F}_o$ denoting the set of all external boundaries and $\mathscr{F}_i$ denote the set of all internal boundaries. We obtain the following discontinuous Galerkin formulation on the entire discretized manifold

$$
\begin{aligned}
\sum_{K\in\mathscr{T}_h} \left\langle v_h^p \mid f_h^p \right\rangle_K &= \sum_{K\in\mathscr{T}_h} - \left\langle e_h^q \mid d_{qp} v_h^p \right\rangle_K + (-1)^{p+1} \sum_{F\in\mathscr{F}_o} \left\langle \frac{1}{2}\mathrm{tr}_q e_h^q - u_h^q \mid \mathrm{tr}_p v_h^p \right\rangle_F \\
&+ (-1)^{p+1} \sum_{F\in\mathscr{F}_i} \left\langle \theta \mathrm{tr}_q e_h^q|_L + (1-\theta)\mathrm{tr}_q e_h^q|_R \mid \mathrm{tr}_p v_h^p|_L + \mathrm{tr}_p v_h^p|_R \right\rangle_F \\
\sum_{K\in\mathscr{T}_h} \left\langle v_h^q \mid f_h^q \right\rangle_K &= \sum_{K\in\mathscr{T}_h} (-1)^p \left\langle d_{pq} v_h^q \mid e_h^p \right\rangle_K + (-1)^{p+1} \sum_{F\in\mathscr{F}_o} \left\langle \mathrm{tr}_q v_h^q \mid \frac{1}{2}\mathrm{tr}_p e_h^p - u_h^p \right\rangle_F \\
&+ (-1)^{p+1} \sum_{F\in\mathscr{F}_i} \left\langle \mathrm{tr}_q v_h^q|_L + \mathrm{tr}_q v_h^q|_R \mid (1-\theta)\mathrm{tr}_p e_h^p|_L + \theta \mathrm{tr}_p e_h^p|_R \right\rangle_F.
\end{aligned}
\tag{6.9}
$$

We can rewrite (6.9) in the following manner

$$
\begin{aligned}
\sum_{K\in\mathscr{T}_h} \left\langle v_h^p \mid f_h^p \right\rangle_K &= \sum_{K\in\mathscr{T}_h} - \left\langle e_h^q \mid d_{qp} v_h^p \right\rangle_K + (-1)^{p+1} \sum_{F\in\mathscr{F}_o} \left\langle \hat{\beta}_h^q \mid \mathrm{tr}_p v_h^p \right\rangle_F \\
&+ (-1)^{p+1} \sum_{F\in\mathscr{F}_i} \langle \hat{\beta}_h^q \mid \mathrm{tr}_p v_h^p|_L + \mathrm{tr}_p v_h^p|_R \rangle_F \\
\sum_{K\in\mathscr{T}_h} \left\langle v_h^q \mid f_h^q \right\rangle_K &= \sum_{K\in\mathscr{T}_h} (-1)^p \left\langle d_{pq} v_h^q \mid e_h^p \right\rangle_K + (-1)^{p+1} \sum_{F\in\mathscr{F}_o} \left\langle \mathrm{tr}_q v_h^q \mid \hat{\beta}_h^p \right\rangle_F \\
&+ (-1)^{p+1} \sum_{F\in\mathscr{F}_i} \left\langle \mathrm{tr}_q v_h^q|_L + \mathrm{tr}_q v_h^q|_R \mid \hat{\beta}_h^p \right\rangle_F,
\end{aligned}
\tag{6.10}
$$

where $\hat{\beta}_h^q$ and $\hat{\beta}_h^p$ can be considered as the numerical fluxes in the discontinuous Galerkin formulation. For $F \in \mathscr{F}_i$ the numerical fluxes are equal to

$$\begin{aligned} \hat{\beta}_h^p &= (1-\theta)\mathrm{tr}_p e_h^p|_L + \theta \mathrm{tr}_p e_h^p|_R, \\ \hat{\beta}_h^q &= \theta \mathrm{tr}_q e_h^q|_L + (1-\theta)\mathrm{tr}_q e_h^q|_R, \end{aligned} \tag{6.11}$$

with $\theta \in [0,1]$ and for $F \in \mathscr{F}_o$ the numerical fluxes are equal to

$$\hat{\beta}_h^p = \frac{1}{2}\mathrm{tr}_p e_h^p - u_h^p, \qquad \hat{\beta}_h^q = \frac{1}{2}\mathrm{tr}_q e_h^q - u_h^q, \tag{6.12}$$

where the input boundary ports u_h^p and u_h^q are to be chosen depending on the external boundary conditions.

6.2 *DG formulation in state space variables*

We start with a pH-system (PDE) represented by the Stokes–Dirac structure (3.12), in which the port variables e^p, e^q, f^p and f^q are related to the energy variables α^p, α^q and co-energy variables β^p, β^q through the relations given by

$$\begin{aligned} f^p &= -\frac{\mathrm{d}\alpha^p}{\mathrm{d}t}, & f^q &= -\frac{\mathrm{d}\alpha^q}{\mathrm{d}t}, \\ e^p &= \beta^p = \delta_p H, & e^q &= \beta^q = \delta_q H, \end{aligned} \tag{6.13}$$

where H is the Hamiltonian of the system.

Upon discretization let the discretized variables corresponding to the energy and co-energy variables be α_h^p, α_h^q and β_h^p, β_h^q, respectively. The generalized Stokes–Dirac structure (4.53) represents the discretized Stokes–Dirac structure over each element, thus, the differential equation expressed in terms of discretized energy and co-energy variables is given by

$$\begin{aligned} \begin{bmatrix} -\frac{d\alpha_h^p}{\mathrm{d}t} \\ -\frac{d\alpha_h^q}{\mathrm{d}t} \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^{r_1} d_{pq} + (-1)^{r_1+q}\frac{1}{2}\mathrm{tr}_p^* Q(\mathrm{tr}_q) \\ \mathrm{d}_{qp} + (-1)^p \frac{1}{2}(Q\mathrm{tr}_q)^*\mathrm{tr}_p & 0 \end{bmatrix} \begin{bmatrix} \beta_h^p \\ \beta_h^q \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & (-1)^{r_1+q}\mathrm{tr}_p^* \\ (-1)^p (Q\mathrm{tr}_q)^* & 0 \end{bmatrix} \begin{bmatrix} u_h^p \\ u_h^{p*} \end{bmatrix}, \\ \begin{bmatrix} y_h^{p*} \\ y_h^p \end{bmatrix} &= \begin{bmatrix} 0 & -Q(\mathrm{tr}_q) \\ -(-1)^p \mathrm{tr}_p & 0 \end{bmatrix} \begin{bmatrix} \beta_h^p \\ \beta_h^q \end{bmatrix}, \end{aligned} \tag{6.14}$$

where $p+q=n+1$, $r_1 = pq+1$ and Q defined as in (4.33) and (4.34). Here on each element we have used

$$\begin{aligned} f_h^p &= -\frac{\mathrm{d}\alpha_h^p}{\mathrm{d}t}, & f_h^q &= -\frac{\mathrm{d}\alpha_h^q}{\mathrm{d}t}, \\ e_h^p &= \beta_h^p, & e_h^q &= \beta_h^q. \end{aligned} \tag{6.15}$$

Following Section 6.1, as the spaces of the energy and co-energy variables are the same as the corresponding port variables, we can rewrite the weak form (6.10) in terms of the energy and co-energy variables

$$
\begin{aligned}
\sum_{K\in\mathscr{T}_h} \langle \nu_h^p \mid \dot{\alpha}_h^p\rangle_K &= \sum_{K\in\mathscr{T}_h} \left\langle \beta_h^q \mid \mathrm{d}_{qp}\nu_h^p\right\rangle_K + (-1)^p \sum_{F\in\mathscr{F}_o} \left\langle \hat{\beta}_h^q \mid \mathrm{tr}_p \nu_h^p\right\rangle_F \\
&\quad + (-1)^p \sum_{F\in\mathscr{F}_i} \left\langle \hat{\beta}_h^q \mid \mathrm{tr}_p \nu_h^p|_L + \mathrm{tr}_p \nu_h^p|_R\right\rangle_F \\
\sum_{K\in\mathscr{T}_h} \langle \nu_h^q \mid \dot{\alpha}_h^q\rangle_K &= (-1)^{p+1} \sum_{K\in\mathscr{T}_h} \left\langle d_{pq}\nu_h^q \mid \beta_h^p\right\rangle_K + (-1)^p \sum_{F\in\mathscr{F}_o} \left\langle \mathrm{tr}_q \nu_h^q \mid \hat{\beta}_h^p\right\rangle_F \\
&\quad + (-1)^p \sum_{F\in\mathscr{F}_i} \left\langle \mathrm{tr}_q \nu_h^q|_L + \mathrm{tr}_q \nu_h^q|_R \mid \hat{\beta}_h^p\right\rangle_F,
\end{aligned}
\tag{6.16}
$$

where $\hat{\beta}_h^q$ and $\hat{\beta}_h^p$ are the numerical fluxes (6.11), (6.12) in the discontinuous Galerkin formulation.

