Port-Hamiltonian Systems and their Discontinuous Galerkin Discretization

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PORT-HAMILTONIAN SYSTEMS AND THEIR DISCONTINUOUS GALERKIN DISCRETIZATION

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Summary

This dissertation discusses port-Hamiltonian formulations and their numerical discretization for several classes of hyperbolic partial differential equations. The thesis focuses on three key topics: Hamiltonian formulations of the incompressible Euler equations with a free surface, port-Hamiltonian formulations of the incompressible Euler equations with a free surface, and port-Hamiltonian discontinuous Galerkin discretizations for a class of linear hyperbolic partial differential equations.

In Chapter 2, based on the classical formulations, we derive generalized Hamiltonian formulations of the incompressible Euler equations with a free surface using the language of differential forms. Three sets of variables, including velocity, solenoidal velocity, potential, vorticity, and free surface, are used to represent the incompressible Euler equations with a free surface. Additionally, we derive the corresponding Poisson bracket for these sets of variables and express the Hamiltonian systems using these Poisson brackets. Next, we extend the generalized Hamiltonian formulations of the incompressible Euler equations with a free surface to include conditions that permit energy exchange at the boundary of the spatial domain. We derive the corresponding Dirac structure and port-Hamiltonian formulations of the incompressible Euler equations with a domain boundary, consisting of a free surface and a fixed surface with inhomogeneous boundary conditions.

In Chapter 3, we first obtain the weak form of the Dirac structure for a class of linear hyperbolic partial differential equations defined in broken Sobolev spaces. Next, by approximating all variables using piecewise polynomial spaces of differential forms, we derive port-Hamiltonian discontinuous Galerkin (PHDG) discretizations and demonstrate their power conservation properties. We also obtain the corresponding pseudo-Poisson brackets and prove they are also Poisson brackets. Finally, we present several numerical experiments to verify the accuracy and capabilities of PHDG methods.

Samenvatting

Dit proefschrift bespreekt poort-Hamiltoniaanse formuleringen en hun numerieke discretisatie voor verschillende klassen van hyperbolische partiële differentiaalvergelijkingen. Drie belangrijke onderwerpen worden besproken, namelijk Hamiltoniaanse formuleringen van de onsamendrukbare Euler-vergelijkingen met een vrij oppervlak, poort-Hamiltoniaanse formuleringen van de onsamendrukbare Euler-vergelijkingen met een vrij oppervlak, en de poort-Hamiltoniaanse discontinue Galerkin discretisatie voor een klasse van lineaire hyperbolische partiële differentiaalvergelijkingen.

In hoofdstuk 2 worden gebaseerd op de klassieke formuleringen gegeneraliseerde Hamiltoniaanse formuleringen van de onsamendrukbare Euler-vergelijkingen met een vrij oppervlak afgeleid met behulp van differentiaalvormen. Drie groepen van variabelen, waaronder de snelheid, solenoïdale snelheid, potentiaal, vorticiteit en het vrije oppervlak, worden gebruikt om de onsamendrukbare Euler-vergelijkingen met een vrij oppervlak te beschrijven. Daarnaast leiden we de bijbehorende Poisson-haak af voor iedere groep van variabelen en formuleren we de Hamiltoniaanse systemen met behulp van deze Poisson-haken. Vervolgens breiden we de gegeneraliseerde Hamiltoniaanse formuleringen van de onsamendrukbare Euler-vergelijkingen met een vrij oppervlak uit om randvoorwaarden toe te staan die energie-uitwisseling aan de grens van het ruimtelijk domein mogelijk maken. We leiden de bijbehorende Dirac-structuur en poort-Hamiltoniaanse formuleringen af van de onsamendrukbare Euler-vergelijkingen voor een domein waarbij de rand uit een vrij oppervlak en een vast oppervlak met inhomogene randvoorwaarden bestaat.

In hoofdstuk 3 wordt eerst de zwakke vorm van de Dirac-structuur voor een klasse van lineaire hyperbolische partiële differentiaalvergelijkingen gedefinieerd in gebroken Sobolev-ruimten. Vervolgens leiden we de poort-Hamiltoniaanse discontinue Galerkin (PHDG) discretisatie af door alle variabelen te benaderen in stuksgewijze polynomiale ruimten van differentiaalvormen en demonstreren we energiebehoud van deze formulering.

Daarnaast wordt ook de bijbehorende pseudo-Poisson-haak afgeleid en bewijzen we dat dit ook een Poisson-haak is. Ten slotte presenteren we verschillende numerieke berekeningen om de nauwkeurigheid en de mogelijkheden van de PHDG-methoden te demonstreren.

Contents

Chapter 1

Introduction

1.1 Euler equations

The Euler equations represent a fundamental type of equations in fluid dynamics, describing the motion of an inviscid, ideal fluid, and are widely used to describe various physical phenomena. The Euler equations can be applied to model gas dynamics [98], shock waves [31], wave propagation in fluids [34], and aerodynamics [3]. These equations capture the evolution of fluid properties such as velocity and pressure over time. Particularly, when viscosity and thermal conductivity are both zero, the Navier-Stokes equations reduce to the Euler equations. A general introduction to the Euler equations and their main properties can be found in [11, 13, 24, 25, 38, 51].

The Euler equations model both compressible and incompressible flows. The compressible Euler equations describe fluids with varying density, including high-speed aerodynamics [80], compressible gas flows [44], wind tunnels [12], and rocket propulsion systems [83]. In contrast, the incompressible Euler equations are a special case where the density remains constant regardless of the changes in pressure. This incompressible model is suitable for describing flows at low Mach numbers [46], where the effects of compressibility are negligible. The incompressible Euler equations provide an important model to study fluid phenomena like turbulence [73] and are fundamental in understanding various fluid motion scenarios, including transport problems [16] and hydraulic engineering [26]. The dynamics of ocean waves and currents are governed by the incompressible Euler equations with a free surface, and describe the motion of water under the influence of gravitational and other forces [40]. They play an important role in modeling and predicting phenomena like wave propagation and the formation of ocean currents. By studying these equations, researchers gain deeper insight into free surface problems in oceanography and related fields.

A crucial relation obtained from the Euler equations is the vorticity equation, which is a fundamental equation in fluid dynamics describing the evolution of vorticity which is defined as the curl of the velocity field. The vorticity equation focuses on the rotational characteristics of fluid motion and is widely used in describing incompressible flow, e.g. turbulence, atmospheric flows, and aerodynamics where the lift force on a wing is directly linked to air rotation [55, 88].

In this work we focus on the incompressible Euler equations with a free surface, which describes e.g. the motion of water waves or the surface of droplets in air, and is governed by surface tension and gravity forces. The Euler equations are defined in a time-dependent domain $\Omega(t) = \Omega(x,t) \subseteq \mathbb{R}^d, d \in \{1,2,3\}$, with the free surface denoted as $\Sigma(t) = \Sigma(x,t) \subseteq \partial\Omega(t)$ and are given by [25, 96]

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla\big(\frac{p}{\rho} + \Phi\big) \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla \cdot \vec{v} = 0 \qquad \text{in } \Omega, \tag{1.1b}$$

$$\frac{\partial \Sigma}{\partial t} = \langle \vec{v}, \vec{n} \rangle \qquad \text{at } \Sigma, \tag{1.1c}$$

$$p - \bar{p} = \tau k \qquad \text{at } \Sigma, \tag{1.1d}$$

$$\langle \vec{v}, \vec{n} \rangle = g \qquad \text{at } \Gamma. \tag{1.1e}$$

Here $\vec{v}$ denotes the fluid velocity, p the pressure with $\bar{p}$ the constant atmospheric pressure, Φ the gravity potential, ρ the fluid density, τ the surface tension, t time and $\nabla \in \mathbb{R}^d$ the nabla operator. The domain $\Omega(t)$ has the boundary $\partial\Omega$ with external normal vector $\vec{n}$ and mean curvature k, and consists of a nonoverlapping free surface Σ and a fixed boundary Γ where the normal velocity g is specified.

In the Euler equations with a free surface (1.1a) represents the momentum equation and (1.1b) the continuity equation, which ensures conservation of mass. The free surface $\Sigma(t)$ is a two-dimensional manifold in $\mathbb{R}^3$ and considered an unparametrized independent variable. The motion and shape of the free surface $\Sigma(t)$ are governed by the balance between the pressure p, surface tension τ, and the mean surface curvature k. In addition, a fluid particle at the free surface should follow the free surface motion, which implies that the free surface velocity must be equal to the normal fluid velocity at the free surface. These conditions are stated, respectively, by (1.1d) and (1.1c). Finally, (1.1e) denotes a prescribed normal velocity at a fixed boundary surface Γ.

A key feature of the Euler equations with a free surface (1.1) compared to potential flow models for free surface flow, see e.g. [61, 96], is the incorporation of vorticity which represents the rotation of the fluid and is an important physical parameter in fluid dynamics. Another important feature of (1.1) is that the normal velocity at the fixed surface Γ allows for energy exchange with the external environment, which is a key element in port-Hamiltonian formulations, which allow general boundary conditions in a Hamiltonian formulation, and is an important topic of this PhD Thesis.

1.2 Hamiltonian systems

Formulating the Euler equations within a Hamiltonian framework is a crucial method for studying these equations. Hamiltonian systems are a class of dynamical systems that arise in classical mechanics and various branches of physics. Many partial differential equations in mathematical physics have an underlying Hamiltonian structure. For example, the Maxwell equations, describing electromagnetic waves, the Schrödinger equations in quantum mechanics, the shallow water equations, and the Euler equations for compressible and incompressible flows, all have a Hamiltonian structure. The Hamiltonian formulation captures the system's time-evolution through a Hamiltonian, representing the total energy of the system, and a Poisson bracket, which defines its dynamics. This formalism serves as the natural mathematical structure for developing the theory of conservative mechanical systems such as the equations of celestial mechanics. The Hamiltonian structure expresses important invariants and symmetries of the system and provides a deep insight into its behaviour. In [1, 2, 11, 37, 56], the authors provide a systematic introduction to the Hamiltonian formulation and discuss important aspects of Hamiltonian theory.

Many of the special properties of Hamiltonian systems are formulated in terms of the Poisson bracket operator, which plays a central role in Hamiltonian systems. For arbitrary functionals $F, G, H : \Omega \to \mathbb{R}$, a Poisson bracket $\{\cdot, \cdot\}$ defined on the domain Ω should satisfy the following three properties:

(1) $\{F, G\}$ is bilinear;

(2) $\{F, G\}$ is skew-symmetric, i.e., $\{F, G\} = -\{G, F\}$;

(3) $\{F, G\}$ satisfies the Jacobi identity:

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0.$$

If a bracket in a dynamical system satisfies properties (1) and (2), the system is termed power-conserving. Furthermore, if this bracket also satisfies the Jacobi identity, it ensures the consistency and closure of the algebraic structure defined by the Poisson brackets [2, 11].

The analysis of the Euler equations with a free surface presented in this Ph.D. thesis builds upon the work presented in [52], where the Hamiltonian structure and associated Poisson brackets for three-dimensional incompressible Euler flows with a free surface are determined. In [52], the authors consider a domain $\Omega(t) \subseteq \mathbb{R}^3$ where the boundary $\partial\Omega(t) \subseteq \mathbb{R}^2$ coincides with the entire free surface $\Sigma(t)$, i.e., $\partial\Omega(t) = \Sigma(t)$. However, in our work the boundary $\partial\Omega(t)$ consists of a non-overlapping free surface $\Sigma(t)$ and a fixed surface Γ with a prescribed normal velocity. This extension is crucial to obtain a port-Hamiltonian formulation and presents an important challenge.

The evolution of any functional $\mathcal{F} = \mathcal{F}(\vec{v}, \Sigma)$ by the flow governed by the incompressible Euler equations with a free surface can be expressed as [52]

$$\dot{\mathcal{F}}(\vec{v}, \Sigma) = \{\mathcal{F}, H\}(\vec{v}, \Sigma), \tag{1.2}$$

where the dot represents differentiation with respect to time, H is the Hamiltonian

$$H(\vec{v}, \Sigma) = \frac{1}{2}\int_{\Omega(t)} |\vec{v}|^2 d^3x + \tau \int_{\Sigma(t)} ds, \tag{1.3}$$

which represents the kinetic and potential energy of the flow, and the Poisson bracket $\{\cdot, \cdot\}(\vec{v}, \Sigma)$ is defined as

$$\{\mathcal{F}, \mathcal{G}\}(\vec{v}, \Sigma) = \int_{\Omega(t)} \langle \nabla \times \vec{v}, \frac{\delta \mathcal{F}}{\delta \vec{v}} \times \frac{\delta \mathcal{G}}{\delta \vec{v}} \rangle d^3x + \int_{\Sigma(t)} (\frac{\delta \mathcal{F}}{\delta \Sigma}\frac{\delta \mathcal{G}}{\delta \phi} - \frac{\delta \mathcal{G}}{\delta \Sigma}\frac{\delta \mathcal{F}}{\delta \phi}) ds, \tag{1.4}$$

where $\frac{\delta \mathcal{F}}{\delta \vec{v}}, \frac{\delta \mathcal{F}}{\delta \Sigma}, \frac{\delta \mathcal{F}}{\delta \phi}$ denote the functional derivatives with respect to the velocity, free surface and surface potential, respectively. More details on the definition of the functional derivatives will be given in Chapter 2.

Beyond this work, numerous other studies have focused on exploring the Hamiltonian structure of Euler flow. In [14], the authors present generalized Hamiltonian formulations including the corresponding Poisson brackets for a number of incompressible fluids, including Euler inviscid fluid, Newtonian viscous fluid, a perfectly elastic medium, the upper-convected Maxwell and Oldroyd-B viscoelastic fluids. In [18] the variational and Hamiltonian formalism for the two-dimensional Euler equations governing an ideal, incompressible and inhomogeneous fluid is established. The focus lies on the investigation of conservation properties of the horizontal momentum. Reference [47] investigates different Poisson structures that have been proposed to obtain a Hamiltonian formulation for the evolution equations in fluid mechanics, including the Euler equations in vorticity form and ideal fluids with a free boundary. In [65], the author presents a framework to describe fluid mechanics from the Hamiltonian perspective, and gives several examples, including compressible fluids. The Hamiltonian structure of the Euler equations for inviscid incompressible fluid flow in vorticity form has been derived in [67] and the author discusses a specific Hamiltonian formulation that depends on vorticity. A Hamiltonian density formulation of an ideal fluid with or without a magnetic field is presented in [66]. These authors also derived three equivalent forms for the Poisson bracket, encompassing the usual physical variables, conserved variables, and dynamical variables. In [60], the authors use special functionals of the vorticity as dynamical variables and derive a noncanonical Hamiltonian structure for a two- or three-dimensional ideal fluid. In [59], a Hamiltonian formulation of adiabatic free boundary inviscid fluid flow is derived in terms of only physical variables. The corresponding noncanonical Poisson bracket in Eulerian representation is obtained by considering it as a reduction from the canonical bracket in Lagrangian representation. The authors in [57] study incompressible fluids from the Lie algebra viewpoint. This work focuses on the Clebsch variables and vortices by using symplectic geometry and the Lie-Poisson structure on the dual of a Lie algebra.

In this work, we study Hamiltonian formulations in a generalized form by introducing the concepts of differential forms and exterior calculus in suitable Sobolev spaces. Differential forms are mathematical objects used in differential geometry and calculus, which generalize scalar fields, vectors, and higher-dimensional entities. It provides an approach to multivariable calculus that is independent of coordinates on infinite dimensional manifolds. In the context of differential forms, Sobolev spaces plays a crucial role in the mathematical formulation and analysis of partial differential equations, especially in analyzing variational formulations of PDEs. Exterior calculus builts upon the concept of differential forms, which extends traditional vector calculus to provide a more general way to deal with complex functions, geometry, and physics. Many important concepts in differential geometry can be nicely expressed in the language of exterior calculus, including the exterior derivative, wedge product, Hodge star operator and many other operators. In [33, 39, 41, 78] a detailed overview is given of differential forms and their applications in various mathematical domains.

1.3 Port-Hamiltonian systems

The standard Hamiltonian formulation introduced in Section 1.2 describes a closed system without energy exchange with the external environment. In order to formulate more general cases, we introduce the concept of a port-Hamiltonian system. This concept helps us construct a more general structure for the incompressible Euler equations with the boundary composed of both a free surface and a fixed boundary that allows energy exchange with the outside.

A port-Hamiltonian system is a mathematical framework used to describe the dynamics of interconnected physical systems, is able to incorporate non-conservative dynamical systems and allows for non-zero energy exchange through the boundary of a domain. It builds upon the theory of port-based modelling, where the components can exchange energy through their ports. From a mathematical point of view, a distributed-parameter port-Hamiltonian system is constructed based on the existing Hamiltonian framework, with the key elements including the Hamiltonian functional, which represents the total energy of the system, and the conjugate power variables, which represent the energy flow. We refer to [36, 76, 90, 93] for a more general overview of port-Hamiltonian systems.

An important extension provided by port-Hamiltonian systems theory is that a dynamical system is modeled as the interconnection of several subsystems. These subsystems can be classified into three types of ideal components: energy-storing elements, energy-dissipating elements and energy-routing elements, which are based on their relation to the energy. In a port-Hamiltonian system, many physical elements can be characterized based on their energy-related characteristics. For example, ideal inductors, capacitors, masses and springs can be regarded as energy-storing elements, whereas resistors and dampers can be regarded as energy-dissipating elements. The simplest examples of energy-routing elements are transformers and gyroscopes.

A crucial element in defining a port-Hamiltonian system is the notion of a Dirac structure, which is a geometric structure generalizing the Poisson structure on the state space. It was originally introduced in [32]. The Dirac structure provides a unified framework for representing the dynamics, energy exchanges, and constraints within the system. In [93], van der

Schaft and Maschke present a framework for deriving port-Hamiltonian formulations of distributed-parameter systems. Let the linear spaces $\mathcal{F}$ and $\mathcal{E}$ be the dual flow and effort spaces, the total space $\mathcal{F} \times \mathcal{E}$ is called the space of power variables, and is equipped with a bilinear and non-degenerate pairing

$$\langle \cdot | \cdot \rangle : \mathcal{F} \times \mathcal{E} \to \mathbb{R}.$$

Hence, we can define a symmetric bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathcal{F} \times \mathcal{E}$ as

$$\langle\!\langle (f_1, e_1), (f_2, e_2) \rangle\!\rangle := \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle, \ (f_i, e_i) \in \mathcal{F} \times \mathcal{E}, \ i = 1, 2.$$

A Dirac structure D is a subspace of $\mathcal{F} \times \mathcal{E}$ such that $D = D^{\perp}$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Therefore, the Dirac structure is defined in a symplectic form, which ensures the power conservation of the system. The Hamiltonian dynamics of the system is contained in the definition of the Dirac structure, which provides a geometric representation of the system's evolution over time. In this Ph.D. thesis, the spaces $\mathcal{F}$ and $\mathcal{E}$ are formalized using Sobolev spaces of differential forms.

A crucial property of the Dirac structure is that any power-conserving interconnection of a Dirac structure is again a Dirac structure, see [21, 91, 58]. This implies that any power-conserving interconnection of a port-Hamiltonian system defined on a domain that is subdivided into several subdomains is again a port-Hamiltonian system, with the Hamiltonian the sum of the Hamiltonians on each subdomain. This property is particularly significant in the study of interconnected physical systems. It allows for a systematic approach to model complex systems with several subsystems, ensuring conservation of energy of the whole system through the interconnection between subsystems.

Another crucial concept in port-Hamiltonian systems is the pseudo-Poisson bracket, which is a generalization of the classical Poisson bracket. For linear problems, the Dirac structure can be directly related to a Poisson bracket. However, for nonlinear problems, it generally only relates to a pseudo-Poisson bracket, which is bilinear and skew-symmetric, but does not automatically satisfy the Jacobi identity. Hence that property needs to be proven separately to obtain a Hamiltonian formulation.

Port-Hamiltonian systems are widely applied to various fields, such as control theory and mechatronics. They provide a mathematical framework for formulating complex physical systems, including the compressible and incompressible Euler equations. In [20], the authors present the port-Hamiltonian formulation of the two-dimensional shallow water equations by taking translations and rotations of the tank into consideration. Reference [36] presents the basic models of electrical systems, mechanical systems, simple elastic models and irreversible thermodynamics. In [54], the authors formulate the model of a Timoshenko beam within the framework of distributed port-Hamiltonian systems. In [63], the authors present a pseudo-port-Hamiltonian formulation for three-dimensional isentropic and non-isentropic compressible fluids, including vorticity effects in the velocity field. Port-Hamiltonian formulations of the compressible Euler equations, both in terms of the velocity-pressure and vorticity-streamfunction formulations are proposed in [72]. In [92, 93], the authors formulate the Euler equations for an ideal isentropic fluid with non-zero energy flow through the boundary of a spatial domain with a modified non-constant Dirac structure. Recently, in [74, 75], the authors discuss the port-Hamiltonian formulation of the incompressible Euler equations in terms of the velocity and pressure, but without a free surface.

1.4 Structure-preserving discretizations

The (port)-Hamiltonian structure expresses important invariants and symmetries of the system and provides a deep insight into its behavior. However, in general, a numerical discretization may not preserve the Hamiltonian structure of partial differential equations. Hence if it is possible to derive a Hamiltonian structure preserving numerical discretization then the numerical solution will inherit many of the properties of the infinite dimensional Hamiltonian system. The complexity in the numerical discretization of (port)-Hamiltonian systems for nonlinear equations arises from the fact that in general the port-Hamiltonian formulation is built upon a nonconstant Dirac structure. Furthermore, also specific aspects of the numerical discretization such as discontinuous function spaces make it difficult to preserve the (port)-Hamiltonian structure. It is therefore very challenging to consider the associated Dirac structure for general nonlinear port-Hamiltonian systems directly. A good starting point is to study first the numerical discretization of a Dirac structure for a class of linear hyperbolic partial differential equations. Several important physical models can be represented by this class of linear hyperbolic equations, including the scalar wave equation, the linear shallow water equations, the Maxwell equations, and the classical Klein-Gordon equation.

It is natural to introduce the concept of finite element exterior calculus here. Finite element exterior calculus (FEEC) is a mathematical framework that extends the traditional finite element method by using differential forms and exterior calculus, which is based on functional analysis, and combines tools from geometry, topology and algebra to develop and analyze numerical methods for various physical systems. Detailed overviews of finite element exterior calculus can be found in several references, including [5, 7, 8]. In these works, the authors present reviews of exterior algebra, exterior calculus, and Hodge theory. In addition, they introduce the polynomial spaces of differential forms and provide the details of creating bases and defining degrees of freedom. For more details, readers can refer to [4, 9].

Finite element exterior calculus methods have wide applications, for instance the Maxwell equations of electromagnetics, the equations of elasticity, the Hodge Laplacian, elliptic eigenvalue problems, and many other problems. Recently, several researchers started employing finite element exterior calculus to the numerical discretization of various port-Hamiltonian systems. The authors in [17] studied the dual form of a constant linear Stokes-Dirac structure, and derived a weak form of the dual port-Hamiltonian system. A mixed Galerkin discretization of the Dirac structure for a hyperbolic system, consisting of two conservation laws, is obtained in [48, 49]. In [69], the authors linearize the two-dimensional flow of water through an open channel and present a spatial discretization of the linear shallow water equations, which is considered in one dimension.

Numerous approaches have been employed to achieve numerical discretization methods that preserve the structure of port-Hamiltonian systems. In [19], the authors proposed a structure-preserving partitioned finite element method for a port-Hamiltonian system consisting of two conservation laws, where one conservation law is solved in weak form and the other in strong form. A structure-preserving model reduction approach for large-scale nonlinear port-Hamiltonian systems is introduced in [23], which is based on Petrov-Galerkin projections. In [71] the authors model and discretize the port-Hamiltonian system of the boundary controlled 3D Maxwell's equations by using the partitioned finite element method. Reference [79] proposes a geometric framework for structure preserving discretizations of distributed-parameter port-Hamiltonian systems by using discrete exterior calculus and considering the matrix representations for linear port-Hamiltonian systems on a simplicial complex.

Considering the similarity between the boundary ports in port-Hamiltonian systems and numerical fluxes in discontinuous Galerkin(DG) finite element methods, DG methods form an interesting type of numerical discretization for port-Hamiltonian systems, for details see [27, 29, 30, 35].

Discontinuous Galerkin (DG) methods are a class of numerical techniques used for solving partial differential equations. They combine features of finite element and finite volume methods, offering advantages for solving problems with complex geometries, discontinuous solutions, and highly nonlinear behavior. In [97], the authors discretized a class of linear Hamiltonian hyperbolic systems using a DG method with a homogeneous boundary condition. A structure-preserving DG approximation of a linear port-Hamiltonian system is proposed in [85], but this study only considers the one-dimensional case. Recently, a novel framework for the DG discretization of port-Hamiltonian systems of conservation laws is proposed in [50]. This paper states the spaces for the boundary ports explicitly, and presents the discrete Dirac structure in a strong way. The work in this thesis follows a similar approach as in [50], but we need weaker boundary conditions for the weak formulation of the Dirac structure.

1.5 Thesis objectives and outline

With the tools of differential forms and exterior calculus, it is natural to develop the classical Hamiltonian systems (1.2)–(1.4) describing the incompressible Euler equations with a free surface in a more general form. However, the generalized Hamiltonian formulations of the incompressible Euler equations only concern the homogeneous boundary condition on a fixed surface. Our aim is to extend this to the incompressible Euler equations with a free surface, allowing for energy exchange on the fixed boundary within the framework of port-Hamiltonian systems. Our subsequent goal is to develop structure-preserving numerical discretizations for the port-Hamiltonian systems of the incompressible Euler equations with a free surface. However, developing such a discretization method is challenging due to the nonconstant Dirac structure for this port-Hamiltonian system. Therefore, we consider a class of linear hyperbolic partial differential equations to develop a general framework for port-Hamiltonian discontinuous Galerkin discretizations.

The main objectives that will be discussed in this PhD thesis are:

(1) **Obtain generalized Hamiltonian formulations for the incompressible Euler equations with a free surface.**

Generalize the classical incompressible Euler equations with a free surface by using the tools of differential forms and exterior calculus. Extend the classical Hamiltonian systems (1.2)–(1.4) by defining all variables and related functional derivatives in suitable Sobolev spaces of differential forms.

(2) **Obtain port-Hamiltonian formulations for the incompressible Euler equations with a free surface.**

Construct port-Hamiltonian formulations for the incompressible Euler equations with a boundary consisting of a free surface and a fixed surface with inhomogeneous boundary conditions, based on the Hamiltonian formulations for the incompressible Euler equations with a free surface. Derive different types of formulations of the incompressible Euler equations with a free surface, including the velocity-pressure formulation and the vorticity equation.

(3) **Develop port-Hamiltonian discontinuous Galerkin finite element discretizations for a class of linear hyperbolic partial differential equations.**

Derive a numerical discretization for port-Hamiltonian systems within broken Sobolev spaces on a tessellation with polyhedral elements. This discretization, combined with the power-conserving property of the Dirac structure, must automatically ensure conservation of power in the entire system. Demonstrate that the port-Hamiltonian discontinuous Galerkin discretization is Hamiltonian. Provide a priori error estimates and several numerical experiments to demonstrate the accuracy and capability of the port-Hamiltonian discontinuous Galerkin discretizations.

This dissertation is organized as follows: in Chapter 2, we address Objective (1) by generalizing the incompressible Euler equations with a free surface and the associated Hamiltonian structure (1.2)–(1.4) in three important sets of variables: (v, Σ), $(\eta, \phi_\partial, \Sigma)$ and $(\omega, \phi_\partial, \Sigma)$, with v the velocity, η the solenoidal velocity, ϕ_∂ a potential, ω the vorticity, and Σ the free surface. Using the Hodge decomposition of the velocity v, we can obtain equivalent relations between these three sets of variables. In addition, we present the corresponding Poisson brackets. Next, we construct in Chapter 2 the Dirac structures for the incompressible Euler equations on a domain with a boundary consisting of a free surface and a fixed surface with inhomogeneous boundary conditions. Building upon the framework of the port-Hamiltonian systems, we obtain the port-Hamiltonian formulations of the incompressible Euler equations with a free surface, thereby addressing Objective (2). Objective (3) is the main topic in Chapter 3.

We derive a structure-preserving port-Hamiltonian discontinuous Galerkin discretization for a class of linear hyperbolic partial differential equations. We also provide a priori error estimates and demonstrate the accuracy and capabilities of the port-Hamiltonian discontinuous Galerkin discretization through several numerical experiments.

Chapter 2

Hamiltonian and Port-Hamiltonian Formulations of the Incompressible Euler Equations with a Free Surface[1]

Abstract

We present port-Hamiltonian formulations of the incompressible Euler equations with a free surface governed by surface tension and gravity forces, modelling e.g. capillary and gravity waves and the evolution of droplets in air. Three sets of variables are considered, namely (v, Σ), $(\eta, \phi_\partial, \Sigma)$ and $(\omega, \phi_\partial, \Sigma)$, with v the velocity, η the solenoidal velocity, ϕ_∂ a potential, ω the vorticity, and Σ the free surface, resulting in the incompressible Euler equations in primitive variables and the vorticity equation. First, the Hamiltonian formulation for the incompressible Euler equations in a domain with a free surface combined with a fixed boundary surface with a homogeneous boundary condition will be derived in the proper Sobolev spaces of differential forms. Next, these results will be extended to port-Hamiltonian formulations allowing inhomogeneous boundary conditions and a nonzero energy flow through the boundaries. Our main results are the construction and proof of Dirac structures in suitable Sobolev spaces of differential forms for each variable set, which provides the core of any port-Hamiltonian formulation. Finally, it is proven that the state dependent Dirac structures are related to Poisson brackets that are linear, skew-symmetric and satisfy the Jacobi identity.

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2.1 Introduction

The evolution of many infinite dimensional systems can be described using a Hamiltonian formulation consisting of a Hamiltonian functional, representing the total energy of the system, and a Poisson bracket defining its dynamics. Examples are inviscid fluid mechanics, ideal magnetohydrodynamics, classical electromagnetism and the Schrödinger equation, see e.g. [1, 2, 11, 39, 56, 65].

In this chapter, we will consider Hamiltonian formulations of the incompressible Euler equations with a free surface governed by surface tension and gravity forces. These equations model a large variety of fluid mechanical problems, e.g. capillary and gravity waves and the behaviour of droplets in air [96]. The incompressible Euler equations in a time-dependent domain $\Omega(t) \subset \mathbb{R}^d$, $d \in \{1,2,3\}$ with free surface $\Sigma(t) \subseteq \partial\Omega(t)$ are given by [25, 96]

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla\Big(\frac{p}{\rho} + \Phi\Big) \qquad \text{in } \Omega, \tag{2.1a}$$

$$\nabla \cdot \vec{v} = 0 \qquad \text{in } \Omega, \tag{2.1b}$$

$$\frac{\partial \Sigma}{\partial t} = \langle \vec{v}, \vec{n} \rangle \qquad \text{at } \Sigma, \tag{2.1c}$$

$$p - \bar{p} = \tau k \qquad \text{at } \Sigma, \tag{2.1d}$$

$$\langle \vec{v}, \vec{n} \rangle = g \qquad \text{at } \Gamma. \tag{2.1e}$$

Here $\vec{v}$ denotes the fluid velocity, p the pressure with $\bar{p}$ a constant atmospheric pressure, Φ the gravity potential, ρ the fluid density, τ the surface tension, t time and $\nabla \in \mathbb{R}^d$ the nabla operator. The domain $\Omega(t)$ has a boundary $\partial\Omega$ with external normal vector $\vec{n}$ and mean curvature k, and consists of a nonoverlapping free surface Σ and a fixed boundary Γ where the normal velocity g is specified. The motion and shape of the free surface $\Sigma(t)$ is governed by a balance between the pressure p, surface tension τ, and the mean surface curvature k. In addition, a fluid particle at the free surface should follow the free surface motion, which implies that the free surface velocity must be equal to the normal fluid velocity at the free surface. These conditions are stated, respectively, by (2.1d) and (2.1c).

The Hamiltonian formulation of the incompressible Euler equations can be expressed using a Hamiltonian functional and a Lie-Poisson bracket [10, 14, 57]. The Hamiltonian formulation of the incompressible Euler equations has a very rich mathematical structure and has been extensively researched, see e.g. [11, 18, 47, 65, 67].

The Hamiltonian formulation of the incompressible Euler equations with a free surface governed by surface tension, which describes the evolution of liquid droplets in an inviscid flow, and is obtained by setting $\Sigma(t) = \partial\Omega(t)$, $\Phi = 0$ in (2.1), is presented in [52], see also [59]. The evolution of any functional $\mathcal{F} = \mathcal{F}(\vec{v}, \Sigma)$ by the flow governed by the incompressible Euler equations with a free surface with $\Sigma = \partial\Omega$ can be expressed as [52]

$$\dot{\mathcal{F}}(\vec{v}, \Sigma) = \{\mathcal{F}, H\}(\vec{v}, \Sigma), \tag{2.2}$$

where the dot represents differentiation with respect to time, H is the Hamiltonian

$$H(\vec{v}, \Sigma) = \frac{1}{2}\int_{\Omega(t)} |\vec{v}|^2 d^3x + \tau \int_{\Sigma(t)} ds, \tag{2.3}$$

and the Poisson bracket $\{\cdot, \cdot\}(\vec{v}, \Sigma)$ is defined as

$$\{\mathcal{F}, \mathcal{G}\}(\vec{v}, \Sigma) = \int_{\Omega(t)} \langle \nabla \times \vec{v}, \frac{\delta \mathcal{F}}{\delta \vec{v}} \times \frac{\delta \mathcal{G}}{\delta \vec{v}} \rangle d^3x + \int_{\Sigma(t)} (\frac{\delta \mathcal{F}}{\delta \Sigma}\frac{\delta \mathcal{G}}{\delta \phi} - \frac{\delta \mathcal{G}}{\delta \Sigma}\frac{\delta \mathcal{F}}{\delta \phi}) ds, \tag{2.4}$$

where $\frac{\delta \mathcal{F}}{\delta \vec{v}}, \frac{\delta \mathcal{F}}{\delta \Sigma}, \frac{\delta \mathcal{F}}{\delta \phi}$ denote, respectively, the functional derivatives with respect to the velocity, free surface and surface potential.

The Hamiltonian formulation (2.2)–(2.4) describes a closed system without energy exchange with the external environment, namely H in (2.3) is constant. In the general case, with $\partial\Omega = \Sigma \cup \Gamma$, with $\mathrm{meas}(\Gamma) > 0$ and $g \neq 0$ in (2.1e), the incompressible Euler equations with a free surface exchange energy with the external environment. In order to describe the Hamiltonian structure of (2.1) we need to use the more general concept of port-Hamiltonian systems. The main objective of this chapter is to formulate the Euler equations with a free surface (2.1) as a port-Hamiltonian system with special emphasis on formulating the port-Hamiltonian system in the proper Sobolev spaces of differential forms. This will significantly extend the number of applications, modelled by (2.1), that can be described as a Hamiltonian system.

A port-Hamiltonian system describes a non-conservative open dynamical system that interacts with the environment through energy flow over boundary ports. Currently, port-Hamiltonian systems are a very active field of research. We refer to [36, 43, 76, 89, 90, 94] for an overview of the theory and applications of port-Hamiltonian systems.

From a mathematical point of view, the distributed-parameter port-Hamiltonian system is constructed based on the existing Hamiltonian framework by using the notion of a Dirac structure [93, 99], which is a geometric structure generalizing both symplectic and Poisson structures, and was originally introduced in [32, 36]. In [93, 99], a Dirac structure D is defined on $\mathcal{F} \times \mathcal{E}$, with $\mathcal{F}$ and $\mathcal{E}$, respectively, the dual flow and effort spaces. The spaces $\mathcal{F}$ and $\mathcal{E}$ are formalized using Sobolev spaces of differential forms. The proper choice of these Sobolev spaces, their duality relations, and the formulation of a bilinear form and Dirac structure for the Euler equations with a free surface will be the main topic of this chapter.