7. Energy conservation

Consider an oriented polyhedral manifold Ω in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$ discretized into shape regular finite elements $K \in \mathscr{T}_h$. Let $\mathscr{F}_h = \mathscr{F}_h^o \cup \mathscr{F}_h^i$ denote the set of all boundaries of the discretized manifold with $\mathscr{F}_h^o$ and $\mathscr{F}_h^i$ being the set of all external and internal faces, respectively. At a given time t, let $\alpha^p(t) \in L^2\Lambda^p(\Omega), \alpha^q(t) \in L^2\Lambda^q(\Omega)$, $\beta^p(t) := \delta_p H \in H^1\Lambda^{n-p}(\Omega)$ and $\beta^q(t) := \delta_q H \in H\Lambda^{n-q}(\Omega)$ be the solution satisfying the partial differential equations represented by the Dirac structure (3.12). The Hamiltonian of this system is given by

$$
H = \sum_{K\in\mathscr{T}_h} \frac{1}{2}\Big(\langle \beta^p \mid \alpha^p\rangle_K + \langle \beta^q \mid \alpha^q\rangle_K\Big). \tag{7.1}
$$

We have the following constitutive relations for the port-Hamiltonian system

$$
\begin{aligned}
\alpha^p &= C_p * \beta^p, \\
\alpha^q &= C_q * \beta^q,
\end{aligned}
\tag{7.2}
$$

where $C_p, C_q \in L^2\Lambda^0(\Omega)$ are coefficient functions that depend on the spatial domain and the physical problem under consideration. Here, $*$ is the Hodge star operator defined in Definition 2.1. Using the constitutive relations we can rewrite the Hamiltonian for the system as

$$
H = \sum_{K\in\mathscr{T}_h} \frac{1}{2}\left(\left\langle \beta^p \mid C_p * \beta^p\right\rangle_K + \left\langle \beta^q \mid C_q * \beta^q\right\rangle_K\right). \tag{7.3}
$$

Given the discontinuous finite element spaces

$$
\begin{aligned}
F_p(\mathscr{T}_h) &:= \left\{\alpha_h^p \in L^2\Lambda^p(\Omega)\,\big|\alpha_h^p\big|_K \in F_p(K) \quad \forall K \in \mathscr{T}_h\right\},\\
F_q(\mathscr{T}_h) &:= \left\{\alpha_h^q \in L^2\Lambda^q(\Omega)\,\big|\alpha_h^q\big|_K \in F_q(K) \quad \forall K \in \mathscr{T}_h\right\},\\
E_p(\mathscr{T}_h) &:= \left\{\beta_h^p \in L^2\Lambda^{n-p}(\Omega)\,\big|\beta_h^p\big|_K \in E_p(K) \quad \forall K \in \mathscr{T}_h\right\},\\
E_q(\mathscr{T}_h) &:= \left\{\beta_h^q \in L^2\Lambda^{n-q}(\Omega)\,\big|\beta_h^q\big|_K \in E_q(K) \quad \forall K \in \mathscr{T}_h\right\}.
\end{aligned}
\tag{7.4}
$$

Let $\alpha_h^p(t) \in F_p(\mathscr{T}_h), \alpha_h^q(t) \in F_q(\mathscr{T}_h), \beta_h^p(t) \in E_p(\mathscr{T}_h)$ and $\beta_h^q(t) \in E_q(\mathscr{T}_h)$ be the discrete solutions satisfying the discrete Stokes–Dirac structure (6.14). The discrete energy of the system is then given by

$$
E_h = \frac{1}{2}\sum_{K\in\mathscr{T}_h} \langle\beta_h^p \mid \alpha_h^p\rangle_K + \langle\beta_h^q \mid \alpha_h^q\rangle_K\,. \tag{7.5}
$$

7.1 *Discrete Hodge star duality*

We extend the L^2 inner product between $\beta^p(t), \nu^p \in L^2\Lambda^{n-p}(\Omega)$ and $\beta^q(t), \nu^q \in L^2\Lambda^{n-q}(\Omega)$, given by (2.11), into a weighted inner product as

$$
\begin{aligned}
\langle\beta^p, \nu^p\rangle_{wL^2\Lambda^{n-p}(\Omega)} &= \int_\Omega C_p * \beta^p \wedge \nu^p = \int_\Omega \beta^p(\mu)\cdot\nu^p(\mu)\cdot C_p(\mu)\,\mathrm{d}\mu,\\
\langle\beta^q, \nu^q\rangle_{wL^2\Lambda^{n-q}(\Omega)} &= \int_\Omega C_q * \beta^q \wedge \nu^q = \int_\Omega \beta^q(\mu)\cdot\nu^q(\mu)\cdot C_q(\mu)\,\mathrm{d}\mu,
\end{aligned}
\tag{7.6}
$$

where $\mathrm{d}\mu$ is the Lebesgue measure on Ω, and the functions C_p, C_q, with $C_p, C_q \geq C > 0$, stated in (7.2). Let $e_h^p, \tilde{e}_h^p \in E_p$ and $e_h^q, \tilde{e}_h^q \in E_q$. Analogously, we define a weighted inner product on E_p and E_q,

$$
\begin{aligned}
g_{wp}\left(e_h^p, \tilde{e}_h^p\right) &= \sum_{f\in\Delta(K),\dim(f)=[p,p+r-1]} \int_f e_c^p(\mu)\cdot\tilde{e}_c^p(\mu)\cdot C_p(\mu)\,\mathrm{d}\mu,\\
g_{wq}\left(e_h^q, \tilde{e}_h^q\right) &= \int_K e_c^q(\mu)\cdot\tilde{e}_c^q(\mu)\cdot C_q(\mu)\,\mathrm{d}\mu,
\end{aligned}
\tag{7.7}
$$

where $\mathrm{d}\mu$ is the Lebesgue measure on element K. Here, e_c^p, e_c^q denote the coefficients for the polynomial differential forms e_h^p, e_h^q, respectively. Analogously, $\tilde{e}_c^p, \tilde{e}_c^q$ denote the coefficients for the polynomial differential forms $\tilde{e}_h^p, \tilde{e}_h^q$, respectively. For fixed $\tilde{e}_h^p$ and $\tilde{e}_h^q$, (7.7) defines a linear functional in e_h^p and e_h^q, respectively. As, E_p and F_p are dual to each other, there exists an $f_h^p \in F_p(K)$ such that $\langle f_h^p \mid e_h^p\rangle_K = g_{wp}(e_h^p, \tilde{e}_h^p), \forall e_h^p \in E_p$. Similarly, there exists an $f_h^q \in F_q(K)$ such that $\langle f_h^q \mid e_h^q\rangle_K = g_{wq}(e_h^q, \tilde{e}_h^q), \forall e_h^q \in E_q$.

We denote $f_h^p = \star_p \tilde{e}_h^p$ and $f_h^q = \star_q \tilde{e}_h^q$, and call $\star_p$ and $\star_q$ discrete Hodge star operators. Thus

$$\begin{aligned} \left\langle e_h^p \mid \star_p \tilde{e}_h^p \right\rangle_K &= g_{wp}\left(e_h^p, \tilde{e}_h^p\right), \quad \forall e_h^p \in E_p(K), \\ \left\langle e_h^q \mid \star_q \tilde{e}_h^q \right\rangle_K &= g_{wq}\left(e_h^q, \tilde{e}_h^q\right), \quad \forall e_h^q \in E_p(K). \end{aligned} \tag{7.8}$$

Hence in the discrete framework the constitutive relationship between the state space variables α_h^p, β_h^p and α_h^q, β_h^q is

$$\begin{aligned} \alpha_h^p &= \star_p \beta_h^p, \\ \alpha_h^q &= \star_q \beta_h^q. \end{aligned} \tag{7.9}$$

Using (6.16) and (7.9) we can now state the port-Hamiltonian DG discretization of (3.12) for $\beta_h^p(t) \in E_p(\mathscr{T}_h), \beta_h^q(t) \in E_q(\mathscr{T}_h)$ as

$$\begin{aligned} \sum_{K\in\mathscr{T}_h} \left\langle \nu_h^p \mid \star_p \dot{\beta}_h^p \right\rangle_K &= \sum_{K\in\mathscr{T}_h} \left\langle \beta_h^q \mid d_{qp}\nu_h^p \right\rangle_K + (-1)^p \sum_{F\in\mathscr{F}_o} \left\langle \hat{\beta}_h^q \mid \mathrm{tr}_p \nu_h^p \right\rangle_F \\ &\quad + (-1)^p \sum_{F\in\mathscr{F}_i} \left\langle \hat{\beta}_h^q \mid \mathrm{tr}_p \nu_h^p|_L + \mathrm{tr}_p \nu_h^p|_R \right\rangle_F, \\ \sum_{K\in\mathscr{T}_h} \left\langle \nu_h^q \mid \star_q \dot{\beta}_h^q \right\rangle_K &= (-1)^{p+1} \sum_{K\in\mathscr{T}_h} \left\langle d_{pq}\nu_h^q \mid \beta_h^p \right\rangle_K + (-1)^p \sum_{F\in\mathscr{F}_o} \left\langle \mathrm{tr}_q \nu_h^q \mid \hat{\beta}_h^p \right\rangle_F \\ &\quad + (-1)^p \sum_{F\in\mathscr{F}_i} \left\langle \mathrm{tr}_q \nu_h^q|_L + \mathrm{tr}_q \nu_h^q|_R \mid \hat{\beta}_h^p \right\rangle_F, \end{aligned} \tag{7.10}$$

where $\hat{\beta}_h^q$ and $\hat{\beta}_h^p$ are given by (6.11) and (6.12) and $\nu_h^p \in E_p(K), \nu_h^q \in E_q(K)$. Using (7.9) along with (7.5), the energy on each element of the system represented by the discrete Dirac structure (6.14) is given by

$$E_K = \frac{1}{2}\left\langle \beta_h^p \mid \star_p \beta_h^p \right\rangle_K + \left\langle \beta_h^q \mid \star_q \beta_h^q \right\rangle_K. \tag{7.11}$$

Choosing the port variables at each common face as stated in (6.5) and using Lemma (5.2), the interconnection between the elements $K \in \mathscr{T}_h$ given by (4.74) is power preserving. Thus, the discrete energy of the total system is

$$E_h = \frac{1}{2}\sum_{K\in\mathscr{T}_h} \left(\left\langle \beta_h^p \mid \star_p \beta_h^p \right\rangle_K + \left\langle \beta_h^q \mid \star_q \beta_h^q \right\rangle_K \right). \tag{7.12}$$

7.2 *Energy conservation*

THEOREM 7.1 (Energy Conservation). Given an n-dimensional polyhedral oriented manifold Ω with Lipschitz continuous boundary $\partial\Omega$, discretized into a set of discontinuous elements $\mathscr{T}_h$. Let $\beta_h^p(t) \in E_p(\mathscr{T}_h)$, $\beta_h^q(t) \in E_q(\mathscr{T}_h)$ and $\hat{\beta}_h^p(t) \in E_p(\mathscr{F}_h)$, $\hat{\beta}_h^q(t) \in E_q(\mathscr{F}_h)$ satisfy (7.10). Let the elements be connected using the interconnection structure (4.74), choosing the port variables and interconnection variables at each face as (6.5). Let the total discrete energy of the discrete system E_h be given by (7.12). Then the rate of change in discrete energy for the whole system is

$$\dot{E}_h = \sum_{F \in \mathscr{F}_o} \left(\langle y_h^q \mid u_h^p \rangle_F + \langle u_h^q \mid y_h^p \rangle_F \right). \tag{7.13}$$

Proof. The energy on each element $K \in \mathscr{T}_h$ of the discretized system represented by the Stokes–Dirac structure (6.14) is given by (7.11). The change in discrete energy on each element $K \in \mathscr{T}_h$ is

$$\dot{E}_K = \frac{1}{2} \left(\left\langle \beta_h^p \mid \star_p \dot{\beta}_h^p \right\rangle_K + \left\langle \dot{\beta}_h^p \mid \star_p \beta_h^p \right\rangle_K + \left\langle \dot{\beta}_h^q \mid \star_q \beta_h^q \right\rangle_K + \left\langle \beta_h^q \mid \star_q \dot{\beta}_h^q \right\rangle_K \right), \tag{7.14}$$

which using the symmetry of g_{wp} and g_{wq}, given by (7.7), results in

$$\dot{E}_K = \left(\left\langle \beta_h^p \mid \star_p \dot{\beta}_h^p \right\rangle_K + \left\langle \beta_h^q \mid \star_q \dot{\beta}_h^q \right\rangle_K \right). \tag{7.15}$$

Using (6.15) and (7.9), the duality product (4.52) can be written in terms of state space variables as

$$\begin{aligned}
&\langle e_h^p \mid f_h^p \rangle_K + \langle e_h^q \mid f_h^q \rangle_K + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&= -\langle \beta_h^p \mid \dot{\alpha}_h^p \rangle_K - \langle \beta_h^q \mid \dot{\alpha}_h^q \rangle_K + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&= -\left\langle \beta_h^p \mid \star_p \dot{\beta}_h^p \right\rangle_K - \left\langle \beta_h^q \mid \star_q \dot{\beta}_h^q \right\rangle_K + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&\quad + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)}.
\end{aligned} \tag{7.16}$$

Next, using the fact that (6.14) with the duality product (7.16) is a Dirac structure, Theorem 4.9 then gives

$$\begin{aligned}
&-\left\langle \beta_h^p \mid \star_p \dot{\beta}_h^p \right\rangle_K - \left\langle \beta_h^q \mid \star_q \dot{\beta}_h^q \right\rangle_K + \left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \\
&+ \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} = 0.
\end{aligned} \tag{7.17}$$

Using (7.17), (7.15) becomes

$$\dot{E}_K = \left(\left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(\partial K)^* \times E_p(\partial K)} \right). \tag{7.18}$$

We have already proved that with the power preserving interconnection Dirac structure (4.74) and interconnection variables (6.5), we can add all energy contributions from the elements and obtain, since

TABLE 1 *Overview of polynomial differential form spaces for Case 1. Note, $p + q = n + 1$ and the spaces E_q, F_q and E_p, F_p are dual pairs of polynomial differential form spaces*

$E_q(K)$	$\mathscr{P}_{r+1}\Lambda^{n-q}(K)$
$F_q(K)$	$\bigoplus_{f\in\Delta(K),\dim(f)\in[n-q,n-q+r]} \mathscr{P}^-_{r+1+n-q-\dim(f)}\Lambda^{\dim(f)-n+q}(f)$
$E_p(K)$	$\bigoplus_{f\in\Delta(K),\dim(f)\in[p,p+r-1]} \mathscr{P}^-_{r+p-\dim(f)}\Lambda^{\dim(f)-p}(f)$
$F_p(K)$	$\mathscr{P}_r\Lambda^p(K)$

the contributions from the internal faces $F \in \mathscr{F}_i$ cancel, see Section 5,

$$\dot{E}_h = \sum_{F\in\mathscr{F}_o} \left(\left\langle y_h^{p*} \mid u_h^p \right\rangle_{E_p(F)^*\times E_p(F)} + \left\langle u_h^{p*} \mid y_h^p \right\rangle_{E_p(F)^*\times E_p(F)} \right). \tag{7.19}$$

As we have assumed that the operator $Q : E_q(\partial K) \to E_p(\partial K)^*$, stated in (4.34), is surjective there exist $y_h^q, u_h^q \in E_q(\partial K)$ such that $y_h^{p*} = Q(y_h^q)$ and $u_h^{p*} = Q(u_h^q)$, thus

$$\begin{aligned}\dot{E}_h &= \sum_{F\in\mathscr{F}_o} \left(\langle Q\left(y_h^q\right) \mid u_h^p\rangle_{E_p(F)^*\times E_p(F)} + \langle Q\left(u_h^q\right) \mid y_h^p\rangle_{E_p(F)^*\times E_p(F)} \right) \\ &= \sum_{F\in\mathscr{F}_o} \left(\langle y_h^q \mid u_h^p\rangle_F + \langle u_h^q \mid y_h^p\rangle_F \right).\end{aligned} \tag{7.20}$$

□

The change in discrete energy for the whole system thus only depends on the energy input–output through the domain boundaries. The choice of proper boundary conditions, or in other words the choice of suitable external input–output boundary port variables for the system, is therefore crucial for the stability of the system. To obtain bounds on the energy of the fully discrete system one could use a symplectic time integrator. For more details, see Hairer *et al.* (2006).

8. Error analysis

In this section we will state an *a priori* error estimate for the port-Hamiltonian discontinuous Galerkin discretization presented in this paper. For quick reference we summarize in Tables 1 and 2 the choices for the dual pairs of the spaces $F_q(K), E_q(K)$ and $F_p(K), E_p(K)$ for Cases 1 and 2, respectively. More details can be found in Sections 4.2, 4.3 and 4.4.

The related discontinuous Galerkin finite element spaces $F_p(\mathscr{T}_h), F_q(\mathscr{T}_h), E_p(\mathscr{T}_h)$ and $E_q(\mathscr{T}_h)$ are stated in (7.4). Using the trace operator tr we define the spaces $E_p(\mathscr{F}_h)$ and $E_q(\mathscr{F}_h)$ on the boundaries of each element in the discretized polyhedral manifold as

$$\begin{aligned}E_p(\mathscr{F}_h) &:= \left\{\lambda_h^p \in L^2\Lambda^{n-p}(\Omega) \mid \lambda_h^p|_{\partial K} \in E_p(\partial K) \quad \forall K \in \mathscr{T}_h\right\}, \\ E_q(\mathscr{F}_h) &:= \left\{\lambda_h^q \in L^2\Lambda^{n-q}(\Omega) \mid \lambda_h^q|_{\partial K} \in E_q(\partial K) \quad \forall K \in \mathscr{T}_h\right\}.\end{aligned} \tag{8.1}$$

TABLE 2 *Overview of polynomial differential form spaces for Case 2. Note, $p + q = n + 1$ and the spaces E_q, F_q and E_p, F_p are dual pairs of polynomial differential form spaces*

$E_q(K)$	$\mathscr{P}_r^- \Lambda^{n-q}(K)$
$F_q(K)$	$\bigoplus_{f\in\Delta(K),\dim(f)\in[n-q,n-q+r-1]} \mathscr{P}_{r+n-q-\dim(f)-1}\Lambda^{\dim(f)-n+q}(f)$
$E_p(K)$	$\bigoplus_{f\in\Delta(K),\dim(f)\in[p,p+r-1]} \mathscr{P}_{r+p-\dim(f)-1}\Lambda^{\dim(f)-p}(f)$
$F_p(K)$	$\mathscr{P}_r^- \Lambda^{p}(K)$

Define the discontinuous Galerkin operators $\mathscr{D}_1$ and $\mathscr{D}_2$ as

$$\mathscr{D}_1 : (H\Lambda^{n-q}(\Omega) + E_q(\mathscr{T}_h)) \times E_p(\mathscr{T}_h) \times (H^{-1/2}\Lambda^{n-q}(\mathscr{F}_h) + E_q(\mathscr{F}_h)) \to \mathbb{R},$$

$$\begin{aligned}\mathscr{D}_1\left(u^q, \lambda_h^p; \hat{u}^q\right) = &- \sum_{K\in\mathscr{T}_h} \left\langle u^q \mid d_{qp}\lambda_h^p\right\rangle_K \\ &+ (-1)^{p+1} \sum_{F\in\mathscr{F}_i} \langle \hat{u}^q \mid \lambda_h^p|_L + \lambda_h^p|_R\rangle_F \\ &+ (-1)^{p+1} \sum_{F\in\mathscr{F}_o} \langle \hat{u}^q \mid \lambda_h^p\rangle_F ,\end{aligned} \tag{8.2a}$$

$$\mathscr{D}_2 : \left(H^1\Lambda^{n-p}(\Omega) + E_p(\mathscr{T}_h)\right) \times E_q(\mathscr{T}_h) \times \left(H^{1/2}\Lambda^{n-p}(\mathscr{F}_h) + E_p(\mathscr{F}_h)\right) \to \mathbb{R},$$

$$\begin{aligned}\mathscr{D}_2\left(v^p, \lambda_h^q; \hat{v}^p\right) = &(-1)^p \sum_{K\in\mathscr{T}_h} \left\langle d_{pq}\lambda_h^q \mid v^p\right\rangle_K \\ &+ (-1)^{p+1} \sum_{F\in\mathscr{F}_i} \langle \lambda_h^q|_L + \lambda_h^q|_R \mid \hat{v}^p\rangle_F \\ &+ (-1)^{p+1} \sum_{F\in\mathscr{F}_o} \langle \lambda_h^q \mid \hat{v}^p\rangle_F .\end{aligned} \tag{8.2b}$$

LEMMA 8.1 (Energy Conservation). The discontinuous Galerkin (DG) operators $\mathscr{D}_1$ and $\mathscr{D}_2$, as defined in (8.2), for $\beta_h^p \in E_p(\mathscr{T}_h)$, $\beta_h^q \in E_q(\mathscr{T}_h)$ and external input $u_h^p, y_h^p \in E_p(\mathscr{F}_h)$, $u_h^q, y_h^q \in E_q(\mathscr{F}_h)$ satisfy the following relation

$$\mathscr{D}_1\left(\beta_h^q, \beta_h^p; \hat{\beta}_h^q\right) + \mathscr{D}_2\left(\beta_h^p, \beta_h^q; \hat{\beta}_h^p\right) = -\sum_{F\in\mathscr{F}_o} \left(\langle y_h^q \mid u_h^p\rangle_F + \langle u_h^q \mid y_h^p\rangle_F\right), \tag{8.3}$$

where $p + q = n + 1$.