Port-Hamiltonian formulations of the compressible Euler equations, both in terms of the velocity-pressure and vorticity-streamfunction formulations are proposed in [72]. Recently, in [74, 75], the authors discuss the port-Hamiltonian formulation of the incompressible Euler equations in terms of the velocity and pressure, but without a free surface.

In this chapter, we will consider the port-Hamiltonian formulation for the incompressible Euler equations in a domain with a free surface combined with a fixed boundary surface where an inhomogeneous boundary condition is imposed. Three sets of variables will be considered, namely (v, Σ), $(\eta, \phi_\partial, \Sigma)$ and $(\omega, \phi_\partial, \Sigma)$, with v the velocity, η the solenoidal velocity, ϕ_∂ a surface potential, ω the vorticity and Σ the free surface. Using these variables several important formulations of the incompressible Euler equations with a free surface can be obtained such as the primitive variable formulation (2.1) and the vorticity equation, with their suitability depending on the type of application. For each of these formulations of (2.1) we will first derive the Hamiltonian formulation with homogeneous boundary conditions at Γ ($g = 0$ in (2.1e)) in the proper setting of Sobolev spaces of differential forms. Next, we will define a bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ and structures D_i ($i = 1, 2, 3$) and prove that these structures are Dirac structures, namely $D_i = D_i^\perp$, with $\perp$ denoting the orthogonal complement with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. These results constitute the core of this chapter. For linear problems, the Dirac structure can be directly related to a Poisson bracket. But for nonlinear problems, it generally only relates to a skew-symmetric pseudo-Poisson bracket [93] since the Jacobi identity needs to be proven separately and the Dirac structure only ensures power conservation. We will prove that the various state dependent Dirac structures D_i ($i = 1, 2, 3$) presented in this chapter also satisfy the Jacobi identity and that the resulting formulations constitute a port-Hamiltonian system, hence allow a non-zero energy exchange with the external environment.

The outline of this chapter is as follows. First, we will give in Section 2.2 a summary of some important results from exterior algebra, Sobolev spaces, trace operators, and Hodge decompositions that will be extensively used in this chapter. In Section 2.3 we will introduce the incompressible

Euler equations with a free surface in terms of the velocity and pressure and discuss the decomposition of the velocity v into three parts, namely an exact form ϕ (potential), co-exact form β and a harmonic form α. Next, in Section 2.4 we will discuss the generalized Hamiltonian formulation of incompressible Euler equations with a free surface in three sets of variables, namely (v, Σ), $(\eta, \phi_\partial, \Sigma)$ and $(\omega, \phi_\partial, \Sigma)$ with the homogeneous boundary condition $g = 0$ at Γ. Finally, in Section 2.5 we will present the Dirac structures and port-Hamiltonian formulations of the incompressible Euler equations in a domain with both a free boundary surface and a fixed boundary surface with the inhomogeneous boundary condition $g \neq 0$ at Γ with respect to the (v, Σ), $(\eta, \phi_\partial, \Sigma)$ and $(\omega, \phi_\partial, \Sigma)$ variables.

2.2 Preliminaries

Since the language of differential forms is crucial to express the geometrical properties of port-Hamiltonian systems, we will introduce in this section some of the geometrical concepts and tools that will be used later on and set the notation used in this chapter. For more details we refer to [1, 2, 39, 56, 78].

2.2.1 Differential forms, function spaces, and related results

Let Ω be an open, bounded and connected oriented n-dimensional manifold with an $(n-1)$-dimensional Lipschitz boundary $\partial\Omega$.

2.2.1.1 Riemannian metric and index operators

Assume that there is a Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle : T\Omega \times T\Omega \to \mathbb{R}$, where $T\Omega = \bigcup_{p \in \Omega} T_p\Omega$, with $T_p\Omega$ the n-dimensional tangent space at the point $p \in \Omega$. At each point $p \in \Omega$, $\langle\langle \cdot, \cdot \rangle\rangle$ is symmetric, bilinear and positive definite. For any vector $X \in T_p\Omega$, the index lowering operator $\flat : T_p\Omega \to T_p^*\Omega$, with $T_p^*\Omega$ the dual space of $T_p\Omega$, is defined as [1, 2]: $X^\flat(\cdot) := \langle\langle X, \cdot \rangle\rangle$. Dually, the index raising operator $\sharp : T_p^*\Omega \to T_p\Omega$ is given by $(X^\flat)^\sharp = X$.

2.2.1.2 Exterior product and inner product

Let $\Lambda^k(\Omega)$ be the space that contains all smooth differential k-forms on Ω. Denote with $\wedge : \Lambda^k(\Omega) \times \Lambda^j(\Omega) \to \Lambda^{k+j}(\Omega)$ the exterior product of differential forms [1], which satisfies the relation

$$\lambda \wedge \mu = (-1)^{kj} \mu \wedge \lambda, \tag{2.5}$$

where $\lambda \in \Lambda^k(\Omega)$, $\mu \in \Lambda^j(\Omega)$. The space $\Lambda^k(\Omega)$ is endowed with the inner product [7]

$$\langle \lambda, \mu \rangle_{\Lambda^k} := \sum_{\sigma \in S(k,n)} \lambda(E_{\sigma(1)}, \dots, E_{\sigma(k)}) \mu(E_{\sigma(1)}, \dots, E_{\sigma(k)}), \tag{2.6}$$

with $(E_1, \dots, E_n)$ an orthonormal frame on Ω with respect to the metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, and $S(k,n)$, with $1 \leq k \leq n$, the set of all permutations of the numbers $\{1, 2, \dots, n\}$ such that $\sigma(1) < \dots < \sigma(k)$.

2.2.1.3 Hodge star operator, exterior derivative and coderivative operators, interior product

Since $\dim \Lambda^k(\Omega) = \dim \Lambda^{n-k}(\Omega)$, there exists an isometry taking k-forms into $(n-k)$-forms, which is called the Hodge star operator. Let $v_\Omega \in \Lambda^n(\Omega)$ be the volume form of Ω. Then the Hodge star operator $* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ is defined by the relation [78]

$$\lambda \wedge * \mu = \langle \lambda, \mu \rangle_{\Lambda^k} v_\Omega, \quad \forall \lambda, \mu \in \Lambda^k(\Omega). \tag{2.7}$$

We have the relation

$$* * \lambda = (-1)^{k(n-k)} \lambda, \quad \lambda \in \Lambda^k(\Omega). \tag{2.8}$$

For all k-forms, we denote by $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ the (weak) exterior derivative operator, taking differential k-forms on the domain Ω to differential $(k+1)$-forms. Given $\alpha \in \Lambda^k(\Omega)$ and any vector fields $X_1, X_2, \dots, X_{k+1}$, the $(k+1)-$form $d\alpha$ is given by the formula [64]

$$\begin{aligned} d\alpha(X_1, \dots, X_{k+1}) = & \frac{1}{k+1} \Big\{ \sum_{i=1}^{k+1} (-1)^{i+1} X_i \Big(\alpha(X_1, \dots, \hat{X}_i, \dots, X_{k+1}) \Big) \\ & + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}) \Big\}, \end{aligned}$$

where the hat is used to indicate a suppressed argument, and $[X_i, X_j] = X_i X_j - X_j X_i$ represents the bracket of vector fields.

Conversely, the coderivative operator $\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ is defined as [7]

$$\delta := (-1)^{n(k+1)+1} * d * . \tag{2.9}$$

Another operator, taking k-forms into $(k-1)$-forms, is called interior product, which is defined as: given $X \in T_p\Omega$ and $\alpha \in \Lambda^k(\Omega)$, then $i_X\alpha$ is the $(k-1)$-form given by

$$i_X\alpha(X_1, \dots, X_{k-1}) := \alpha(X, X_1, \dots, X_{k-1}), \tag{2.10}$$

where $X_1, \dots, X_{k-1}$ are any vectors that belong to $T_p\Omega$. We have the following relation between the Hodge star operator and the interior product [2]

$$i_X\alpha = *(X^\flat \wedge *\alpha). \tag{2.11}$$

A proof of (2.11) is given in Appendix 2.A.

2.2.1.4 Sobolev spaces of differential forms

The spaces $L^2\Lambda^k(\Omega)$ and $H^s\Lambda^k(\Omega)$ with $k \in \mathbb{N} \cup \{0\}, s \geq 0$ are the spaces of differential k-forms on the domain $\Omega \subseteq \mathbb{R}^d, d \geq 1$, whose coefficient functions belong, respectively, to the Sobolev spaces $L^2(\Omega)$ and $H^s(\Omega)$, with $H^0(\Omega) = L^2(\Omega)$, [7, 8]. For a definition of these Sobolev spaces on general manifolds, see [78, Definition 1.3.2]. We define the corresponding L^2-inner product and norm as

$$\langle \lambda, \mu \rangle_{L^2\Lambda^k(\Omega)} := \int_\Omega \lambda \wedge *\mu, \ \ \|\lambda\|^2_{L^2\Lambda^k(\Omega)} := \langle \lambda, \lambda \rangle_{L^2\Lambda^k(\Omega)}, \tag{2.12}$$

for any $\lambda, \mu \in L^2\Lambda^k(\Omega)$. There is another important Sobolev space [7, 8] given by

$$H\Lambda^k(\Omega) := \{\lambda \in L^2\Lambda^k(\Omega) \mid d\lambda \in L^2\Lambda^{k+1}(\Omega)\}, \tag{2.13}$$

with norm defined as

$$\|\lambda\|^2_{H\Lambda^k(\Omega)} := \|\lambda\|^2_{L^2\Lambda^k(\Omega)} + \|d\lambda\|^2_{L^2\Lambda^{k+1}(\Omega)}.$$

Analogously, we define [7, 77]

$$H^*\Lambda^k(\Omega) := *H\Lambda^{n-k}(\Omega) = \{\lambda \in L^2\Lambda^k(\Omega) \mid \delta\lambda \in L^2\Lambda^{k-1}(\Omega)\}.$$

2.2.2 Trace operator and Stokes' theorem

2.2.2.1 Trace operator

Given a differential k-form α, then $\alpha|_{\partial\Omega}$, the boundary value of $\alpha \in \Lambda^k(\Omega)$, is defined by a skew-symmetric map [78]

$$\alpha \,|_{\partial\Omega} \colon \Gamma(T\Omega \,|_{\partial\Omega}) \times \ldots \times \Gamma(T\Omega \,|_{\partial\Omega}) \to C^\infty(\Omega),$$

where $\Gamma(T\Omega)$ is the space of all smooth vector fields, and we can write $\alpha|_{\partial\Omega} \in \Lambda^k(\Omega)|_{\partial\Omega}$. Any k-form α at $\partial\Omega$ can be decomposed into a tangential component $\mathbf{t}\alpha : \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$ and a normal component $\boldsymbol{n}\alpha := \alpha|_{\partial\Omega} - \mathbf{t}\alpha$, i.e. $\alpha|_{\partial\Omega} = \mathbf{t}\alpha + \boldsymbol{n}\alpha$ [78]. The trace operator is defined as $\mathrm{tr} := \mathbf{t}$.

Choose any vector fields $X_1, \ldots, X_k \in \Gamma(T\Omega|_{\partial\Omega})$, decomposed into their tangential and normal parts $X_i = X_i^{\parallel} + X_i^{\perp}$, $i = 1, \ldots, k$. We write then

$$\mathrm{tr}(\alpha)(X_1, \ldots, X_k) = \alpha(X_1^{\parallel}, \ldots, X_k^{\parallel}), \tag{2.14}$$

hence,

$$\boldsymbol{n}(\alpha)(X_1, \ldots, X_k) = \alpha(X_1^{\perp}, \ldots, X_k^{\perp}). \tag{2.15}$$

The trace operator tr has the following important properties.

Lemma 2.2.1. *[7, 78] For $\alpha \in \Lambda^k(\Omega)$ and $\beta \in \Lambda^j(\Omega)$, there hold the following relations*

$$\mathrm{tr}(\alpha \wedge \beta) = \mathrm{tr}(\alpha) \wedge \mathrm{tr}(\beta), \quad \mathrm{tr}(d\alpha) = d(\mathrm{tr}(\alpha)), \quad \boldsymbol{n}(\delta\alpha) = \delta\boldsymbol{n}(\alpha), \tag{2.16}$$

$$* \, \boldsymbol{n}(\alpha) = \mathrm{tr}(*\alpha), \quad *\mathrm{tr}(\alpha) = \boldsymbol{n}(*\alpha). \tag{2.17}$$

Lemma 2.2.2. *[56, Stokes' Theorem] On an n-dimensional oriented manifold Ω with boundary $\partial\Omega$, we have for any $\alpha \in \Lambda^{n-1}(\Omega)$*

$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \mathrm{tr}(\alpha). \tag{2.18}$$

2.2.2.2 Boundary traces of Sobolev spaces $H^1\Lambda^k(\Omega)$

We can extend the trace operator $\mathrm{tr} : \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$ to a bounded linear operator from $H^1\Lambda^k(\Omega)$ onto $H^{\frac{1}{2}}\Lambda^k(\partial\Omega)$, where the space $H^{\frac{1}{2}}\Lambda^k(\Omega)$ and the norm are defined as

$$\begin{aligned} &H^{\frac{1}{2}}\Lambda^k(\partial\Omega) := \{\mu \in L^2\Lambda^k(\partial\Omega) \mid \exists \lambda \in H^1\Lambda^k(\Omega), \ s.t. \ \mathrm{tr}(\lambda) = \mu\}, \\ &\|\mu\|_{H^{\frac{1}{2}}\Lambda^k(\partial\Omega)} := \inf_{\mathrm{tr}(\lambda)=\mu, \lambda \in H^1\Lambda^k(\Omega)} \|\lambda\|_{H^1\Lambda^k(\Omega)}. \end{aligned}$$

2.2.2.3 Boundary traces of Sobolev spaces $H\Lambda^k(\Omega)$

If $k = 0$, since $H\Lambda^0(\Omega) = H^1\Lambda^0(\Omega)$, we have that $H^{\frac{1}{2}}\Lambda^0(\partial\Omega)$ is the trace space of $H\Lambda^0(\Omega)$.

If $0 \le k < n$, we denote the dual space of $H^{\frac{1}{2}}\Lambda^k(\partial\Omega)$ by $H^{-\frac{1}{2}}\Lambda^k(\partial\Omega)$. In [7, p19], it is shown that the trace operator $\mathrm{tr} : \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$ can be extended to a bounded linear operator that maps $H\Lambda^k(\Omega)$ onto $H^{-\frac{1}{2}}\Lambda^k(\partial\Omega)$ for $0 < k < n$. Since $H\Lambda^n(\Omega) = L^2\Lambda^n(\Omega)$, there is no trace for $k = n$.

Since $H^*\Lambda^k(\Omega) = *H\Lambda^{n-k}(\Omega)$, then for any $\mu \in H\Lambda^{n-k}(\Omega)$ there exists a $\lambda \in H^*\Lambda^k(\Omega)$, such that $\mu = (-1)^{k(n-k)} * \lambda$. For $0 < k < n$, $\mathrm{tr}(\mu) \in H^{-\frac{1}{2}}\Lambda^{n-k}(\partial\Omega)$, it immediately follows that $\mathrm{tr}(*\lambda) \in H^{-\frac{1}{2}}\Lambda^{n-k}(\partial\Omega)$. Similarly, for $\lambda \in H^*\Lambda^n(\Omega)$, $\mathrm{tr}(*\lambda) \in H^{\frac{1}{2}}\Lambda^0(\partial\Omega)$.

We can now state the integration by parts formula, see [7, 8, 78]: for either $\lambda \in H\Lambda^{k-1}(\Omega)$, $\mu \in H^1\Lambda^k(\Omega)$ or $\lambda \in H^1\Lambda^{k-1}(\Omega)$, $\mu \in H^*\Lambda^k(\Omega)$,

$$\begin{aligned}\langle d\lambda, \mu\rangle_{L^2\Lambda^k(\Omega)} &= \langle \lambda, \delta\mu\rangle_{L^2\Lambda^{k-1}(\Omega)} + \int_{\partial\Omega} \mathrm{tr}(\lambda) \wedge \mathrm{tr}(*\mu) \\ &= \langle \lambda, \delta\mu\rangle_{L^2\Lambda^{k-1}(\Omega)} + \int_{\partial\Omega} \mathrm{tr}(\lambda) \wedge *\boldsymbol{n}(\mu),\end{aligned} \tag{2.19}$$

where the integral on the boundary is interpreted via the pairing of $H^{-\frac{1}{2}}\Lambda^{k-1}(\partial\Omega)$ and $H^{\frac{1}{2}}\Lambda^{n-k}(\partial\Omega)$.

Furthermore, the boundary term $\mathrm{tr}(\lambda) \wedge *\boldsymbol{n}(\mu)$ can be represented as:

Lemma 2.2.3. *[78, p27] Given the differential forms* $\lambda \in H\Lambda^{k-1}(\Omega)$ *and* $\mu \in H^1\Lambda^k(\Omega)$ *or* $\lambda \in H^1\Lambda^{k-1}(\Omega)$ *and* $\mu \in H^*\Lambda^k(\Omega)$. *Let* $\gamma := \mathrm{tr}(\lambda)\wedge*\boldsymbol{n}(\mu)$. *Then*

$$\gamma = \langle \lambda, i_{\mathcal{N}}\mu\rangle_{\Lambda^{k-1}} v_{\partial\Omega}, \tag{2.20}$$

where $v_{\partial\Omega} = i_{\mathcal{N}} v_\Omega|_{\partial\Omega} \in \Lambda^{n-1}(\partial\Omega)$ *is the area form for the boundary* $\partial\Omega$, *with* $\mathcal{N}$ *the unit outward normal vector to the boundary* $\partial\Omega$, *and* $v_\Omega \in \Lambda^n(\Omega)$ *the volume form of* Ω.

Lemma 2.2.4. *Given* $f_1, f_2 \in H^1\Lambda^0(\Omega)$. *Let* $\mu \in H\Lambda^1(\Omega)$ *or* $\mu \in H^*\Lambda^1(\Omega)$, *with* $\boldsymbol{n}(\mu) = 0$ *at* $\partial\Omega$, *then*

$$\int_\Omega df_1 \wedge *\delta(\mu \wedge df_2) = \int_{\partial\Omega} \big(\langle df_1, \mu\rangle_{\Lambda^1} i_{\mathcal{N}} df_2\big) v_{\partial\Omega}. \tag{2.21}$$

Proof. Using the integration by parts formula (2.19), we have

$$\int_\Omega df_1 \wedge *\delta(\mu \wedge df_2) = -\int_{\partial\Omega} \mathrm{tr}(df_1) \wedge *\boldsymbol{n}(\mu \wedge df_2).$$

From (2.20) we have that

$$\begin{aligned}\mathrm{tr}(df_1) \wedge *\boldsymbol{n}(\mu \wedge df_2) &= \langle df_1, i_{\mathcal{N}}(\mu \wedge df_2)\rangle_{\Lambda^1} v_{\partial\Omega} \\ &= \langle df_1, i_{\mathcal{N}}\mu \wedge df_2 - \mu \wedge i_{\mathcal{N}} df_2\rangle_{\Lambda^1} v_{\partial\Omega}.\end{aligned} \tag{2.22}$$

Using the definition of $\boldsymbol{n}(\mu)$ in (2.15) and applying the boundary condition $\boldsymbol{n}(\mu) = 0$, we have

$$i_{\mathcal{N}}\mu = \mu(\mathcal{N}) = \boldsymbol{n}(\mu)(\mathcal{N}) = 0. \tag{2.23}$$

Hence, by using (2.22) and (2.23), we have that

$$\begin{aligned}\int_\Omega df_1 \wedge *\delta(\mu \wedge df_2) &= \int_{\partial\Omega} \langle df_1, \mu \wedge i_{\mathcal{N}} df_2\rangle_{\Lambda^1} v_{\partial\Omega} \\ &= \int_{\partial\Omega} \big(\langle df_1, \mu\rangle_{\Lambda^1} i_{\mathcal{N}} df_2\big) v_{\partial\Omega}.\end{aligned}$$

□

2.2.3 Hodge decompositions

In order to define the Hodge decompositions, we first define the spaces $H\Lambda^k(\Omega)$ and $H^*\Lambda^k(\Omega)$ with a zero boundary trace denoted by

$$\begin{aligned}\mathring{H}\Lambda^k(\Omega) &:= \{\mu \in H\Lambda^k(\Omega) \mid \operatorname{tr}(\mu) = 0\}, \\ \mathring{H}^*\Lambda^k(\Omega) &:= \{\mu \in H^*\Lambda^k(\Omega) \mid \operatorname{tr}(*\mu) = 0\}.\end{aligned}$$

For the spaces $H\Lambda^k(\Omega)$, $H^*\Lambda^k(\Omega)$, $\mathring{H}\Lambda^k(\Omega)$ and $\mathring{H}^*\Lambda^k(\Omega)$, we have the corresponding spaces of cycles, boundaries, cocycles and coboundaries. For details, we refer to [7, 8]. The k-cocycles are denoted as

$$\mathfrak{Z}^k := \{\mu \in H\Lambda^k(\Omega) \mid d\mu = 0\}, \quad \mathring{\mathfrak{Z}}^k := \{\mu \in \mathring{H}\Lambda^k(\Omega) \mid d\mu = 0\}, \tag{2.24}$$

and the k-coboundaries as

$$\mathfrak{B}^k := dH\Lambda^{k-1}(\Omega), \quad \mathring{\mathfrak{B}}^k := d\mathring{H}\Lambda^{k-1}(\Omega). \tag{2.25}$$

The k-cycles are denoted as

$$\mathfrak{Z}^{*k} := \{\mu \in H^*\Lambda^k(\Omega) \mid \delta\mu = 0\}, \quad \mathring{\mathfrak{Z}}^{*k} := \{\mu \in \mathring{H}^*\Lambda^k(\Omega) \mid \delta\mu = 0\}, \tag{2.26}$$

and the k-boundaries as

$$\mathfrak{B}^{*k} := \delta H^*\Lambda^{k+1}(\Omega), \quad \mathring{\mathfrak{B}}^{*k} := \delta\mathring{H}^*\Lambda^{k+1}(\Omega). \tag{2.27}$$

Each of the spaces of cycles is closed in $H\Lambda^k(\Omega)$, $H^*\Lambda^k(\Omega)$ or $L^2\Lambda^k(\Omega)$. The spaces of boundaries are closed in $L^2\Lambda^k(\Omega)$ due to the Poincaré inequality [7]. Finally, the k-th harmonic forms are defined as

$$\begin{aligned}\mathfrak{H}^k &:= \{\mu \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) \mid d\mu = 0,\ \delta\mu = 0\}, \\ \mathring{\mathfrak{H}}^k &:= \{\mu \in \mathring{H}\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega) \mid d\mu = 0,\ \delta\mu = 0\}.\end{aligned} \tag{2.28}$$

Lemma 2.2.5. *[7, p22] The spaces of cycles, cocycles, boundaries, and coboundaries, and their orthogonal complement in* $L^2\Lambda^k(\Omega)$*, indicated with* $\perp$*, satisfy the relations*

$$\begin{aligned} \mathfrak{Z}^{k\perp} \subset \mathfrak{B}^{k\perp} = \mathring{\mathfrak{Z}}^{*k}, \qquad \mathfrak{Z}^{*k\perp} \subset \mathfrak{B}^{*k\perp} = \mathring{\mathfrak{Z}}^{k}, \\ \mathring{\mathfrak{Z}}^{k\perp} \subset \mathring{\mathfrak{B}}^{k\perp} = \mathfrak{Z}^{*k}, \qquad \mathring{\mathfrak{Z}}^{*k\perp} \subset \mathring{\mathfrak{B}}^{*k\perp} = \mathfrak{Z}^{k}. \end{aligned} \tag{2.29}$$

Based on the spaces that we defined above, there exist two types of Hodge decompositions of $L^2\Lambda^k(\Omega)$, each with different boundary conditions [7, 78, 8],

$$\text{(i): } L^2\Lambda^k = \mathfrak{B}^k \oplus \mathring{\mathfrak{B}}^{*k} \oplus \mathfrak{H}^k, \tag{2.30a}$$

$$\text{(ii): } L^2\Lambda^k = \mathring{\mathfrak{B}}^k \oplus \mathfrak{B}^{*k} \oplus \mathring{\mathfrak{H}}^k. \tag{2.30b}$$

2.2.4 Shape derivatives

Let $V \subseteq \Omega$ with $\dim V = m \leq n = \dim \Omega$ be an oriented compact submanifold. Given a flow $\xi_t : \Omega \to \Omega$ defined in a neighborhood of V for small time t, and define the submanifold $V(t) = \xi_t V$ [39]. Obviously, $V(0) = \xi_0 V = V$.

Definition 2.2.1. *([95]) Let* $X = \frac{d\xi_t(x)}{dt}\,|_{t=0}$ *be the velocity field. Let the functional* I *be defined as* $I(V(t);\alpha) = \int_{V(t)} \alpha$, $\alpha \in \Lambda^m(\Omega)$*. Then the shape derivative of the functional* I *at* V *in the direction of the vector field* X *is defined as*

$$dI(V; X, \alpha) := \lim_{t\to 0} \frac{1}{t}\Big(I(V(t);\alpha) - I(V(0);\alpha)\Big). \tag{2.31}$$

From (2.31), we have that

$$\begin{aligned} dI(V; X, \alpha) &= \lim_{t\to 0} \frac{1}{t}\Big(\int_{V(t)} \alpha(y) - \int_V \alpha(x)\Big) \\ &= \lim_{t\to 0} \frac{1}{t}\Big(\int_V \xi_t^* \alpha(\xi_t x) - \int_V \alpha(x)\Big) \\ &= \int_V \lim_{t\to 0} \frac{1}{t}\Big(\xi_t^* \alpha(\xi_t x) - \alpha(x)\Big) \\ &= \int_V \mathfrak{L}_X \alpha, \end{aligned} \tag{2.32}$$

where $x \in V, y = \xi_t x \in V(t)$ and $\xi_t^* : \Lambda^m(V(t)) \to \Lambda^m(V)$ is a pull-back map and $\mathfrak{L}_X$ is the Lie derivative.

2.3 Incompressible Euler equations, function spaces, and Hodge decompositions

In this section, we will first define several Sobolev spaces of differential forms that will be extensively used in this chapter. Next, we will state the incompressible Euler equations with a free surface in terms of differential forms, discuss the Hodge decomposition of the velocity, and give some auxiliary results.

2.3.1 Definition of Sobolev spaces

For the analysis of the incompressible Euler equations, we introduce the following notation

$$
\begin{aligned}
&P\Lambda^k(\Omega) := \{\gamma \in H\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega) \mid d\gamma = 0 \text{ in } \Omega\},\\
&\mathring{P}\Lambda^k(\Omega) := \{\gamma \in H\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega) \mid d\gamma = 0 \text{ in } \Omega,\ \mathrm{tr}(\gamma) = 0 \text{ at } \Gamma\};\\
\\
&P^*\Lambda^k(\Omega) := \{\gamma \in H\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega) \mid \delta\gamma = 0 \text{ in } \Omega\},\\
&\mathring{P}^*\Lambda^k(\Omega) := \{\gamma \in H\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega) \mid \delta\gamma = 0 \text{ in } \Omega,\ \mathrm{tr}(*\gamma) = 0 \text{ at } \Gamma\};\\
\\
&\mathring{V}\Lambda^k(\Omega) := \{\mu \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) \mid d\mu = 0 \text{ in } \Omega\},\\
&\mathring{V}^*\Lambda^k(\Omega) := \{\mu \in \mathring{H}\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega) \mid \delta\mu = 0 \text{ in } \Omega\}.
\end{aligned}
$$

Note, $P^*\Lambda^k(\Omega) = *P\Lambda^{n-k}(\Omega)$, $\mathring{P}^*\Lambda^k(\Omega) = *\mathring{P}\Lambda^{n-k}(\Omega)$ and $*\mathring{V}\Lambda^k(\Omega) = \mathring{V}^*\Lambda^{n-k}(\Omega)$. For $k = 1$ elements of $P^*\Lambda^1(\Omega)$ are proxies (vector fields associated to one-forms [7, p26-27]) of divergence-free vector fields.

2.3.2 Incompressible Euler equations with a free surface

Let $\Omega(t) \subset \mathbb{R}^n, n \in \{1, 2, 3\}$ be a time dependent oriented manifold with $(n-1)$-dimensional Lipschitz continuous boundary $\partial\Omega$. The boundary $\partial\Omega$ is split into a free surface part Σ and a possibly empty fixed part $\Gamma = \partial\Omega\backslash\Sigma$. The incompressible Euler equations with a free surface, including gravity as an external force and surface tension at the free surface, see e.g. [96, p447-455], stated in differential forms [2, 11] are

$$v_t = -i_{v^\sharp} dv - d\Big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \frac{\tilde{p}}{\rho} + \Phi\Big) = -i_{v^\sharp} dv - dh \quad \text{in } \Omega, \tag{2.33a}$$

$$\delta v = 0 \quad \text{in } \Omega, \tag{2.33b}$$

$$\Sigma_t = *\boldsymbol{n}v \quad \text{at } \Sigma, \tag{2.33c}$$

$$\text{tr}(\tilde{p}) = \tau k \quad \text{at } \Sigma, \tag{2.33d}$$

$$*\,\boldsymbol{n}v = g \quad \text{at } \Gamma. \tag{2.33e}$$

The variables in (2.33) are defined as follows:

- $v \in P^*\Lambda^1(\Omega)$ denotes the fluid velocity; $p \in H^1\Lambda^0(\Omega)$ the pressure, $\tilde{p} := p - \bar{p} \in H^1\Lambda^0(\Omega)$, with $\bar{p} \in \mathbb{R}^+$ a constant atmospheric pressure; $\rho \in \mathbb{R}^+$ a constant fluid density; $\Phi \in H^1\Lambda^0(\Omega)$ the gravity potential, which is a linear harmonic function with $d\Phi = g_0$, with g_0 the gravity acceleration and $\delta g_0 = 0$, e.g. in Cartesian coordinates $\Phi = g_0 z$, with z the free surface height above the reference $x - y$ plane. We also define
$$h := \frac{1}{2}\langle\!\langle v^\sharp, v^\sharp \rangle\!\rangle + \frac{\tilde{p}}{\rho} + \Phi \in H^1\Lambda^0(\Omega). \tag{2.34}$$

- Given a flow $\xi_t : \bar{\Omega} \to \bar{\Omega}$, we define the free surface as $\Sigma(t) = \xi_t\Sigma$. Obviously, $\Sigma(0) = \Sigma$. We define the time derivative of Σ as
$$\Sigma_t := \Big\{ \frac{\partial}{\partial t}\xi_t(\Sigma(0)) \in \mathbb{R}^n, x \in \Sigma \subseteq \partial\Omega \Big\}.$$
Hence, (2.33c) expresses that the velocity of points at the free surface Σ equals the normal fluid velocity.

- k is the mean curvature of Σ, defined as $k := \text{div}(\mathcal{N})$, with $\mathcal{N}$ the unit outward normal vector at $\partial\Omega$. The divergence div satisfies $\text{div}(X)v_\Omega = d(i_X v_\Omega)$ for any vectors $X \in T_p\Omega$ and volume form v_Ω; $\tau \in \mathbb{R}, \tau \geq 0$ is the surface tension, which is assumed to be a constant. Hence (2.33d) expresses that at the free surface Σ the pressure jump $p - \bar{p}$ at Σ is equal to the surface tension.

- $g \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$ is a prescribed normal velocity at Γ and is equal to zero at Σ.

The Euler equations with a free surface (2.33) model free surface gravity waves, including waves around an object, capillary waves, and also for instance the shape of droplets in the air.

2.3.3 Hodge decomposition of the velocity

In this section, we will discuss the Hodge decomposition of the velocity. This will introduce three new variables, namely an exact form ϕ (potential), a co-exact form β and a harmonic form α. This Hodge decomposition will be an essential tool in the analysis of the different Hamiltonian formulations that will be discussed in the subsequent sections.

According to (2.30a) and (2.30b), there are two different Hodge decompositions for $v \in P^*\Lambda^1(\Omega)$, namely

$$v = d\phi + \delta\beta + \alpha, \text{ where} \tag{2.35}$$

$$(i) : d\phi \in \mathfrak{B}^1,\ \delta\beta \in \mathring{\mathfrak{B}}^{*1} \text{ and } \alpha \in \mathfrak{H}^1,$$
$$(ii) : d\phi \in \mathring{\mathfrak{B}}^1,\ \delta\beta \in \mathfrak{B}^{*1} \text{ and } \alpha \in \mathring{\mathfrak{H}}^1.$$

Since in general, an inviscid incompressible fluid must satisfy an inhomogeneous boundary condition for the normal component of the velocity at the domain boundary $\partial\Omega = \Sigma \cup \Gamma$, we will use Hodge decomposition (i) for the velocity v.

Lemma 2.3.1. *The exact form ϕ, co-exact form β and harmonic form α in the Hodge decomposition (2.35) $v = d\phi + \delta\beta + \alpha$, where $v \in P^*\Lambda^1(\Omega)$, $d\phi \in \mathfrak{B}^1$, $\delta\beta \in \mathring{\mathfrak{B}}^{*1}$ and $\alpha \in \mathfrak{H}^1$, solve the following boundary value problems separately*

$$\begin{cases} \delta d\phi = 0 \quad \text{in } \Omega, \\ *\boldsymbol{n}(d\phi) = *\boldsymbol{n}v = \Sigma_t \quad \text{on } \Sigma, \\ *\boldsymbol{n}(d\phi) = *\boldsymbol{n}v = g \quad \text{on } \Gamma, \end{cases} \tag{2.36}$$

$$\begin{cases} d\delta\beta = dv \quad \text{in } \Omega, \\ d\beta = 0 \quad \text{in } \Omega, \\ \operatorname{tr}(*\beta) = 0 \quad \text{on } \Sigma \cup \Gamma, \end{cases} \tag{2.37}$$

$$\begin{cases} d\alpha = \delta\alpha = 0 \quad \text{in } \Omega, \\ \operatorname{tr}(*\alpha) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.38}$$

Proof. From the Hodge decomposition $v = d\phi + \delta\beta + \alpha$ and the divergence-free condition (2.33b) since $v \in P^*\Lambda^1(\Omega)$, it is obvious that

$$\delta v = \delta d\phi = 0, \quad dv = d\delta\beta.$$

Since $d\delta\beta = dv \in \mathfrak{B}^2$, and from Lemma 2.2.5 we have $\mathfrak{B}^2 \subset \mathfrak{Z}^2$, with $\mathfrak{Z}^2$ given by (2.24), which implies that $d\beta = 0$. Also, since $\delta\beta \in \mathring{\mathfrak{B}}^{*1}, \alpha \in \mathfrak{H}^1$, it directly follows that $\operatorname{tr}(*\beta) = \operatorname{tr}(*\alpha) = 0$, and $d\alpha = \delta\alpha = 0$. Then using

(2.16) and (2.17), we obtain

$$\begin{aligned}&\boldsymbol{n}(\delta\beta) = (-1)^{n-1}\boldsymbol{n}(*d*\beta) = (-1)^{n-1} * \mathrm{tr}(d*\beta) = (-1)^{n-1} * d\mathrm{tr}(*\beta) = 0,\\ &\boldsymbol{n}(\alpha) = (-1)^{n-1} * \mathrm{tr}(*\alpha) = 0.\end{aligned} \tag{2.39}$$

Thus,

$$\boldsymbol{n}(d\phi) = \boldsymbol{n}v - \boldsymbol{n}(\delta\beta) - \boldsymbol{n}(\alpha) = \boldsymbol{n}v.$$

□

In case $\Gamma = \partial\Omega$, hence a domain without a free surface, the harmonic form α in the Hodge decomposition (2.35) is uncoupled from $d\phi$ and $\delta\beta$. It is then straightforward to consider a connected domain and include α in the Hamiltonian formulation. At the free surface Σ, α is, however, coupled to Σ through the boundary condition $\boldsymbol{n}(\alpha) = 0$ at $\partial\Omega$, which significantly complicates the Hamiltonian formulation. For clarity of exposition, we assume therefore in the mathematical formulations that use a Hodge decomposition, in Sections 2.4 and 2.5 that the domain Ω is simply connected, which implies $\alpha = 0$.

For the analysis in the subsequent sections we need explicit solutions of the equations for the exact form ϕ (2.36) and the co-exact form β (2.37). For the Poisson equation (2.36), we define the solution operator N_ϕ, which is stated in the following lemma.

Lemma 2.3.2. *Given the space*

$$V := \{\phi \in H\Lambda^0(\Omega) \mid \int_\Omega \phi v_\Omega = 0\}.$$

The weak formulation of the Poisson equation (2.36): Find $\phi \in V$, such that

$$\langle d\phi, d\psi\rangle_{L^2\Lambda^1(\Omega)} = \int_{\partial\Omega} \mathrm{tr}(\psi) \wedge g_\partial, \quad \forall\psi \in H\Lambda^0(\Omega), \tag{2.40}$$

with $g_\partial \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$ defined as

$$g_\partial = *\boldsymbol{n}(d\phi) = \begin{cases}\Sigma_t & \text{on } \Sigma,\\ g & \text{on } \Gamma,\end{cases}$$

with $\int_{\partial\Omega} g_\partial = 0$, has a unique solution

$$\phi(x) = N_\phi(g_\partial), \tag{2.41}$$

with solution operator $N_\phi : H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega) \to H\Lambda^0(\Omega)$.

The proof of this lemma is standard and therefore omitted. For (2.37) we define the solution operator N_β, which is stated in the following lemma.

Lemma 2.3.3. *The weak formulation of (2.37): Find* $\beta \in \mathring{V}\Lambda^2(\Omega)$, *such that*

$$\langle \delta\beta, \delta\gamma \rangle_{L^2\Lambda^1(\Omega)} = \langle \omega, \gamma \rangle_{L^2\Lambda^2(\Omega)}, \quad \forall \gamma \in \mathring{V}\Lambda^2(\Omega), \tag{2.42}$$

with $\omega = dv \in \mathring{V}\Lambda^2(\Omega)$, *has a unique solution*

$$\beta = N_\beta(\omega), \tag{2.43}$$

with solution operator $N_\beta(\cdot) : \mathring{V}\Lambda^2(\Omega) \to \mathring{V}\Lambda^2(\Omega)$.

Proof. Using the Poincaré inequality [7, Theorem 2.2], there exists a positive constant c such that

$$\|\beta\|_{L^2\Lambda^2(\Omega)} \leq c\big(\|d\beta\|_{L^2\Lambda^3(\Omega)} + \|\delta\beta\|_{L^2\Lambda^1(\Omega)}\big), \quad \forall \beta \in H\Lambda^2(\Omega) \cap \mathring{H}^*\Lambda^2(\Omega).$$

Taking $\gamma = \beta$ in (2.42) and using $\beta \in \mathring{V}\Lambda^2(\Omega)$, which implies $d\beta = 0$, gives the result. □

2.4 Hamiltonian formulations of the incompressible Euler equations with a free surface

In this section, we will formulate the classical Hamiltonian system for the incompressible Euler equations with a free surface (2.33) by applying the differential geometric tools introduced in Section 2.2. Hamiltonian formulations for the (v, Σ), $(\eta, \phi_\partial, \Sigma)$ and $(\omega, \phi_\partial, \Sigma)$ variables will be given. These Hamiltonian formulations will provide the mathematical framework for the port-Hamiltonian formulations that will be discussed in Section 2.5. As an extension of the Hamiltonian system for dynamic free surface problems governed by surface tension considered in [52], we will take a gravity force and, next to the free surface Σ, also a fixed boundary surface Γ into consideration. For the Hamiltonian formulations discussed in this section, we will assume the homogeneous boundary condition $g = 0$ at Γ in (2.33e). In Section 2.5 we will then consider in the port-Hamiltonian formulations the inhomogeneous boundary condition $g \neq 0$ at Γ.

2.4.1 Velocity-free surface Hamiltonian formulation

In order to define the Poisson bracket for the incompressible Euler equations with a free surface in terms of the (v, Σ) variables, we state the corresponding functional derivatives $(\frac{\delta \mathcal{F}}{\delta v}, \frac{\delta \mathcal{F}}{\delta \Sigma})$.

Definition 2.4.1. *Given a functional $\mathcal{F} : P^*\Lambda^1(\Omega) \to \mathbb{R}$, the functional derivative $\frac{\delta \mathcal{F}}{\delta v} \in P\Lambda^{n-1}(\Omega)$ is defined as [1, 2, 65]*

$$\int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge \partial v = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(\mathcal{F}(v + \epsilon \partial v) - \mathcal{F}(v)\big), \ \forall \partial v \in P^*\Lambda^1(\Omega), \tag{2.44}$$

if the limit on the right side exits.

Using the shape derivative stated in Section 2.2.4, we can give the following definition of the functional derivative with respect to the free surface boundary Σ.

Definition 2.4.2. *Let $\partial\Sigma = \partial\Sigma_v v_\Sigma \in H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma)$, where v_Σ denotes the surface form, $v_\Sigma = v_{\partial\Omega}|_\Sigma \in \Lambda^{n-1}(\Sigma)$, and $\partial\Sigma_v \in H^{-\frac{1}{2}}\Lambda^0(\Sigma)$ represent infinitesimal volume preserving variations of $\Sigma \subseteq \partial\Omega$ in its normal direction, which are extended to zero at $\Gamma \subseteq \partial\Omega$. Given arbitrary functionals $\mathcal{F} = \int_V f$, where $f \in L^2\Lambda^m(\Omega)$ is the density function, with $V = \Sigma \subseteq \partial\Omega$ or $V = \Omega$, $m = \dim(V)$. Assume that the velocity v remains constant when Σ varies. Then the functional derivative $\frac{\delta \mathcal{F}}{\delta \Sigma} \in H^{\frac{1}{2}}\Lambda^0(\Sigma)$ is defined as*

$$\int_V \frac{\delta \mathcal{F}}{\delta \Sigma} \wedge \partial\Sigma = d\mathcal{F}(V; \partial\Sigma_v \mathcal{N}, f), \ \forall \partial\Sigma \in H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma),$$

where the shape derivative $d\mathcal{F}$ is given by (2.31).

2.4.1.1 Hamiltonian and corresponding functional derivatives with respect to (v, Σ)

The Hamiltonian $H : P^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ for the incompressible Euler equations with a free surface (2.33) is given by, see [52, 65]

$$H(v, \Sigma) = \int_\Omega \big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp \rangle\!\rangle + \Phi\big) v_\Omega + \frac{\tau}{\rho}\int_\Sigma v_\Sigma. \tag{2.45}$$

Lemma 2.4.1. *The functional derivatives of the Hamiltonian (2.45) with respect to (v, Σ) are*

$$\frac{\delta H}{\delta v} = (-1)^{n-1} * v \qquad \text{in } \Omega, \tag{2.46a}$$

$$\frac{\delta H}{\delta \Sigma} = \operatorname{tr}\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi\big) + \frac{\tau k}{\rho} \qquad at\ \Sigma. \tag{2.46b}$$

Proof. For any $v, \partial v \in P^*\Lambda^1(\Omega)$, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(H(v+\epsilon\partial v) - H(v)\big) &= \int_\Omega \langle\!\langle v^\sharp, \partial v^\sharp\rangle\!\rangle v_\Omega = \langle v, \partial v\rangle_{L^2\Lambda^1(\Omega)} \\ &= \int_\Omega (-1)^{n-1} * v \wedge \partial v.\end{aligned}$$

Therefore, using (2.44) the functional derivative of the Hamiltonian with respect to v is given by (2.46a). Next, we turn to compute the functional derivative with respect to Σ. Given

$$H_1\big(\Omega; (\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi)v_\Omega\big) = \int_\Omega \big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi\big)v_\Omega.$$

Using (2.32), Cartan's formula $\mathfrak{L}_X = i_X d + d i_X$, and Stokes' theorem, we obtain since $\partial\Sigma_v = 0$ at Γ that

$$\begin{aligned}dH_1\big(\Omega; \partial\Sigma_v\mathcal{N}, (\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi)v_\Omega\big) &= \int_\Omega d\big((\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi)i_{\partial\Sigma_v\mathcal{N}}v_\Omega\big) \\ &= \int_\Sigma \operatorname{tr}(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi) \wedge i_{\partial\Sigma_v\mathcal{N}}v_\Omega.\end{aligned}$$

Using the relation $i_{fX}\alpha = f i_X\alpha$, where $f \in \Lambda^0(\Omega)$, $X \in T_p\Omega$ and $\alpha \in \Lambda^k(\Omega)$, we have

$$i_{\partial\Sigma_v\mathcal{N}}v_\Omega|_\Sigma = \partial\Sigma_v i_{\mathcal{N}}v_\Omega|_\Sigma = \partial\Sigma_v v_\Sigma = \partial\Sigma,$$

with $i_{\mathcal{N}}v_\Omega|_\Sigma = v_\Sigma$ and $\mathcal{N}$ the outward pointing normal vector to the boundary Σ. Thus,

$$dH_1\big(\Omega; \partial\Sigma_v\mathcal{N}, (\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi)v_\Omega\big) = \int_\Sigma \operatorname{tr}\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi\big) \wedge \partial\Sigma. \tag{2.47}$$

Given

$$H_2(\Sigma; \tau v_\Sigma) = \frac{\tau}{\rho}\int_\Sigma v_\Sigma.$$

Using (2.32), Cartan's formula and the definition of the mean curvature k, the shape derivative of H_2 is given by

$$dH_2(\Sigma; \partial\Sigma_v\mathcal{N}, \frac{\tau}{\rho}v_\Sigma) = \frac{\tau}{\rho}\int_\Sigma \mathfrak{L}_{\partial\Sigma_v\mathcal{N}}v_\Sigma = \frac{\tau}{\rho}\int_\Sigma \big(d i_{\partial\Sigma_v\mathcal{N}}v_\Sigma + i_{\partial\Sigma_v\mathcal{N}}dv_\Sigma\big).$$

Since $i_{\partial\Sigma_v\mathcal{N}} v_\Sigma = \partial\Sigma_v i_\mathcal{N} i_\mathcal{N} v_\Omega|_\Sigma = 0$, the shape derivative of H_2 becomes

$$\begin{aligned} dH_2(\Sigma; \partial\Sigma_v\mathcal{N}, \frac{\tau}{\rho} v_\Sigma) &= \frac{\tau}{\rho}\int_\Sigma i_{\partial\Sigma_v\mathcal{N}} dv_\Sigma \\ &= \frac{\tau}{\rho}\int_\Sigma \partial\Sigma_v i_\mathcal{N} \mathrm{div}(\mathcal{N}) v_\Omega \\ &= \frac{\tau}{\rho}\int_\Sigma k\partial\Sigma. \end{aligned} \tag{2.48}$$

From (2.47) and (2.48), we obtain (2.46b). □

2.4.1.2 Hamiltonian formulation with respect to (v, Σ) variables

Inspired by the classical form of the Poisson bracket for the incompressible Euler equations, see e.g. [14, 52], the generalized Poisson bracket for the incompressible Euler equations with a free surface can be formulated as

Definition 2.4.3. *Let $\mathcal{F}$, $\mathcal{G}$: $\mathring{P}^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ be arbitrary functionals. The Poisson bracket $\{\cdot,\cdot\}(v,\Sigma) : \mathcal{F} \times \mathcal{G} \to \mathbb{R}$ in terms of the velocity v and free surface Σ is defined as*

$$\begin{aligned} \{\mathcal{F},\mathcal{G}\}(v,\Sigma) =& (-1)^{n-1}\int_\Omega (*dv) \wedge (*\frac{\delta\mathcal{G}}{\delta v}) \wedge (*\frac{\delta\mathcal{F}}{\delta v}) \\ &+ (-1)^{n-1}\int_\Sigma \Big(\frac{\delta\mathcal{F}}{\delta\Sigma} \wedge \mathrm{tr}(\frac{\delta\mathcal{G}}{\delta v}) - \frac{\delta\mathcal{G}}{\delta\Sigma} \wedge \mathrm{tr}(\frac{\delta\mathcal{F}}{\delta v})\Big). \end{aligned} \tag{2.49}$$

Remark 2.4.2. *The bracket $\{\cdot,\cdot\}(v,\Sigma)$ defined in (2.49) is bilinear, skew-symmetric and satisfies the Jacobi identity [52].*

Remark 2.4.3. *Using the product rule for functional derivatives, see Appendix 2.B, it is straightforward to show that the bracket $\{\cdot,\cdot\}(v,\Sigma)$ also satisfies the Leibniz rule, $\{FG, H\} = F\{G,H\} + G\{F,H\}$ for any functionals F, G, H.*

Theorem 2.4.4. *Given a connected, oriented domain Ω with Lipschitz continuous boundary $\partial\Omega$, which consists of a non-overlapping free surface boundary $\Sigma \subseteq \partial\Omega$ and fixed boundary $\Gamma \subseteq \partial\Omega$, $\Sigma \cup \Gamma = \partial\Omega$. For any functional $\mathcal{F}(v,\Sigma) : \mathring{P}^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$, and Hamiltonian functional (2.45), the incompressible Euler equations with a free surface (2.33), and $g = 0$ at Γ, can be expressed as the Hamiltonian system*

$$\dot{\mathcal{F}}(v,\Sigma) = \{\mathcal{F}, H\}(v,\Sigma). \tag{2.50}$$

Proof. First, we will prove that (2.33) implies (2.50). Inserting the functional derivatives (2.46) with respect to (v, Σ) into the Poisson bracket (2.49), we have

$$\begin{aligned}
&\{\mathcal{F}, H\}(v, \Sigma) \\
=&(-1)^{n-1} \int_\Omega (*dv) \wedge v \wedge (*\frac{\delta \mathcal{F}}{\delta v}) + (-1)^{n-1} \int_\Sigma \frac{\delta \mathcal{F}}{\delta \Sigma} \wedge (-1)^{n-1} \mathrm{tr}(*v) \\
-&(-1)^{n-1} \int_\Sigma \mathrm{tr}(\frac{1}{2} \langle\langle v^\sharp, v^\sharp \rangle\rangle + \Phi) \wedge \mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}) - (-1)^{n-1} \int_\Sigma \frac{\tau k}{\rho} \wedge \mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}).
\end{aligned}$$

Using $\frac{\delta \mathcal{F}}{\delta v} \in \mathring{P}\Lambda^{n-1}(\Omega)$, which implies that $d(\frac{\delta \mathcal{F}}{\delta v}) = 0$ in Ω and $\mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}) = 0$ at Γ, and Stokes' theorem, gives

$$\int_\Sigma \mathrm{tr}(\frac{1}{2} \langle\langle v^\sharp, v^\sharp \rangle\rangle + \Phi) \wedge \mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}) = \int_\Omega d\big(\frac{1}{2} \langle\langle v^\sharp, v^\sharp \rangle\rangle + \Phi\big) \wedge \frac{\delta \mathcal{F}}{\delta v}.$$

Then, $\{\mathcal{F}, H\}$ can be rewritten as

$$\begin{aligned}
\{\mathcal{F}, H\}(v, \Sigma) = &- \int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge \Big(i_{v^\sharp} dv + d(\frac{1}{2} \langle\langle v^\sharp, v^\sharp \rangle\rangle + \Phi)\Big) + \int_\Sigma \frac{\delta \mathcal{F}}{\delta \Sigma} \wedge *\boldsymbol{n} v \\
&+(-1)^n \int_\Sigma \frac{\tau k}{\rho} \wedge \mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}),
\end{aligned} \tag{2.51}$$

where we used that $*(v \wedge *dv) = i_{v^\sharp} dv$, which follows from (2.11). From $\frac{\delta \mathcal{F}}{\delta v} \in \mathring{P}\Lambda^{n-1}(\Omega)$ and Stokes' theorem, together with (2.33d), we obtain

$$\int_\Sigma \frac{\tau k}{\rho} \wedge \mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}) = \int_{\partial\Omega} \mathrm{tr}(\frac{\tilde{p}}{\rho}) \wedge \mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}) = (-1)^{n-1} \int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge d(\frac{\tilde{p}}{\rho}). \tag{2.52}$$

Using (2.52), and subsequently (2.33a), (2.33c) and the functional chain rule, we obtain from (2.51)

$$\begin{aligned}
&\{\mathcal{F}, H\}(v, \Sigma) \\
&= - \int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge \Big(i_{v^\sharp} dv + d(\frac{1}{2} \langle\langle v^\sharp, v^\sharp \rangle\rangle + \frac{\tilde{p}}{\rho} + \Phi)\Big) + \int_\Sigma \frac{\delta \mathcal{F}}{\delta \Sigma} \wedge *\boldsymbol{n} v \\
&= \int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge v_t + \int_\Sigma \frac{\delta \mathcal{F}}{\delta \Sigma} \wedge \Sigma_t \\
&= \dot{\mathcal{F}}(v, \Sigma).
\end{aligned}$$

Next, we will prove that (2.50) implies (2.33). If we consider (2.50) with the right hand side written as (2.51) and add and subtract $\int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge d(\frac{\tilde{p}}{\rho})$

to the right hand side of (2.50), then we obtain using $\frac{\delta\mathcal{F}}{\delta v} \in \mathring{P}\Lambda^{n-1}(\Omega)$ and Stokes' theorem

$$\int_\Omega \frac{\delta\mathcal{F}}{\delta v} \wedge \big(v_t + i_{v^\sharp} dv + d(\frac{1}{2}\langle\langle v^\sharp, v^\sharp\rangle\rangle + \frac{\tilde{p}}{\rho} + \Phi)\big) + \int_\Sigma \frac{\delta\mathcal{F}}{\delta\Sigma} \wedge \big(\Sigma_t - *\boldsymbol{n}v\big)$$
$$+ (-1)^n \int_\Sigma \operatorname{tr}(\frac{\delta\mathcal{F}}{\delta v}) \wedge \big(\operatorname{tr}(\frac{\tilde{p}}{\rho}) - \frac{\tau k}{\rho}\big) = 0.$$

Taking $\frac{\delta\mathcal{F}}{\delta v} \in \mathring{P}\Lambda^{n-1}(\Omega)$, $\frac{\delta\mathcal{F}}{\delta\Sigma} \in H^{\frac{1}{2}}\Lambda^0(\Omega)$ arbitrary gives (2.33a), (2.33c) and (2.33d). Equations (2.33b) and (2.33e), with $g = 0$ at Γ, are automatically satisfies if $v \in \mathring{P}^*\Lambda^1(\Omega)$. ☐

Corollary 2.4.1. *For the Hamiltonian $H : P^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ stated in (2.45), we have*

$$\dot{H} = \{H, H\}(v, \Sigma) = 0.$$

Proof. This result follows directly from Theorem 2.4.4 using the skew-symmetry of the Poisson bracket (2.49). ☐

This means that the total energy of an inviscid fluid with free surface boundary Σ and homogeneous boundary condition $g = 0$ at the fixed boundary Γ is conserved when there is no fluid flow through the fixed boundary Γ.

In Section 2.5.2, we will extend the Hamiltonian system stated in Theorem 2.4.4 to a port-Hamiltonian system, which also incorporates the inhomogeneous boundary condition (2.33e) with $g \neq 0$, and accounts for a net energy flow through the open surface Γ.

2.4.2 Solenoidal velocity-potential-free surface Hamiltonian formulation

Based on the Poisson bracket discussed in Section 2.4.1 and the Hodge decompositions given in Section 2.3, we will discuss in this section the Hamiltonian formulation of the incompressible Euler equations with a free surface in terms of a potential function ϕ, a solenoidal velocity field η and the free surface Σ.

We will denote the restriction of the potential function $\phi \in H\Lambda^0(\Omega)$ to the boundary $\partial\Omega$ as $\phi_\partial = \operatorname{tr}(\phi) \in H^{\frac{1}{2}}\Lambda^0(\partial\Omega)$. The solenoidal velocity field is defined as $\eta = \delta\beta \in \mathring{\mathfrak{B}}^{*1}$, which is tangential to $\partial\Omega$. This follows directly from (2.39), which states that $\boldsymbol{n}(\eta) = 0$ at $\partial\Omega$.

In the computation of the functional derivatives of $\mathcal{F}$ in Section 2.4.1, we can use the fact that the variations of v and Σ are not restricted at the free surface. This is not the case for η and Σ since they are coupled through the boundary condition $\boldsymbol{n}(\eta) = 0$.

Given the volume-preserving diffeomorphism $g_t = \varphi_t \circ \xi_t : \Omega \to \Omega$, where $\xi_t, \varphi_t : \Omega \to \Omega$ are both also volume-preserving diffeomorphisms, with ξ_t divergence-free on Ω and parallel to $\partial\Omega$. Differentiating the diffeomorphism g_t and using the relation $\partial\Sigma = *\boldsymbol{n}(\partial\varphi)$, we obtain that variations $\partial\eta$ have the form [52]

$$\partial\eta = \eta' + [\eta, u]_1, \tag{2.53}$$

where $\eta' \in P^*\Lambda^1(\Omega)$ has a trace parallel to $\partial\Omega$, which implies $\boldsymbol{n}(\eta') = 0$ at $\partial\Omega$, $u \in P^*\Lambda^1(\Omega)$ and is coupled to the surface perturbations $\partial\Sigma$ through $*\boldsymbol{n}(u) = \partial\Sigma$. At the fixed surface Γ, $*\boldsymbol{n}(u) = 0$. The bracket $[\cdot,\cdot]_1 : P^*\Lambda^1(\Omega) \times P^*\Lambda^1(\Omega) \to P^*\Lambda^1(\Omega)$ is a Lie-bracket, which satisfies for $\alpha, \beta \in P^*\Lambda^1(\Omega)$, see e.g. [39, 74],

$$[\alpha, \beta]_1 := [\alpha^\sharp, \beta^\sharp]^\flat = (-1)^{n-1}\delta(\alpha \wedge \beta), \tag{2.54}$$

where $[\cdot,\cdot] : T\Omega \times T\Omega \to T\Omega$ is the classical Lie bracket for vector fields.

We now consider the dynamic equations for η and ϕ. Following the same approach as used for (2.53), we have that $\eta \in \mathring{\mathfrak{B}}^{*1}$ satisfies the equation

$$\eta_t = [\eta, u']_1, \tag{2.55}$$

where $*\boldsymbol{n}(u') = \Sigma_t$ at Σ and $*\boldsymbol{n}(u') = 0$ at Γ. We therefore choose $u' = dN_\phi(\Sigma_t)$. Using (2.11), (2.33a) and (2.55), it follows that

$$\begin{aligned} d\phi_t &= v_t - \eta_t = v_t - [\eta, u']_1 \\ &= - * (v \wedge *d\eta) - d\Big(\frac{1}{2}\langle\langle v^\sharp, v^\sharp\rangle\rangle + \frac{\tilde{p}}{\rho} + \Phi\Big) - [\eta, u']_1. \end{aligned} \tag{2.56}$$

Hence, the incompressible Euler equations with a free surface (2.33) can be expressed in terms of the $(\eta, \phi_\partial, \Sigma)$ variables as

$$\begin{cases} \eta_t = [\eta, u']_1 \quad \text{in } \Omega & (2.57\text{a}) \\ d\phi_t = - * (v \wedge *d\eta) - d\Big(\frac{1}{2}\langle\langle v^\sharp, v^\sharp\rangle\rangle + \frac{\tilde{p}}{\rho} + \Phi\Big) - [\eta, u']_1 \text{ in } \Omega, & (2.57\text{b}) \\ \delta d\phi = 0 \text{ in } \Omega, & (2.57\text{c}) \\ * \boldsymbol{n}(d\phi) = \Sigma_t \text{ at } \Sigma, & (2.57\text{d}) \\ * \boldsymbol{n}(d\phi) = g \text{ at } \Gamma, & (2.57\text{e}) \\ \text{tr}(\frac{\tilde{p}}{\rho}) = \frac{\tau k}{\rho} \text{ at } \Sigma, & (2.57\text{f}) \end{cases}$$

with

$$v = d\phi + \eta \quad \text{in } \Omega. \tag{2.58}$$

Note, taking the divergence of (2.57b) we obtain

$$\delta d(\frac{\tilde{p}}{\rho}) = \delta i_{v^\sharp} d\eta + \delta d\big(\frac{1}{2}\langle\langle v^\sharp, v^\sharp \rangle\rangle\big), \tag{2.59}$$

hence (2.57) can be reduced to two Poisson equations, namely (2.59), (2.57f) for the pressure $\frac{\tilde{p}}{\rho}$ and (2.57b)–(2.57e) for the potential ϕ, and a transport equation for η (2.57a).

2.4.2.1 Hodge decomposition of the functional derivative with respect to velocity

For the analysis in the following sections, we will also need a Hodge decomposition of the functional derivative $\frac{\delta \mathcal{F}}{\delta v} \in P\Lambda^{n-1}(\Omega)$. We will use the Hodge decomposition (2.30b) for a simply connected domain, which results in

$$\frac{\delta \mathcal{F}}{\delta v} = \frac{\delta_\beta \mathcal{F}}{\delta v} + \frac{\delta_\phi \mathcal{F}}{\delta v}, \tag{2.60}$$

where $\frac{\delta_\beta \mathcal{F}}{\delta v} \in \mathring{\mathfrak{B}}^{(n-1)}$ and $\frac{\delta_\phi \mathcal{F}}{\delta v} \in \mathfrak{B}^{*(n-1)}$.

Lemma 2.4.5. *Given a simply-connected, oriented domain Ω with Lipschitz continuous boundary $\partial\Omega$. For $\frac{\delta \mathcal{F}}{\delta v} \in P\Lambda^{n-1}(\Omega)$, the functional derivative $\frac{\delta_\phi \mathcal{F}}{\delta v} \in \mathfrak{B}^{*(n-1)}$ can be expressed as*

$$\frac{\delta_\phi \mathcal{F}}{\delta v} = - * dw, \tag{2.61}$$

where

$$w = -N_\phi\big(\mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v})\big) \in H\Lambda^0(\Omega), \tag{2.62}$$

with $N_\phi : H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega) \to H\Lambda^0(\Omega)$ the solution operator of the Poisson equation (2.36).

Proof. Since $\frac{\delta_\beta \mathcal{F}}{\delta v} \in \mathring{\mathfrak{B}}^{(n-1)}$, from definition (2.25), there exists an $w' \in \mathring{H}\Lambda^{n-2}(\Omega)$ such that

$$\frac{\delta_\beta \mathcal{F}}{\delta v} = dw', \tag{2.63}$$

with boundary condition

$$\mathrm{tr}(\frac{\delta_\beta \mathcal{F}}{\delta v}) = \mathrm{tr}(dw') = 0.$$

Similarly, from $\frac{\delta_\phi \mathcal{F}}{\delta v} \in \mathfrak{B}^{*(n-1)}$ and definition (2.27), there exists an $w \in H\Lambda^0(\Omega)$ such that

$$\frac{\delta_\phi \mathcal{F}}{\delta v} = \delta(*w) = - * dw. \tag{2.64}$$

From $\frac{\delta \mathcal{F}}{\delta v} \in P\Lambda^{n-1}(\Omega)$ we have $d(\frac{\delta \mathcal{F}}{\delta v}) = 0$ and using subsequently (2.60), (2.63) and (2.64), we obtain that w satisfies the Laplace equation (2.62), since

$$0 = *d(\frac{\delta \mathcal{F}}{\delta v}) = (-1)^n \delta(* \frac{\delta_\beta \mathcal{F}}{\delta v} + * \frac{\delta_\phi \mathcal{F}}{\delta v}) = (-1)^n \big(\delta(*dw') - \delta(**dw)\big) = \delta dw,$$

with boundary condition

$$*\boldsymbol{n}(dw) = \mathrm{tr}(*dw) = -\mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}) + \mathrm{tr}(\frac{\delta_\beta \mathcal{F}}{\delta v}) = -\mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}).$$

Using the solution operator for the Laplace equation as stated in Lemma 2.3.2, we obtain (2.62). □

2.4.2.2 Functional derivatives with respect to $(\eta, \phi_\partial, \Sigma)$ variables

The first step in the derivation of the solenoidal velocity-potential-free surface $(\eta, \phi_\partial, \Sigma)$ Hamiltonian formulation of the incompressible Euler equations with a free surface is to obtain the functional derivatives with respect to η, ϕ_∂ and Σ.

Given arbitrary functionals $\mathcal{F}(v, \Sigma) : P^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ and $\tilde{\mathcal{F}}(\eta, \phi_\partial, \Sigma) : \mathring{\mathfrak{B}}^{*1} \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$, which satisfy the relation

$$\tilde{\mathcal{F}}(\eta, \phi_\partial, \Sigma) = \mathcal{F}(v, \Sigma), \tag{2.65}$$

we can define the following functional derivatives.

Definition 2.4.4. *The functional derivative* $\frac{\delta \tilde{\mathcal{F}}}{\delta \eta} \in \mathring{\mathfrak{B}}^{(n-1)}$ *is defined as*

$$\int_\Omega \frac{\delta \tilde{\mathcal{F}}}{\delta \eta} \wedge \partial\eta = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(\tilde{\mathcal{F}}(\eta + \epsilon \partial\eta) - \tilde{\mathcal{F}}(\eta)\big), \ \forall \partial\eta \in \mathring{\mathfrak{B}}^{*1}.$$

The functional derivative $\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial} \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$ *is defined as*

$$\int_{\partial\Omega} \frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial} \wedge \partial\phi_\partial = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(\tilde{\mathcal{F}}(\phi_\partial + \epsilon \partial\phi_\partial) - \tilde{\mathcal{F}}(\phi_\partial)\big), \ \forall \partial\phi_\partial \in H^{\frac{1}{2}}\Lambda^0(\partial\Omega).$$

The functional derivative $\frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma}$ is stated in Definition 2.4.2 with $\mathcal{F}$ replaced by $\tilde{\mathcal{F}}$.

The next lemma provides the relation between the functional derivatives $(\frac{\delta\mathcal{F}}{\delta v}, \frac{\delta\mathcal{F}}{\delta\Sigma})$ and $(\frac{\delta\tilde{\mathcal{F}}}{\delta\eta}, \frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}, \frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma})$. Here, we choose $u = dN_\phi(\partial\Sigma) \in H\Lambda^1(\Omega)$, with $\partial\Sigma = 0$ at Γ and N_ϕ the solution operator of the Laplace equation stated in Lemma 2.3.2.

Lemma 2.4.6. *For any functionals* $\tilde{\mathcal{F}}(\eta, \phi_\partial, \Sigma) : \mathring{\mathfrak{B}}^{*1} \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ *and* $\mathcal{F}(v, \Sigma) : P^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$, *the functional derivatives have the following relations*

$$\begin{cases} \dfrac{\delta\tilde{\mathcal{F}}}{\delta\eta} = \dfrac{\delta_\beta\mathcal{F}}{\delta v}, & (2.66a) \\ \dfrac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial} = (-1)^{n-1}\mathrm{tr}(\dfrac{\delta\mathcal{F}}{\delta v}), & (2.66b) \\ \dfrac{\delta\tilde{\mathcal{F}}}{\delta\Sigma} = \dfrac{\delta\mathcal{F}}{\delta\Sigma} + (-1)^{n-1}\langle dN_\phi(\dfrac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}), \eta\rangle_{\Lambda^1}. & (2.66c) \end{cases}$$

Furthermore, we have

$$\frac{\delta\mathcal{F}}{\delta v} = \frac{\delta\tilde{\mathcal{F}}}{\delta\eta} + (-1)^{n-1} * dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}). \tag{2.67}$$

Proof. The variation ∂v has the form

$$\partial v = d(\partial\phi) + \partial\eta.$$

Hence,

$$\int_\Omega \frac{\delta\mathcal{F}}{\delta v} \wedge \partial v = \int_\Omega \frac{\delta\mathcal{F}}{\delta v} \wedge \big(d(\partial\phi) + \partial\eta\big).$$

Applying the decomposition (2.60), with (2.61) and (2.53), we obtain

$$\begin{aligned} \int_\Omega \frac{\delta\mathcal{F}}{\delta v} \wedge \partial v = \int_\Omega \Big(& \frac{\delta\mathcal{F}}{\delta v} \wedge d(\partial\phi) + \frac{\delta_\beta\mathcal{F}}{\delta v} \wedge \partial\eta - *dw \wedge \eta' \\ & - *dw \wedge [\eta, u]_1 \Big). \end{aligned} \tag{2.68}$$

Using the integration by parts formula (2.19) and $\frac{\delta\mathcal{F}}{\delta v} \in P\Lambda^{n-1}(\Omega)$, which follows from (2.44) and implies $d(\frac{\delta\mathcal{F}}{\delta v}) = 0$, hence $\delta * (\frac{\delta\mathcal{F}}{\delta v}) = 0$, we have that

$$\begin{aligned} \int_\Omega \frac{\delta\mathcal{F}}{\delta v} \wedge d(\partial\phi) =& \big\langle * \frac{\delta\mathcal{F}}{\delta v}, d(\partial\phi)\big\rangle_{L^2\Lambda^1(\Omega)} \\ =& (-1)^{n-1} \int_{\partial\Omega} \mathrm{tr}(\frac{\delta\mathcal{F}}{\delta v}) \wedge \partial\phi_\partial. \end{aligned} \tag{2.69}$$

For $w \in H\Lambda^0(\Omega)$, as stated in Lemma 2.4.5, and $\eta' \in P^*\Lambda^1(\Omega)$, we have using (2.19)

$$\begin{aligned}\int_\Omega *dw \wedge \eta' =&(-1)^{n-1}\langle w, \delta\eta'\rangle_{L^2\Lambda^0(\Omega)} + (-1)^{n-1}\int_{\partial\Omega} \mathrm{tr}(w) \wedge *\boldsymbol{n}(\eta') \\ =&0,\end{aligned} \tag{2.70}$$

since $*\boldsymbol{n}(\eta') = 0$ at $\partial\Omega$. Using (2.54), Lemma 2.2.4 and the fact that $\eta \in P^*\Lambda^1(\Omega)$, $u = dN_\phi(\partial\Sigma)$, we obtain

$$\int_\Omega *dw \wedge [\eta, u]_1 = \int_\Omega dw \wedge *\delta(\eta \wedge u) = \int_{\partial\Omega} (\langle dw, \eta\rangle_{\Lambda^1} i_{\mathcal{N}} u) v_{\partial\Omega}. \tag{2.71}$$

Since $u = dN_\phi(\partial\Sigma)$ with $\partial\Sigma = 0$ at Γ, we have using Lemma 2.3.2 that $*\boldsymbol{n}(u) = \partial\Sigma$ at Σ and $*\boldsymbol{n}(u) = 0$ at Γ. From (2.11) and the relation $v_\Sigma = i_{\mathcal{N}} v_\Omega|_\Sigma$, it follows that

$$* v_\Sigma = *i_{\mathcal{N}} v_\Omega|_\Sigma = * * (\mathcal{N}^\flat \wedge *v_\Omega)|_\Sigma = (-1)^{n-1}\mathcal{N}^\flat, \tag{2.72}$$

which, using $\partial\Sigma = \partial\Sigma_v v_\Sigma$, results in

$$\boldsymbol{n}(u) = (-1)^{n-1} * \partial\Sigma = (-1)^{n-1}\partial\Sigma_v(*v_\Sigma) = (\partial\Sigma_v)^\flat \quad \text{at } \Sigma. \tag{2.73}$$

Finally, using (2.15), it follows that

$$\begin{aligned}(i_{\mathcal{N}} u) v_{\partial\Omega} &= (i_{\mathcal{N}} u) v_\Sigma = u(\mathcal{N}) v_\Sigma = \boldsymbol{n}(u)(\mathcal{N}) v_\Sigma \\ &= (\partial\Sigma_v)^\flat(\mathcal{N}) v_\Sigma = \partial\Sigma,\end{aligned} \tag{2.74}$$

and (2.71) becomes

$$\int_\Omega *dw \wedge [\eta, u]_1 = \int_\Sigma \langle dw, \eta\rangle_{\Lambda^1} \wedge \partial\Sigma. \tag{2.75}$$

From (2.65), using the chain rule, we now obtain that

$$\begin{aligned}\int_\Omega \frac{\delta\mathcal{F}}{\delta v} \wedge \partial v + \int_\Sigma \frac{\delta\mathcal{F}}{\delta\Sigma} \wedge \partial\Sigma = \int_\Omega \frac{\delta\tilde{\mathcal{F}}}{\delta\eta} \wedge \partial\eta + \int_{\partial\Omega} \frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial} \wedge \partial\phi_\partial \\ + \int_\Sigma \frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma} \wedge \partial\Sigma.\end{aligned} \tag{2.76}$$

Substituting (2.68)-(2.70) and (2.75) into (2.76), we have

$$\begin{aligned}&\int_\Omega \frac{\delta_\beta\mathcal{F}}{\delta v} \wedge \partial\eta + (-1)^{n-1}\int_{\partial\Omega} \mathrm{tr}(\frac{\delta\mathcal{F}}{\delta v}) \wedge \partial\phi_\partial + \int_\Sigma (\frac{\delta\mathcal{F}}{\delta\Sigma} - \langle dw, \eta\rangle_{\Lambda^1}) \wedge \partial\Sigma \\ &= \int_\Omega \frac{\delta\tilde{\mathcal{F}}}{\delta\eta} \wedge \partial\eta + \int_{\partial\Omega} \frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial} \wedge \partial\phi_\partial + \int_\Sigma \frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma} \wedge \partial\Sigma,\end{aligned}$$

which implies using (2.62) that (2.66a)–(2.66c) hold. Finally, (2.67) follows using (2.60)–(2.62). □

Using the fact that ϕ is a harmonic function, the Hamiltonian $H(v, \Sigma)$, stated in (2.45), can be expressed as $\tilde{H} : \mathring{\mathfrak{B}}^{*1} \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$,

$$\begin{aligned}\tilde{H}(\eta, \phi_\partial, \Sigma) = &\int_\Omega \big(\frac{1}{2}\langle\!\langle \eta^\sharp, \eta^\sharp \rangle\!\rangle + \Phi\big) v_\Omega + \frac{1}{2}\int_{\partial\Omega} \mathrm{tr}(\phi) \wedge *\boldsymbol{n}(d\phi) \\ &+ \frac{\tau}{\rho}\int_\Sigma v_\Sigma,\end{aligned} \tag{2.77}$$

with $*\boldsymbol{n}(d\phi)$ given by (2.36). The functional derivatives of $\tilde{H}$ with respect to the $(\eta, \phi_\partial, \Sigma)$ variables can be obtained using (2.66a)–(2.66c), together with (2.46a)–(2.46b),

$$\begin{cases} \dfrac{\delta \tilde{H}}{\delta \eta} = (-1)^{n-1} * \eta, & (2.78a) \\ \dfrac{\delta \tilde{H}}{\delta \phi_\partial} = *\boldsymbol{n}(d\phi), & (2.78b) \\ \dfrac{\delta \tilde{H}}{\delta \Sigma} = \mathrm{tr}\big(\dfrac{1}{2}\langle\!\langle v^\sharp, v^\sharp \rangle\!\rangle + \Phi\big) + \dfrac{\tau k}{\rho} + (-1)^{n-1}\langle d\phi, \eta\rangle_{\Lambda^1}. & (2.78c) \end{cases}$$

2.4.2.3 Hamiltonian formulation with respect to $(\eta, \phi_\partial, \Sigma)$ variables

Using Lemma 2.4.6, we can express the Poisson bracket (2.49) in terms of the $(\eta, \phi_\partial, \Sigma)$ variables, which is stated in the following lemma.