Proof. The proof follows from (6.16) with numerical fluxes (6.11)–(6.12) and uses the same steps as in the proof of Theorem 3. □

DEFINITION 8.2 (Canonical Projection Operator π_h^l). Let $K \in \mathscr{T}_h$ be a simplex and let $\Delta(K)$ denote the set of all subsimplices f of K with $\dim(f) \geq l$. For $r \in \mathbb{N}$, $r \geq 1$, the projection operator π_h^l is the mapping, (Section 4.9, Arnold *et al.*, 2006),

$$\pi_h^l : C^0\Lambda^l(K) \to \mathscr{P}_r\Lambda^l(K), \tag{8.4}$$

satisfying, for all $\omega^l \in C^0\Lambda^l(K)$

$$\int_f \mathrm{tr}_{K,f}\left(\omega^l - \pi_h^l\omega^l\right) \wedge \eta = 0, \quad \forall \eta \in \mathscr{P}^-_{r+l-\dim(f)}\Lambda^{\dim(f)-l}(f), \quad \forall f \in \Delta(K). \tag{8.5}$$

DEFINITION 8.3 (Canonical Projection Operator π_{h-}^l). Let $K \in \mathscr{T}_h$ be a simplex and let $\Delta(K)$ denote the set of all subsimplices f of K with $\dim(f) \geq l$. For $r \in \mathbb{N}$, $r \geq 1$, the projection operator π_{h-}^l is the mapping, Section 4.9, Arnold *et al.* (2006),

$$\pi_{h-}^l : C^0\Lambda^l(K) \to \mathscr{P}_r^-\Lambda^l(K), \tag{8.6}$$

satisfying, for all $\omega^l \in C^0\Lambda^l(K)$

$$\int_f \mathrm{tr}_{K,f}\left(\omega^l - \pi_{h-}^l\omega^l\right) \wedge \eta \quad = 0, \quad \forall \eta \in \mathscr{P}_{r+l-\dim(f)-1}\Lambda^{\dim(f)-l}(f), \quad \forall f \in \Delta(K). \tag{8.7}$$

8.1 *Interpolation Error Bounds*

The canonical projections stated in Definitions 8.2 and 8.3 depend on traces on the subsimplices. This requires function spaces with more regularity than is available in the spaces $\mathscr{F}_{p,q}$ and $\mathscr{E}_{p,q}$, (3.3) and (3.5), which are used in the definition of the Stokes–Dirac structure stated in Section 3.2. This can be remediated by modifying the canonical projections into smoothed projections, which we also denote as π_h^l and $\pi_{h,-}^l$. The details of the construction of these smoothed projections can be found in Section 5.4 (Arnold *et al.*, 2006) and Section 5.5 (Arnold *et al.*, 2010). Based on Theorem 5.9 (Arnold *et al.*, 2010), we can state the following lemmas:

LEMMA 8.4 (Interpolation Error Bounds). Let $\pi_h^l : L^2\Lambda^l(\Omega) \to \mathscr{P}_r\Lambda^l(\mathscr{T}_h), r \geq 1$ and $\pi_{h,-}^l : L^2\Lambda^l(\Omega) \to \mathscr{P}^-_{r+1}\Lambda^l(\mathscr{T}_h), r \geq 0$ be, respectively, the smoothed projection operators based on the canonical projections stated by Definitions 8.2 and 8.3. Then π_h^l is a projection onto $\mathscr{P}_r\Lambda^l(\mathscr{T}_h)$ for $r \geq 1$ and satisfies

$$\left|\left|\omega - \pi_h^l\omega\right|\right|_{L^2\Lambda^l(\Omega)} \leq ch^s||\omega||_{H^s\Lambda^l(\Omega)}, \quad \omega \in H^s\Lambda^l(\Omega), \tag{8.8}$$

for $0 \leq s \leq r+1$. Moreover, for all $\omega \in L^2\Lambda^l(\Omega)$, $\pi_h^l\omega \to \omega$ in $L^2\Lambda^l(\Omega)$ as $h \to 0$ and $\mathrm{d}\pi_h^{l-1} = \pi_h^l d$. Similarly, $\pi_{h,-}^l$ is a projection onto $\mathscr{P}^-_{r+1}\Lambda^l(\mathscr{T}_h)$ for $r \geq 0$ and also satisfies (8.8) for $0 \leq s \leq r+1$. Moreover, for all $\omega \in L^2\Lambda^l(\Omega)$, $\pi_{h,-}^l\omega \to \omega$ in $L^2\Lambda^l(\Omega)$ as $h \to 0$ and $\mathrm{d}\pi_{h,-}^{l-1} = \pi_{h,-}^l d$.

LEMMA 8.5 The discontinuous Galerkin discretization operators $\mathscr{D}_1$ and $\mathscr{D}_2$, as defined in (8.2) for $\beta^p \in H^1\Lambda^{n-p}(\Omega), \hat{\beta}^p \in H^{\frac{1}{2}}\Lambda^{n-p}(\mathscr{F}_h), \beta^q \in H\Lambda^{n-q}(\Omega), \hat{\beta}^q \in H^{-\frac{1}{2}}\Lambda^{n-q}(\mathscr{F}_h), v_h^p \in E_p(\mathscr{T}_h), v_h^q \in E_q(\mathscr{T}_h)$, along with the smoothed canonical projection operators defined in Definitions 8.2 and 8.3, satisfy

$$\mathscr{D}_1\left(\beta^q - \pi_h^{n-q}\beta^q, v^p; \beta^q - \pi_h^{n-q}\beta^q\right) = 0, \tag{8.9a}$$

$$\mathscr{D}_2\left(\beta^p - \pi_{h-}^{n-p}\beta^p, v^q; \beta^p - \pi_{h-}^{n-p}\beta^p\right) = 0. \tag{8.9b}$$

Proof. Using Definition 8.2 for the projection operator π_h^l together with (8.2a), (4.20), (8.9a) is immediate. Similarly, using Definition 8.3 for the projection operator π_{h-}^l together with (8.2b), (4.18) gives (8.9b). □

8.2 *Energy equation*

For the error analysis of the port-Hamiltonian discontinuous Galerkin finite element formulation (6.16) with numerical fluxes (6.11)–(6.12), we define the following bilinear forms. For $u^p \in (H^1\Lambda^{n-p}(\Omega) + E_p(\mathscr{T}_h)), v^q \in (H\Lambda^{n-q}(\Omega) + E_q(\mathscr{T}_h)), v^p \in E_p(\mathscr{T}_h), v^q \in E_q(\mathscr{T}_h)$ and corresponding boundary port variables $\hat{u}^p \in (H^{1/2}\Lambda^{n-p}(\mathscr{F}_h) + E_p(\mathscr{F}_h)), \hat{v}^q \in (H^{-1/2}\Lambda^{n-q}(\mathscr{F}_h) + E_q(\mathscr{F}_h))$ we define the bilinear forms

$$\begin{aligned}\mathscr{A}(u^p, v^q; v^p, v^q) &= \sum_{K\in\mathscr{T}_h}\left(\left\langle v^p \mid C_p * u^p\right\rangle_K + \left\langle v^q \mid C_q * v^q\right\rangle_K\right) \\ &= \sum_{K\in\mathscr{T}_h}\left(\int_K u^p(\mu)\cdot v^p(\mu)\cdot C_p(\mu)\,\mathrm{d}\mu \right. && (8.10a)\\ &\qquad \left. + \int_K v^q(\mu)\cdot v^q(\mu)\cdot C_q(\mu)\,\mathrm{d}\mu\right) && (8.10b)\end{aligned}$$

$$\mathscr{B}(u^p, v^q, \hat{u}^p, \hat{v}^q; v^p, v^q) = \mathscr{D}_1(v^q, v^p; \hat{v}^q) + \mathscr{D}_2(u^p, v^q; \hat{u}^p), \tag{8.10c}$$

with C_p and C_q the coefficients in the constitutive relation between the energy and co-energy variables, (7.2). In the discrete setting $\mathscr{A}$ satisfies the following relation:

Lemma 8.6 For $u_h^p, v^p \in E_p(\mathscr{T}_h), v_h^q, v^q \in E_q(\mathscr{T}_h)$ and corresponding boundary port variables $\hat{u}_h^p \in E_p(\mathscr{F}_h), \hat{v}_h^q \in E_q(\mathscr{F}_h)$ the bilinear forms satisfy

$$\begin{aligned}\mathscr{A}\left(u_h^p, v_h^q; v^p, v^q\right) &= \sum_{K\in\mathscr{T}_h}\left(\left\langle v^p \mid C_p * u^p\right\rangle_K + \left\langle v^q \mid C_q * v^q\right\rangle_K\right) \\ &= \sum_{K\in\mathscr{T}_h}\sum_{f\in\Delta(K),\dim(f)\in[p,r+p-1]}\int_f u_h^p(\mu)\cdot v^p(\mu)\cdot C_p(\mu)|_f\,\mathrm{d}\mu \\ &\quad + \sum_{K\in\mathscr{T}_h}\int_K v_h^q(\mu)\cdot v^q(\mu)\cdot C_q(\mu)|_K\,\mathrm{d}\mu\end{aligned}$$

with C_p and C_q the coefficients in the constitutive relation between the energy and co-energy variables, (7.2).