Lemma 2.4.7. *Let $\tilde{\mathcal{F}}, \tilde{\mathcal{G}} : \mathring{\mathfrak{B}}^{*1} \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ be arbitrary functionals. The Poisson bracket (2.49) in the $(\eta, \phi_\partial, \Sigma)$ variables is equal to*

$$\begin{aligned}&\{\tilde{\mathcal{F}}, \tilde{\mathcal{G}}\}(\eta, \phi_\partial, \Sigma) \\ =&(-1)^{n-1}\int_\Omega (*d\eta) \wedge \big(* \frac{\delta\tilde{\mathcal{G}}}{\delta\eta} + dN_\phi(\frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial})\big) \wedge \big(* \frac{\delta\tilde{\mathcal{F}}}{\delta\eta} + dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial})\big) \\ &+ \int_\Sigma \Big(\frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma} + (-1)^n\langle dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}), \eta\rangle_{\Lambda^1}\Big) \wedge \frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial} \\ &- \int_\Sigma \Big(\frac{\delta\tilde{\mathcal{G}}}{\delta\Sigma} + (-1)^n\langle dN_\phi(\frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial}), \eta\rangle_{\Lambda^1}\Big) \wedge \frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}.\end{aligned} \tag{2.79}$$

Proof. After substituting (2.67) into the Poisson bracket (2.49) the result is immediate. □

The Hamiltonian system for the Euler equations with a free surface (2.33) in terms of the $(\eta, \phi_\partial, \Sigma)$ variables is now stated in the following theorem.

Theorem 2.4.8. *Given a simply-connected oriented domain Ω with Lipschitz continuous boundary $\partial\Omega$, which consists of a nonoverlapping free surface $\Sigma \subseteq \partial\Omega$ and fixed boundary $\Gamma \subseteq \partial\Omega$, $\Sigma \cup \Gamma = \partial\Omega$. For any functional $\tilde{\mathcal{F}}(\eta, \phi_\partial, \Sigma) : \mathring{\mathfrak{B}}^{*1} \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$, and Hamiltonian functional $\tilde{H}$ (2.77), the incompressible Euler equations with a free surface in the solenoidal velocity-potential-free surface variables $(\eta, \phi_\partial, \Sigma)$ (2.57)–(2.58), with $g = 0$ at Γ, can be expressed as the Hamiltonian system*

$$\dot{\tilde{\mathcal{F}}}(\eta, \phi_\partial, \Sigma) = \{\tilde{\mathcal{F}}, \tilde{H}\}(\eta, \phi_\partial, \Sigma). \tag{2.80}$$

Proof. Inserting the Hamiltonian $\tilde{H}$ (2.77) into the bracket (2.79), we have

$$\begin{aligned}
&\{\tilde{\mathcal{F}}, \tilde{H}\}(\eta, \phi_\partial, \Sigma)\\
=&(-1)^{n-1}\int_\Omega (*d\eta) \wedge \big(*\frac{\delta \tilde{H}}{\delta \eta} + dN_\phi(\frac{\delta \tilde{H}}{\delta \phi_\partial})\big) \wedge \big(*\frac{\delta \tilde{\mathcal{F}}}{\delta \eta} + dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial})\big)\\
&+\int_\Sigma \Big(\frac{\delta \tilde{\mathcal{F}}}{\delta \Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial}), \eta\rangle_{\Lambda^1}\Big) \wedge \frac{\delta \tilde{H}}{\delta \phi_\partial}\\
&-\int_\Sigma \Big(\frac{\delta \tilde{H}}{\delta \Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \tilde{H}}{\delta \phi_\partial}), \eta\rangle_{\Lambda^1}\Big) \wedge \frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial}\\
:=&T_1 + T_2 + T_3.
\end{aligned}$$

If (2.33) holds with $g = 0$ at Γ, we can compute from (2.66), (2.67), and (2.78), the terms T_1, T_2 and T_3 as follows

$$\begin{aligned}
T_1 =&(-1)^{n-1}\int_\Omega (*d\eta) \wedge v \wedge *\big(\frac{\delta \tilde{\mathcal{F}}}{\delta \eta} + (-1)^{n-1} * dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial})\big)\\
=&-\int_\Omega \big(\frac{\delta \tilde{\mathcal{F}}}{\delta \eta} + (-1)^{n-1} * dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial})\big) \wedge i_{v^\sharp} dv,\\
T_2 =&\int_\Sigma \Big(\frac{\delta \tilde{\mathcal{F}}}{\delta \Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial}), \eta\rangle_{\Lambda^1}\Big) \wedge *\boldsymbol{n}(d\phi),\\
T_3 =&-\int_\Sigma \big(\mathrm{tr}(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi) + \frac{\tau k}{\rho}\big) \wedge (-1)^{n-1}\mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v})\\
=&-\int_\Omega \big(\frac{\delta \tilde{\mathcal{F}}}{\delta \eta} + (-1)^{n-1} * dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial})\big) \wedge d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big),
\end{aligned}$$

where for T_3 we used $\frac{\delta\mathcal{F}}{\delta v} \in \mathring{P}\Lambda^{n-1}(\Omega)$ since $v \in \mathring{P}^*\Lambda^1(\Omega)$, (2.67) and (2.52), which follows from (2.33d). Collecting all terms gives

$$
\begin{aligned}
&\{\tilde{\mathcal{F}}, \tilde{H}\}(\eta, \phi_\partial, \Sigma) \\
=&\int_\Omega \Big(\frac{\delta\tilde{\mathcal{F}}}{\delta\eta} + (-1)^{n-1} * dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial})\Big) \wedge \big(- i_{v^\sharp} dv - d(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \frac{\tilde{p}}{\rho} + \Phi)\big) \\
&+ \int_\Sigma \big(\frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma} + (-1)^n \langle dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}), \eta\rangle_{\Lambda^1}\big) \wedge *\boldsymbol{n}(d\phi) \\
=&\int_\Omega \Big(\frac{\delta\tilde{\mathcal{F}}}{\delta\eta} + (-1)^n * dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial})\Big) \wedge v_t \\
&+ \int_\Sigma \big(\frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma} + (-1)^n \langle dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}), \eta\rangle_{\Lambda^1}\big) \wedge \Sigma_t \\
=&\dot{\tilde{\mathcal{F}}}(\eta, \phi_\partial, \Sigma),
\end{aligned}
\tag{2.81}
$$

which gives (2.80). Note in the last step we used the functional chain rule with $\frac{\delta\mathcal{F}}{\delta v}$ and $\frac{\delta\mathcal{F}}{\delta\Sigma}$ replaced by, respectively, (2.67) and (2.66c), and (2.33a), (2.33c).

Next, we assume that (2.80) holds. If we use (2.66c) and (2.67) for the functional derivatives of $\tilde{\mathcal{F}}$ in (2.81), then the proof is identical to the second part of Theorem 2.4.4. □

In Section 2.5.3 we will extend the Hamiltonian system stated in Theorem 2.4.8 to a port-Hamiltonian system which also incorporates the inhomogeneous boundary condition (2.33e) with $g \neq 0$.

2.4.3 Vorticity-potential-free surface Hamiltonian formulation

The objective of this section is to derive the Hamiltonian formulation of the Euler equations with a free surface in terms of the vorticity ω, the potential function ϕ_∂ and the free surface Σ.

We start with defining the vorticity $\omega = dv \in \mathring{V}\Lambda^2(\Omega)$. After applying the exterior derivative d to both sides of (2.33a) and using (2.11) we obtain the vorticity equation

$$\omega_t + d * (v \wedge *\omega) = 0. \tag{2.82}$$

Therefore, the dynamic equations for the incompressible Euler equations with a free surface in the $(\omega, \phi_\partial, \Sigma)$ variables are

$$\begin{cases} \omega_t + d*\big(v\wedge *\omega\big) = 0 \text{ in } \Omega \qquad\qquad , & (2.83a)\\ d\omega = 0 \text{ in } \Omega, & (2.83b)\\ *\boldsymbol{n}(\omega) = 0 \text{ at } \Gamma\cup\Sigma, & (2.83c)\end{cases}$$

$$\begin{cases} d\phi_t = -*(v\wedge *\omega) - d\big(\frac{1}{2}\langle\langle v^\sharp, v^\sharp\rangle\rangle + \frac{\tilde{p}}{\rho} + \Phi\big) \\ \qquad\qquad - [\delta N_\beta(\omega), u']_1 \text{ in } \Omega, & (2.84a)\\ \delta d\phi = 0 \text{ in } \Omega, & (2.84b)\\ *\boldsymbol{n}(d\phi) = \Sigma_t \text{ at } \Sigma, & (2.84c)\\ *\boldsymbol{n}(d\phi) = g \text{ at } \Gamma, & (2.84d)\\ \mathrm{tr}(\frac{\tilde{p}}{\rho}) = \frac{\tau k}{\rho} \text{ at } \Sigma, & (2.84e)\end{cases}$$

with

$$v = d\phi + \delta N_\beta(\omega) \quad \text{in } \Omega, \tag{2.85}$$

and $u' = dN_\phi(\Sigma_t)$, with $\Sigma_t = 0$ at Γ.

2.4.3.1 Functional derivatives with respect to $(\omega, \phi_\partial, \Sigma)$ variables

Our aim is now to formulate the Hamiltonian formulation in terms of the $(\omega, \phi_\partial, \Sigma)$ variables. We start by defining the functional $\bar{\mathcal{F}}(\omega, \phi_\partial, \Sigma) : \mathring{V}\Lambda^2(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ as

$$\bar{\mathcal{F}}(\omega, \phi_\partial, \Sigma) = \tilde{\mathcal{F}}(\eta, \phi_\partial, \Sigma) = \mathcal{F}(v, \Sigma).$$

Definition 2.4.5. *The functional derivative* $\frac{\delta\bar{\mathcal{F}}}{\delta\omega} \in \mathring{V}^*\Lambda^{n-2}(\Omega)$ *is defined as*

$$\int_\Omega \frac{\delta\bar{\mathcal{F}}}{\delta\omega}\wedge\partial\omega = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\bar{\mathcal{F}}(\omega+\epsilon\partial\omega) - \bar{\mathcal{F}}(\omega)\big),\ \forall \partial\omega \in \mathring{V}\Lambda^2(\Omega).$$

Since ω is independent of ϕ_∂ and Σ, it's obvious that

$$\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial} = \frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial},\quad \frac{\delta\bar{\mathcal{F}}}{\delta\Sigma} = \frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma} = \frac{\delta\mathcal{F}}{\delta\Sigma} + (-1)^{n-1}\langle dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}), \eta\rangle_{\Lambda^1}. \tag{2.86}$$

We now present the relations between the functional derivatives in terms of ω and η.

Lemma 2.4.9. *For an arbitrary an functional* $\bar{\mathcal{F}}(\omega, \phi_\partial, \Sigma) : \mathring{V}\Lambda^2(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$, *we have*

$$\frac{\delta\tilde{\mathcal{F}}}{\delta\eta} = (-1)^{n-1} d\big(\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\big). \tag{2.87}$$

Furthermore, we have

$$\frac{\delta\mathcal{F}}{\delta v} = (-1)^{n-1} d(\frac{\delta\bar{\mathcal{F}}}{\delta\omega}) + (-1)^{n-1} * dN_\phi(\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial}). \tag{2.88}$$

Proof. Using the relation $\omega = dv = d\eta$ and the chain rule, we have used the integration by parts formula (2.19)

$$\begin{aligned}\int_\Omega \frac{\delta\tilde{\mathcal{F}}}{\delta\eta} \wedge \partial\eta &= \int_\Omega \frac{\delta\bar{\mathcal{F}}}{\partial\omega} \wedge \partial\omega = \int_\Omega \frac{\delta\bar{\mathcal{F}}}{\partial\omega} \wedge d(\partial\eta) = \langle d(\partial\eta), *\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\rangle_{L^2\Lambda^2(\Omega)} \\ &= \langle \delta * \frac{\delta\bar{\mathcal{F}}}{\delta\omega}, \partial\eta\rangle_{L^2\Lambda^1(\Omega)} + \int_{\partial\Omega} \mathrm{tr}(\partial\eta) \wedge \mathrm{tr}(\frac{\delta\bar{\mathcal{F}}}{\delta\omega}).\end{aligned}$$

Since $\frac{\delta\bar{\mathcal{F}}}{\delta\omega} \in \mathring{V}^*\Lambda^{n-2}(\Omega)$, it follows that $\mathrm{tr}(\frac{\delta\bar{\mathcal{F}}}{\delta\omega}) = 0$. Thus,

$$\int_\Omega \frac{\delta\tilde{\mathcal{F}}}{\delta\eta} \wedge \partial\eta = \int_\Omega \partial\eta \wedge *\delta * \big(\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\big) = (-1)^{n-1} \int_\Omega d\big(\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\big) \wedge \partial\eta.$$

Hence,

$$\frac{\delta\tilde{\mathcal{F}}}{\delta\eta} = (-1)^{n-1} d\big(\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\big). \tag{2.89}$$

Substituting (2.89) into (2.67) and using (2.86), we immediately obtain (2.88). □

Lemma 2.4.10. *The Hamiltonian functional* $\bar{H}(\omega, \phi_\partial, \Sigma) : \mathring{V}\Lambda^2(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ *is given by*

$$\bar{H}(\omega, \phi_\partial, \Sigma) = \frac{1}{2}\int_\Omega \beta\wedge*\omega + \int_\Omega \Phi v_\Omega + \frac{1}{2}\int_{\partial\Omega} \mathrm{tr}(\phi)\wedge*\boldsymbol{n}(d\phi) + \frac{\tau}{\rho}\int_\Sigma v_\Sigma. \tag{2.90}$$

The functional derivatives in terms of $(\omega, \phi_\partial, \Sigma)$ *are*

$$\begin{cases} \dfrac{\delta\bar{H}}{\delta\omega} = *\beta, & (2.91\text{a}) \\ \dfrac{\delta\bar{H}}{\delta\phi_\partial} = *\boldsymbol{n}(d\phi), & (2.91\text{b}) \\ \dfrac{\delta\bar{H}}{\delta\Sigma} = \mathrm{tr}\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi\big) + \dfrac{\tau k}{\rho} + (-1)^{n-1}\langle d\phi, \eta\rangle_{\Lambda^1}. & (2.91\text{c}) \end{cases}$$

Proof. With (2.77), and after integration by parts of the first term in (2.77) using (2.37), we obtain (2.90). Introducing $\bar{H}$ into (2.87) and using (2.78a), we obtain that $\frac{\delta \bar{H}}{\delta \omega}$ satisfies

$$d\frac{\delta \bar{H}}{\delta \omega} = (-1)^{n-1}\frac{\delta \tilde{H}}{\delta \eta} = *\eta = *\delta\beta = d(*\beta),$$

with the boundary condition

$$\mathrm{tr}(\frac{\delta \bar{H}}{\delta \omega}) = 0.$$

Thus, we obtain (2.91a). Finally, (2.91b) and (2.91c) hold directly from (2.78b), (2.78c) and (2.86). □

2.4.3.2 Poisson bracket with respect to $(\omega, \phi_\partial, \Sigma)$ variables

The Poisson bracket for the Hamiltonian formulation in terms of the $(\omega, \phi_\partial, \Sigma)$ variables are stated in the following lemma.

Lemma 2.4.11. *Let* $\bar{\mathcal{F}}, \bar{\mathcal{G}} : \mathring{V}\Lambda^2(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$. *The Poisson bracket with respect to* $(\omega, \phi_\partial, \Sigma)$ *can be expressed as*

$$\begin{aligned}
&\{\bar{\mathcal{F}}, \bar{\mathcal{G}}\}(\omega, \phi_\partial, \Sigma)\\
=&(-1)^{n-1}\int_\Omega (*\omega) \wedge \Big((-1)^{n-1} * d\frac{\delta \bar{\mathcal{G}}}{\delta \omega} + dN_\phi(\frac{\delta \bar{\mathcal{G}}}{\delta \phi_\partial})\Big) \wedge \big((-1)^{n-1} * d\frac{\delta \bar{\mathcal{F}}}{\delta \omega}\\
&+ dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial})\Big) + \int_\Sigma \Big(\frac{\delta \bar{\mathcal{F}}}{\delta \Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}), \delta N_\beta(\omega)\rangle_{\Lambda^1}\Big) \wedge \frac{\delta \bar{\mathcal{G}}}{\delta \phi_\partial}\\
&- \int_\Sigma \Big(\frac{\delta \bar{\mathcal{G}}}{\delta \Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \bar{\mathcal{G}}}{\delta \phi_\partial}), \delta N_\beta(\omega)\rangle_{\Lambda^1}\Big) \wedge \frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}. \qquad (2.92)
\end{aligned}$$

Proof. After substituting (2.86) and (2.87) into the Poisson bracket (2.79) and using (2.43), which gives $\eta = \delta N_\beta(\omega)$, the result is immediate. □

Theorem 2.4.12. *Given a simply-connected oriented domain* Ω *with Lipschitz continuous boundary* $\partial\Omega$*, which consists of a nonoverlapping free surface* $\Sigma \subseteq \partial\Omega$ *and fixed boundary* $\Gamma \subseteq \partial\Omega$, $\Sigma \cup \Gamma = \partial\Omega$. *For any functionals* $\bar{\mathcal{F}}(\omega, \phi_\partial, \Sigma) : \mathring{V}\Lambda^2(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ *and Hamiltonian functional* $\bar{H}$ *(2.90), the incompressible Euler equations with a free surface in the vorticity-potential-free surface variables* $(\omega, \phi_\partial, \Sigma)$ *(2.83)–(2.85), with* $g = 0$ *at* Γ*, can be expressed as the Hamiltonian system*

$$\dot{\bar{\mathcal{F}}}(\omega, \phi_\partial, \Sigma) = \{\bar{\mathcal{F}}, \bar{H}\}(\omega, \phi_\partial, \Sigma). \qquad (2.93)$$

Proof. First, we will prove that (2.83)–(2.85) imply (2.93). Inserting the functional derivatives of the Hamiltonian $\bar{H}$ (2.91) into the Poisson bracket (2.92) gives

$$
\begin{aligned}
&\{\bar{\mathcal{F}}, \bar{H}\}(\omega, \phi_\partial, \Sigma) \\
=&(-1)^{n-1}\int_\Omega (*\omega) \wedge \big((-1)^{n-1} * d\frac{\delta \bar{H}}{\delta\omega} + dN_\phi(\frac{\delta \bar{H}}{\delta \phi_\partial})\big) \wedge \big((-1)^{n-1} * d\frac{\delta \bar{\mathcal{F}}}{\delta\omega} \\
&+dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial})\big) + \int_\Sigma \Big(\frac{\delta \bar{\mathcal{F}}}{\delta\Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}), \delta N_\beta(\omega)\rangle_{\Lambda^1}\Big) \wedge \frac{\delta \bar{H}}{\delta \phi_\partial} \\
&-\int_\Sigma \Big(\frac{\delta \bar{H}}{\delta\Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \bar{H}}{\delta \phi_\partial}), \delta N_\beta(\omega)\rangle_{\Lambda^1}\Big) \wedge \frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial} \\
:=& T_1 + T_2 + T_3.
\end{aligned}
$$

The term T_1 can be computed using $*d\frac{\delta \bar{H}}{\delta\omega} = *d{*\beta} = (-1)^{n-1}\delta\beta$, $dN_\phi\big(\frac{\delta \bar{H}}{\delta\phi_\partial}\big) = dN_\phi\big(* \boldsymbol{n}(d\phi)\big) = d\phi$, the Hodge decomposition (2.35) and (2.19), yielding

$$
\begin{aligned}
T_1 =&(-1)^{n-1}\int_\Omega (*\omega) \wedge v \wedge \big((-1)^{n-1} * d\frac{\delta \bar{\mathcal{F}}}{\delta\omega} + dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial})\big) \\
=&\int_\Omega \frac{\delta \bar{\mathcal{F}}}{\delta\omega} \wedge *\delta\big((*\omega)\wedge v\big) + (-1)^{n-1}\int_\Omega (*\omega)\wedge v \wedge dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}) \\
=&-\int_\Omega \frac{\delta \bar{\mathcal{F}}}{\delta\omega} \wedge d*(v \wedge *\omega) + (-1)^n \int_\Omega *dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}) \wedge *(v\wedge *\omega).
\end{aligned}
$$

The term T_2 can be computed using (2.91b), resulting in

$$
T_2 = \int_\Sigma \big(\frac{\delta \bar{\mathcal{F}}}{\delta\Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}), \delta N_\beta(\omega)\rangle_{\Lambda^1}\big) \wedge *\boldsymbol{n}(d\phi).
$$

From (2.73) and (2.74), we have that

$$
* \boldsymbol{n}(d\phi) = \Sigma_t = (i_{\mathcal{N}} u')v_\Sigma \text{ at } \Sigma. \tag{2.94}
$$

Using (2.94), Lemma 2.2.4, which is possible since $\delta N_\beta(\omega)$, $u' \in P^*\Lambda^1(\Omega)$ and, using (2.16), (2.37), $\boldsymbol{n}(\delta N_\beta(\omega)) = \delta\boldsymbol{n}(\beta) = \delta * \mathrm{tr}(*\beta) = 0$ at $\partial\Omega$, $*\boldsymbol{n}(u') = 0$ at Γ and (2.54) we obtain

$$
\begin{aligned}
T_2 =& \int_\Sigma \frac{\delta \bar{\mathcal{F}}}{\delta\Sigma} \wedge *\boldsymbol{n}(d\phi) + (-1)^n \int_{\partial\Omega} \langle dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}), \delta N_\beta(\omega)\rangle_{\Lambda^1} i_{\mathcal{N}}(u')v_\Sigma \\
=& \int_\Sigma \frac{\delta \bar{\mathcal{F}}}{\delta\Sigma} \wedge *\boldsymbol{n}(d\phi) + (-1)^n \int_\Omega dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}) \wedge *\delta\big(\delta N_\beta(\omega)\wedge u'\big) \\
=& \int_\Sigma \frac{\delta \bar{\mathcal{F}}}{\delta\Sigma} \wedge *\boldsymbol{n}(d\phi) + (-1)^n \int_\Omega *dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}) \wedge [\delta N_\beta(\omega), u']_1.
\end{aligned}
$$

The term T_3 can be computed using (2.91b) and (2.91c), together with (2.84e)

$$\begin{aligned} T_3 =& -\int_\Sigma \big(\mathrm{tr}(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi) + \frac{\tau k}{\rho}\big) \wedge \frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial} \\ =&(-1)^n \int_\Sigma \mathrm{tr}\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big) \wedge \mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}). \end{aligned}$$

Since $\frac{\delta \mathcal{F}}{\delta v} \in \mathring{P}\Lambda^{n-1}(\Omega)$, which implies $d\big(\frac{\delta \mathcal{F}}{\delta v}\big) = 0$ in Ω and $\mathrm{tr}\big(\frac{\delta \mathcal{F}}{\delta v}\big) = 0$ at Γ, we obtain using Stokes' theorem and (2.88)

$$\begin{aligned} T_3 =& -\int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big) \\ =&(-1)^n \int_\Omega \Big(d\big(\frac{\delta \bar{\mathcal{F}}}{\delta \omega}\big) + *dN_\phi\big(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}\big)\Big) \wedge d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big). \end{aligned}$$

Consider now

$$\begin{aligned} &\int_\Omega d\big(\frac{\delta \bar{\mathcal{F}}}{\delta \omega}\big) \wedge d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big) \\ &= (-1)^{n-1}\langle d\big(\frac{\delta \bar{\mathcal{F}}}{\delta \omega}\big), *d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big)\rangle_{L^2\Lambda^{n-1}(\Omega)} \\ &= (-1)^{n-1}\langle \frac{\delta \bar{\mathcal{F}}}{\delta \omega}, \delta * d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big)\rangle_{L^2\Lambda^{n-2}(\Omega)} \\ &= 0, \end{aligned}$$

where we used $\frac{\delta \bar{\mathcal{F}}}{\delta \omega} \in \mathring{V}^*\Lambda^{n-2}(\Omega)$ and $\delta * d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big) = 0$. Hence T_3 is equal to

$$T_3 = (-1)^n \int_\Omega *dN_\phi\big(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}\big) \wedge d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big).$$

Collecting all terms and using (2.83a), (2.84a), (2.84c) and the functional chain rule we obtain

$$\begin{aligned} \{\bar{\mathcal{F}}, \bar{H}\}(\omega, \phi_\partial, \Sigma) =& \int_\Omega \frac{\delta \bar{\mathcal{F}}}{\delta \omega} \wedge \big(-d*(v \wedge *\omega)\big) + \int_\Sigma \frac{\delta \bar{\mathcal{F}}}{\delta \Sigma} \wedge *\boldsymbol{n}(d\phi) \\ &+(-1)^{n-1} \int_\Omega *dN_\phi\big(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}\big) \wedge \Big(-*(v \wedge *\omega) \\ &-d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big) - [\delta N_\beta(\omega), u']_1\Big) \\ =& \int_\Omega \frac{\delta \bar{\mathcal{F}}}{\delta \omega} \wedge \omega_t + \int_{\partial\Omega} \frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial} \wedge (\phi_\partial)_t + \int_\Sigma \frac{\delta \bar{\mathcal{F}}}{\delta \Sigma} \wedge \Sigma_t \\ =& \dot{\bar{\mathcal{F}}}(\omega, \phi_\partial, \Sigma), \end{aligned} \tag{2.95}$$

which gives (2.93). Here we used in the second step of (2.95) the relation

$$
\begin{aligned}
&(-1)^{n-1}\int_\Omega *dN_\phi(\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial})\wedge d\phi_t = \int_\Omega d\phi_t\wedge\big(d(\frac{\delta\bar{\mathcal{F}}}{\delta\omega}) + *dN_\phi(\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial})\big)\\
&= (-1)^{n-1}\int_\Omega d\phi_t\wedge\frac{\delta\mathcal{F}}{\delta v} = (-1)^{n-1}\int_{\partial\Omega}\mathrm{tr}(\phi_t)\wedge\mathrm{tr}(\frac{\delta\mathcal{F}}{\delta v}) \qquad (2.96)\\
&= \int_{\partial\Omega}\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial}\wedge(\phi_\partial)_t,
\end{aligned}
$$

since $\int_\Omega d\phi_t\wedge d\big(\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\big) = \langle\frac{\delta\bar{\mathcal{F}}}{\delta\omega},\delta{*}d\phi_t\rangle_{L^2\Lambda^{n-2}(\Omega)} + (-1)^{n-1}\int_{\partial\Omega}\mathrm{tr}(\frac{\delta\bar{\mathcal{F}}}{\delta\omega})\wedge\mathrm{tr}(d\phi_t) = 0$ for $\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\in\mathring{V}^*\Lambda^{n-2}(\Omega)$ and $\delta * d\phi_t = *d^2\phi_t = 0$.

Next, we assume that (2.93) holds. Using the same steps as in the evaluation of T_1, T_2 and T_3, but without using (2.84c), we obtain that (2.93) can be expressed as

$$
\begin{aligned}
&\int_\Omega\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\wedge\big(\omega_t + d*(v\wedge *\omega)\big) + \int_\Sigma\frac{\delta\bar{\mathcal{F}}}{\delta\Sigma}\wedge\big(\Sigma_t - *\boldsymbol{n}(d\phi)\big)\\
&+ (-1)^{n-1}\int_\Omega *dN_\phi(\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial})\wedge\big(d\phi_t + *(v\wedge *\omega) + d(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi)\big)\\
&+ [\delta N_\beta(\omega), u']_1\big) + \int_\Sigma\frac{\tau k}{\rho}\wedge\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial} = 0,
\end{aligned}
$$

where we used (2.96) for the third term. Then by adding and subtracting $(-1)^{n-1}\int_\Omega *dN_\phi(\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial})\wedge d(\frac{\tilde{p}}{\rho})$, using the relation

$$
(-1)^{n-1}\int_\Omega *dN_\phi(\frac{\delta\bar{\mathcal{F}}}{\delta\phi})\wedge d(\frac{\tilde{p}}{\rho}) = \int_\Sigma\mathrm{tr}(\frac{\tilde{p}}{\rho})\wedge\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial},
$$

which can be derived in the same way as (2.96), we obtain

$$
\begin{aligned}
&\int_\Omega\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\wedge\big(\omega_t + d*(v\wedge *\omega)\big) + \int_\Sigma\frac{\delta\bar{\mathcal{F}}}{\delta\Sigma}\wedge\big(\Sigma_t - *\boldsymbol{n}(d\phi)\big)\\
&+ (-1)^{n-1}\int_\Omega *dN_\phi(\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial})\wedge\Big(d\phi_t + *(v\wedge *\omega) + d(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp\rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho})\\
&+ [\delta N_\beta(\omega), \eta]_1\Big) + \int_\Sigma\big(\mathrm{tr}(\frac{\tilde{p}}{\rho}) - \frac{\tau k}{\rho}\big)\wedge\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial} = 0.
\end{aligned}
$$

Taking $\frac{\delta\bar{\mathcal{F}}}{\delta\omega}\in\mathring{V}^*\Lambda^{n-2}(\Omega)$, $\frac{\delta\bar{\mathcal{F}}}{\delta\phi_\partial}\in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$ and $\frac{\delta\bar{\mathcal{F}}}{\delta\Sigma}\in H^{\frac{1}{2}}\Lambda^0(\Sigma)$ arbitrary gives (2.83a), (2.84a), (2.84c) and (2.84e). The other conditions in (2.83)–(2.84) are automatically satisfied due to the choice of the function spaces. □

Corollary 2.4.2. *Given a simply-connected oriented domain Ω with Lipschitz continuous boundary $\partial\Omega$, which consists of a nonoverlapping free surface $\Sigma \subseteq \partial\Omega$ and fixed boundary $\Gamma \subseteq \partial\Omega$, $\Sigma \cup \Gamma = \partial\Omega$. For any functionals $\bar{\mathcal{F}}_p(\phi_\partial, \Sigma) : H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$, with $\bar{\mathcal{F}}_p(\phi_\partial, \Sigma) = \bar{\mathcal{F}}(0, \phi_\partial, \Sigma)$ the potential flow free surface water waves (2.84) with $g \neq 0$ and $\tau = 0$ can be expressed as the Hamiltonian system*

$$\dot{\bar{\mathcal{F}}}_p(\phi_\partial, \Sigma) = \{\bar{\mathcal{F}}_p, \bar{H}_p\},$$

with the canonical Poisson bracket given by

$$\{\bar{\mathcal{F}}_p, \bar{\mathcal{G}}_p\}(\phi_\partial, \Sigma) = \int_\Sigma \Big(\frac{\delta\bar{\mathcal{F}}_p}{\delta\Sigma} \wedge \frac{\delta\bar{\mathcal{G}}_p}{\delta\phi_\partial} - \frac{\delta\bar{\mathcal{G}}_p}{\delta\Sigma} \wedge \frac{\delta\bar{\mathcal{F}}_p}{\delta\phi_\partial}\Big),$$

and the Hamiltonian functional

$$\bar{H}_p = \frac{1}{2}\int_{\partial\Omega} \phi_\partial \wedge *\boldsymbol{n}(d\phi).$$

Proof. The result is immediate after introducing $\omega = 0$ and $\tau = 0$ in (2.93). Note at the free surface $\tilde{p} = 0$ since $p = \bar{p}$ the atmospheric pressure. □

Corollary 2.4.2 states a coordinate-free Hamiltonian formulation of the Hamilton principle for surface waves stated in [61] for $\Omega \subseteq \mathbb{R}^2$.

In Section 2.5.4, we will extend the Hamiltonian system stated in Theorem 2.4.12 to a port-Hamiltonian system that also incorporates the inhomogeneous boundary condition (2.84d) with $g \neq 0$ at Γ.

2.5 Port-Hamiltonian system of the incompressible Euler equations with a free surface

Based on the Hamiltonian framework for the incompressible Euler equations with a free surface discussed in Section 2.4, we will now present the port-Hamiltonian formulation for these equations. The port-Hamiltonian framework allows for inhomogeneous boundary conditions and the exchange of the energy of the system through its boundaries. The port-Hamiltonian framework, therefore, provides a direct extension of Hamiltonian systems to open systems. We will consider three formulations:

i) a (v, Σ) formulation with velocity v and free surface Σ;

ii) an $(\eta, \phi_\partial, \Sigma)$ formulation with the solenoidal velocity η, boundary potential ϕ_∂ and free surface Σ;

iii) an $(\omega, \phi_\partial, \Sigma)$ formulation with vorticity ω, boundary potential ϕ_∂ and free surface Σ.