Proof. This relation is immediate using (4.19) for $E_p(K)$. □

LEMMA 8.7 The DG formulation (7.10) with $\beta^p(t) \in H^1\Lambda^{n-p}(\Omega)$ and $\beta^q(t) \in H\Lambda^{n-q}(\Omega)$ the exact solution of the partial differential equations represented by the Dirac structure (3.12) satisfies $\forall \nu^p \in E_p(\mathscr{T}_h), \forall \nu^q \in E_q(\mathscr{T}_h)$

$$\mathscr{A}(\dot\beta^p, \dot\beta^q; \nu^p, \nu^q) + \mathscr{B}(\beta^p, \beta^q, \beta^p, \beta^q; \nu^p, \nu^q) = 0, \tag{8.11}$$

and the orthogonality condition

$$\mathscr{A}\left(\dot\beta^p - \dot\beta^p_h, \dot\beta^q - \dot\beta^q_h; \nu^p, \nu^q\right) + \mathscr{B}\left(\beta^p - \beta^p_h, \beta^q - \beta^q_h, \beta^p - \hat\beta^p_h, \beta^q - \hat\beta^q_h; \nu^p, \nu^q\right) = 0. \tag{8.12}$$

8.3 A priori *error estimate for port-Hamiltonian DG discretization*

In this section we will prove an *a priori* error estimate for the port-Hamiltonian discontinuous Galerkin discretization.

THEOREM 8.8 (Error Estimate). Given the port-Hamiltonian discontinuous Galerkin formulation (7.10) with (6.11)-(6.12) on the polyhedron domain $\Omega \subset \mathbb{R}^n$, which is based on the generalized Stokes–Dirac structure (4.53) with numerical solutions $\beta^p_h(t) \in E_p(\mathscr{T}_h)$ and $\beta^q_h(t) \in E_q(\mathscr{T}_h)$ for $t \in (0, T]$. Assume that the exact solutions $\beta^p(t) \in H^1\Lambda^{n-p}(\Omega)$ and $\beta^q(t) \in H\Lambda^{n-q}_h(\Omega)$ with $t \in (0, T]$ are sufficiently smooth and the boundary conditions are applied exactly at $\partial\Omega$, then we have the following *a priori* error estimate

$$\begin{aligned}
&||\beta^p(t) - \beta^p_h(t)||^2_{L^2\Lambda^{n-p}(\Omega)} + ||\beta^q(t) - \beta^q_h(t)||^2_{L^2\Lambda^{n-q}(\Omega)} \\
&\leq Ch^{2(r+1)}\Big(e^{2\epsilon T}\left(||\beta^p(0)||^2_{H^{r+1}\Lambda^{n-p}(\Omega)} + ||\beta^q(0)||^2_{H^{r+1}\Lambda^{n-q}(\Omega)}\right) \\
&\quad + \int_0^T e^{2\epsilon(T-t)}\left(||\dot\beta^p(t)||^2_{H^{r+1}\Lambda^{n-p}(\Omega)} + ||\dot\beta^q(t)||^2_{H^{r+1}\Lambda^{n-q}(\Omega)}\right)\mathrm{d}t\Big),
\end{aligned} \tag{8.13}$$

with C and ϵ strictly positive constants independent of β^p, β^q and the mesh size h.

Proof. For Case 1 in Section 4.2, the error contributions satisfy

$$\begin{aligned}
er^p &= \beta^p - \beta^p_h = \beta^p - \pi^{n-p}_{h-}\beta^p + \pi^{n-p}_{h-} er^p, \\
er^q &= \beta^q - \beta^q_h = \beta^q - \pi^{n-q}_{h}\beta^q + \pi^{n-q}_{h} er^q,
\end{aligned} \tag{8.14}$$

where we used here the properties $\pi^{n-p}_{h-}\beta^p_h = \beta^p_h$ and $\pi^{n-q}_h\beta^q_h = \beta^q_h$ of the projection operators π^{n-p}_{h-} and π^{n-q}_h, which follow directly from Definitions 8.2 and 8.3. Note for Case 2 in Section 4.2 the projection operator $\pi_{h,-}$ and π^h need to be interchanged in the proof of Theorem 8.8. Introducing the test forms as

TABLE 3 *Order of accuracy of the pH-DGFEM for the scalar wave equation, $V_h \in \mathscr{P}_0^- \Lambda^0(K)$ and $\sigma_h \in \mathscr{P}_1 \Lambda^1(K)$ for Dirichlet boundary conditions*

		L^2error				L^∞error			
θ	h	Velocity	Order	Stress	Order	Velocity	Order	Stress	Order
0	0.0625	0.184760	–	0.001473	–	2.1484 e-06	–	0.001930	–
	0.03125	0.092513	0.9979	7.4556e−04	0.9826	1.0338e−06	1.055	9.7758e−04	0.9717
	0.015625	0.046273	0.9994	3.7454e−04	0.9932	5.0462e−07	1.034	4.9035e−04	0.9954
1/2	0.0625	0.184760	–	9.7832e−04	–	9.8968e−07	–	0.001252	–
	0.03125	0.092513	0.9979	4.9335e−04	0.9877	4.7913e−07	1.029	6.3741e−04	0.9718
	0.015625	0.046273	0.9994	2.4735e−04	0.9960	2.3464e−07	1.046	3.2016e−04	0.9929
1	0.0625	0.184760	–	0.001473	–	2.1484e−06	–	0.001930	–
	0.03125	0.092513	0.9979	7.4556e−04	0.9826	1.0338e−06	1.055	9.8423e−04	0.9717
	0.015625	0.046273	0.9994	3.7454e−04	0.9932	5.0462e−07	1.034	4.9553e−04	0.9900

$v^p = \pi_{h-}^{n-p} er^p \in E_p(\mathscr{T}_h)$ and $v^q = \pi_h^{n-q} er^q \in E_q(\mathscr{T}_h)$, into (8.12) gives

$$\mathscr{A}\left(\dot\beta^p - \dot\beta_h^p, \dot\beta^q - \dot\beta_h^q; \pi_{h-}^{n-p} er^p, \pi_h^{n-q} er^q\right) + \mathscr{B}\left(\beta^p - \beta_h^p, \beta^q - \beta_h^q, \beta^p - \hat\beta_h^p, \beta^q - \hat\beta_h^q; \pi_{h-}^{n-p} er^p, \pi_h^{n-q} er^q\right) = 0. \quad (8.15)$$

Using the linearity of $\mathscr{A}$ and (8.14), we obtain

$$\begin{aligned}&\mathscr{A}\left(\dot\beta^p - \dot\beta_h^p, \dot\beta^q - \dot\beta_h^q; \pi_{h-}^{n-p} er^p, \pi_h^{n-q} er^q\right)\\ &= \mathscr{A}\left(\dot\beta^p - \pi_{h-}^{n-p}\dot\beta^p, \dot\beta^q - \pi_h^{n-q}\dot\beta^q; \pi_{h-}^{n-p} er^p, \pi_h^{n-q} er^q\right)\\ &\quad + \mathscr{A}\left(\pi_{h-}^{n-p}\dot{er}^p, \pi_h^{n-q}\dot{er}^q; \pi_{h-}^{n-p} er^p, \pi_h^{n-q} er^q\right).\end{aligned} \quad (8.16)$$

Also, using the linearity of $\mathscr{D}_1, \mathscr{D}_2$ and $\mathscr{B}$ with $\pi_h^{n-q}\hat\beta^q, \pi_h^{n-q}\hat{er}^q$ and $\pi_{h-}^{n-p}\hat\beta^p$, $\pi_{h-}^{n-p}\hat{er}^p$ the projections on the faces $f \in \partial\Delta(K)$, we obtain

$$\begin{aligned}&\mathscr{B}\left(\beta^p - \beta_h^p, \beta^q - \beta_h^q, \beta^p - \hat\beta_h^p, \beta^q - \hat\beta_h^q; \pi_{h-}^{n-p} er^p, \pi_h^{n-q} er^q\right)\\ &= \mathscr{B}\left(\beta^p - \pi_{h-}^{n-p}\beta^p, \beta^q - \pi_h^{n-q}\beta^q, \beta^p - \pi_{h-}^{n-p}\hat\beta_h^p, \beta^q - \pi_h^{n-q}\hat\beta_h^q; \pi_{h-}^{n-p} er^p, \pi_h^{n-q} er^q\right)\\ &\quad + \mathscr{B}\left(\pi_{h-}^{n-p} er^p, \pi_h^{n-q} er^q, \pi_{h-}^{n-p}\hat{er}^p, \pi_h^{n-q}\hat{er}^q; \pi_{h-}^{n-p} er^p, \pi_h^{n-q} er^q\right).\end{aligned} \quad (8.17)$$

Substituting (8.16) and (8.17) into (8.15), we obtain

$$\begin{aligned}
&\mathscr{A}\left(\dot{\beta}^p - \pi_{h-}^{n-p}\dot{\beta}^p, \dot{\beta}^q - \pi_h^{n-q}\dot{\beta}^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right) \\
&\quad + \mathscr{A}\left(\pi_{h-}^{n-p}\dot{er}^p, \pi_h^{n-q}\dot{er}^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right) \\
&\quad + \mathscr{B}\left(\beta^p - \pi_{h-}^{n-p}\beta^p, \beta^q - \pi_h^{n-q}\beta^q, \beta^p - \pi_{h-}^{n-p}\hat{\beta}_h^p, \beta^q - \pi_h^{n-q}\hat{\beta}_h^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right) \\
&\quad + \mathscr{B}\left(\pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q, \pi_{h-}^{n-p}\hat{er}^p, \pi_h^{n-q}\hat{er}^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right) \\
&\quad = 0.
\end{aligned} \tag{8.18}$$

Using Lemma 8.6 and (7.6), we obtain

$$\begin{aligned}
&\mathscr{A}\left(\pi_{h-}^{n-p}\dot{er}^p, \pi_h^{n-q}\dot{er}^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right) \\
&= \sum_{K\in\mathscr{T}_h}\left(\sum_{f\in\Delta(K),\dim(f)\in[p,r+p-1]}\int_f \pi_{h-}^{n-p}\dot{er}^p(\mu)\cdot\pi_{h-}^{n-p}er^p(\mu)\cdot C_p(\mu)|_f\,\mathrm{d}\mu\right. \\
&\qquad\left. + \int_K \pi_h^{n-q}\dot{er}^q(\mu)\cdot\pi_h^{n-q}er^q(\mu)\cdot C_q(\mu)\,\mathrm{d}\mu\right) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\pi_{h-}^{n-p}er^p\right\|^2_{wL^2\Lambda^{n-p}(\Omega)} + \left\|\pi_h^{n-q}er^q\right\|^2_{wL^2\Lambda^{n-q}(\Omega)}\right).
\end{aligned} \tag{8.19}$$

Using Lemma 8.5, we obtain

$$\begin{aligned}
&\mathscr{B}\left(\beta^p - \pi_{h-}^{n-p}\beta^p, \beta^q - \pi_h^{n-q}\beta^q, \beta^p - \pi_{h-}^{n-p}\hat{\beta}_h^p, \beta^q - \pi_h^{n-q}\hat{\beta}_h^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right) \\
&= \mathscr{D}_1\left(\beta^q - \pi_h^{n-q}\beta^q, \pi_{h-}^{n-p}er^p; \beta^q - \pi_h^{n-q}\hat{\beta}^q\right) \\
&\quad + \mathscr{D}_2\left(\beta^p - \pi_{h-}^{n-p}\beta^p, \pi_h^{n-q}er^q; \beta^p - \pi_{h-}^{n-p}\hat{\beta}^p\right) \\
&= 0.
\end{aligned} \tag{8.20}$$

Lemma 8.1 gives

$$\begin{aligned}
&\mathscr{B}\left(\pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q, \pi_{h-}^{n-p}\hat{er}^p, \pi_h^{n-q}\hat{er}^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right) \\
&\qquad = -\sum_{F\in\mathscr{F}_o}\left(\langle y_h^q \mid u_h^p\rangle_F + \langle u_h^q \mid y_h^p\rangle_F\right),
\end{aligned} \tag{8.21}$$

where $y_h^p, u_h^p \in E_p(\mathscr{F}_h)$ and $y_h^q, u_h^q \in E_q(\mathscr{F}_h)$. Using the definition of the external boundary port variables y_h^p and y_h^q as stated in the generalized Stokes–Dirac structure (4.53) with $y_h^{p*} = Q(y_h^q)$, we can rewrite

(8.21) as

$$\mathscr{B}\left(\pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q, \pi_{h-}^{n-p}\hat{er}^p, \pi_h^{n-q}\hat{er}^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right)$$
$$= \sum_{F\in\mathscr{F}_o}\left(\left\langle \mathrm{tr}_q\pi_h^{n-q}er^q \mid u_h^p\right\rangle_F + (-1)^p\left\langle u_h^q \mid \mathrm{tr}_p\pi_{h-}^{n-p}er^p\right\rangle_F\right). \tag{8.22}$$

Using $\pi_{h-}^{n-p}er^p = \pi_{h-}^{n-p}\beta^p - \beta_h^p$ and $\pi_h^{n-q}er^q = \pi_h^{n-q}\beta^q - \beta_h^q$, we obtain

$$\begin{aligned}\mathrm{tr}_p\pi_{h-}^{n-p}er^p &= \mathrm{tr}_p\pi_{h-}^{n-p}\beta^p - \mathrm{tr}_p\beta_h^p,\\ \mathrm{tr}_q\pi_h^{n-q}er^q &= \mathrm{tr}_q\pi_h^{n-q}\beta^q - \mathrm{tr}_q\beta_h^q.\end{aligned} \tag{8.23}$$

Assume that the boundary conditions are applied exactly, then (8.23) gives $\mathrm{tr}_p\pi_{h-}^{n-p}er^p = 0$ and $\mathrm{tr}_q\pi_h^{n-q}er^q = 0$. Thus,

$$\mathscr{B}\left(\pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q, \pi_{h-}^{n-p}\hat{er}^p, \pi_h^{n-q}\hat{er}^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right) = 0. \tag{8.24}$$

Furthermore, we have

$$\mathscr{A}\left(\dot{\beta}^p - \pi_{h-}^{n-p}\dot{\beta}^p, \dot{\beta}^q - \pi_h^{n-q}\dot{\beta}^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right)$$
$$= \left\langle \dot{\beta}^p - \pi_{h-}^{n-p}\dot{\beta}^p, \pi_{h-}^{n-p}er^p\right\rangle_{wL^2\Lambda^{n-p}(\Omega)} + \left\langle \dot{\beta}^q - \pi_h^{n-q}\dot{\beta}^q \mid \pi_h^{n-q}er^q\right\rangle_{wL^2\Lambda^{n-q}(\Omega)}. \tag{8.25}$$

Using Cauchy's Schwarz inequality with $\epsilon > 0$ and Young's inequality with exponent 2, we can simplify (8.25) as

$$\begin{aligned}&\mathscr{A}\left(\dot{\beta}^p - \pi_{h-}^{n-p}\dot{\beta}^p, \dot{\beta}^q - \pi_h^{n-q}\dot{\beta}^q; \pi_{h-}^{n-p}er^p, \pi_h^{n-q}er^q\right)\\ &\le \frac{1}{4\epsilon}\left|\left|\dot{\beta}^p - \pi_{h-}^{n-p}\dot{\beta}^p\right|\right|^2_{wL^2\Lambda^{n-p}(\Omega)} + \epsilon\left|\left|\pi_{h-}^{n-p}er^p\right|\right|^2_{wL^2\Lambda^{n-p}(\Omega)}\\ &\quad + \frac{1}{4\epsilon}\left|\left|\dot{\beta}^q - \pi_h^{n-q}\dot{\beta}^q\right|\right|^2_{wL^2\Lambda^{n-q}(\Omega)} + \epsilon\left|\left|\pi_h^{n-q}er^q\right|\right|^2_{wL^2\Lambda^{n-q}(\Omega)}.\end{aligned} \tag{8.26}$$

Using Lemma 8.4, the following bounds hold

$$\begin{aligned}\left|\left|\dot{\beta}^p - \pi_{h-}^{n-p}\dot{\beta}^p\right|\right|^2_{wL^2\Lambda^{n-p}(\Omega)} &\le Ch^{2(r+1)}\left|\left|\dot{\beta}^p(t)\right|\right|^2_{H^{r+1}\Lambda^{n-p}(\Omega)},\\ \left|\left|\dot{\beta}^q - \pi_h^{n-q}\dot{\beta}^q\right|\right|^2_{wL^2\Lambda^{n-q}(\Omega)} &\le Ch^{2(r+1)}||\dot{\beta}^q(t)||^2_{H^{r+1}\Lambda^{n-q}(\Omega)},\end{aligned} \tag{8.27}$$

where r is the lowest order of the polynomial differential form spaces E_p, E_q and h is the element size. Finally, (8.26) simplifies to

$$\begin{aligned}&\mathscr{A}\left(\dot{\beta}^p-\pi_{h-}^{n-p}\dot{\beta}^p,\dot{\beta}^q-\pi_h^{n-q}\dot{\beta}^q;\pi_{h-}^{n-p}er^p,\pi_h^{n-q}er^q\right)\\&\le Ch^{2(r+1)}\left(||\dot{\beta}^p(t)||^2_{H^{r+1}\Lambda^{n-p}(\Omega)}+||\dot{\beta}^q(t)||^2_{H^{r+1}\Lambda^{n-q}(\Omega)}\right)\\&\quad+\epsilon\left(||\pi_{h-}^{n-p}er^p||^2_{wL^2\Lambda^{n-p}(\Omega)}+||\pi_h^{n-q}er^q||^2_{wL^2\Lambda^{n-q}(\Omega)}\right).\end{aligned}\tag{8.28}$$

Using (8.18), with (8.19), (8.20), (8.24) and (8.28), gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\left|\pi_{h-}^{n-p}er^p\right|\right|^2_{wL^2\Lambda^{n-p}(\Omega)}+\left|\left|\pi_h^{n-q}er^q\right|\right|^2_{wL^2\Lambda^{n-q}(\Omega)}\right)\\&\le Ch^{2(r+1)}\left(||\dot{\beta}^p(t)||^2_{H^{r+1}\Lambda^{n-p}(\Omega)}+||\dot{\beta}^q(t)||^2_{H^{r+1}\Lambda^{n-q}(\Omega)}\right)\\&\quad+\epsilon\left(\left|\left|\pi_{h-}^{n-p}er^p\right|\right|^2_{wL^2\Lambda^{n-p}(\Omega)}+\left|\left|\pi_h^{n-q}er^q\right|\right|^2_{wL^2\Lambda^{n-q}(\Omega)}\right).\end{aligned}\tag{8.29}$$

Next, we will use Gronwall's inequality, which states (Evans, 2010): let $\eta(\cdot)$ be a non-negative absolutely continuous function on $[0,T]$, which satisfies for all t the differential inequality

$$\eta'(t)\le\phi(t)\eta(t)+\psi(t),\tag{8.30}$$

where $\phi(t)$ and $\psi(t)$ are non-negative measurable functions on $[0,T]$. Then

$$\eta(t)\le e^{\int_0^t\phi(s)\,\mathrm{d}s}\left[\eta(0)+\int_0^t\psi(s)\,\mathrm{d}s\right],\tag{8.31}$$

for all $0\le t\le T$. Using the above-stated Gronwall's inequality, we obtain

$$\begin{aligned}&\left|\left|\pi_{h-}^{n-p}er^p\right|\right|^2_{wL^2\Lambda^{n-p}(\Omega)}+\left|\left|\pi_h^{n-q}er^q\right|\right|^2_{wL^2\Lambda^{n-q}(\Omega)}\le e^{2\epsilon T}\left(\left|\left|\pi_{h-}^{n-p}er^p(0)\right|\right|^2_{wL^2\Lambda^{n-p}(\Omega)}\right.\\&\left.+\left|\left|\pi_h^{n-q}er^q(0)\right|\right|^2_{wL^2\Lambda^{n-q}(\Omega)}\right)+Ch^{2(r+1)}\int_0^T e^{2\epsilon(T-t)}\left(||\dot{\beta}^p(t)||^2_{H^{r+1}\Lambda^{n-p}(\Omega)}\right.\\&\left.+||\dot{\beta}^q(t)||^2_{H^{r+1}\Lambda^{n-q}(\Omega)}\right)\mathrm{d}t.\end{aligned}\tag{8.32}$$