For all three formulations, we will prove that they can be described by a Dirac structure and we will give the related port-Hamiltonian system. The analysis of nonlinear port-Hamiltonian systems for the incompressible Euler equations with a free surface presented in this section will build on the mathematical framework for distributed-parameter port-Hamiltonian systems discussed in [93]. In the next section, we will first introduce some basic notations and the concept of a Dirac structure. The port-Hamiltonian formulations will be discussed in Sections 2.5.2–2.5.4.

2.5.1 Introduction to Dirac structures

The central part in defining port-Hamiltonian systems is the definition of a Dirac structure. We will first give a brief introduction to general Dirac structures, see [90, 93].

Let the linear spaces $\mathcal{F}$ and $\mathcal{E}$ denote, respectively, the space of flows and the space of efforts. The total space $\mathcal{F} \times \mathcal{E}$ is called the space of power variables, and is equipped with a linear and non-degenerate pairing

$$\langle \cdot \mid \cdot \rangle : \mathcal{F} \times \mathcal{E} \to \mathbb{R}.$$

Using the pairing $\langle \cdot | \cdot \rangle$, we can define a symmetric bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathcal{F} \times \mathcal{E}$ as

$$\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle := \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle, \ (f_i, e_i) \in \mathcal{F} \times \mathcal{E}, \ i = 1, 2.$$

Definition 2.5.1. *[90] Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces with the pairing $\langle \cdot | \cdot \rangle$. A Dirac structure is a linear subspace $D \subset \mathcal{F} \times \mathcal{E}$ such that $D = D^\perp$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$.*

Remark 2.5.1. *From Definition 2.5.1 we have that for any $(f, e) \in D$*

$$0 = \langle\langle (f, e), (f, e) \rangle\rangle = 2\langle e | f \rangle \ \textit{implies} \ \langle e | f \rangle = 0, \tag{2.97}$$

which means that if (f, e) is a pair of power variables, the Dirac structure is power-conserving.

2.5.2 Dirac structure in terms of velocity and free surface variables

In this section, we will present the Dirac structure with respect to the state variables (v, Σ) for the n-dimensional incompressible Euler equations with a free surface (2.33). We start with defining the following pairings, extension and trace-lifting operators.

Definition 2.5.2. *Given the differential forms $\alpha \in L^2\Lambda^k(\Omega)$ and $\beta \in L^2\Lambda^{n-k}(\Omega)$, we define at Ω the pairing between α and β as*

$$\langle \alpha | \beta \rangle := \int_\Omega \alpha \wedge \beta.$$

Similarly, we define at $\partial\Omega$ the pairing between $\alpha \in L^2\Lambda^k(\partial\Omega)$ and $\beta \in L^2\Lambda^{n-k-1}(\partial\Omega)$ as

$$\langle \alpha | \beta \rangle := \int_{\partial\Omega} \alpha \wedge \beta.$$

Definition 2.5.3. *[53, Theorem 11.7] The function space $H_{00}^{\frac{1}{2}}\Lambda^0(\Gamma_1)$, with $\Gamma_1 \subset \partial\Omega$, is defined as*

$$\begin{aligned} H_{00}^{\frac{1}{2}}\Lambda^0(\Gamma_1) :=& \{\mu \in H^{\frac{1}{2}}\Lambda^0(\Gamma_1) \mid \rho^{-\frac{1}{2}}\mu \in L^2\Lambda^0(\Gamma_1), \rho = \operatorname{dist}(x, \partial\Omega)\} \\ =& \{\mu \in H^{\frac{1}{2}}\Lambda^0(\Gamma_1) \mid \mu_0 \in H^{\frac{1}{2}}\Lambda^0(\partial\Omega)\}, \end{aligned} \tag{2.98}$$

with μ_0 the extension of μ by zero outside Γ_1.

Remark 2.5.2. *Note, $H_{00}^{\frac{1}{2}}\Lambda^0(\Gamma_1) \subset H^{\frac{1}{2}}\Lambda^0(\Gamma_1)$ with strict inclusion.*

Lemma 2.5.3. *[22, p45] Given the extension operator $E : H^{\frac{1}{2}}\Lambda^0(\Sigma) \to H^{\frac{1}{2}}\Lambda^0(\partial\Omega)$, defined as*

$$E(\mu_\Sigma)|_\Sigma = \mu_\Sigma, \quad \forall \mu_\Sigma \in H^{\frac{1}{2}}\Lambda^0(\Sigma), \tag{2.99}$$

then $H^{\frac{1}{2}}\Lambda^0(\partial\Omega)$ can be decomposed into the spaces

$$H^{\frac{1}{2}}\Lambda^0(\partial\Omega) = H^{\frac{1}{2}}\Lambda^0(\Sigma) \oplus H_{00}^{\frac{1}{2}}\Lambda^0(\Gamma). \tag{2.100}$$

Definition 2.5.4. *The trace-lifting operator* $\operatorname{li} : H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \to H^1\Lambda^0(\Omega)$ *is defined as*

$$\operatorname{tr}\big(\operatorname{li}(\mu)\big) = \mu, \quad \forall \mu \in H^{\frac{1}{2}}\Lambda^0(\partial\Omega). \tag{2.101}$$

Next, we define the function spaces used in the definition of the Dirac structure of the incompressible Euler equations with a free surface in the (v, Σ) variables

$$\begin{aligned}\mathcal{F}_1 &:= P^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Gamma),\\ \mathcal{E}_1 &:= P\Lambda^{n-1}(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\Sigma) \times H^{\frac{1}{2}}\Lambda^0(\Gamma),\end{aligned} \tag{2.102}$$

where the spaces $\mathcal{F}_1$ and $\mathcal{E}_1$ are dual to each other with a non-degenerate pairing using L^2 as pivot space.

The Dirac structure for the incompressible Euler equations with a free surface in terms of the state variables (v, Σ) is stated in the following theorem.

Theorem 2.5.4. *Let $\Omega \subset \mathbb{R}^n$, $n \in \{1, 2, 3\}$ be a n-dimensional oriented connected manifold with Lipschitz continuous boundary $\partial\Omega = \Sigma \cup \Gamma$ with $\Sigma \cap \Gamma = \emptyset$. Assume $v \in P^*\Lambda^1(\Omega)$ is the solution of the incompressible Euler equations with a free surface (2.33). Given the function spaces $\mathcal{F}_1$ and $\mathcal{E}_1$, defined in (2.102), together with the bilinear form:*

$$\begin{aligned}\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle = &\int_\Omega \left(e_v^1 \wedge f_v^2 + e_v^2 \wedge f_v^1\right) + \int_\Sigma \left(e_\Sigma^1 \wedge f_\Sigma^2 + e_\Sigma^2 \wedge f_\Sigma^1\right)\\ &+ \int_\Gamma \left(e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1\right),\end{aligned} \tag{2.103}$$

where

$$f^i = (f_v^i, f_\Sigma^i, f_b^i) \in \mathcal{F}_1, \quad e^i = (e_v^i, e_\Sigma^i, e_b^i) \in \mathcal{E}_1, \quad i = 1, 2.$$

Then $D_1 \subset \mathcal{F}_1 \times \mathcal{E}_1$, defined as

$$\begin{aligned}D_1 := \Bigg\{ &(f_v, f_\Sigma, f_b, e_v, e_\Sigma, e_b) \in \mathcal{F}_1 \times \mathcal{E}_1 \mid\\ &\begin{pmatrix} f_v \\ f_\Sigma \end{pmatrix} = \begin{pmatrix} d\big(\mathrm{li}(\tilde{e}_\Sigma)\big) + i_{(*e_v)^\sharp} dv \\ (-1)^n \mathrm{tr}_\Sigma(e_v) \end{pmatrix},\\ &\begin{pmatrix} f_b \\ e_b \end{pmatrix} = \begin{pmatrix} (-1)^n \mathrm{tr}_\Gamma(e_v) \\ \tilde{e}_\Sigma|_\Gamma \end{pmatrix} \Bigg\}\end{aligned} \tag{2.104}$$

is a Dirac structure, where $\mathrm{tr}_\Sigma(\cdot)$ and $\mathrm{tr}_\Gamma(\cdot)$ are, respectively, the boundary traces at Σ and Γ, and $\tilde{e}_\Sigma = E(e_\Sigma)$.

Before we prove that (2.104) is a Dirac structure, we first prove that f_v is well-defined.

Lemma 2.5.5. *Given* $e_\Sigma \in H^{\frac{1}{2}}\Lambda^0(\Sigma)$, $e_v \in P\Lambda^{n-1}(\Omega)$, *then* $f_v \in P^*\Lambda^1(\Omega)$ *in the Dirac structure (2.104) is well-defined for the lifting operator* $\mathrm{li}(\cdot)$.

Proof. Assume that there are two trace-lifting operators $\mathrm{li}, \mathrm{li}' : H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \to H^1\Lambda^0(\Omega)$ such that

$$\mathrm{tr}\big(\mathrm{li}(\tilde{\mu})\big) = \mathrm{tr}\big(\mathrm{li}'(\tilde{\mu})\big) = \tilde{\mu} = E(\mu_\Sigma), \quad \forall \mu_\Sigma \in H^{\frac{1}{2}}\Lambda^0(\Sigma).$$

For any $e_v \in P\Lambda^{n-1}(\Omega)$ and $e_\Sigma \in H^{\frac{1}{2}}\Lambda^0(\Sigma)$, the variables $f_v \in P^*\Lambda^1(\Omega)$ and $f_v' \in P^*\Lambda^1(\Omega)$ are given as

$$\begin{aligned} f_v &= d\big(\mathrm{li}(\tilde{e}_\Sigma)\big) + i_{(*e_v)^\sharp} dv, \\ f_v' &= d\big(\mathrm{li}(\tilde{e}_\Sigma)\big) + i_{(*e_v)^\sharp} dv, \end{aligned}$$

with $\tilde{e}_\Sigma = E(e_\Sigma)$. Then,

$$f_v - f_v' = d\big(\mathrm{li}(\tilde{e}_\Sigma) - \mathrm{li}'(\tilde{e}_\Sigma)\big).$$

Since $\delta f_v = \delta f_v' = 0$, we have

$$\delta d\big(\mathrm{li}(\tilde{e}_\Sigma) - \mathrm{li}'(\tilde{e}_\Sigma)\big) = 0,$$

with boundary condition

$$\mathrm{tr}\big(\mathrm{li}(\tilde{e}_\Sigma) - \mathrm{li}'(\tilde{e}_\Sigma)\big) = E(e_\Sigma) - E(e_\Sigma) = 0 \quad \text{at } \partial\Omega.$$

So, $\mathrm{li}(\tilde{e}_\Sigma) - \mathrm{li}'(\tilde{e}_\Sigma)$ satisfies the Laplace equation in Ω with homogeneous Dirichlet boundary condition at $\partial\Omega$, hence

$$\mathrm{li}(\tilde{e}_\Sigma) - \mathrm{li}'(\tilde{e}_\Sigma) = 0 \quad \text{in } \Omega,$$

which implies that $f_v = f_v'$. □

Next, we give a proof of Theorem 2.5.4.

Proof. (1) First we check that $i_{(*e_v^1)^\sharp} dv \wedge e_v^2$ is skew-symmetric. For any $e_v^{1,2} \in P\Lambda^{n-1}(\Omega)$, we have using (2.11)

$$\begin{aligned} i_{(*e_v^1)^\sharp} dv \wedge e_v^2 &= * \big((*e_v^1) \wedge (*dv)\big) \wedge e_v^2 \\ &= - *\big((*e_v^2) \wedge (*dv)\big) \wedge e_v^1 \\ &= - i_{(*e_v^2)^\sharp} dv \wedge e_v^1. \end{aligned} \tag{2.105}$$

(2) Next, we show that $D_1 \subset D_1^{\perp}$. Let $(f^1, e^1) \in D_1$ be fixed, and consider any $(f^2, e^2) \in D_1$. From (2.105), e_v^1, $e_v^2 \in P\Lambda^{n-1}(\Omega)$, hence $de_v^1 = de_v^2 = 0$, and ' Theorem, we obtain that

$$\begin{aligned}
&\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle \\
&= \int_\Omega \Big(e_v^1 \wedge \big(d(\mathrm{li}(\tilde{e}_\Sigma^2)) + i_{(*e_v^2)^\sharp} dv \big) + e_v^2 \wedge \big(d(\mathrm{li}(\tilde{e}_\Sigma^1)) + i_{(*e_v^1)^\sharp} dv \big) \Big) \\
&+(-1)^n \int_\Sigma \Big(e_\Sigma^1 \wedge \mathrm{tr}_\Sigma(e_v^2) + e_\Sigma^2 \wedge \mathrm{tr}_\Sigma(e_v^1) \Big) \\
&+(-1)^n \int_\Gamma \Big(\tilde{e}_\Sigma^1 \wedge \mathrm{tr}_\Gamma(e_v^2) + \tilde{e}_\Sigma^2 \wedge \mathrm{tr}_\Gamma(e_v^1) \Big) \\
&= 0.
\end{aligned}$$

Hence, we have $(f^1, e^1) \in D_1^{\perp}$, which implies $D_1 \subset D_1^{\perp}$.

(3) Finally, we show that $D_1^{\perp} \subset D_1$. Let $(f^1, e^1) \in D_1^{\perp} \subset \mathcal{F}_1 \times \mathcal{E}_1$. Then

$$\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle = 0, \quad \forall (f^2, e^2) \in D_1,$$

or equivalently for all $(f^2, e^2) \in D_1$ we have

$$\begin{aligned}
&\int_\Omega \Big(e_v^1 \wedge \big(d(\mathrm{li}(\tilde{e}_\Sigma^2)) + i_{(*e_v^2)^\sharp} dv \big) + e_v^2 \wedge f_v^1 \Big) + \int_\Sigma \Big((-1)^n e_\Sigma^1 \wedge \mathrm{tr}_\Sigma(e_v^2) \\
&+ e_\Sigma^2 \wedge f_\Sigma^1 \Big) + \int_\Gamma \Big(e_b^1 \wedge (-1)^n \mathrm{tr}_\Gamma(e_v^2) + \tilde{e}_\Sigma^2 \wedge f_b^1 \Big) = 0. \qquad (2.106)
\end{aligned}$$

Take $e_v^2 \in P\Lambda^{n-1}(\Omega)$ such that $\mathrm{tr}_\Gamma(e_v^2) = 0$ and choose $\tilde{e}_\Sigma^2 \in H_{00}^{\frac{1}{2}}\Lambda^0(\Sigma)$, which from (2.98) implies that $\tilde{e}_\Sigma^2|_\Gamma = 0$, then we obtain using the skew-symmetry of $i_{(*e_v^2)^\sharp} dv \wedge e_v^1$ and $e_v^1 \wedge d\big(\mathrm{li}(\tilde{e}_\Sigma^2)\big) = (-1)^{n-1} d\big(e_v^1 \wedge \mathrm{li}(\tilde{e}_\Sigma^2)\big)$ since $e_v^1 \in P\Lambda^{n-1}(\Omega)$

$$\begin{aligned}
&\int_\Omega \Big(- e_v^2 \wedge i_{(*e_v^1)^\sharp} dv + (-1)^{n-1} d\big(e_v^1 \wedge \mathrm{li}(\tilde{e}_\Sigma^2)\big) + e_v^2 \wedge f_v^1 \Big) \\
&+ \int_{\partial\Omega} (-1)^n \tilde{e}_\Sigma^1 \wedge \mathrm{tr}(e_v^2) + \int_\Sigma e_\Sigma^2 \wedge f_\Sigma^1 = 0,
\end{aligned}$$

where we extended $\int_\Sigma (-1)^n e_\Sigma^1 \wedge \mathrm{tr}_\Sigma(e_v^2)$ to $\partial\Omega$ since $\mathrm{tr}_\Gamma(e_v^2) = 0$. Using Stokes' theorem and $\tilde{e}_\Sigma^2|_\Gamma = 0$, we then obtain

$$\begin{aligned}
&\int_\Omega \Big(- e_v^2 \wedge i_{(*e_v^1)^\sharp} dv + (-1)^n d\big(e_v^2 \wedge \mathrm{li}(\tilde{e}_\Sigma^1)\big) + e_v^2 \wedge f_v^1 \Big) \\
&+ \int_\Sigma \Big((-1)^{n-1} e_\Sigma^2 \wedge \mathrm{tr}_\Sigma(e_v^1) + e_\Sigma^2 \wedge f_\Sigma^1 \Big) = 0.
\end{aligned}$$

Using $d\big(e_v^2 \wedge \mathrm{li}(\tilde{e}_\Sigma^1)\big) = (-1)^{n-1} e_v^2 \wedge d\big(\mathrm{li}(\tilde{e}_\Sigma^1)\big)$ for $e_v^2 \in P\Lambda^{n-1}(\Omega)$, we have

$$\int_\Omega e_v^2 \wedge \big(- i_{(*e_v^1)^\sharp} dv - d\big(\mathrm{li}(\tilde{e}_\Sigma^1)\big) + f_v^1\big) + \int_\Sigma e_\Sigma^2 \wedge \big((-1)^{n-1}\mathrm{tr}_\Sigma(e_v^1) + f_\Sigma^1\big)$$
$$= 0.$$

Since e_v^2 and e_Σ^2 are arbitrary, we have

$$\begin{cases} f_v^1 = d(\mathrm{li}(\tilde{e}_\Sigma^1)) + i_{(*e_v^1)^\sharp} dv & \text{in } \Omega, \quad (2.107\text{a}) \\ f_\Sigma^1 = (-1)^n \mathrm{tr}_\Sigma(e_v^1) & \text{at } \Sigma. \quad (2.107\text{b}) \end{cases}$$

Next, we consider the relations for the port variables f_b, e_b. After introducing (2.107) into (2.106), we obtain

$$\begin{aligned} &\int_\Omega \Big(e_v^1 \wedge \big(d(\mathrm{li}(\tilde{e}_\Sigma^2)) + i_{(*e_v^2)^\sharp} dv\big) + e_v^2 \wedge \big(d(\mathrm{li}(\tilde{e}_\Sigma^1)) + i_{(*e_v^1)^\sharp} dv\big)\Big) \\ &+ (-1)^n \int_\Sigma \Big(e_\Sigma^1 \wedge \mathrm{tr}_\Sigma(e_v^2) + e_\Sigma^2 \wedge \mathrm{tr}_\Sigma(e_v^1)\Big) \\ &+ \int_\Gamma \Big(e_b^1 \wedge (-1)^n \mathrm{tr}_\Gamma(e_v^2) + \tilde{e}_\Sigma^2 \wedge f_b^1 \Big) \\ &= \int_\Gamma \Big(\tilde{e}_\Sigma^2 \wedge \big((-1)^{n-1}\mathrm{tr}_\Gamma(e_v^1) + f_b^1\big) + (-1)^n \mathrm{tr}_\Gamma(e_v^2) \wedge \big(- \tilde{e}_\Sigma^1 + e_b^1\big)\Big) \\ &= \ 0. \end{aligned}$$

Since e_Σ^2 and e_v^2 are arbitrary, we have

$$\begin{cases} f_b^1 = (-1)^n \mathrm{tr}_\Gamma(e_v^1), & (2.108\text{a}) \\ e_b^1 = \tilde{e}_\Sigma^1|_\Gamma. & (2.108\text{b}) \end{cases}$$

Hence $D_1^\perp \subset D_1$, which together with $D_1 \subset D_1^\perp$ gives that $D_1 = D_1^\perp$, thus D_1 is a Dirac structure.

□

Remark 2.5.6. *Notice that by the power-conserving property (2.97), together with the bilinear form (2.103), it immediately follows that for any* $(f_v, e_\Sigma, f_b, e_v, e_\Sigma, e_b) \in D_1 \subset \mathcal{F}_1 \times \mathcal{E}_1$ *we have*

$$\int_\Omega e_v \wedge f_v + \int_\Sigma e_\Sigma \wedge f_\Sigma + \int_\Gamma e_b \wedge f_b = 0.$$

With the Hamiltonian (2.45), flow variables and energy variables, together with the chain rule, we have

$$\dot{H} = \int_\Gamma e_b \wedge f_b. \tag{2.109}$$

Next, we aim at deriving the (v, Σ) distributed-parameter port-Hamiltonian formulation of the incompressible Euler equations with a free surface. Define the flow variables in the Dirac structure (2.104) as

$$f_v = -v_t \in P^*\Lambda^1(\Omega), \quad f_\Sigma = -\Sigma_t \in H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma),$$

and the energy variables as

$$e_v = \frac{\delta H}{\delta v} \in P\Lambda^{n-1}(\Omega), \quad e_\Sigma = \frac{\delta H}{\delta \Sigma} \in H^{\frac{1}{2}}\Lambda^0(\Sigma),$$

with the Hamiltonian H given in (2.45). Using Theorem 2.5.4, the port-Hamiltonian formulation can be stated in the following corollary.

Corollary 2.5.1. *Given an oriented n-dimensional connected manifold Ω with Lipschitz continuous boundary $\partial\Omega$. Given the essential boundary condition (2.33d) and the port variable $f_b = -g$. The distributed-parameter port-Hamiltonian system for the incompressible Euler equations with a free surface (2.33) with respect to the state space variables $(v, \Sigma) \in P^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma)$, Dirac structure D_1 (2.104), and Hamiltonian (2.45), is given as*

$$\begin{pmatrix} -v_t \\ -\Sigma_t \end{pmatrix} = \begin{pmatrix} i_{(*\cdot)^\sharp} dv & d\big(\mathrm{li}(E(\cdot))\big) \\ (-1)^n \mathrm{tr}_\Sigma(\cdot) & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta \Sigma} \end{pmatrix}, \tag{2.110}$$

with port variables f_b, e_b defined as,

$$\begin{pmatrix} f_b \\ e_b \end{pmatrix} = \begin{pmatrix} (-1)^n \mathrm{tr}_\Gamma(\cdot) & 0 \\ 0 & E(\cdot)|_\Gamma \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta \Sigma} \end{pmatrix}. \tag{2.111}$$

Proof. Using (2.46) and the essential boundary condition (2.33d), we have

$$\begin{aligned} \frac{\delta H}{\delta v} &= (-1)^{n-1} * v \in P\Lambda^{n-1}(\Omega), \\ \frac{\delta H}{\delta \Sigma} &= \mathrm{tr}\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp \rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big) \in H^{\frac{1}{2}}\Lambda^0(\Sigma). \end{aligned} \tag{2.112}$$

Thus, from Lemma 2.5.4 we obtain

$$d\big(\mathrm{li}(E(\frac{\delta H}{\delta \Sigma}))\big) = d\big(\frac{1}{2}\langle\!\langle v^\sharp, v^\sharp \rangle\!\rangle + \Phi + \frac{\tilde{p}}{\rho}\big). \tag{2.113}$$

Substituting (2.112) and (2.113) into the port-Hamiltonian formulation (2.110), we obtain the incompressible Euler equations (2.33a) and (2.33c). Next, the port variable $f_b = -g = (-1)^n \mathrm{tr}_\Gamma(\frac{\delta H}{\delta v}) = -*\boldsymbol{n}(v)$ gives the inhomogeneous boundary condition (2.33e) at Γ, and $v \in P^*\Lambda^1(\Omega)$ implies (2.33b). □

We will show that the Dirac structure (2.103), (2.104) is associated to a Poisson bracket $\{\cdot,\cdot\}_D(v,\Sigma)$. Define the bilinear form (see [93])

$$[e_1, e_2]_D := (e_1, f_2) = \int_\Omega e_v^1 \wedge f_v^2 + \int_\Sigma e_\Sigma^1 \wedge f_\Sigma^2 + \int_\Gamma e_b^1 \wedge f_b^2.$$

For $\frac{\delta \mathcal{F}}{\delta v}, \frac{\delta \mathcal{G}}{\delta v} \in P\Lambda^{n-1}(\Omega)$ and $\frac{\delta \mathcal{F}}{\delta \Sigma}, \frac{\delta \mathcal{G}}{\delta \Sigma} \in H^{\frac{1}{2}}\Lambda^0(\Sigma)$, the bilinear form $\{\cdot,\cdot\}_D(v,\Sigma) : \mathcal{F} \times \mathcal{G} \to \mathbb{R}$ associated to the Dirac structure (2.104) is

$$\begin{aligned}
&\{\mathcal{F},\mathcal{G}\}_D(v,\Sigma) := [\delta\mathcal{F}, \delta\mathcal{G}]_D \\
&= \int_\Omega \frac{\delta \mathcal{G}}{\delta v} \wedge \Big(d\big(\mathrm{li}(E(\frac{\delta \mathcal{F}}{\delta \Sigma}))\big) + i_{(*\frac{\delta \mathcal{F}}{\delta v})^\sharp} dv \Big) + \int_\Sigma \frac{\delta \mathcal{G}}{\delta \Sigma} \wedge (-1)^n \mathrm{tr}_\Sigma(\frac{\delta \mathcal{F}}{\delta v}) \\
&+ \int_\Gamma E(\frac{\delta \mathcal{G}}{\delta \Sigma}) \wedge (-1)^n \mathrm{tr}_\Gamma(\frac{\delta \mathcal{F}}{\delta v}).
\end{aligned} \tag{2.114}$$

Remark 2.5.7. *With (2.11), Stokes' theorem and the condition* $d(\frac{\delta \mathcal{G}}{\delta v}) = 0$*, which follows from* $\frac{\delta \mathcal{G}}{\delta v} \in P\Lambda^{n-1}(\Omega)$*, we have that (2.114) can be rewritten as*

$$\begin{aligned}
\{\mathcal{F},\mathcal{G}\}_D(v,\Sigma) =&(-1)^{n-1} \int_\Omega (*dv) \wedge (*\frac{\delta \mathcal{G}}{\delta v}) \wedge (*\frac{\delta \mathcal{F}}{\delta v}) \qquad (2.115)\\
&+(-1)^{n-1} \int_{\partial\Omega} \Big(E(\frac{\delta \mathcal{F}}{\delta \Sigma}) \wedge \mathrm{tr}(\frac{\delta \mathcal{G}}{\delta v}) - E(\frac{\delta \mathcal{G}}{\delta \Sigma}) \wedge \mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}) \Big).
\end{aligned}$$

Since the bracket $\{\cdot,\cdot\}_D(v,\Sigma)$ has the same structure as the Poisson bracket (2.49), it is straightforward to check that $\{\cdot,\cdot\}_D(v,\Sigma)$ is linear, anti-symmetric and satisfies the Jacobi identity. The bracket $\{\cdot,\cdot\}_D(v,\Sigma)$ is thus a Poisson bracket and the system generated by the Dirac structure D_1 is a Poisson structure.

Theorem 2.5.8. *Let* $\Omega \subset \mathbb{R}^n, n \in \{1,2,3\}$ *be an oriented n-dimensional connected manifold with Lipschitz continuous boundary* $\partial\Omega = \Sigma \cup \Gamma$ *with free surface* Σ *and fixed boundary* Γ*,* $\Sigma \cap \Gamma = \emptyset$*. For any functionals* $\mathcal{F}(v,\Sigma) : P^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$*, the port-Hamiltonian formulation* *of the incompressible Euler equations with a free surface (2.110) and (2.111) is equivalent to*

$$\dot{\mathcal{F}}(v,\Sigma) = \{\mathcal{F}, H\}_D(v,\Sigma) - \int_\Gamma E(\frac{\delta\mathcal{F}}{\delta\Sigma}) \wedge *\boldsymbol{n}(v), \tag{2.116}$$

with the Hamiltonian H given by (2.45).

Proof. Using the Poisson bracket (2.114) in terms of the (v,Σ) variables, together with (2.11), we have

$$\begin{aligned}
&\{\mathcal{F}, H\}_D(v,\Sigma) = -\{H, \mathcal{F}\}_D(v,\Sigma) \\
&= -\int_\Omega \frac{\delta\mathcal{F}}{\delta v} \wedge \Big(i_{(*\frac{\delta H}{\delta v})^\sharp} dv + d\big(\mathrm{li}(E(\frac{\delta H}{\delta\Sigma}))\big)\Big) + (-1)^{n-1}\int_\Sigma \frac{\delta\mathcal{F}}{\delta\Sigma} \wedge \mathrm{tr}(\frac{\delta H}{\delta v}) \\
&+(-1)^{n-1}\int_\Gamma E(\frac{\delta\mathcal{F}}{\delta\Sigma}) \wedge \mathrm{tr}(\frac{\delta H}{\delta v}).
\end{aligned}$$

Hence, the right-hand side term of (2.116) is

$$\begin{aligned}
&\{\mathcal{F}, H\}_D(v,\Sigma) - \int_\Gamma E(\frac{\delta\mathcal{F}}{\delta\Sigma}) \wedge *\boldsymbol{n}(v) \\
&= -\int_\Omega \frac{\delta\mathcal{F}}{\delta v} \wedge \Big(i_{(*\frac{\delta H}{\delta v})^\sharp} dv + d\big(\mathrm{li}(E(\frac{\delta H}{\delta\Sigma}))\big)\Big) + (-1)^{n-1}\int_\Sigma \frac{\delta\mathcal{F}}{\delta\Sigma} \wedge \mathrm{tr}(\frac{\delta H}{\delta v}).
\end{aligned} \tag{2.117}$$

Next, using the functional chain rule

$$\dot{\mathcal{F}}(v,\Sigma) = \int_\Omega \frac{\delta\mathcal{F}}{\delta v} \wedge v_t + \int_\Sigma \frac{\delta\mathcal{F}}{\delta\Sigma} \wedge \Sigma_t, \tag{2.118}$$

and combining (2.117) with (2.118), we obtain (2.116) if (2.110) holds.

Conversely, if (2.116) holds, together with (2.117) and (2.118), (2.110) directly follows since $\mathcal{F}$ is arbitrary. Since the Poisson bracket $\{\cdot,\cdot\}_D$ is skew-symmetric, we have from (2.116)

$$\dot{H}(v,\Sigma) = -\int_\Gamma E(\frac{\delta H}{\delta\Sigma}) \wedge *\boldsymbol{n}(v) = \int_\Gamma E(\frac{\delta H}{\delta\Sigma}) \wedge (-1)^n \mathrm{tr}(\frac{\delta H}{\delta v}). \tag{2.119}$$

Using (2.109), we obtain the relation for the port variables (2.111). □

Corollary 2.5.2. *Given the Hamiltonian $H : P^*\Lambda^1(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ stated in (2.45) and the essential boundary conditions (2.33d),(2.33e). The rate of change of the Hamiltonian for the incompressible Euler equations with a free surface (2.33) is*

$$\dot{H}(v,\Sigma) = -\int_\Gamma \mathrm{tr}(h) \wedge g. \tag{2.120}$$

Proof. From (2.119), with the functional derivative $\frac{\delta H}{\delta \Sigma} = \mathrm{tr}(h)$, which follows from (2.46b), and (2.33d), and the inhomogeneous boundary condition (2.33e) for the velocity, $*\boldsymbol{n}(v) = g$ at Γ, we obtain (2.120). □

Note, the power balance depends only on the boundary values $\mathrm{tr}(h)$ and g at the fixed boundary Γ. This means that if the boundary of the domain Ω only consists of a free surface Σ, then the Hamiltonian system is power-conserving.

Remark 2.5.9. *Consider the homogeneous boundary condition, which means that for any functional $\mathcal{F}$, $\frac{\delta \mathcal{F}}{\delta v} \in \mathring{P}\Lambda^k(\Omega)$, we have that $\mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}) = 0$. Substituting the boundary condition into (2.115) and (2.116), we obtain the same results as (2.49) and (2.50) shown.*

2.5.3 Dirac structure in terms of the $(\eta, \phi_\partial, \Sigma)$ variables

Based on the Hamiltonian formulation presented in Section 2.4.2, we will present in this section the port-Hamiltonian formulation of the incompressible Euler equations with a free surface with respect to $(\eta, \phi_\partial, \Sigma)$ variables with, η a solenoidal velocity field, ϕ_∂ a potential function and free surface Σ. We will first define the trace-lifting operators and function spaces that will be used in the definition of the Dirac structure.

Definition 2.5.5. *The harmonic trace-lifting operator* $\mathrm{li}_\phi : H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \to H^1\Lambda^0(\Omega)$ *is defined as*

$$\mathrm{tr}\big(\mathrm{li}_\phi(\mu)\big) = \mu, \ \forall \mu \in H^{\frac{1}{2}}\Lambda^0(\partial\Omega), \tag{2.121}$$

with

$$\langle d(\mathrm{li}_\phi(\mu)), d\psi \rangle_{L^2\Lambda^1(\Omega)} = 0, \quad \forall \psi \in \mathring{H}^1\Lambda^0(\Omega).$$

Next, we define the following function spaces for the Dirac structure in the $(\eta, \phi_\partial, \Sigma)$ variables as

$$\begin{aligned}\mathcal{F}_2 :=& \mathring{\mathfrak{B}}^{*1} \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Gamma),\\ \mathcal{E}_2 :=& \mathring{\mathfrak{B}}^{(n-1)} \times H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega) \times H^{\frac{1}{2}}\Lambda^0(\Sigma) \times H^{\frac{1}{2}}\Lambda^0(\Gamma),\end{aligned} \tag{2.122}$$

where the spaces $\mathcal{F}_2$ and $\mathcal{E}_2$ are dual to each other with a non-degenerate pairing using L^2 as pivot space.

The following theorem states the Dirac structure for the $(\eta, \phi_\partial, \Sigma)$ formulation.

Theorem 2.5.10. *Let $\Omega \subset \mathbb{R}^n$, $n \in \{1,2,3\}$ be a n-dimensional oriented and simply connected manifold with Lipschitz continuous boundary $\partial\Omega = \Sigma \cup \Gamma$ with $\Sigma \cap \Gamma = \emptyset$. Given the function spaces $\mathcal{F}_2$ and $\mathcal{E}_2$, defined in (2.122), together with the bilinear form:*

$$\begin{aligned}\langle\!\langle (f^1,e^1),(f^2,e^2)\rangle\!\rangle = &\int_\Omega \left(e^1_\eta \wedge f^2_\eta + e^2_\eta \wedge f^1_\eta\right) + \int_{\partial\Omega} \left(e^1_\phi \wedge f^2_\phi + e^2_\phi \wedge f^1_\phi\right) \\ &+ \int_\Sigma \left(e^1_\Sigma \wedge f^2_\Sigma + e^2_\Sigma \wedge f^1_\Sigma\right) + \int_\Gamma \left(e^1_b \wedge f^2_b + e^2_b \wedge f^1_b\right),\end{aligned} \tag{2.123}$$

where

$$f^i = (f^i_\eta, f^i_\phi, f^i_\Sigma, f^i_b) \in \mathcal{F}_2,\ \ e^i = (e^i_\eta, e^i_\phi, e^i_\Sigma, e^i_b) \in \mathcal{E}_2,\ \ i = 1,2.$$

Then $D_2 \subset \mathcal{F}_2 \times \mathcal{E}_2$, defined as

$$\begin{aligned} D_2 := \Bigg\{ &(f_\eta, f_\phi, f_\Sigma, f_b, e_\eta, e_\phi, e_\Sigma, e_b) \in \mathcal{F}_2 \times \mathcal{E}_2 \mid \\ &\begin{pmatrix} f_\eta \\ d\big(\mathrm{li}_\phi(f_\phi)\big) \\ \\ \\ f_\Sigma|_\Sigma \end{pmatrix} = \begin{pmatrix} -\,[\eta, dN_\phi(e_\phi)]_1, \\ *\Big(\big(*\,e_\eta + dN_\phi(e_\phi)\big) \wedge (*d\eta)\Big) \\ +\,[\eta, dN_\phi(e_\phi)]_1 \\ +\,d\Big(\mathrm{li}\big(\tilde{e}_\Sigma\big) + (-1)^n \mathrm{li}\big(\langle dN_\phi(e_\phi), \eta\rangle_{\Lambda^1}\big)\Big) \\ -\,e_\phi, \end{pmatrix} \\ &\begin{pmatrix} f_b \\ e_b \end{pmatrix} = \begin{pmatrix} -e_\phi|_\Gamma \\ \tilde{e}_\Sigma|_\Gamma \end{pmatrix} \Bigg\}, \end{aligned} \tag{2.124}$$

with $\eta \in \mathring{\mathfrak{B}}^{*1}$ and $\tilde{e}_\Sigma = E(e_\Sigma)$, is a Dirac structure.