Let $\beta_h^p(x,0)=\pi_{h-}^{n-p}\beta^p(x,0)$ and $\beta_h^q(x,0)=\pi_h^{n-q}\beta^q(x,0)$ for all $x\in\Omega$, be the projections of the initial conditions to the discontinuous Galerkin finite element spaces. Using the fact that $\pi_{h,-}^{n-p}$ and π_h^{n-q} are projections and $||\pi_h u||_{L^2}=||u||_{L^2}$ gives (8.13). □

TABLE 4 *Order of accuracy of the pH-DGFEM for the scalar wave equation,* $V_h \in \mathscr{P}_1^- \Lambda^0(K)$ *and* $\sigma_h \in \mathscr{P}_2 \Lambda^1(K)$ *for Dirichlet boundary conditions*

		L^2 error				L^∞ error			
θ	h	Velocity	Order	Stress	Order	Velocity	Order	Stress	Order
0	0.0625	0.008886	–	1.021e−04	–	1.211e−06	–	5.9490e−04	–
	0.03125	0.002224	1.9980	2.5540e−05	1.9995	3.0701e−07	1.9802	1.5026e−04	1.9851
	0.015625	5.5639e−04	1.9995	6.3882e−06	1.9992	7.7017e−08	1.9950	3.7662e−05	1.9962
1/2	0.0625	0.008886	–	7.5472e−05	–	1.8361e−07	–	4.3570e−04	–
	0.03125	0.002224	1.9980	1.8886e−05	1.9985	4.6103e−08	1.9937	1.0979e−04	1.9884
	0.015625	5.5639e−04	1.9995	4.7222e−06	1.9998	1.1555e−08	1.9962	2.7504e−05	1.9971
1	0.0625	0.008886	–	1.0224e−04	–	1.1824e−06	–	5.8582e−04	–
	0.03125	0.002224	1.9980	2.5555e−05	2.0003	3.0517e−07	1.9540	1.4969e−04	1.9684
	0.015625	5.5639e−04	1.9995	6.3901e−06	1.9997	7.6901e−08	1.9885	3.7626e−05	1.9921

TABLE 5 *Order of accuracy of the pH-DGFEM for the scalar wave equation,* $V_h \in \mathscr{P}_2^- \Lambda^0(K)$ *and* $\sigma_h \in \mathscr{P}_3 \Lambda^1(K)$ *for Dirichlet boundary conditions*

		L^2 error				L^∞ error			
θ	h	Velocity	Order	Stress	Order	Velocity	Order	Stress	Order
0	0.0625	2.8790e−04	–	5.7062e−06	–	1.7594e−07	–	6.7039e−04	–
	0.03125	3.6031e−05	2.9982	7.2018e−07	2.9861	2.2527e−08	2.9653	8.4647e−05	2.9854
	0.015625	4.5053e−06	2.9995	9.0376e−08	2.9943	2.8328e−09	2.9913	1.0607e−05	2.9964
1/2	0.0625	2.8790e−04	–	3.0651e−06	–	4.6243e−08	–	3.7029e−04	–
	0.03125	3.6031e−05	2.9982	3.8035e−07	3.0105	5.9055e−09	2.9691	4.6908e−05	2.9807
	0.015625	4.5053e−06	2.9995	4.7392e−08	3.0046	7.4213e−10	2.9923	5.8831e−06	2.9952
1	0.0625	2.8790e−04	–	5.7071e−06	–	1.7496e−07	–	6.8605e−04	–
	0.03125	3.6031e−05	2.9982	7.2023e−07	2.9862	2.2492e−08	2.9594	8.5146e−05	3.0103
	0.015625	4.5053e−06	2.9995	9.0380e−08	2.9944	2.8317e−09	2.9896	1.0622e−05	3.0027

TABLE 6 *Order of accuracy of the pH-DGFEM for the scalar wave equation,* $V_h \in \mathscr{P}_0^- \Lambda^0(K)$ *and* $\sigma_h \in \mathscr{P}_1 \Lambda^1(K)$ *for Neumann boundary conditions*

		L^2 error				L^∞ error			
θ	h	Velocity	Order	Stress	Order	Velocity	Order	Stress	Order
0	0.0625	0.187760	–	0.001513	–	2.1510 e−06	–	0.001890	–
	0.03125	0.094016	0.9979	7.6567e−04	0.9826	1.0352e−06	1.055	9.6372e−04	0.9717
	0.015625	0.047027	0.9994	3.8464e−04	0.9932	5.0554e−07	1.034	4.8339e−04	0.9954
1/2	0.0625	0.187760	–	0.01273	–	9.8968e−07	–	0.001252	–
	0.03125	0.094016	0.9979	6.4194e−04	0.9877	4.7913e−07	1.029	6.3741e−04	0.9718
	0.015625	0.047027	0.9994	3.2186e−04	0.9960	2.3464e−07	1.046	3.2016e−04	0.9929
1	0.0625	0.187760	–	0.001513	–	2.1510 e−06	–	0.001890	–
	0.03125	0.094016	0.9979	7.6567e−04	0.9826	1.0352e−06	1.055	9.6372e−04	0.9717
	0.015625	0.047027	0.9994	3.8464e−04	0.9932	5.0554e−07	1.034	4.8339e−04	0.9954

9. Results

To support our theory we apply the formulation for the DG discretization of linear port-Hamiltonian systems, presented in Section 6, to the scalar wave equation. For an n-dimensional oriented manifold Ω

TABLE 7 *Order of accuracy of the pH-DGFEM for the scalar wave equation, $V_h \in \mathscr{P}_1^- \Lambda^0(K)$ and $\sigma_h \in \mathscr{P}_2 \Lambda^1(K)$ for Neumann boundary conditions*

		L^2error				L^∞error			
θ	h	Velocity	Order	Stress	Order	Velocity	Order	Stress	Order
0	0.0625	0.008912	–	1.0212e−04	–	1.1988e−06	–	5.9511e−04	–
	0.03125	0.002231	1.9980	2.5308e−05	1.9995	3.0384e−07	1.9802	1.5032e−04	1.9851
	0.015625	5.5796e−04	1.9995	6.3307e−06	1.9992	7.6224e−08	1.9950	3.7679e−05	1.9962
1/2	0.0625	0.008912	–	7.5512e−05	–	1.8361e−07	–	4.3570e−04	–
	0.03125	0.002231	1.9980	1.8897e−05	1.9985	4.6103e−08	1.9937	1.0979e−04	1.9884
	0.015625	5.5796e−04	1.9995	4.7250e−06	1.9998	1.1555e−08	1.9962	2.7504e−05	1.9971
1	0.0625	0.008912	–	1.0212e−04	–	1.1988e−06	–	5.9511e−04	–
	0.03125	0.002231	1.9980	2.5308e−05	1.9995	3.0384e−07	1.9802	1.5032e−04	1.9851
	0.015625	5.5796e−04	1.9995	6.3307e−06	1.9992	7.6224e−08	1.9950	3.7679e−05	1.9962

TABLE 8 *Order of accuracy of the pH-DGFEM for the scalar wave equation, $V_h \in \mathscr{P}_2^- \Lambda^0(K)$ and $\sigma_h \in \mathscr{P}_3 \Lambda^1(K)$ for Neumann boundary conditions*

		L^2error				L^∞error			
θ	h	Velocity	Order	Stress	Order	Velocity	Order	Stress	Order
0	0.0625	2.8810e−04	–	5.6995e−06	–	1.7594e−07	–	6.7039e−04	–
	0.03125	3.6057e−05	2.9982	7.1933e−07	2.9861	2.2527e−08	2.9653	8.4647e−05	2.9854
	0.015625	4.5087e−06	2.9995	8.9948e−08	2.9943	2.8328e−09	2.9913	1.0607e−05	2.9964
1/2	0.0625	2.8790e−04	–	3.0651e−06	–	4.6243e−08	–	3.7029e−04	–
	0.03125	3.6031e−05	2.9982	3.8035e−07	3.0105	5.9055e−09	2.9691	4.6908e−05	2.9807
	0.015625	4.5053e−06	2.9995	4.7392e−08	3.0046	7.4213e−10	2.9923	5.8831e−06	2.9952
1	0.0625	2.8810e−04	–	5.6995e−06	–	1.7594e−07	–	6.7039e−04	–
	0.03125	3.6057e−05	2.9982	7.1933e−07	2.9861	2.2527e−08	2.9653	8.4647e−05	2.9854
	0.015625	4.5087e−06	2.9995	8.9948e−08	2.9943	2.8328e−09	2.9913	1.0607e−05	2.9964

the scalar wave equation is given in vector notation by

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \frac{1}{\mu}\nabla \cdot (Y \nabla u) \quad \text{in } \Omega, \tag{9.1}$$

along with appropriate boundary conditions at the boundary $\partial\Omega$. Here $u \in \mathbb{R}^n$ is the displacement, $\mu \in \mathbb{R}$ the mass density and $Y \in \mathbb{R}^{n\times n}$ the modulus of rigidity, ∇ the nabla operator and t time. In a port-Hamiltonian formulation (9.1) is written as, see Talasila *et al.* (2002); van der Schaft & Maschke (2002); Golo *et al.* (2004); van der Schaft *et al.* (2014),

$$\begin{aligned} \begin{bmatrix} \dot{\rho} \\ \dot{\epsilon} \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^n d \\ -d & 0 \end{bmatrix} \begin{bmatrix} V \\ \sigma \end{bmatrix} \\ \begin{bmatrix} V^b \\ \sigma^b \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & (-1)^{n-1} \end{bmatrix} \begin{bmatrix} \mathrm{tr}(V) \\ \mathrm{tr}(\sigma) \end{bmatrix}. \end{aligned} \tag{9.2}$$