Lemma 2.5.11. *Given $e_\eta \in \mathring{\mathfrak{B}}^{(n-1)}$, $e_\phi \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$ and $e_\Sigma \in H^{\frac{1}{2}}\Lambda^0(\Sigma)$, then $d\big(\mathrm{li}_\phi(f_\phi)\big)$ in (2.124) is well-defined.*

Proof. Assume that there exist $\mathrm{li}_\phi(f_\phi),\ \mathrm{li}'_\phi(f_\phi) \in H^1\Lambda^0(\Omega)$ such that for two operators $\mathrm{li},\ \mathrm{li}' : H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \to H^1\Lambda^0(\Omega)$, we have

$$\begin{aligned} d\big(\mathrm{li}_\phi(f_\phi)\big) &= *\Big(\big(*\,e_\eta + dN_\phi(e_\phi)\big) \wedge (*d\eta)\Big) \\ &\quad + d\Big(\mathrm{li}\big(\tilde{e}_\Sigma\big) + (-1)^n \mathrm{li}\big(\langle dN_\phi(e_\phi), \eta\rangle_{\Lambda^1}\big)\Big), \\ d\big(\mathrm{li}'_\phi(f_\phi)\big) &= *\Big(\big(*\,e_\eta + dN_\phi(e_\phi)\big) \wedge (*d\eta)\Big) \\ &\quad + d\Big(\mathrm{li}'\big(\tilde{e}_\Sigma\big) + (-1)^n \mathrm{li}'\big(\langle dN_\phi(e_\phi), \eta\rangle_{\Lambda^1}\big)\Big). \end{aligned}$$

Let $\tilde{f} := \mathrm{li}_\phi(f_\phi) - \mathrm{li}'_\phi(f_\phi)$, then with (2.121), we directly have that

$$\mathrm{tr}(\tilde{f}) = \mathrm{tr}\big(\mathrm{li}_\phi(f_\phi) - \mathrm{li}'_\phi(f_\phi)\big) = 0. \tag{2.125}$$

Since $\mathrm{li}_\phi(\mu)$, $\mathrm{li}'_\phi(\mu)$ both satisfy the Laplace equation, see Definition 2.5.5, we have

$$\delta d\big(\mathrm{li}_\phi(f_\phi) - \mathrm{li}'_\phi(f_\phi)\big) = \delta d\tilde{f} = 0, \tag{2.126}$$

hence, from (2.125) and (2.126) we have

$$\tilde{f} = \mathrm{li}_\phi(f_\phi) - \mathrm{li}'_\phi(f_\phi) = 0.$$

Next, let

$$\hat{f} = \mathrm{li}(\tilde{e}_\Sigma) - \mathrm{li}'(\tilde{e}_\Sigma) + (-1)^n \mathrm{li}(\langle dN_\phi(e_\phi), \eta\rangle_{\Lambda^1}) - (-1)^n \mathrm{li}'(\langle dN_\phi(e_\phi), \eta\rangle_{\Lambda^1}).$$

Using (2.101), we have

$$\mathrm{tr}(\hat{f}) = 0 \quad \text{at } \partial\Omega. \tag{2.127}$$

Since $\tilde{f}$ is harmonic we obtain

$$0 = \delta d\tilde{f} = \delta d\hat{f}, \tag{2.128}$$

which together with (2.127) implies $\hat{f} = 0$ in Ω. □

In the following, we give the proof of Theorem 2.5.10.

Proof. (1) From the definition of the trace-lifting operator li_ϕ (2.121), and taking $\phi = N_\phi(e^i_\phi)$, $\psi = \mathrm{li}_\phi(f^j_\phi)$, $i, j \in \{1, 2\}$ in (2.40) with $f^j_\phi \in H^{\frac{1}{2}}\Lambda^0(\partial\Omega)$, $e^i_\phi \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$, we obtain

$$\langle dN_\phi(e^i_\phi), d\big(\mathrm{li}_\phi(f^j_\phi)\big)\rangle_{L^2\Lambda^1(\Omega)} = \int_{\partial\Omega} e^i_\phi \wedge f^j_\phi. \tag{2.129}$$

Thus, (2.123) can be rewritten as

$$\begin{aligned}\langle\!\langle (f^1, e^1), (f^2, e^2)\rangle\!\rangle = \int_\Omega \Big(& e^1_\eta \wedge f^2_\eta + e^2_\eta \wedge f^1_\eta + dN_\phi(e^1_\phi) \wedge *d\big(\mathrm{li}_\phi(f^2_\phi)\big) \\ & + dN_\phi(e^2_\phi) \wedge *d\big(\mathrm{li}_\phi(f^1_\phi)\big)\Big) + \int_\Sigma \Big(e^1_\Sigma \wedge f^2_\Sigma + e^2_\Sigma \wedge f^1_\Sigma\Big) \\ & + \int_\Gamma \Big(e^1_b \wedge f^2_b + e^2_b \wedge f^1_b\Big).\end{aligned} \tag{2.130}$$

(2) Since $e_\eta \in \mathring{\mathfrak{B}}^{(n-1)} = d\mathring{H}\Lambda^{n-2}(\Omega)$, it follows that $de_\eta = 0$ and $\mathrm{tr}(e_\eta) = 0$. Then using (2.8) and (2.19), together with $\delta(*e_\eta) = (-1)^n * (de_\eta) = 0$, we obtain for any $f \in L^2\Lambda^0(\Omega)$ that

$$\int_\Omega e_\eta \wedge df = \langle f, \delta(*e_\eta)\rangle_{L^2\Lambda^0(\Omega)} + (-1)^{n-1}\int_{\partial\Omega} \mathrm{tr}(f) \wedge \mathrm{tr}(e_\eta) = 0.$$

Choosing $f = \mathrm{li}_\phi(f^j_\phi) - \mathrm{li}\big(\tilde{e}^j_\Sigma\big) + (-1)^{n-1}\mathrm{li}\big(\langle dN_\phi(e^j_\phi), \eta\rangle_{\Lambda^1(\Omega)}\big)$ for $i, j \in \{1, 2\}$, then gives

$$\int_\Omega e^i_\eta \wedge d\Big(\mathrm{li}_\phi(f^j_\phi) - \mathrm{li}(\tilde{e}^j_\Sigma) + (-1)^{n-1}\mathrm{li}\big(\langle dN_\phi(e^j_\phi), \eta\rangle_{\Lambda^1}\big)\Big) = 0,$$

which implies using the expression for f_ϕ in (2.124) that

$$\begin{aligned} &\int_\Omega e^i_\eta \wedge [\eta, dN_\phi(e^j_\phi)]_1 \\ &= -\int_\Omega e^i_\eta \wedge *\Big((*e^j_\eta + dN_\phi(e^j_\phi)) \wedge (*d\eta)\Big). \end{aligned} \tag{2.131}$$

(3) Next, we show that $D_2 \subset D_2^\perp$. Let $(f^1, e^1) \in D_2$ be fixed, and consider any $(f^2, e^2) \in D_2$. From (2.130), we have that

$$\begin{aligned} &\langle\!\langle (f^1, e^1), (f^2, e^2)\rangle\!\rangle \\ &= \int_\Omega \Big(-e^1_\eta \wedge [\eta, dN_\phi(e^2_\phi)]_1 - e^2_\eta \wedge [\eta, dN_\phi(e^1_\phi)]_1 \\ &+ (-1)^{n-1} dN_\phi(e^1_\phi) \wedge \big(* e^2_\eta + dN_\phi(e^2_\phi)\big) \wedge (*d\eta) \\ &+ dN_\phi(e^1_\phi) \wedge *d\Big(\mathrm{li}\big(\tilde{e}^2_\Sigma\big) + (-1)^n \mathrm{li}\big(\langle dN_\phi(e^2_\phi), \eta\rangle_{\Lambda^1}\big)\Big) \\ &+ dN_\phi(e^1_\phi) \wedge *[\eta, dN_\phi(e^2_\phi)]_1 \\ &+ (-1)^{n-1} dN_\phi(e^2_\phi) \wedge \big(* e^1_\eta + dN_\phi(e^1_\phi)\big) \wedge (*d\eta) \\ &+ dN_\phi(e^2_\phi) \wedge *d\Big(\mathrm{li}\big(\tilde{e}^1_\Sigma\big) + (-1)^n \mathrm{li}\big(\langle dN_\phi(e^1_\phi), \eta\rangle_{\Lambda^1}\big)\Big) \\ &+ dN_\phi(e^2_\phi) \wedge *[\eta, dN_\phi(e^1_\phi)]_1 \Big) \\ &- \int_\Sigma \big(e^1_\Sigma \wedge e^2_\phi + e^2_\Sigma \wedge e^1_\phi\big) - \int_\Gamma \big(\tilde{e}^1_\Sigma \wedge e^2_\phi + \tilde{e}^2_\Sigma \wedge e^1_\phi\big). \end{aligned} \tag{2.132}$$

Using (2.40) with $g_\partial = e^i_\phi$ and the definition of the lifting operator (2.101), we obtain for $i, j \in \{1, 2\}$

$$\int_\Omega dN_\phi(e^i_\phi) \wedge *d\Big(\mathrm{li}\big(\tilde{e}^j_\Sigma\big) + (-1)^n \mathrm{li}\big(\langle dN_\phi(e^j_\phi), \eta\rangle_{\Lambda^1}\big)\Big)$$

$$
\begin{aligned}
=&\Big\langle dN_\phi(e^i_\phi), d\Big(\mathrm{li}\big(\tilde{e}^j_\Sigma\big) + (-1)^n \mathrm{li}\big(\langle dN_\phi(e^j_\phi), \eta\rangle_{\Lambda^1}\big)\Big)\Big\rangle_{L^2\Lambda^1(\Omega)} \\
=&\int_{\partial\Omega} \tilde{e}^j_\Sigma \wedge e^i_\phi + (-1)^n \int_{\partial\Omega} \langle dN_\phi(e^j_\phi), \eta\rangle_{\Lambda^1} \wedge e^i_\phi.
\end{aligned}
\tag{2.133}
$$

Next, from (2.54) and Lemma 2.2.4, we have for $i, j \in \{1, 2\}$

$$
\begin{aligned}
&\int_\Omega dN_\phi(e^i_\phi) \wedge *[\eta, dN_\phi(e^j_\phi)]_1 \\
=&(-1)^{n-1} \int_\Omega dN_\phi(e^i_\phi) \wedge *\delta\big(\eta \wedge dN_\phi(e^j_\phi)\big) \\
=&(-1)^{n-1} \int_{\partial\Omega} \big(\langle dN_\phi(e^i_\phi), \eta\rangle_{\Lambda^1} i_{\mathcal{N}} dN_\phi(e^j_\phi)\big) v_\Sigma \\
=&(-1)^{n-1} \int_{\partial\Omega} \langle dN_\phi(e^i_\phi), \eta\rangle_{\Lambda^1} \wedge e^j_\phi,
\end{aligned}
\tag{2.134}
$$

where in the first step we use that fact that $\delta dN_\phi(e^i_\phi) = 0, i = 1, 2$ and $\delta\eta = 0$ for $\eta \in \mathring{\mathfrak{B}}^{*1}$. To show the last step, let $u = dN_\phi(e^j_\phi)$. Then u satisfies

$$
\delta u = 0 \text{ in } \Omega, \quad *\boldsymbol{n}(u) = e^j_\phi \text{ at } \partial\Omega. \tag{2.135}
$$

By applying [33, Proposition 3.20], we have that

$$
\boldsymbol{n}(u) = \mathcal{N}^\flat \wedge (i_{\mathcal{N}} u).
$$

With (2.72) and $i_{\mathcal{N}} u \in L^2\Lambda^0(\partial\Omega)$, we have

$$
*\boldsymbol{n}(u) = *\big(\mathcal{N}^\flat \wedge (i_{\mathcal{N}} u)\big) = (i_{\mathcal{N}} u) \wedge (*\mathcal{N}^\flat) = (i_{\mathcal{N}} u) v_\Sigma.
$$

Finally, using (2.135), we obtain that $i_{\mathcal{N}} dN_\phi(e^j_\phi) v_\Sigma = e^j_\phi$. Substituting (2.131), (2.133) and (2.134) into (2.132), we have

$$
\begin{aligned}
&\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle \\
=&\int_\Omega \Big((-1)^{n-1}\big(*e^1_\eta + dN_\phi(e^1_\phi)\big) \wedge \big(*e^2_\eta + dN_\phi(e^2_\phi)\big) \wedge (*d\eta) \\
&+(-1)^{n-1}\big(*e^2_\eta + dN_\phi(e^2_\phi)\big) \wedge \big(*e^1_\eta + dN_\phi(e^1_\phi)\big) \wedge (*d\eta)\Big) \\
&+\int_{\partial\Omega} \Big(\tilde{e}^2_\Sigma \wedge e^1_\phi + (-1)^n \langle dN_\phi(e^2_\phi), \eta\rangle_{\Lambda^1} \wedge e^1_\phi\Big) \\
&+\int_{\partial\Omega} \Big(\tilde{e}^1_\Sigma \wedge e^2_\phi + (-1)^n \langle dN_\phi(e^1_\phi), \eta\rangle_{\Lambda^1} \wedge e^2_\phi\Big) \\
&+(-1)^{n-1} \int_{\partial\Omega} \Big(\langle dN_\phi(e^1_\phi), \eta\rangle_{\Lambda^1} \wedge e^2_\phi + \langle dN_\phi(e^2_\phi), \eta\rangle_{\Lambda^1} \wedge e^1_\phi\Big)
\end{aligned}
$$

$$- \int_{\partial\Omega} (\tilde{e}^1_\Sigma \wedge e^2_\phi + \tilde{e}^2_\Sigma \wedge e^1_\phi)$$
$$= 0.$$

Therefore, $(f^2, e^2) \in D_2^\perp$ which implies that $D_2 \subset D_2^\perp$.

(4) Finally, we show that $D_2^\perp \subset D_2$. Let $(f^1, e^1) \in D_2^\perp \subset \mathcal{F}_2 \times \mathcal{E}_2$. Then,

$$\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle = 0, \quad \forall (f^2, e^2) \in D_2,$$

which equals

$$\int_\Omega \Big(- e^1_\eta \wedge [\eta, dN_\phi(e^2_\phi)]_1 + e^2_\eta \wedge f^1_\eta + (-1)^{n-1} dN_\phi(e^1_\phi) \wedge \big(* e^2_\eta + dN_\phi(e^2_\phi)\big) \wedge (*d\eta) + dN_\phi(e^1_\phi) \wedge *d\big(\mathrm{li}(\tilde{e}^2_\Sigma) + (-1)^n \mathrm{li}(\langle dN_\phi(e^2_\phi), \eta\rangle_{\Lambda^1})\big) + dN_\phi(e^1_\phi) \wedge *[\eta, dN_\phi(e^2_\phi)]_1 + dN(e^2_\phi) \wedge *d(\mathrm{li}_\phi(f^1_\phi))\Big)$$
$$+ \int_\Sigma (-e^1_\Sigma \wedge e^2_\phi + e^2_\Sigma \wedge f^1_\Sigma) + \int_\Gamma (-e^1_b \wedge e^2_\phi + \tilde{e}^2_\Sigma \wedge f^1_b) = 0. \tag{2.136}$$

Next, we take $\tilde{e}^2_\Sigma \in H^{\frac{1}{2}}_{00}\Lambda^0(\Sigma)$ which implies $\tilde{e}^2_\Sigma|_\Gamma = 0$, $e^2_\phi \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$ such that $e^2_\phi|_\Gamma = 0$. Using (2.131), (2.133) and (2.134), then (2.136) can be rewritten as

$$\int_\Omega \Big(e^1_\eta \wedge *\big((*e^2_\eta + dN_\phi(e^2_\phi)) \wedge (*d\eta)\big) + e^2_\eta \wedge f^1_\eta + dN_\phi(e^2_\phi) \wedge *d(\mathrm{li}_\phi(f^1_\phi)) + (-1)^{n-1} dN_\phi(e^1_\phi) \wedge \big(* e^2_\eta + dN_\phi(e^2_\phi)\big) \wedge (*d\eta)\Big) + \int_{\partial\Omega} \Big(\tilde{e}^2_\Sigma \wedge e^1_\phi + (-1)^n \langle dN_\phi(e^2_\phi), \eta\rangle_{\Lambda^1} \wedge e^1_\phi \Big) + (-1)^{n-1} \int_{\partial\Omega} \langle dN_\phi(e^1_\phi), \eta\rangle_{\Lambda^1} \wedge e^2_\phi$$
$$+ \int_\Sigma (-e^1_\Sigma \wedge e^2_\phi + e^2_\Sigma \wedge f^1_\Sigma) = 0. \tag{2.137}$$

Using the boundary conditions $\tilde{e}^2_\Sigma|_\Gamma = e^2_\phi|_\Gamma = 0$, we have

$$\int_{\partial\Omega} \tilde{e}^2_\Sigma \wedge e^1_\phi - \int_\Sigma e^1_\Sigma \wedge e^2_\phi = \int_\Sigma e^2_\Sigma \wedge e^1_\phi - \int_{\partial\Omega} \tilde{e}^1_\Sigma \wedge e^2_\phi.$$

and (2.137) becomes

$$\int_\Omega \Big((-1)^{n-1} \big(* e^1_\eta + dN_\phi(e^1_\phi)\big) \wedge \big(* e^2_\eta + dN_\phi(e^2_\phi)\big) \wedge (*d\eta) + e^2_\eta \wedge f^1_\eta + dN(e^2_\phi) \wedge *d(\mathrm{li}_\phi(f^1_\phi))\Big) + \int_{\partial\Omega} \big(- \tilde{e}^1_\Sigma \wedge e^2_\phi + (-1)^{n-1} \langle dN_\phi(e^1_\phi), \eta\rangle_{\Lambda^1} \wedge e^2_\phi\big) + (-1)^n \int_{\partial\Omega} \langle dN_\phi(e^2_\phi), \eta\rangle_{\Lambda^1} \wedge e^1_\phi + \int_\Sigma \big(e^2_\Sigma \wedge f^1_\Sigma + e^2_\Sigma \wedge e^1_\phi\big) = 0.$$

Again, applying (2.131), (2.133) and (2.134), we have that

$$\int_\Omega \Big((-1)^{n-1}\big(* e_\eta^1 + dN_\phi(e_\phi^1)\big) \wedge \big(* e_\eta^2 + dN_\phi(e_\phi^2)\big) \wedge (*d\eta) + e_\eta^2 \wedge f_\eta^1 \\ + dN(e_\phi^2) \wedge *d(\mathrm{li}_\phi(f_\phi^1)) - dN_\phi(e_\phi^2) \wedge *d\big(\mathrm{li}(\tilde{e}_\Sigma^1) \\ + (-1)^n \mathrm{li}(\langle dN_\phi(e_\phi^1), \eta\rangle_{\Lambda^1})\big) - dN_\phi(e_\phi^2) \wedge *[\eta, dN_\phi(e_\phi^1)]_1 \Big) \\ + \int_\Sigma \big(e_\Sigma^2 \wedge f_\Sigma^1 + e_\Sigma^2 \wedge e_\phi^1 \big) = 0. \tag{2.138}$$

Since the first term in (2.138) can be split into two parts, it can be further evaluated as

$$\begin{aligned} &\big(* e_\eta^1 + dN_\phi(e_\phi^1)\big) \wedge \big(* e_\eta^2 + dN_\phi(e_\phi^2)\big) \wedge (*d\eta) \\ =&(-1)^n e_\eta^2 \wedge *\Big(\big(* e_\eta^1 + dN_\phi(e_\phi^1)\big) \wedge (*d\eta) \Big) \\ -&dN_\phi(e_\phi^2) \wedge \big(* e_\eta^1 + dN_\phi(e_\phi^1)\big) \wedge (*d\eta). \end{aligned} \tag{2.139}$$

Finally, using (2.131) and (2.139), then (2.138) can be transformed into

$$\int_\Omega \Big(e_\eta^2 \wedge \big(f_\eta^1 + [\eta, dN_\phi(e_\phi^1)]_1 \big) + dN_\phi(e_\phi^2) \wedge *\Big(d\big(\mathrm{li}_\phi(f_\phi^1)\big) \\ - *\big(\big(* e_\eta^1 + dN_\phi(e_\phi^1)\big) \wedge (*d\eta) \big) - d\big(\mathrm{li}(\tilde{e}_\Sigma^1) + (-1)^n \mathrm{li}(\langle dN_\phi(e_\phi^1), \eta\rangle_{\Lambda^1})\big) \\ - [\eta, dN_\phi(e_\phi^1)]_1 \Big) + \int_\Sigma \big(e_\Sigma^2 \wedge f_\Sigma^1 + e_\Sigma^2 \wedge e_\phi^1 \big) = 0.$$

Since e_η^2, e_ϕ^2, e_Σ^2 are arbitrary, the integral can only be zero if

$$\begin{cases} f_\eta^1 &= -[\eta, dN_\phi(e_\phi^1)]_1, \\ d\big(\mathrm{li}_\phi(f_\phi^1)\big) &= *\Big((*e_\eta^1 + dN_\phi(e_\phi^1)) \wedge (*d\eta) \Big) + d\Big(\mathrm{li}(\tilde{e}_\Sigma^1) \\ & \quad + (-1)^n \mathrm{li}(\langle dN_\phi(e_\phi^1), \eta\rangle_{\Lambda^1})\Big) + [\eta, dN_\phi(e_\phi^1)]_1, \\ f_\Sigma^1 &= -e_\phi^1. \end{cases} \tag{2.140}$$

Substituting (2.140) into (2.136), following the same steps as in Step (3), we obtain

$$\int_\Gamma \big((-e_b^1 + \tilde{e}_\Sigma^1) \wedge e_\phi^2 + \tilde{e}_\Sigma^2 \wedge (f_b^1 + e_\phi^1) \big) = 0.$$

Hence,

$$f_b^1 = -e_\phi^1|_\Gamma, \quad e_b^1 = \tilde{e}_\Sigma^1|_\Gamma.$$

□

Corollary 2.5.3. *Given an oriented and simply connected n-dimensional manifold Ω with Lipschitz continuous boundary $\partial\Omega$. Given the essential boundary condition (2.57f) and the port variable $f_b = -g$. The distributed-parameter port-Hamiltonian system for the incompressible Euler equations with a free surface (2.57) with respect to the variables $(\eta, \phi_\partial, \Sigma) \in \mathring{\mathfrak{B}}^{*1} \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$, Dirac structure D_2 (2.124) and Hamiltonian (2.77), is given by*

$$\begin{pmatrix} -\eta_t \\ -d\phi_t \\ -\Sigma_t \end{pmatrix} = \begin{pmatrix} 0 & -[\eta, dN_\phi(\cdot)]_1 & 0 \\ *\big((*\cdot) \wedge *d\eta\big) & A_{22} & d\big(\mathrm{li}_\Sigma(E(\cdot))\big) \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \tilde{H}}{\delta \eta} \\ \frac{\delta \tilde{H}}{\delta \phi_\partial} \\ \frac{\delta \tilde{H}}{\delta \Sigma} \end{pmatrix}, \tag{2.141}$$

where $A_{22} = *\big(dN_\phi(\cdot) \wedge *d\eta\big) + (-1)^n d\big(\mathrm{li}(\langle dN_\phi(\cdot), \eta\rangle_{\Lambda^1})\big) + [\eta, dN_\phi(\cdot)]_1$, with port variables f_b, e_b defined as

$$\begin{pmatrix} f_b \\ e_b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & E(\cdot) \end{pmatrix} \begin{pmatrix} \frac{\delta \tilde{H}}{\delta \phi_\partial} \\ \frac{\delta \tilde{H}}{\delta \Sigma} \end{pmatrix}. \tag{2.142}$$

Proof. Substituting the functional derivatives

$$e_\eta = \frac{\delta \tilde{H}}{\delta \eta}, \quad e_\phi = \frac{\delta \tilde{H}}{\delta \phi_\partial}, \quad e_\Sigma = \frac{\delta \tilde{H}}{\delta \Sigma},$$

given in (2.78) into (2.141), with the essential boundary condition (2.57f) and the port variable $f_b = -g$, we immediately obtain (2.57). □

From the bilinear form (2.130), we can define the following bilinear form

$$[e_1, e_2]_D = \int_\Omega \Big(e^1_\eta \wedge f^2_\eta + dN_\phi(e^1_\phi) \wedge *d\big(\mathrm{li}_\phi(f^2_\phi)\big)\Big) + \int_\Sigma e^1_\Sigma \wedge f^2_\Sigma + \int_\Gamma e^1_b \wedge f^2_b.$$

For $\frac{\delta \tilde{\mathcal{F}}}{\delta \eta}, \frac{\delta \tilde{\mathcal{G}}}{\delta \eta} \in \mathring{\mathfrak{B}}^{(n-1)}$, $\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial}, \frac{\delta \tilde{\mathcal{G}}}{\delta \phi_\partial} \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$, $\frac{\delta \tilde{\mathcal{F}}}{\delta \Sigma}, \frac{\delta \tilde{\mathcal{G}}}{\delta \Sigma} \in H^{\frac{1}{2}}\Lambda^0(\Sigma)$, the Poisson bracket $\{\cdot, \cdot\}_D(\eta, \phi_\partial, \Sigma)$ associated to the Dirac structure (2.124) is defined as

$$\begin{aligned}
&\{\tilde{\mathcal{F}}, \tilde{\mathcal{G}}\}_D(\eta, \phi_\partial, \Sigma) \\
&= -\int_\Omega \frac{\delta \tilde{\mathcal{G}}}{\delta \eta} \wedge [\eta, dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial})]_1 + \int_\Omega dN_\phi(\frac{\delta \tilde{\mathcal{G}}}{\delta \phi_\partial}) \wedge *\Big(* \big((*\frac{\delta \tilde{\mathcal{F}}}{\delta \eta} + dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial})) \\
&\wedge *d\eta\big) + (-1)^n d\big(\mathrm{li}(\langle dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial}), \eta\rangle_{\Lambda^1})\big) + [\eta, dN_\phi(\frac{\delta \tilde{\mathcal{F}}}{\delta \phi_\partial})]_1 + d\big(\mathrm{li}(E(\frac{\delta \tilde{\mathcal{F}}}{\delta \Sigma}))\Big)
\end{aligned}$$

$$-\int_{\Sigma}\frac{\delta\tilde{\mathcal{G}}}{\delta\Sigma}\wedge\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}-\int_{\Gamma}E(\frac{\delta\tilde{\mathcal{G}}}{\delta\Sigma})\wedge\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}, \tag{2.143}$$

with $\eta\in\mathring{\mathfrak{B}}^{*1}$.

Remark 2.5.12. *Using (2.131), (2.133) and (2.134), we have following equalities*

(1)

$$\begin{aligned}&-\int_{\Omega}\frac{\delta\tilde{\mathcal{G}}}{\delta\eta}\wedge[\eta,dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial})]_1\\&=\int_{\Omega}\frac{\delta\tilde{\mathcal{G}}}{\delta\eta}\wedge*\Big(\big(*\frac{\delta\tilde{\mathcal{F}}}{\delta\eta}+dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial})\big)\wedge(*d\eta)\Big),\end{aligned} \tag{2.144}$$

(2)

$$\begin{aligned}&\int_{\Omega}dN_\phi(\frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial})\wedge*\Big((-1)^n d\big(\mathrm{li}(\langle dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}),\eta\rangle_{\Lambda^1})\big)+d\big(\mathrm{li}(E(\frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma}))\big)\Big)\\&=\int_{\partial\Omega}\Big(E(\frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma})+(-1)^n\langle dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}),\eta\rangle_{\Lambda^1}\Big)\wedge\frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial},\end{aligned} \tag{2.145}$$

(3)

$$\begin{aligned}&\int_{\Omega}dN_\phi(\frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial})\wedge*[\eta,dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial})]_1\\&=(-1)^{n-1}\int_{\partial\Omega}\langle dN_\phi(\frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial}),\eta\rangle_{\Lambda^1}\wedge\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}.\end{aligned} \tag{2.146}$$

Substituting (2.144), (2.145) and (2.146) into (2.143), we can rewrite the Poisson bracket as

$$\begin{aligned}&\{\tilde{\mathcal{F}},\tilde{\mathcal{G}}\}_D(\eta,\phi_\partial,\Sigma)\\&=(-1)^{n-1}\int_{\Omega}(*d\eta)\wedge\big(*\frac{\delta\tilde{\mathcal{G}}}{\delta\eta}+dN_\phi(\frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial})\big)\wedge\big(*\frac{\delta\tilde{\mathcal{F}}}{\delta\eta}+dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial})\big)\\&+\int_{\partial\Omega}\Big(E(\frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma})+(-1)^n\langle dN_\phi(\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}),\eta\rangle_{\Lambda^1}\Big)\wedge\frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial}\\&-\int_{\partial\Omega}\Big(E(\frac{\delta\tilde{\mathcal{G}}}{\delta\Sigma})+(-1)^n\langle dN_\phi(\frac{\delta\tilde{\mathcal{G}}}{\delta\phi_\partial}),\eta\rangle_{\Lambda^1}\Big)\wedge\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_\partial}.\end{aligned} \tag{2.147}$$

Remark 2.5.13. *The bracket $\{\cdot,\cdot\}_D(\eta,\phi_\partial,\Sigma)$ (2.147) is linear, skew-symmetric and satisfies the Jacobi identity, which follows directly from the* *fact that $\{\cdot,\cdot\}_D(\eta,\phi_\partial,\Sigma)$ has the same structure as the Poisson bracket $\{\cdot,\cdot\}(\eta,\phi_\partial,\Sigma)$ stated in Lemma 2.4.7. The bracket $\{\cdot,\cdot\}_D(\eta,\phi_\partial,\Sigma)$ is thus a Poisson bracket and the system generated by the Dirac structure D_2 is a Poisson structure.*

Theorem 2.5.14. *Let $\Omega \subset \mathbb{R}^n, n \in \{1,2,3\}$ be an oriented n-dimensional connected manifold with Lipschitz continuous boundary $\partial\Omega = \Sigma\cup\Gamma$ with free surface Σ and fixed boundary Γ, $\Sigma\cap\Gamma = \emptyset$. For any functionals $\tilde{\mathcal{F}}(\eta,\phi_\partial,\Sigma)$: $\mathring{\mathfrak{B}}^{*1} \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$, the port-Hamiltonian formulation of the incompressible Euler equations with a free surface (2.141),(2.142) is equivalent to*

$$\dot{\tilde{\mathcal{F}}}(\eta,\phi_\partial,\Sigma) = \{\tilde{\mathcal{F}},\tilde{H}\}_D(\eta,\phi_\partial,\Sigma) - \int_\Gamma E(\frac{\delta\tilde{\mathcal{F}}}{\delta\Sigma}) \wedge *\boldsymbol{n}(d\phi), \tag{2.148}$$

with the Hamiltonian $\tilde{H}$ given by (2.77).

The proof to Theorem 2.5.14 is identical to Theorem 2.5.8.

Corollary 2.5.4. *Given the Hamiltonian functional $\tilde{H} : \mathring{\mathfrak{B}}^{*1} \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega)\times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ stated in (2.77), and the essential boundary conditions (2.57e), (2.57f), the rate of change of the Hamiltonian for the incompressible Euler equations with a free surface in the $(\eta,\phi_\partial,\Sigma)$ variables (2.57) is equal to*

$$\dot{\tilde{H}} = -\int_\Gamma \mathrm{tr}(h) \wedge g.$$

2.5.4 Dirac structure in terms of the $(\omega,\phi_\partial,\Sigma)$ variables

Using the results presented in Section 2.4.3, we will present in this section the Dirac structure and port-Hamiltonian formulation for the n-dimensional ($n \in \{2,3\}$) inviscid incompressible Euler equations with a free surface in terms of the $(\omega,\phi_\partial,\Sigma)$ variables. We will first define the following linear spaces for the Dirac structure

$$\begin{aligned}\mathcal{F}_3 &:= \mathring{V}\Lambda^2(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Gamma),\\ \mathcal{E}_3 &:= \mathring{V}^*\Lambda^{n-2}(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega) \times H^{\frac{1}{2}}\Lambda^0(\Sigma) \times H^{\frac{1}{2}}\Lambda^0(\Gamma),\end{aligned} \tag{2.149}$$

where the spaces $\mathcal{F}_3$ and $\mathcal{E}_3$ are dual to each other with a non-degenerate pairing using L^2 as pivot space.

The following theorem states the Dirac structure with respect to the $(\omega,\phi_\partial,\Sigma)$ variables.

Theorem 2.5.15. *Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$ be a n-dimensional oriented and simply connected manifold with boundary $\partial\Omega = \Sigma \cup \Gamma$ and $\Sigma \cap \Gamma = \emptyset$. Assume $\omega = dv \in \mathring{V}\Lambda^2(\Omega)$ is the solution of the incompressible Euler equations with a free surface in the $(\omega, \phi_\partial, \Sigma)$ variables (2.83)–(2.85). Given the function spaces $\mathcal{F}_3$ and $\mathcal{E}_3$, defined in (2.149), together with the bilinear form*

$$\langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle = \int_\Omega \left(e^1_\omega \wedge f^2_\omega + e^2_\omega \wedge f^1_\omega\right) + \int_{\partial\Omega} \left(e^1_\phi \wedge f^2_\phi + e^2_\phi \wedge f^1_\phi\right) + \int_\Sigma \left(e^1_\Sigma \wedge f^2_\Sigma + e^2_\Sigma \wedge f^1_\Sigma\right) + \int_\Gamma \left(e^1_b \wedge f^2_b + e^2_b \wedge f^1_b\right), \tag{2.150}$$

where

$$f^i = (f^i_\omega, f^i_\phi, f^i_\Sigma, f^i_b) \in \mathcal{F}_3,\ e^i = (e^i_\omega, e^i_\phi, e^i_\Sigma, e^i_b) \in \mathcal{E}_3,\ i = 1, 2.$$

Then $D_3 \subset \mathcal{F}_3 \times \mathcal{E}_3$, defined as

$$D_3 := \Bigg\{ (f_\omega, f_\phi, f_\Sigma, f_b, e_\omega, e_\phi, e_\Sigma, e_b) \in \mathcal{F}_3 \times \mathcal{E}_3 \ |$$

$$\begin{pmatrix} f_\omega \\ d\big(\mathrm{li}_\phi(f_\phi)\big) \\ \\ \\ f_\Sigma|_\Sigma \end{pmatrix} = \begin{pmatrix} d * \Big(\big((-1)^{n-1} * de_\omega + dN_\phi(e_\phi)\big) \wedge *\omega \Big) \\ *\Big(\big((-1)^{n-1} * de_\omega + dN_\phi(e_\phi)\big) \wedge *\omega \Big) \\ +[\delta N_\beta(\omega), dN_\phi(e_\phi)]_1 + d\big(\mathrm{li}(\tilde{e}_\Sigma) \\ +(-1)^n \mathrm{li}(\langle dN_\phi(e_\phi), \delta N_\beta(\omega)\rangle_{\Lambda^1})\big) \\ -e_\phi \end{pmatrix} \tag{2.151}$$

$$\begin{pmatrix} f_b \\ e_b \end{pmatrix} = \begin{pmatrix} -e_\phi|_\Gamma \\ \tilde{e}_\Sigma|_\Gamma \end{pmatrix} \Bigg\},$$

with $\tilde{e}_\Sigma = E(e_\Sigma)$, is a Dirac structure.

Before proving this theorem, we first need to check if $d\big(\mathrm{li}(f_\phi)\big)$ is well-defined.