Here, the kinetic momentum $\rho = \mu * \frac{\mathrm{d}u}{\mathrm{d}t} \in L^2\Lambda^n(\Omega)$, with $u \in L^2\Lambda^0(\Omega)$ and the elastic strain $\epsilon = \mathrm{d}u \in L^2\Lambda^1(\Omega)$ are the energy variables. The velocity $V = \frac{\mathrm{d}u}{\mathrm{d}t} \in H^1\Lambda^0(\Omega)$ and elastic stress $\sigma = (Y * \mathrm{d}u) \in$

$H\Lambda^{n-1}(\Omega)$ are the co-energy variables. Note, d is the exterior derivative for differential forms and $*$ the Hodge star operator. The Hamiltonian energy for the wave equation is given by

$$H = \frac{1}{2}\int_{\Omega}(\epsilon \wedge \sigma + \rho \wedge V). \tag{9.3}$$

In $2D$ the energy variables ρ is a 2-form and ϵ a 1-form, whereas the co-energy variable V is a 0-form and σ a 1-form. So, we take $p = 2, q = 1, n = 2$ and consider the domain $\Omega = [-1,1]\times[-1,1]$. We choose the discrete energy variables as $V_h \in E_p(\mathscr{T}_h), \sigma_h \in E_q(\mathscr{T}_h), \rho_h \in F_p(\mathscr{T}_h)$ and $\epsilon \in F_q(\mathscr{T}_h)$. Following the procedure discussed in Section 6, see also (7.10), we obtain the following port-Hamiltonian discontinuous Galerkin (pHDG) formulation of the wave equation: find $V_h \in E_p(\mathscr{T}_h), \sigma_h \in E_q(\mathscr{T}_h)$ such that for all $\nu^p \in E_p(\mathscr{T}_h), \nu^q \in E_q(\mathscr{T}_h)$,

$$\begin{aligned}
\sum_{K\in\mathscr{T}_h}\left\langle \nu^p \mid \star_p \dot{V}_h\right\rangle_K &= \sum_{K\in\mathscr{T}_h}\left\langle \sigma_h \mid d_{qp}\nu_h^p\right\rangle_K + (-1)^p\sum_{F\in\mathscr{F}_o}\left\langle \hat{\sigma}_h \mid \mathrm{tr}_p \nu_h^p\right\rangle_F \\
&\quad + (-1)^p\sum_{F\in\mathscr{F}_i}\left\langle \hat{\sigma}_h \mid (\mathrm{tr}_p\nu_h^p|_L + \mathrm{tr}_p\nu_h^p|_R)\right\rangle_F, \\
\sum_{K\in\mathscr{T}_h}\left\langle \nu^q \mid \star_q \dot{\sigma}_h\right\rangle_K &= (-1)^{p+1}\sum_{K\in\mathscr{T}_h}\left\langle d_{pq}\nu_h^q \mid V_h\right\rangle_K + (-1)^p\sum_{F\in\mathscr{F}_o}\left\langle \mathrm{tr}_q\nu_h^q \mid \hat{V}_h\right\rangle_F \\
&\quad + (-1)^p\sum_{F\in\mathscr{F}_i}\langle\left(\mathrm{tr}_q\nu_h^q|_L + \mathrm{tr}_q\nu_h^q|_R\right) \mid \hat{V}_h\rangle_F,
\end{aligned} \tag{9.4}$$

where for $F \in \mathscr{F}_i$

$$\begin{aligned}
\hat{V}_h &= (1-\theta)\mathrm{tr}_p V_h|_L + \theta\,\mathrm{tr}_p V_h|_R, \\
\hat{\sigma}_h &= \theta\,\mathrm{tr}_q\sigma_h|_L + (1-\theta)\mathrm{tr}_q\sigma_h|_R,
\end{aligned} \tag{9.5}$$

with $\theta \in [0,1]$ and for $F \in \mathscr{F}_o$,

$$\hat{V}_h = \mathrm{tr}_p V_h, \qquad \hat{\sigma}_h = \mathrm{tr}_q\sigma_h. \tag{9.6}$$

Note that if $\theta = 1$ then we have an alternating type of flux, which is similar to the flux considered for LDG methods (Yan & Shu, 2012). As model problem we use the exact solution for the displacement u, velocity V and elastic stress σ given by

$$\begin{aligned}
u &= \frac{1}{2\pi}\sin(2\pi t)(\sin(2\pi x) + \sin(2\pi y)), && (x,y) \in [-1,1]\times[-1,1], \\
V &= \cos(2\pi t)(\sin(2\pi x) + \sin(2\pi y)), && (x,y) \in [-1,1]\times[-1,1], \\
\sigma &= \sin(2\pi t)(\cos(2\pi x)\,\mathrm{d}x + \cos(2\pi y)\,\mathrm{d}y), && (x,y) \in [-1,1]\times[-1,1],
\end{aligned} \tag{9.7}$$

with two types of boundary conditions. Example 1: Dirichlet boundary condition on the whole boundary

$$\begin{aligned}
V &= \cos(2\pi t)\sin(2\pi x), \quad x \in [-1,1], y \in \{-1,1\}, \\
V &= \cos(2\pi t)\sin(2\pi y), \quad x \in \{-1,1\}, y \in [1,-1].
\end{aligned} \tag{9.8}$$

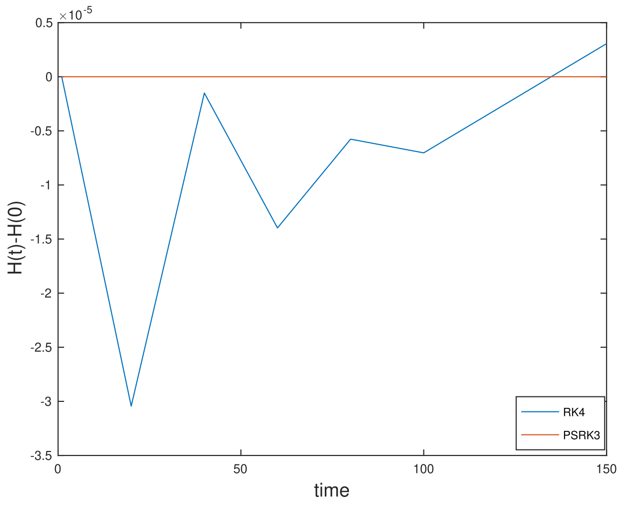

FIG. 2. Comparison of long term evolution of energy for scalar wave equation using fourth-order Runge–Kutta time integrator (blue) and partitioned symplectic third-order Runge–Kutta time integrator (red).

Example 2: Dirichlet and Neumann boundary conditions at different parts of the boundary

$$\begin{aligned} V &= \cos(2\pi t)\sin(2\pi x), \quad & x \in [-1,1], y = -1,\\ V &= \cos(2\pi t)\sin(2\pi y), \quad & x = -1, y \in [1,-1],\\ \sigma\cdot\hat{n} &= \sin(2\pi t)\cos(2\pi x), \quad & x \in [-1,1], y = 1,\\ \sigma\cdot\hat{n} &= \sin(2\pi t)\cos(2\pi y), \quad & x = 1, y \in [1,-1], \end{aligned} \tag{9.9}$$

where $\hat{n}$ is the unit outward normal at the domain boundary. The mass density and modulus of rigidity are taken as 1. Using $p = 2, q = 1$ in the expression for the spaces E_p and E_q, shown in Table 1, the finite element space V_h identifies with the space of Lagrange finite elements of order r and σ_h with the space of Brezzi–Douglas–Marini finite elements of order $r+1$. The time integration is done with a fourth-order explicit Runge–Kutta method. The final time in the simulations is $T = 1$. The port-Hamiltonian DG finite element algorithm used for the computations is implemented in Matlab. The mesh used in the numerical experiments is a rectangular domain Ω subdivided into standard right-angled triangles, which are further refined into smaller right-angled triangles in subsequent mesh refinements. The L^2 and L^∞ errors in the numerical solution at time $T = 1$ were obtained for $\theta \in \{0, \frac{1}{2}, 1\}$ for the discrete differential form spaces $V_h \in \mathscr{P}_0^-\Lambda^0(K)$ and $\sigma_h \in \mathscr{P}_1\Lambda^1(K)$, $V_h \in \mathscr{P}_1^-\Lambda^0(K), \sigma_h \in \mathscr{P}_2\Lambda^1(K)$ and $V_h \in \mathscr{P}_2^-\Lambda^0(K), \sigma_h \in \mathscr{P}_3\Lambda^1(K)$. For Example 1 (9.7) with Dirichlet boundary conditions on the whole boundary (9.8), see Tables 3, 4 and 5. For Example 2 (9.7) with Dirichlet and Neumann boundary conditions on different parts of the boundary (9.9), see Tables 6, 7 and 8.

In the computations values of $\theta = \{0, 1\}$, require less computing time as compared to values of $\theta \in (0, 1)$ without any significant improvement in the convergence results, which is confirmed by computations for $\theta = 1/3$ and $\theta = 2/3$. Referring to Tables 3–8, we observe that the L^2 and L^∞ errors are restricted by the polynomial order r of the space V_h, resulting in convergence order $r+1$. Note that although σ_h belongs to an $r+1$th order finite element space the convergence order is $r+1$ and not $r+2$.

The reason being that in the port-Hamiltonian formulation the conservation laws are coupled, hence the convergence rate of both the co-energy variables is restricted by the convergence rate of the co-energy variable V, which is discretized using an rth order accurate finite element space. We also observe that the numerical results are consistent with the *a priori* error bounds, given in (8.13). Figure 2 shows the evolution of discrete energy over a long period of time when the velocity $V_h \in \mathscr{P}_2^- \Lambda^0(K)$, and the stress $\sigma_h \in \mathscr{P}_3 \Lambda^1(K)$. The plot shows that the energy difference is oscillating in case of nonsymplectic fourth-order Runge–Kutta time integrator (Romeo, 2020), but is constant in case of partitioned symplectic third-order Runge–Kutta time integrator (Geng, 1993).

Acknowledgements

We thank the Shell-NWO/FOM PhD-75 program, project 15CSER49, for giving us the opportunity and funding to work on this project.

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