Lemma 2.5.16. *Given $e_\omega \in \mathring{V}^*\Lambda^{n-2}(\Omega)$, $e_\phi \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$ with $\omega \in \mathring{V}\Lambda^2(\Omega)$, then $d\big(\mathrm{li}_\phi(f_\phi)\big)$ in (2.151) is well-defined.*

Proof. Following the same approach as we did in Lemma 2.5.11, we can prove Lemma 2.5.16. □

Next, we will give a proof of Theorem 2.5.15.

Proof. (1) Using the definition of the trace-lifting operator li_ϕ (2.121), the Laplace solution solution operator N_ϕ (2.41), and (2.129) we can rewrite (2.150) as

$$\begin{aligned}\langle\!\langle (f^1,e^1),(f^2,e^2)\rangle\!\rangle = &\int_\Omega \Big(e^1_\omega \wedge f^2_\omega + e^2_\omega \wedge f^1_\omega + dN_\phi(e^1_\phi)\wedge *d\big(\mathrm{li}_\phi(f^2_\phi)\big)\\ &+dN_\phi(e^2_\phi)\wedge *d\big(\mathrm{li}_\phi(f^1_\phi)\big)\Big) + \int_\Sigma \Big(e^1_\Sigma \wedge f^2_\Sigma + e^2_\Sigma \wedge f^1_\Sigma\Big)\\ &+\int_\Gamma \Big(e^1_b \wedge f^2_b + e^2_b \wedge f^1_b\Big). \qquad (2.152)\end{aligned}$$

(2) We check that $\int_\Omega e^1_\omega \wedge d*\big((*de^2_\omega)\wedge *\omega\big)$ is skew-symmetric in e^1_ω and e^2_ω. For arbitrary $e^i_\omega \in \mathring{V}^*\Lambda^{n-2}(\Omega),\ i=1,2$, which implies the boundary condition $\mathrm{tr}(e^i_\omega)=0$, by using the integration by parts formula (2.19), we obtain

$$\begin{aligned}\int_\Omega e^1_\omega \wedge d*\big((*de^2_\omega)\wedge(*\omega)\big) =&(-1)^{n-1}\int_\Omega \delta(*e^1_\omega)\wedge(*de^2_\omega)\wedge(*\omega)\\ =&(-1)^n\langle de^2_\omega,(*de^1_\omega)\wedge(*\omega)\rangle_{L^2\Lambda^{n-1}(\Omega)}\\ =&-\int_\Omega e^2_\omega \wedge d*\big((*de^1_\omega)\wedge(*\omega)\big).\end{aligned}$$

(3) We show that $D_3 \subset D_3^\perp$. Let $(f^1,e^1)\in D_3$ be fixed, and consider any $(f^2,e^2)\in D_3$. From (2.151) and (2.152), we have that

$$\begin{aligned}&\langle\!\langle (f^1,e^1),(f^2,e^2)\rangle\!\rangle\\ &=\int_\Omega \Big((-1)^{n-1}e^1_\omega \wedge d*\big((*de^2_\omega)\wedge(*\omega)\big) + e^1_\omega\wedge d*\big(dN_\phi(e^2_\phi)\wedge *\omega\big)\\ &+(-1)^{n-1}e^2_\omega \wedge d*\big((*de^1_\omega)\wedge(*\omega)\big) + e^2_\omega\wedge d*\big(dN_\phi(e^1_\phi)\wedge *\omega\big)\\ &+dN_\phi(e^1_\phi)\wedge(*de^2_\omega)\wedge(*\omega) + (-1)^{n-1}dN_\phi(e^1_\phi)\wedge dN_\phi(e^2_\phi)\wedge(*\omega)\\ &+dN_\phi(e^2_\phi)\wedge(*de^1_\omega)\wedge(*\omega) + (-1)^{n-1}dN_\phi(e^2_\phi)\wedge dN_\phi(e^1_\phi)\wedge(*\omega)\\ &+dN_\phi(e^1_\phi)\wedge *[\delta N_\beta(\omega), dN_\phi(e^2_\phi)]_1 + dN_\phi(e^2_\phi)\wedge *[\delta N_\beta(\omega), dN_\phi(e^1_\phi)]_1\\ &+dN_\phi(e^1_\phi)\wedge *d\big(\mathrm{li}(\tilde e^2_\Sigma) + (-1)^n\mathrm{li}(\langle dN_\phi(e^2_\phi),\delta N_\beta(\omega)\rangle_{\Lambda^1})\big)\\ &+dN_\phi(e^2_\phi)\wedge *d\big(\mathrm{li}(\tilde e^1_\Sigma) + (-1)^n\mathrm{li}(\langle dN_\phi(e^1_\phi),\delta N_\beta(\omega)\rangle_{\Lambda^1})\big)\Big)\\ &-\int_\Sigma \Big(e^1_\Sigma\wedge e^2_\phi + e^2_\Sigma\wedge e^1_\phi\Big) - \int_\Gamma \Big(\tilde e^1_\Sigma \wedge e^2_\phi + \tilde e^2_\Sigma\wedge e^1_\phi\Big). \qquad (2.153)\end{aligned}$$

Using the integration by parts formula (2.19) and the condition $\mathrm{tr}(e^i_\omega) = 0,\ i = 1, 2$, since $e^i_\omega \in \mathring{V}^*\Lambda^{n-2}(\Omega)$, we have for $i, j \in \{1, 2\}$ that

$$\begin{aligned}
&\int_\Omega e^i_\omega \wedge d * \left(dN_\phi(e^j_\phi) \wedge *\omega\right) \\
=&(-1)^{n-1} \int_\Omega e^i_\omega \wedge *\delta\left(dN_\phi(e^j_\phi) \wedge *\omega\right) \\
=&(-1)^{n-1} \langle de^i_\omega, dN_\phi(e^j_\phi) \wedge *\omega)\rangle_{L^2\Lambda^{n-1}(\Omega)} \\
=&-\int_\Omega dN_\phi(e^j_\phi) \wedge (*de^i_\omega) \wedge (*\omega).
\end{aligned} \tag{2.154}$$

Following (2.133) and (2.134), with $\eta = \delta N_\beta(\omega)$, we have Using (2.54) and Lemma 2.2.4, we have for $i, j \in \{1, 2\}$

$$\begin{aligned}
&\int_\Omega dN_\phi(e^i_\phi) \wedge *[\delta N_\beta(\omega), dN_\phi(e^j_\phi)]_1 \\
=&(-1)^{n-1} \int_{\partial\Omega} \langle dN_\phi(e^i_\phi), \delta N_\beta(\omega)\rangle_{\Lambda^1} \wedge e^j_\phi,
\end{aligned} \tag{2.155}$$

and

$$\begin{aligned}
&\int_\Omega dN(e^i_\phi) \wedge *d\Big(\mathrm{li}\big(\tilde{e}^j_\Sigma\big) + (-1)^n \mathrm{li}\big(\langle dN_\phi(e^j_\phi), \delta N_\beta(\omega)\rangle_{\Lambda^1}\big)\Big) \\
=&\int_{\partial\Omega} \tilde{e}^j_\Sigma \wedge e^i_\phi + (-1)^n \int_{\partial\Omega} \langle dN_\phi(e^j_\phi), \delta N_\beta(\omega)\rangle_{\Lambda^1} \wedge e^i_\phi.
\end{aligned} \tag{2.156}$$

By the skew-symmetry of term $\int_\Omega e^1_\omega \wedge d * \big((*e^2_\omega) \wedge *\omega\big)$, together with (2.154), (2.155), (2.156), we obtain that (2.153) is equal to

$$\begin{aligned}
\langle\!\langle (f^1, e^1), (f^2, e^2)\rangle\!\rangle &= \int_{\partial\Omega} \left(e^1_\phi \wedge \tilde{e}^2_\Sigma + e^2_\phi \wedge \tilde{e}^1_\Sigma\right) - \int_{\partial\Omega} \left(e^1_\phi \wedge \tilde{e}^2_\Sigma + e^2_\phi \wedge \tilde{e}^1_\Sigma\right) \\
&= 0.
\end{aligned}$$

Therefore $(f^1, e^1) \in D_3^\perp$, which implies that $D_3 \subset D_3^\perp$.

(4) Finally, we show that $D_3^\perp \subset D_3$. Let $(f^1, e^1) \in D_3^\perp \subset \mathcal{F}_3 \times \mathcal{E}_3$. Then

$$\langle\!\langle (f^1, e^1), (f^2, e^2)\rangle\!\rangle = 0, \quad \forall (f^2, e^2) \in D_3,$$

which equals

$$\int_\Omega \Big((-1)^{n-1} e^1_\omega \wedge d * \big((*de^2_\omega) \wedge *\omega\big) + e^1_\omega \wedge d * \big(dN_\phi(e^2_\phi) \wedge *\omega\big)$$

$$\begin{aligned}
&+ e_\omega^2 \wedge f_\omega^1 + (-1)^{n-1} dN_\phi(e_\phi^1) \wedge dN_\phi(e_\phi^2) \wedge (*\omega) \\
&dN_\phi(e_\phi^1) \wedge (*de_\omega^2) \wedge (*\omega) + dN_\phi(e_\phi^2) \wedge *d(\mathrm{li}_\phi(f_\phi^1)) \\
&+ dN_\phi(e_\phi^1) \wedge *[\delta N_\beta(\omega), dN_\phi(e_\phi^2)]_1 \\
&+ dN_\phi(e_\phi^1) \wedge *d\big(\mathrm{li}(\tilde{e}_\Sigma^2) + (-1)^n \mathrm{li}(\langle dN_\phi(e_\phi^2), \delta N_\beta(\omega)\rangle_{\Lambda^1})\big)\Big) \\
&+ \int_\Sigma \big(- e_\Sigma^1 \wedge e_\phi^2 + e_\Sigma^2 \wedge f_\Sigma^1 \big) + \int_\Gamma \big(- e_b^1 \wedge e_\phi^2 + \tilde{e}_\Sigma^2 \wedge f_b^1 \big) = 0.
\end{aligned} \tag{2.157}$$

Taking $\tilde{e}_\Sigma^2 = E(e_\Sigma^2) \in H_{00}^{\frac{1}{2}} \Lambda^0(\partial\Omega)$, which implies that $\tilde{e}_\Sigma^2 = 0$ at Γ, and $e_\phi^2 \in H^{-\frac{1}{2}} \Lambda^{n-1}(\partial\Omega)$ with $e_\phi^2 = 0$ at Γ, then from the skew-symmetry of $\int_\Omega e_\omega^1 \wedge d * \big((*e_\omega^2) \wedge *\omega\big)$, together with (2.154), (2.155), and (2.156), then (2.157) can be rewritten as

$$\begin{aligned}
&\int_\Omega e_\omega^2 \wedge \Big(f_\omega^1 + (-1)^n d * \big((*de_\omega^1) \wedge (*\omega)\big) - d * \big(dN_\phi(e_\phi^1) \wedge *\omega\big)\Big) \\
&+ \int_\Omega dN_\phi(e_\phi^2) \wedge \Big(* d\big(\mathrm{li}_\phi(f_\phi^1)\big) - (*de_\omega^1) \wedge (*\omega) \\
&+ (-1)^n dN_\phi(e_\phi^1) \wedge (*\omega)\Big) + \int_{\partial\Omega} (-1)^{n-1} \langle dN_\phi(e_\phi^1), \delta N_\beta(\omega)\rangle_{\Lambda^1} \wedge e_\phi^2 \\
&+ \int_{\partial\Omega} \tilde{e}_\Sigma^2 \wedge e_\phi^1 + \int_{\partial\Omega} (-1)^n \langle dN_\phi(e_\phi^2), \delta N_\beta(\omega)\rangle_{\Lambda^1} \wedge e_\phi^1 \\
&+ \int_\Sigma \Big(- e_\Sigma^1 \wedge e_\phi^2 + e_\Sigma^2 \wedge f_\Sigma^1 \Big) = 0.
\end{aligned}$$

Again applying (2.155) and (2.156), with $e_\phi^2 = \tilde{e}_\Sigma^2 = 0$ at Γ, we further have

$$\begin{aligned}
&\int_\Omega e_\omega^2 \wedge \Big(f_\omega^1 + (-1)^n d * \big((*de_\omega^1) \wedge (*\omega)\big) - d * \big(dN_\phi(e_\phi^1) \wedge *\omega\big)\Big) \\
&+ \int_\Omega dN_\phi(e_\phi^2) \wedge \Big(* d\big(\mathrm{li}_\phi(f_\phi^1)\big) - (*de_\omega^1) \wedge (*\omega) + (-1)^n dN_\phi(e_\phi^1) \wedge (*\omega) \\
&- *[\delta N_\beta(\omega), dN_\phi(e_\phi^1)]_1 - *d\big(\mathrm{li}(\tilde{e}_\Sigma^1) + (-1)^n \mathrm{li}(\langle dN_\phi(e_\phi^1), \delta N_\beta(\omega)\rangle_{\Lambda^1})\big)\Big) \\
&+ \int_\Sigma \Big(e_\Sigma^2 \wedge e_\phi^1 + e_\Sigma^2 \wedge f_\Sigma^1 \Big) = 0.
\end{aligned}$$

This integral can be zero only if

$$
\begin{cases}
f^1_\omega = d * \Big(\big((-1)^{n-1} * de^1_\omega + dN_\phi(e^1_\phi)\big) \wedge *\omega \Big), \\
d\big(\mathrm{li}_\phi(f^1_\phi)\big) = *\Big(\big((-1)^{n-1} * de^1_\omega + dN_\phi(e^1_\phi)\big) \wedge *\omega \Big) \\
\qquad\qquad + [\delta N_\beta(\omega), dN_\phi(e^1_\phi)]_1 + d\big(\mathrm{li}(\tilde{e}^1_\Sigma)\big) \\
\qquad\qquad + (-1)^n d\big(\mathrm{li}(\langle dN_\phi(e^1_\phi), \delta N_\beta(\omega)\rangle_{\Lambda^1})\big) \\
f^1_\Sigma|_\Sigma = -e^1_\phi.
\end{cases}
\tag{2.158}
$$

Substituting (2.158) into (2.157) yields

$$
\int_{\partial\Omega} \Big(\tilde{e}^1_\Sigma \wedge e^2_\phi + \tilde{e}^2_\Sigma \wedge e^1_\phi \Big) - \int_\Sigma \Big(e^1_\Sigma \wedge e^2_\phi + e^2_\Sigma \wedge e^1_\phi \Big)
+ \int_\Gamma \Big(f^1_b \wedge \tilde{e}^2_\Sigma - e^2_\phi \wedge e^1_b \Big) = 0.
$$

Hence,

$$
f^1_b = -e^1_\phi|_\Gamma, \quad e^1_b = \tilde{e}^1_\Sigma|_\Gamma,
$$

and $D_3^\perp \subset D_3$, which together with $D_3 \subset D_3^\perp$ implies that D_3 is a Dirac structure.

□

Corollary 2.5.5. *Given an oriented and simply connected n-dimensional manifold Ω with Lipschitz continuous boundary $\partial\Omega$. Given the essential boundary condition (2.84e) and the port variable $f_b = -g$. The distributed-parameter port-Hamiltonian system for the incompressible Euler equations with a free surface (2.83)–(2.85) with respect to the variables $(\omega, \phi_\partial, \Sigma) \in \mathring{V}\Lambda^2(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma)$, Dirac structure (2.151), and Hamiltonian (2.89), is given by*

$$
\begin{pmatrix} -\omega_t \\ -d\phi_t \\ -\Sigma_t \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \bar{H}}{\delta\omega} \\ \frac{\delta \bar{H}}{\delta\phi_\partial} \\ \frac{\delta \bar{H}}{\delta\Sigma} \end{pmatrix},
\tag{2.159}
$$

where

$$
\begin{aligned}
B_{11} &= (-1)^{n-1} d * \big(* d(\cdot) \wedge *\omega \big), \\
B_{12} &= d * \big(dN_\phi(\cdot) \wedge *\omega \big), \\
B_{21} &= (-1)^{n-1} * \big(* d(\cdot) \wedge *\omega \big), \\
B_{22} &= * \big(dN_\phi(\cdot) \wedge *\omega \big) + (-1)^n d\big(\mathrm{li}(\langle dN_\phi(\cdot), \delta N_\beta(\omega)\rangle_{\Lambda^1})\big) \\
&\quad + [\delta N_\beta(\omega), dN_\phi(\cdot)]_1, \\
B_{23} &= d\big(\mathrm{li}(E(\cdot))\big),
\end{aligned}
$$

with port variables f_b, e_b *defined as*

$$\begin{pmatrix} f_b \\ e_b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & E(\cdot) \end{pmatrix} \begin{pmatrix} \frac{\delta \bar{H}}{\delta \phi_\partial} \\ \frac{\delta \bar{H}}{\delta \Sigma} \end{pmatrix}. \tag{2.160}$$

Proof. After defining the flow and energy variables in (2.151), respectively, as

$$f_\omega = -\omega_t, \quad f_\phi = -(\phi_\partial)_t, \quad f_\Sigma = -\Sigma_t,$$

and

$$e_\omega = \frac{\delta \bar{H}}{\delta \omega}, \quad e_\phi = \frac{\delta \bar{H}}{\delta \phi_\partial}, \quad e_\Sigma = \frac{\delta \tilde{H}}{\delta \Sigma},$$

we immediately obtain (2.159). Using the functional derivatives of the Hamiltonian (2.91), and the essential boundary condition (2.84e) we obtain (2.83a), (2.84a), (2.84c). The port variable $f_b = \frac{\delta \bar{H}}{\delta \phi_\partial} = -g$, gives the essential boundary condition (2.84d) at Γ. Finally, (2.83b), (2.83c) follow from $\omega \in \mathring{V}\Lambda^2(\Omega)$ and (2.84b) follows from $v \in P^*\Lambda^1(\Omega)$ together with (2.85). □

From the bilinear form (2.152), we can define the following bilinear form

$$[e_1, e_2]_D = \int_\Omega \Big(e_\omega^1 \wedge f_\omega^2 + dN(e_\phi^1) \wedge *d\big(\mathrm{li}_\phi(f_\phi^2)\big)\Big) + \int_\Sigma e_\Sigma^1 \wedge f_\Sigma^2 + \int_\Gamma e_b^1 \wedge f_b^2.$$

For $\frac{\delta \bar{\mathcal{F}}}{\delta \omega}, \frac{\delta \bar{\mathcal{G}}}{\delta \omega} \in \mathring{V}^*\Lambda^{n-2}(\Omega)$, $\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}, \frac{\delta \bar{\mathcal{G}}}{\delta \phi_\partial} \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$ and $\frac{\delta \bar{\mathcal{F}}}{\delta \Sigma}, \frac{\delta \bar{\mathcal{G}}}{\delta \Sigma} \in H^{\frac{1}{2}}\Lambda^0(\Sigma)$, the Poisson bracket $\{\cdot,\cdot\}_D(\omega, \phi_\partial, \Sigma)$ associated to the Dirac structure (2.151) is defined as

$$\begin{aligned}
&\{\bar{\mathcal{F}}, \bar{\mathcal{G}}\}_D(\omega, \phi_\partial, \Sigma) \\
&= \int_\Omega \frac{\delta \bar{\mathcal{G}}}{\delta \omega} \wedge d* \Big(\big((-1)^{n-1} * d(\frac{\delta \bar{\mathcal{F}}}{\delta \omega}) + dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial})\big) \wedge *\omega \Big) \\
&+ \int_\Omega dN_\phi(\frac{\delta \bar{\mathcal{G}}}{\delta \phi_\partial}) \wedge \Big(\big(* d(\frac{\delta \bar{\mathcal{F}}}{\delta \omega}) + (-1)^{n-1} dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial})\big) \wedge *\omega \\
&+ * [\delta N_\beta(\omega), dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial})]_1 \\
&+ * d\big(\mathrm{li}(E(\frac{\delta \bar{\mathcal{F}}}{\delta \Sigma})) + (-1)^n \mathrm{li}(\langle dN_\phi(\frac{\delta \mathcal{F}}{\delta \phi_\partial}), \delta N_\beta(\omega)\rangle_{\Lambda^1})\big)\Big) \\
&- \int_\Sigma \frac{\delta \bar{\mathcal{G}}}{\delta \Sigma} \wedge \frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial} - \int_\Gamma E(\frac{\delta \bar{\mathcal{G}}}{\delta \Sigma}) \wedge \frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}.
\end{aligned} \tag{2.161}$$

Remark 2.5.17. *Using (2.155), (2.156) and Stokes' theorem, (2.161) can be represented as*

$$\begin{aligned}
&\{\bar{\mathcal{F}}, \bar{\mathcal{G}}\}_D(\omega, \phi_\partial, \Sigma) \\
=&(-1)^{n-1} \int_\Omega (*\omega) \wedge \Big((-1)^{n-1} * d\frac{\delta \bar{\mathcal{G}}}{\delta \omega} + dN_\phi(\frac{\delta \bar{\mathcal{G}}}{\delta \phi_\partial})\Big) \wedge \big((-1)^{n-1} * d\frac{\delta \bar{\mathcal{F}}}{\delta \omega} \\
&+dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial})\Big) + \int_{\partial\Omega} \Big(\frac{\delta \bar{\mathcal{F}}}{\delta \Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}), \delta N_\beta(\omega)\rangle_{\Lambda^1}\Big) \wedge \frac{\delta \bar{\mathcal{G}}}{\delta \phi_\partial} \\
&- \int_{\partial\Omega} \Big(\frac{\delta \bar{\mathcal{G}}}{\delta \Sigma} + (-1)^n \langle dN_\phi(\frac{\delta \bar{\mathcal{G}}}{\delta \phi_\partial}), \delta N_\beta(\omega)\rangle_{\Lambda^1}\Big) \wedge \frac{\delta \bar{\mathcal{F}}}{\delta \phi_\partial}.
\end{aligned} \tag{2.162}$$

Remark 2.5.18. *The bracket $\{\cdot,\cdot\}_D(\omega, \phi_\partial, \Sigma)$ (2.161) is linear, skew-symmetric and satisfies the Jacobi identity, which follows directly from the fact that $\{\cdot,\cdot\}_D(\omega, \phi_\partial, \Sigma)$ has the same structure as the Poisson bracket $\{\cdot,\cdot\}(\omega, \phi_\partial, \Sigma)$ stated in Lemma 2.4.11. The bracket $\{\cdot,\cdot\}_D(\omega, \phi_\partial, \Sigma)$ is thus a Poisson bracket and the system generated by the Dirac structure D_3 is a Poisson structure.*

Theorem 2.5.19. *Let $\Omega \subset \mathbb{R}^n, n \in \{2,3\}$ be an n-dimensional oriented connected manifold with Lipschitz continuous boundary $\partial\Omega = \Sigma \cup \Gamma$ with free surface Σ and fixed boundary Γ, $\Sigma \cap \Gamma = \emptyset$. For any functionals $\bar{\mathcal{F}}(\omega, \phi_\partial, \Sigma) : \mathring{V}\Lambda^2(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$, the port-Hamiltonian formulation of the incompressible Euler equations with a free surface (2.141) can be expressed as*

$$\dot{\bar{\mathcal{F}}}(\omega, \phi_\partial, \Sigma) = \{\bar{\mathcal{F}}, \bar{H}\}_D(\omega, \phi_\partial, \Sigma) - \int_\Gamma E(\frac{\delta \bar{\mathcal{F}}}{\delta \Sigma}) \wedge *\boldsymbol{n}(d\phi), \tag{2.163}$$

with the Hamiltonian $\bar{H}$ given by (2.90).

Proof. Substituting $\bar{H}$ (2.90) into (2.161), we can obtain the representation of $\{\bar{H}, \bar{\mathcal{F}}\}_D$. With the relation $\{\bar{\mathcal{F}}, \bar{H}\}_D = -\{\bar{H}, \bar{\mathcal{F}}\}_D$, together with the functional chain rule and (2.159), (2.163) holds. □

Corollary 2.5.6. *Given the Hamiltonian functional $\bar{H} : \mathring{V}\Lambda^2(\Omega) \times H^{\frac{1}{2}}\Lambda^0(\partial\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-1}(\Sigma) \to \mathbb{R}$ stated in (2.90), and the essential boundary conditions (2.84d), (2.84e), the rate of change the Hamiltonian for the incompressible Euler equations with a free surface in the $(\omega, \phi_\partial, \Sigma)$ variables (2.83)–(2.85) is equal to*

$$\dot{\bar{H}} = -\int_\Gamma \mathrm{tr}(h) \wedge g.$$

2.A Proof of Equation (2.11)

From the definition of the interior product i_X (2.10) for a vector field $X \in T_p\Omega$, we have

(1) $i_{X_1+X_2}\alpha = i_{X_1}\alpha + i_{X_2}\alpha$, where $X_1, X_2 \in T_p\Omega$ and $\alpha \in \Lambda^k(\Omega)$;

(2) $i_X(\alpha+\beta) = i_X\alpha + i_X\beta$, where $X \in T_p\Omega$ and $\alpha, \beta \in \Lambda^k(\Omega)$.

Firstly, we will consider the special case $\alpha = dx^1 \wedge \ldots \wedge dx^k$, $X = (f_1\frac{\partial}{\partial x_1}, f_2\frac{\partial}{\partial x_2}, \ldots, f_n\frac{\partial}{\partial x_n})$, and we assume that $T_p\Omega$ has an orthonormal basis, then

$$i_X\alpha = \sum_{i=1}^{k}(-1)^{i-1} i_X(dx^i) dx^1 \wedge \ldots \widehat{dx^i} \wedge \ldots dx^k, \tag{2.164}$$

where $\widehat{dx^i}$ indicates the field that must be deleted. The 1-form related to $X \in T_p\Omega$ then is

$$X^\flat = \sum_{i=1}^{n} f_i dx^i,$$

which induces

$$X^\flat \wedge \alpha = \sum_{i=k+1}^{n} f_i dx^i \wedge dx^1 \wedge \ldots \wedge dx^k,$$

since $dx^i \wedge dx^i = 0,\ i = 1, \ldots k$. Hence,

$$* \left(X^\flat \wedge \alpha\right) = \sum_{i=k+1}^{n} (-1)^i f_i dx^{k+1} \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n. \tag{2.165}$$

Alternatively, we know that $*\alpha = dx^{k+1} \wedge \ldots \wedge dx^n$. From (2.164), we further have

$$i_X(*\alpha) = \sum_{i=k+1}^{n} (-1)^{i-1} i_X(dx^i) dx^{k+1} \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n. \tag{2.166}$$

Since $dx^i(\frac{\partial}{\partial x_j}) = \delta_{ij}$, then $i_X(dx^i) = dx^i(X) = f_i$, (2.166) can be written as

$$i_X(*\alpha) = \sum_{i=k+1}^{n} (-1)^{i-1} f_i dx^{k+1} \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n. \tag{2.167}$$

From (2.165) and (2.167), we obtain that $i_X(*\alpha) = *(X^\flat \wedge \alpha)$.

Next we consider an arbitrary element $\alpha \in \Lambda^k(\Omega)$, which can be represented in the form

$$\alpha = \sum_\sigma a_\sigma dx^{\sigma(1)} \wedge \ldots \wedge dx^{\sigma(k)} =: \sum_\sigma \alpha_\sigma,$$

where $\sigma : \{\sigma(1), \ldots, \sigma(k)\} \in \{1, 2, \ldots, n\}$, with $\sigma(1) < \ldots < \sigma(n)$, are all permutations in the set σ. For every permutation σ, similar as in the proof of $i_X(*\alpha) = *(X^\flat \wedge \alpha)$ with $\alpha = dx^1 \wedge \ldots \wedge dx^k$, we obtain

$$i_X(*\alpha_\sigma) = *(X^\flat \wedge \alpha_\sigma), \ \forall \sigma \in \{1, \ldots, k\} \to \{1, \ldots, n\},$$

which induces that

$$\sum_\sigma i_X(*\alpha_\sigma) = \sum_\sigma *(X^\flat \wedge \alpha_\sigma). \tag{2.168}$$

Using Property (2), (2.168) can be rewritten as

$$i_X(*\sum_\sigma \alpha_\sigma) = *(X^\flat \wedge \sum_\sigma \alpha_\sigma),$$

which means that (2.11) holds.

2.B Proof of product rule for functional derivatives

Firstly, we introduce a general definition for functional derivatives.

Definition 2.B.1. *Given a functional $F : \Lambda^k(\Omega) \to \mathbb{R}$, the functional derivative $\frac{\delta F}{\delta \mu} \in L^2\Lambda^{n-k}(\Omega)$ is defined by*

$$\int_\Omega \frac{\delta F}{\delta \mu} \wedge \partial\mu = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(F(\mu + \epsilon\partial\mu) - F(\mu)\big), \ \forall \partial\mu \in L^2\Lambda^k(\Omega). \tag{2.169}$$

Then, we have

$$\begin{aligned}\int_\Omega \frac{\delta (FG)}{\delta \mu} \wedge \partial\mu =& \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(FG(\mu + \epsilon\partial\mu) - FG(\mu)\big) \\ =& \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(F(\mu + \epsilon\partial\mu)G(\mu + \epsilon\partial\mu) - F(\mu)G(\mu + \epsilon\partial\mu) \\ &+ F(\mu)G(\mu + \epsilon\partial\mu) - F(\mu)G(\mu)\big)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\big(F(\mu+\epsilon\partial\mu)-F(\mu)\big)G(\mu+\epsilon\mu) \\
&\quad + F(\mu)\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(G(\mu+\epsilon\partial\mu)-G(\mu)\big) \\
&= \int_\Omega \big(G(\mu)\frac{\delta F}{\delta\mu} + F(\mu)\frac{\delta G}{\delta\mu}\big)\wedge\partial\mu,
\end{aligned}
$$

where in the last step we used the facts that $F(\mu), G(\mu) \in \mathbb{R}$. Hence, it holds that

$$\frac{\delta(FG)}{\delta\mu} = G(\mu)\frac{\delta F}{\delta\mu} + F(\mu)\frac{\delta G}{\delta\mu}.$$

2.C Proof of Equation (2.53)

Assume $g_s : \Omega \to \Omega$ is a volume-preserving diffeomorphism associated to a divergence-free field η on Ω, which implies that

$$g_0(x) = x, \quad \frac{d}{ds}|_{s=0} g_s(x) = \eta. \tag{2.170}$$

Given a divergence-free vector field ξ, where ξ is assumed to be tangent to $\partial\Omega$.

The variation $\delta\xi$ of the field ξ under the infinitesimal diffeomorphism g_s, which is associated to η by (2.170), is given by

$$
\begin{aligned}
\delta\xi &= \lim_{s\to 0}\frac{\xi(g_s(x)) - \xi(g_0(x))}{s} = \frac{d}{ds}|_{s=0} g_s^*\xi \\
&= \mathcal{L}_\eta\xi = [\eta, \xi],
\end{aligned}
\tag{2.171}
$$

where $g_s^* : T_x^*\Omega \to T_{g_s(x)}^*\Omega$ is the pull back, $T_x^*\Omega$ the dual space of $T_x\Omega$, and $[\cdot,\cdot]$ the Lie bracket.

From [11, Remark 2.5], we know that

$$\frac{d\xi(t)}{dt} = [\eta, \xi(t)]. \tag{2.172}$$

Lemma 2.C.1. *Given a domain $\Omega \subset \mathbb{R}^d$ with the boundary $\partial\Omega = \Sigma \cup \Gamma$, where Σ is a free boundary and Γ a fixed boundary. Given a divergence-free vector field w, tangent to the boundary $\partial\Omega$, hence $\langle w, n\rangle = 0$, with n the external unit normal vector at $\partial\Omega$. The variation δw has the form*

$$\delta w = w' + [w, u], \tag{2.173}$$

where w' is a divergence-free vector field and parallel to boundary $\partial\Omega$, and u a divergence-free vector field satisfying $\langle u, n\rangle = \delta\Sigma$.

Proof. Consider now two volume-preserving diffeomorphisms $\eta_s, \xi_s : \Omega \to \Omega$, with η_s, ξ_s divergence-free vector fields and ξ_s is parallel to $\partial\Omega$, defined as

$$\begin{aligned}
&\eta_0(x) = \xi_0(x) = x, \quad \langle \delta\eta_s, n\rangle = \delta\Sigma, \quad \langle \delta\xi_s, n\rangle = 0, \\
&\frac{d}{ds}|_{s=0}\eta_s(x) = u, \quad \frac{d}{ds}|_{s=0}\xi_s(x) = \xi,
\end{aligned} \tag{2.174}$$

where ξ is divergence-free and tangent to $\partial\Omega$, and $\delta\Sigma = \langle u, n\rangle$. Differentiating the curve $w_s = \eta_s \circ \xi_s$, gives the relation

$$\begin{aligned}
\delta w =&\frac{d}{ds}|_{s=0}w_s^* w = \frac{d}{ds}|_{s=0}(\eta_s \circ \xi_s)^* w = \lim_{s\to 0} \frac{w(\eta_s \circ \xi_s(x)) - w(\eta_0 \circ \xi_0(x))}{s} \\
=&\lim_{s\to 0} \frac{1}{s}\Big(w(\eta_s \circ \xi_s(x)) - w(\eta_s \circ \xi_0(x)) + w(\eta_s \circ \xi_0(x)) - w(\eta_0 \circ \xi_0(x))\Big) \\
=&\lim_{s\to 0} \Big(w(\eta_s \circ \xi_0(x)) - w(\eta_0 \circ \xi_0(x))\Big) \\
&+ \lim_{s\to 0} \frac{1}{s}\Big(w(\eta_s \circ \xi_s(x)) - w(\eta_s \circ \xi_0(x))\Big) \\
:=&T_1 + T_2.
\end{aligned} \tag{2.175}$$

We firstly compute T_1. Since $\xi_0(x) = x$, we have

$$T_1 = \lim_{s\to 0} \frac{1}{s}\Big(w(\eta_s(x)) - w(\eta_0(x))\Big) = \frac{d}{ds}|_{s=0}\eta_s^* w = [w, u]. \tag{2.176}$$

Next, we consider T_2.

$$\begin{aligned}
T_2 =& \lim_{s\to 0} \frac{1}{s}\Big(w(\eta_s \circ \xi_s(x)) - w(\eta_s \circ \xi_0(x))\Big) \\
=& \lim_{s\to 0} \frac{1}{s}\Big(w\big(\eta_s(\xi_s(x)\big) - w\big(\eta_s(\xi_0(x))\big)\Big)
\end{aligned}$$

Since $\lim_{s\to 0} \eta_s(x) = \eta_0(x) = x$, then

$$\begin{aligned}
T_2 =& \lim_{s\to 0} \frac{1}{s}\big(w(\xi_s(x)) - w(\xi_0(x))\big) = \frac{d}{ds}|_{s=0}\xi_s^* w \\
=&[w, \xi] = \text{curl}(w \times \xi),
\end{aligned} \tag{2.177}$$

where w and ξ are both divergence-free and tangent to $\partial\Omega$. Denote $w' = \text{curl}(w \times \xi)$, then

$$\text{div}(w') = 0, \tag{2.178}$$

since the divergence of the curl of any vector field is always zero.

Since the Lie bracket of two tangent vectors on a submanifold is also a tangent vector to a submanifold, w' is also divergence-free and tangent to $\partial\Omega$. Combining (2.176), (2.C) and (2.178), we obtain (2.173). □

From Lemma 2.C.1, we have (2.53) holds.

2.D Proof of Equation (2.54)

From [11, Example 2.12], we know that for divergence-free vector fields $v, u \in T_p\Omega$, the Lie bracket is given as

$$[v, u] = \nabla \times (v \times u). \tag{2.179}$$

We denote the Lie algebra by $\mathfrak{g} = T_p\Omega$, and its corresponding Lie bracket $[\cdot,\cdot]_1$. Assume there is a diffeomorphism $\Phi : \mathfrak{g} \to \mathfrak{g}^\flat$, then the bracket on $\mathfrak{g}$ can be constructed by

$$[v^\flat, u^\flat]_1 = \Phi\big([\Phi^{-1}(v^\flat), \Phi^{-1}(u^\flat)]\big) = \big(\nabla \times (v \times u)\big)^\flat, \tag{2.180}$$

where $v^\flat \in P^*\Lambda^1(\Omega)$ and $u^\flat \in P^*\Lambda^1(\Omega)$.

From [56, P135], we have for any vector X,

$$\nabla \times X = \big(* (d(X^\flat))\big)^\sharp, \quad v \times u = \big(* (v^\flat \wedge u^\flat)\big)^\sharp. \tag{2.181}$$

Substituting (2.181) into (2.180), we have that

$$\big(\nabla \times (v \times u)\big)^\flat = *\big(d(v \times u)^\flat\big) = *d * (v^\flat \wedge u^\flat) = (-1)^{n-1}\delta(\tilde{v} \wedge \tilde{u}).$$

2.E Proof of the chain rule

By the Hodge decomposition, the divergence-free velocity field $v \in P^*\Lambda^1(\Omega)$ can be decomposed into

$$v = d\phi + \eta + \alpha, \tag{2.182}$$

with $d\phi \in \mathfrak{B}^1$, $\eta \in \mathring{\mathfrak{B}}^{*1}$ and the harmonic form $\alpha \in \mathfrak{H}^1$. If Ω is simply connected, then $\alpha = 0$, and we only take $d\phi$ and η into consideration. We have proven in Section 2.4 that $\delta v = \delta(d\phi) = 0$ and $d(d\phi) = 0$.

The functional with respect to v is given by

$$\mathcal{F}(v) = \mathcal{F}\big(v(\eta, \phi)\big) := \tilde{\mathcal{F}}(\eta, \phi).$$

Applying the classical chain rule, it holds

$$\int_\Omega \frac{\delta \mathcal{F}}{\partial v} \wedge \partial v = \int_\Omega \frac{\delta \tilde{\mathcal{F}}}{\delta \eta} \wedge \partial \eta + \int_\Omega \frac{\delta \tilde{\mathcal{F}}}{\delta \phi} \wedge \partial \phi. \tag{2.183}$$

Next, define the lifting operator $R : H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega) \to H^1\Lambda^n(\Omega)$ with

$$\langle R(\lambda), *\mu \rangle_{L^2\Lambda^n(\Omega)} = \langle \lambda, \bar{*}\mathrm{tr}(\mu) \rangle_{L^2\Lambda^{n-1}(\partial\Omega)}, \tag{2.184}$$

where $\lambda \in H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega)$, $\mu \in H^1\Lambda^0(\Omega)$ and $\bar{*}$ is the Hodge star operator with respect to the boundary $\partial\Omega$. Hence,

$$\int_\Omega R(\lambda) \wedge \mu = \int_{\partial\Omega} \lambda \wedge \mathrm{tr}(\mu). \tag{2.185}$$

From $v = d\phi + \eta$, it follows that

$$\int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge \partial v = \int_\Omega \Big(\frac{\delta \mathcal{F}}{\delta v} \wedge \partial \eta + \frac{\delta \mathcal{F}}{\delta v} \wedge d(\partial\phi) \Big). \tag{2.186}$$

The term with respect to ϕ can be rewritten as

$$\begin{aligned}\int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge d(\partial\phi) =& (-1)^{n-1} \int_\Omega * * \frac{\delta \mathcal{F}}{\delta v} \wedge d(\partial\phi) = \int_\Omega d(\partial\phi) \wedge * * \frac{\delta \mathcal{F}}{\delta \phi} \\ =& \langle d(\partial\phi), * \frac{\delta \mathcal{F}}{\delta v} \rangle_{L^2\Lambda^1(\Omega)} \\ =& \langle \partial\phi, \delta(* \frac{\delta \mathcal{F}}{\delta v}) \rangle_{L^2\Lambda^0(\Omega)} + \int_{\partial\Omega} \mathrm{tr}(\partial\phi) \wedge * \boldsymbol{n}(* \frac{\delta \mathcal{F}}{\delta v}).\end{aligned}$$

Using the fact that $*\frac{\delta\mathcal{F}}{\delta v} \in P\Lambda^1(\Omega)$, which implies that $\delta(*\frac{\delta\mathcal{F}}{\delta v}) = 0$, and we obtain

$$\int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge d(\partial\phi) = \int_{\partial\Omega} * \boldsymbol{n}(* \frac{\delta \mathcal{F}}{\delta v}) \wedge \mathrm{tr}(\partial\phi),$$

thus

$$\int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge \partial v = \int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge \partial \eta + \int_{\partial\Omega} * \boldsymbol{n}(* \frac{\delta \mathcal{F}}{\delta v}) \wedge \mathrm{tr}(\partial\phi).$$

Using the lifting operator $R : H^{-\frac{1}{2}}\Lambda^{n-1}(\partial\Omega) \to H^1\Lambda^n(\Omega)$, then

$$\int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge \partial v = \int_\Omega \frac{\delta \mathcal{F}}{\delta v} \wedge \partial \eta + \int_\Omega R\big(* \boldsymbol{n}(* \frac{\delta \mathcal{F}}{\delta v})\big) \wedge \partial\phi.$$

Hence,

$$\frac{\delta \mathcal{F}}{\delta \phi} = R\big(* \boldsymbol{n}(* \frac{\delta \mathcal{F}}{\delta v})\big) \text{ in } \Omega,$$

which implies that

$$\mathrm{tr}(\frac{\delta \mathcal{F}}{\delta \phi}) = \mathrm{tr}\Big(R\big(* \boldsymbol{n}(* \frac{\delta \mathcal{F}}{\delta v})\big)\Big) = * \boldsymbol{n}(* \frac{\delta \mathcal{F}}{\delta v}) = (-1)^{n-1} \mathrm{tr}(\frac{\delta \mathcal{F}}{\delta v}).$$

2.F Time derivative of a functional

Since the domain $\Omega = \Omega(t)$ and free surface $\Sigma(t) \subset \partial\Omega(t)$ are time-dependent, we need to compute the time derivative of a functional on a time-dependent domain, which requires the use of the chain rule. The functional derivative can be interpreted as a Gateaux derivative, but the Gateaux derivative in general does not satisfy a chain rule. We will use therefore the concept of the chain differential [15], which satisfies a chain rule, but is less restrictive than the Fréchet derivative.

Definition 2.F.1. *[15] Given a topological vector space X. The function f has a chain differential $df(x;\xi)$ at x in the direction ξ if, for any sequence $\xi_n \to \xi \in X$ and any sequence of real numbers $\theta_n \to 0$ it holds that*

$$df(x;\xi) = \lim_{n\to\infty} \frac{1}{\theta_n}\big(f(x+\theta_n\xi_n) - f(x)\big).$$

Lemma 2.F.1. *[15, Proposition 3] Let X, Y and Z be three topological vector spaces, and let $f : X \times Y \to Z$. Assume that both partial functions $x \mapsto f(x,y)$ and $y \mapsto f(x,y)$ have chain differentials $d_1 f$ and $d_2 f$, respectively, for all (x,y) in an open set $\Omega \subset X \times Y$ and in any direction, and assume that $d_1 f$ is jointly continuous in its three arguments over $\Omega \times X$. Then f admits a chain differential at any (x,y) in that domain, in any direction (ξ,η), given by*

$$df(x,y;\xi,\eta) = d_1 f(x,y;\xi) + d_2 f(x,y;\eta). \tag{2.187}$$

Lemma 2.F.2. *[15, Theorem 1] Let X, Y and Z be three topological vector spaces, $f : X \to Y$, $g : Y \to Z$ and f has a chain differential at x in the direction ξ and g at $f(x)$ in the direction $df(x;\xi)$, respectively. Let $h = g \circ f$. Then h has a chain differential at x in the direction ξ, given by the chain rule*

$$dh(x;\xi) = dg(f(x); df(x;\xi)). \tag{2.188}$$

Since the domain is time-dependent, which means $\Omega = \Omega(t)$, using [39, Theorem 4.42], for any n-form $\alpha \in \Lambda^n(\Omega)$, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega(t)} \alpha &= \int_{\Omega(t)} \big(\frac{\partial \alpha}{\partial t} + d i_X \alpha + i_X d\alpha\big) \\
&= \int_{\Omega(t)} \frac{\partial \alpha}{\partial t} + \int_{\Sigma(t)} \mathrm{tr}(i_X \alpha) \\
&= \int_{\Omega(t)} \frac{\partial \alpha}{\partial t} + \int_{\Sigma(t)} (*\alpha)\mathrm{tr}(i_X v_\Omega),
\end{aligned} \tag{2.189}$$

where $\Omega(t) = \xi_t\Omega(0)$ and $X = \frac{d\xi_t}{dt}$ at $\Sigma(t)$, with $\alpha = (*\alpha)v_\Omega$, $d\alpha = 0$ for $\alpha \in \Lambda^n(\Omega)$ and $X = 0$ at Γ. Then

$$\frac{d}{dt}\int_{\Omega(t)} \alpha = \int_{\Omega(t)} \frac{\partial \alpha}{\partial t} + \int_{\Sigma(t)} (*\alpha) \wedge \Sigma_t. \tag{2.190}$$

Consider now the functional in the form

$$\mathcal{F}\big(v(t)\big) = \int_{\Omega(t)} f\big(v(t)\big), \quad f\big(v(t)\big) \in \Lambda^n(\Omega). \tag{2.191}$$

From (2.190), we know that

$$\frac{d\mathcal{F}}{dt}\big(v(t)\big) = \int_{\Omega(t)} \frac{\partial f}{\partial t} + \int_{\Sigma(t)} (*f) \wedge \Sigma_t.$$

Since

$$\int_{\Omega(t)} \frac{\partial f}{\partial t} = \int_{\Omega(t)} \frac{\partial f}{\partial v} \wedge \frac{\partial v}{\partial t} = \int_{\Omega(t)} \frac{\delta \mathcal{F}}{\delta v} \wedge \frac{\partial v}{\partial t},$$

hence

$$\frac{d\mathcal{F}}{dt}\big(v(t)\big) = \int_{\Omega(t)} \frac{\delta \mathcal{F}}{\delta v} \wedge \frac{\partial v}{\partial t} + \int_{\Sigma(t)} (*f) \wedge \Sigma_t. \tag{2.192}$$

Lemma 2.F.3. *Let X, Y be topological vector soaces. For any functional $\mathcal{F} : X \times Y \to \mathbb{R}$, the chain rule*

$$d\mathcal{F}(v, \Sigma; \partial v, \partial\Sigma) = \int_{\Omega(t)} \frac{\delta \mathcal{F}}{\delta v} \wedge v_t + \int_{\Sigma(t)} \frac{\delta \mathcal{F}}{\delta \Sigma} \wedge \Sigma_t. \tag{2.193}$$

holds.

Proof. From Lemma 2.F.1, we have

$$d\mathcal{F}(v, \Sigma; \partial v, \partial\Sigma) = d_1\mathcal{F}(v, \Sigma; \partial v) + d_2\mathcal{F}(v, \Sigma; \partial\Sigma).$$

Then using the chain rule Lemma 2.F.2, we obtain that

$$d_1\mathcal{F}(v, \Sigma; \partial v) = d_1\mathcal{F}(v(t), \Sigma; dv(t; \partial v)),$$
$$d_2\mathcal{F}(v, \Sigma; \partial\Sigma) = d_2\mathcal{F}(v, \Sigma(t); d\Sigma(t; \partial\Sigma)).$$

By the definition of chain differential, it holds

$$d_1\mathcal{F}(v(t), \Sigma; dv(t; \partial v)) = \int_{\Omega(t)} \frac{\delta \mathcal{F}}{\delta v} \wedge dv(t; \partial v),$$
$$d_2\mathcal{F}(v, \Sigma(t); d\Sigma(t; \partial\Sigma)) = \int_{\Sigma(t)} \frac{\delta \mathcal{F}}{\delta \Sigma} \wedge d\Sigma(t; \partial\Sigma).$$

Since

$$dv(t;\partial v) = \lim_{\epsilon\to 0} \frac{v(t+\epsilon\partial v) - v(t)}{\epsilon} = v_t,$$

$$d\Sigma(t;\partial\Sigma) = \lim_{\epsilon\to 0} \frac{\Sigma(t+\epsilon\partial\Sigma) - \Sigma(t)}{\epsilon} = \Sigma_t,$$

hence (2.193) holds.

□

Chapter 3

Discontinuous Galerkin Finite Element Methods for Linear Port-Hamiltonian Dynamical Systems[1]

Abstract

In this chapter, we present discontinuous Galerkin (DG) finite element discretizations for a class of linear hyperbolic port-Hamiltonian dynamical systems. The key point in constructing a port-Hamiltonian system is a Stokes-Dirac structure. Instead of following the traditional approach of defining the strong form of the Dirac structure, we define a Dirac structure in weak form, specifically in the input-state-output form. This is implemented within broken Sobolev spaces on a tessellation with polyhedral elements. After that, we state the weak port-Hamiltonian formulation and prove that it relates to a Poisson bracket. In our work, a crucial aspect of constructing the above-mentioned Dirac structure is that we provide a conservative relation between the boundary ports. Next, we state DG discretizations of the port-Hamiltonian system by using the weak form of the Dirac structure and broken polynomial spaces of differential forms, and we provide a priori error estimates for the structure-preserving port-Hamiltonian discontinuous Galerkin (PHDG) discretizations. The accuracy and capabilities of the methods developed in this chapter are demonstrated by presenting several numerical experiments.

3.1 Introduction

The port-Hamiltonian formulation provides a systematic and energy-based approach to describe the dynamics of open physical systems, such as heat

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conduction and acoustic and electric-magnetic wave propagation in heterogeneous media. We refer to [36, 76, 90] for a general overview of port-Hamiltonian systems. The general theory for distributed-parameter port-Hamiltonian systems was originally formulated in [93], in which the port-Hamiltonian framework is extended to handle systems with spatially distributed variables.

Port-Hamiltonian systems are defined by using the notion of a Dirac structure, which is a geometric structure generalizing both the symplectic and Poisson structure on the state space, and was originally introduced in [32]. A crucial property of the Dirac structure is that any power-conserving interconnection of a Dirac structure is again a Dirac structure, see [21, 58, 91]. This implies that any power-conserving interconnections of a port-Hamiltonian system defined on a domain divided into several subdomains is again a port-Hamiltonian system, where the Hamiltonian is the sum of the Hamiltonians on each subdomain. Using the Dirac structure, the dynamics of a port-Hamiltonian system can be expressed by using a pseudo-Poisson bracket, which is bilinear and skew-symmetric. The pseudo-Poisson bracket [93], however, does not automatically satisfy the Jacobi identity, which property needs to be proven separately to obtain a Hamiltonian formulation.

In general, a numerical discretization will not preserve the Hamiltonian structure of partial differential equations. But if it is possible to derive a Hamiltonian structure-preserving numerical discretization, then the numerical solution will inherit many of the properties of the infinite-dimensional Hamiltonian system. This results in numerical solutions with very good (long-time) accuracy and stability properties. Recently, this has motivated many studies on structure-preserving numerical discretizations of port-Hamiltonian systems. In [86, 87] a finite difference discretization is proposed that preserves the geometric Dirac structure of the heat and wave equation. Using discrete finite element exterior calculus [8, 7, 79], the authors in [17] studied the dual form of a constant linear Stokes-Dirac structure, and derived a weak form of the dual port-Hamiltonian system. In [69] the authors proposed a structure-preserving partitioned finite element method for a port-Hamiltonian system consisting of two conservation laws. In [48, 49] a geometric approximation of the Dirac structure of a hyperbolic system consisting of two conservation laws is obtained by using a mixed Galerkin discretization of the distributed parameter port-Hamiltonian system and power-conserving maps.

In this chapter, we will present a novel approach to obtain structure-preserving discretizations of port-Hamiltonian formulations using discontinuous Galerkin (DG) finite element discretizations. The DG finite element method is a numerical method used to solve partial differential equations. Unlike traditional finite element methods, where globally continuous basis functions are used, DG methods use piecewise polynomial basis functions that may have discontinuities at element boundaries [28, 30, 35]. One of the benefits of DG methods is that they allow for the use of unstructured locally refined meshes and different polynomial order in each element (hp-adaptation), and can be easily adapted to complex geometries. Further, DG methods are able to achieve high-order accuracy by applying high-degree polynomials within each element. In addition, DG methods ensure that quantities such as mass, momentum and energy are conserved across element boundaries by choosing conservative fluxes.

In [97], the authors discretized a class of linear Hamiltonian hyperbolic systems using a DG method. This discretization ensures power conservation by defining suitable numerical fluxes, but it does not take the energy exchange at the domain boundary into consideration. A structure-preserving DG approximation of a linear one-dimensional port-Hamiltonian system is proposed in [85], which is power-conserving by using conservative numerical fluxes, but this study does not concern higher-order systems. Recently in [50], a novel framework for the DG discretization of port-Hamiltonian systems of conservation laws is proposed. This paper states the spaces for the boundary ports explicitly, and presents the discrete Dirac structure in a strong way. For computational purposes, however, some of the theories in [50] are complex and redundant. Also, compared to [50], we need weaker boundary continuity in our work.

Using the close analogy between the numerical fluxes in a DG discretization and the port variables in a port-Hamiltonian system, we establish a new mathematical framework to derive Hamiltonian structure-preserving DG discretizations of linear dynamical systems. The novelty of our work is that we derive a weak form of the Dirac structure for linear dynamical equations on a tessellation with polyhedral elements and prove that this is again a Dirac structure. This implies that the energy flow only occurs at the domain boundary, ensuring internal energy conservation. Building upon the above-mentioned weak Dirac structure, we derive the corresponding pseudo-Poisson bracket for this system. Next, we prove that the pseudo-Poisson bracket also satisfies the Jacobi identity, which proves that the bracket is actually a Poisson bracket. By expressing the dynamical system using the Poisson bracket, we obtain the weak form of the port-Hamiltonian formulation of linear dynamical systems on a tessellation with polyhedral elements. Using broken Sobolev spaces of differential forms together with piecewise polynomial spaces of differential forms, we obtain then a structure-preserving port-Hamiltonian discontinuous Galerkin (PHDG) discretization.

This chapter is organized as follows. In Section 3.2, we present a linear dynamical system written in differential forms and discuss some of its applications. In Section 3.3, we derive the strong form of the Dirac structure and obtain the corresponding port-Hamiltonian formulation. The weak form of the input-state-output Dirac structure and port-Hamiltonian formulation on a tessellation of polyhedral elements are presented in Section 3.4. Using the space of broken polynomial differential forms, we obtain the DG discretization of the port-Hamiltonian system in Section 3.5. Section 3.6 discusses several numerical experiments which show the accuracy and capabilities of the novel structure-preserving DG finite element method for linear port-Hamiltonian dynamical systems.

3.2 Linear dynamical systems

In this section, we introduce a class of linear dynamical systems in differential forms and their applications. We first introduce some important operators and notation that will be used later in this chapter.

3.2.1 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a n-dimensional oriented manifold with Lipschitz continuous boundary $\partial\Omega$. Let $\Lambda^k(\Omega)$ be the space of all smooth differential k-forms on Ω. The inner product defined on $\Lambda^k(\Omega)$ is [78, Definition 1.2.2]

$$\langle \lambda, \mu \rangle_{\Lambda^k(\Omega)} := \sum_{\sigma \in S(k,n)} \lambda(E_{\sigma(1)}, \ldots, E_{\sigma(k)}) \mu(E_{\sigma(1)}, \ldots, E_{\sigma(k)}), \tag{3.1}$$

where $(E_1, \ldots, E_n)$ is an orthonormal basis on Ω with respect to the Riemannian metric, and $S(k,n)$ with $1 \leq k \leq n$ is the set of all permutations of the numbers $\{1, 2, \ldots, n\}$ such that $\sigma(1) < \ldots < \sigma(k)$.

Definition 3.2.1 (**Hodge star operator** [78]). *The Hodge star operator* $* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$*, which is an isometry, is defined as*

$$\lambda \wedge *\mu = \langle \lambda, \mu \rangle_{\Lambda^k(\Omega)} \mathrm{vol}_\Omega, \;\; \lambda, \mu \in \Lambda^k(\Omega), \tag{3.2}$$

where $\wedge$ *is the exterior product and* $\mathrm{vol}_\Omega \in \Lambda^n(\Omega)$ *is the Riemannian volume form.*

We have the relation

$$**\lambda = (-1)^{k(n-k)}\lambda, \quad \lambda \in \Lambda^k(\Omega). \tag{3.3}$$

If $(E_1, \dots, E_n)$ is a local orthonormal frame on Ω and $\sigma \in S(k,n)$, the Hodge operator computes as [78, Proposition 1.2.3]

$$(*\omega)(E_{\sigma(k+1)}, \dots, E_{\sigma(n)}) = (\text{sign } \sigma)\omega(E_{\sigma(1)}, \dots, E_{\sigma(k)}), \ \omega \in \Lambda^k(\Omega). \tag{3.4}$$

Definition 3.2.2 (Exterior derivative and coderivative). *The exterior derivative $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ takes differential k-forms on the domain Ω to differential $(k+1)$-forms. The coderivative operator $\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ is defined as*

$$\delta := (-1)^{n(k+1)+1} * d * . \tag{3.5}$$

3.2.2 Model description

In this chapter, we consider the following class of linear dynamical systems stated in differential forms as

$$\begin{aligned} &\frac{\partial \eta}{\partial t} + (-1)^{q(n-q)} d(h \wedge *u) = 0, \\ &\frac{\partial u}{\partial t} - cv + (-1)^{p+1} d(g \wedge *\eta) = 0, \\ &\frac{\partial v}{\partial t} + cu = 0. \end{aligned} \tag{3.6}$$

We identify the variable η with a p-form, and the variables u, v with q-forms, i.e. $\eta \in \Lambda^p(\Omega)$, $u, v \in \Lambda^q(\Omega)$ and $g, c \in \Lambda^0(\Omega), h \in \Lambda^0(\Omega), h \neq 0$, are given functions. Furthermore, d is the exterior derivative, t denotes time, and we assume that $p + q = n + 1$.

3.2.3 Applications

In this section, we present four important examples of the class of linear dynamical systems (3.6).

3.2.3.1 Scalar wave equation

Let $\Omega \subseteq \mathbb{R}^n$ be an n-dimensional oriented manifold. The n-dimensional scalar wave equation is given as [84, 93]

$$\frac{\partial^2 w}{\partial t^2} - d(g \wedge \delta w) = 0,$$

with $w \in \Lambda^n(\Omega)$ the displacement, $g \in \Lambda^0(\Omega)$ with $g(x) \geq g_0 > 0$ a given function. The n-dimensional scalar wave equation can be represented as the first-order hyperbolic system

$$\begin{aligned} \frac{\partial \eta}{\partial t} - d(*u) &= 0, \\ \frac{\partial u}{\partial t} + d(g \wedge *\eta) &= 0, \\ \frac{\partial v}{\partial t} &= 0, \end{aligned} \tag{3.7}$$

which is a special case of (3.6) when c is taken to be zero and $h = -1$. Here, $\eta = d(*w) \in \Lambda^1(\Omega)$ denotes the auxiliary variable, $u = \frac{\partial w}{\partial t} \in \Lambda^n(\Omega)$ the velocity field, and $v \in \Lambda^n(\Omega)$ the mass, which is assumed to be constant with respect to time.

3.2.3.2 Two-dimensional linear shallow water equations

The dynamics of linear shallow water flow can be modeled by (3.6) [69, 70] as

$$\begin{aligned} \frac{\partial \eta}{\partial t} + d(h_l \wedge *u) &= 0, \\ \frac{\partial u}{\partial t} - fv + d(g \wedge *\eta) &= 0, \\ \frac{\partial v}{\partial t} + fu &= 0, \end{aligned} \tag{3.8}$$

where $\Omega \subseteq \mathbb{R}^2$ is a two-dimensional oriented manifold, the variable $\eta \in \Lambda^1(\Omega)$ denotes the height of the water level, $u \in \Lambda^2(\Omega)$ is the component of the water velocity in the x-direction, and $v \in \Lambda^2(\Omega)$ in the y-direction, $h_l \in \Lambda^0(\Omega), h_l(x, y) \geq h_{l_0} > 0$ is a given height, f is the Coriolis parameter, and g is the acceleration due to gravity.

3.2.3.3 Three-dimensional Maxwell equations

Let $\Omega \subseteq \mathbb{R}^3$ be a three-dimensional oriented manifold. The three-dimensional Maxwell equations with zero source term are [39, 62, 93]

$$\frac{\partial B}{\partial t} + d(\frac{1}{\epsilon} \wedge *D) = 0, \tag{3.9a}$$

$$\frac{\partial D}{\partial t} - d(\frac{1}{\mu} \wedge *B) = 0, \tag{3.9b}$$

where $B \in \Lambda^2(\Omega)$ is the magnetic field induction, $D \in \Lambda^2(\Omega)$ is the electric field induction, $\epsilon \in \Lambda^0(\Omega)$, $\epsilon(x) \geq \epsilon_0 > 0$ are the electric permittivity, and $\mu \in \Lambda^0(\Omega)$, $\mu(x) \geq \mu_0 > 0$ are the magnetic permeability. We used the constitutive relations in (3.9)

$$\begin{aligned} * D &= \epsilon E, \\ * B &= \mu H, \end{aligned}$$

with $E \in \Lambda^1(\Omega)$ the electric field and $H \in \Lambda^1(\Omega)$ the magnetic field.

The Maxwell equations (3.9) are the special case of (3.6) with $c = 0, g = \frac{1}{\epsilon}$ and $h = -\frac{1}{\mu}$, where (3.9a) represents Faraday's law and (3.9b) Ampere's law. Using the relation $d \circ d = 0$, we obtain the constraints

$$\begin{aligned} \frac{\partial}{\partial t} dB &= 0, \\ \frac{\partial}{\partial t} dD &= 0. \end{aligned}$$

Hence if the fields B and D satisfy $dB = dD = 0$ at initial time $t = t_0$, then this holds for all later times.

3.2.3.4 Classical Klein–Gordon equation

Let $\Omega \subseteq \mathbb{R}^n$ be a n-dimensional oriented manifold. The n-dimensional classical Klein–Gordon equation in relativistic particle physics is given as [82]

$$\frac{\partial^2 v}{\partial t^2} + mv + d(g \wedge \delta v) = 0, \tag{3.10}$$

with $v \in \Lambda^n(\Omega)$ the displacement, $m \in \Lambda^0(\Omega), m(x) \geq m_0 > 0$ a positive mass, and $g \in \Lambda^0(\Omega)$, $g(x) \geq g_0 > 0$ a given function. Introducing the variables

$$\begin{aligned} u &= -\frac{1}{\sqrt{m}} \frac{\partial v}{\partial t} \in \Lambda^n(\Omega), \\ \eta &= (-1)^n \frac{1}{\sqrt{m}} * \delta v = \frac{1}{\sqrt{m}} d(*v) \in \Lambda^1(\Omega), \end{aligned} \tag{3.11}$$

the first-order form of (3.10) is

$$\begin{aligned} \frac{\partial \eta}{\partial t} + d(*u) &= 0, \\ \frac{\partial u}{\partial t} - \sqrt{m} v + d(g \wedge *\eta) &= 0, \\ \frac{\partial v}{\partial t} + \sqrt{m} u &= 0. \end{aligned} \tag{3.12}$$

3.3 Port-Hamiltonian formulation for linear dynamical systems

In this section, we derive the port-Hamiltonian formulation for the linear dynamical system (3.6). First, we introduce several Sobolev spaces of differential forms and then discuss the Dirac structure and port-Hamiltonian formulation.

3.3.1 Sobolev spaces of differential forms

The Sobolev spaces of differential forms are defined as follows.

Definition 3.3.1. *The spaces* $L^2\Lambda^k(\Omega)$ *and* $H^s\Lambda^k(\Omega)$ *with* $k \in \mathbb{N}^+ := \mathbb{N} \cup \{0\}, s \geq 0,$ *real, are the spaces of differential k-forms on the domain* $\Omega \subseteq \mathbb{R}^d, d \geq 1,$ *whose coefficient functions belong to the Sobolev spaces* $L^2(\Omega)$ *and* $H^s(\Omega)$*, respectively, with* $H^0(\Omega) = L^2(\Omega)$ *[7, 8]. For a definition of these Sobolev spaces on general manifolds, see [78, Definition 1.3.2]. The* L^2*-inner product and norm are defined as [7, 8]*

$$\langle \lambda, \mu \rangle_{L^2\Lambda^k(\Omega)} := \int_\Omega \lambda \wedge *\mu, \ \|\lambda\|^2_{L^2\Lambda^k(\Omega)} := \langle \lambda, \lambda \rangle_{L^2\Lambda^k(\Omega)}, \tag{3.13}$$

for any $\lambda, \mu \in L^2\Lambda^k(\Omega)$.

Definition 3.3.2. *The Hilbert space* $H\Lambda^k(\Omega)$ *is defined as [7, 8]*

$$H\Lambda^k(\Omega) := \{\lambda \in L^2\Lambda^k(\Omega) \mid d\lambda \in L^2\Lambda^{k+1}(\Omega)\}, \tag{3.14}$$

with the norm given by

$$\|\lambda\|^2_{H\Lambda^k(\Omega)} := \|\lambda\|^2_{L^2\Lambda^k(\Omega)} + \|d\lambda\|^2_{L^2\Lambda^{k+1}(\Omega)}. \tag{3.15}$$

Definition 3.3.3. *The Hilbert space* $H^*\Lambda^k(\Omega)$ *is defined as*

$$H^*\Lambda^k(\Omega) := \{\lambda \in L^2\Lambda^k(\Omega) \mid \delta\lambda \in L^2\Lambda^{k-1}(\Omega)\}. \tag{3.16}$$

It's obvious that $H^*\Lambda^k(\Omega) = *H\Lambda^{n-k}(\Omega)$.

3.3.2 Trace operators

In this section, we define operators and Sobolev spaces of differential forms with respect to the boundary.

Given a differential k-form α, then $\alpha|_{\partial\Omega}$, the boundary value of $\alpha \in \Lambda^k(\Omega)$, is defined by a skew-symmetric map [78]

$$\alpha|_{\partial\Omega} : \underbrace{\Gamma(T\Omega|_{\partial\Omega}) \times \ldots \times \Gamma(T\Omega|_{\partial\Omega})}_{k-\text{times}} \to C^\infty(\Omega),$$

where $\Gamma(T\Omega)$ is the space of all smooth vector fields, and we can write $\alpha|_{\partial\Omega} \in \Lambda^k(\Omega)|_{\partial\Omega}$. Any k-form α at $\partial\Omega$ can be decomposed into a tangential component $\mathbf{t}\alpha : \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$ and a normal component $\boldsymbol{n}\alpha := \alpha|_{\partial\Omega} - \mathbf{t}\alpha$, i.e. $\alpha|_{\partial\Omega} = \mathbf{t}\alpha + \boldsymbol{n}\alpha$ [78, Section 1.2]. The tangential trace operator is defined as $\mathrm{tr} := \mathbf{t}$.

Given any vector fields $X_1, \ldots, X_k \in \Gamma(T\Omega|_{\partial\Omega})$, decomposed into their tangential and normal parts $X_i = X_i^{\parallel} + X_i^{\perp},\ i = 1, \ldots, k$. We write the tangential trace as

$$\mathrm{tr}(\alpha)(X_1, \ldots, X_k) = \alpha(X_1^{\parallel}, \ldots, X_k^{\parallel}), \tag{3.17}$$

and the normal trace

$$\boldsymbol{n}(\alpha)(X_1, \ldots, X_k) = \alpha(X_1^{\perp}, \ldots, X_k^{\perp}). \tag{3.18}$$

With the definition of the trace operator $\mathrm{tr} : \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$, using the trace theorem in Sobolev spaces [42], we can obtain the Sobolev spaces for the boundary traces of $H^1\Lambda^k(\Omega)$ as [7]

$$H^{\frac{1}{2}}\Lambda^k(\partial\Omega) := \{\mu \in L^2\Lambda^k(\partial\Omega) \mid \exists\lambda \in H^1\Lambda^k(\Omega),\ \text{s.t. } \mathrm{tr}(\lambda) = \mu\}, \tag{3.19}$$

with norm given as

$$\|\mu\|_{H^{\frac{1}{2}}\Lambda^k(\partial\Omega)} := \inf_{\mathrm{tr}(\lambda)=\mu, \lambda\in H^1\Lambda^k(\Omega)} \|\lambda\|_{H^1\Lambda^k(\Omega)}. \tag{3.20}$$

Next, we define the boundary traces of the Sobolev spaces $H\Lambda^k(\Omega)$. If $k = 0$, since $H\Lambda^0(\Omega) = H^1\Lambda^0(\Omega)$, we have that $H^{\frac{1}{2}}\Lambda^0(\partial\Omega)$ is the trace space of $H\Lambda^0(\Omega)$. If $0 \le k < n$, we denote the dual space of $H^{\frac{1}{2}}\Lambda^k(\partial\Omega)$ by $H^{-\frac{1}{2}}\Lambda^k(\partial\Omega)$. From [7], we know that for $0 < k < n$ the trace operator tr can be extended to a bounded linear operator that maps $H\Lambda^k(\Omega)$ onto $H^{-\frac{1}{2}}\Lambda^k(\partial\Omega)$. Since $H\Lambda^n(\Omega) = L^2\Lambda^n(\Omega)$, there is no trace for $k = n$.

Theorem 3.3.1 (**Stokes' Theorem**[78])**.** *On an n-dimensional oriented manifold Ω with boundary $\partial\Omega$, we have for any $\alpha \in \Lambda^{n-1}(\Omega)$*

$$\int_\Omega d\alpha = \int_{\partial\Omega} \mathrm{tr}(\alpha). \tag{3.21}$$

With the Stokes theorem, we now state the integration by parts formula

$$\int_\Omega d\lambda \wedge \mu = (-1)^{k+1} \int_\Omega \lambda \wedge d\mu + \int_{\partial\Omega} \mathrm{tr}(\lambda) \wedge \mathrm{tr}(\mu), \tag{3.22}$$

for $\lambda \in H\Lambda^k(\Omega)$ and $\mu \in H^1\Lambda^{n-k-1}(\Omega)$.

3.3.3 Dirac structure

In [7], van der Schaft and Maschke presented a framework for port-Hamiltonian formulations of distributed-parameter systems. Following the approach presented in [7], we derive the Dirac structure for the linear dynamical system (3.6). We first give a brief introduction to Dirac structures.

Let the linear spaces $\mathcal{F}$ and $\mathcal{E}$ be the space of flows and efforts, respectively. The total space $\mathcal{F} \times \mathcal{E}$ is called the space of power variables, and is equipped with a bilinear and non-degenerate pairing

$$\langle \cdot \mid \cdot \rangle : \mathcal{F} \times \mathcal{E} \to \mathbb{R}.$$

Hence, we can define a symmetric bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathcal{F} \times \mathcal{E}$ as

$$\langle\!\langle (f_1, e_1), (f_2, e_2) \rangle\!\rangle := \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle, \ (f_i, e_i) \in \mathcal{F} \times \mathcal{E}, \ i = 1, 2. \tag{3.23}$$

Definition 3.3.4 (**Dirac structure** [7])**.** *Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces with the pairing $\langle \cdot | \cdot \rangle$. A Dirac structure is a linear subspace $D \subset \mathcal{F} \times \mathcal{E}$ such that $D = D^\perp$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.*

We define the function spaces

$$\begin{aligned} \mathcal{F} &:= L^2\Lambda^p(\Omega) \times L^2\Lambda^q(\Omega) \times L^2\Lambda^q(\Omega) \times H^{\frac{1}{2}}\Lambda^{n-q}(\partial\Omega), \\ \mathcal{E} &:= H\Lambda^{n-p}(\Omega) \times H^1\Lambda^{n-q}(\Omega) \times L^2\Lambda^{n-q}(\Omega) \times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial\Omega), \end{aligned} \tag{3.24}$$

with $\Omega \subset \mathbb{R}^n$ and $p + q = n + 1$. The bilinear form defined on $\mathcal{F} \times \mathcal{E}$ is defined as

$$\begin{aligned} \langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle := \int_\Omega \Big(& e^1_\eta \wedge f^2_\eta + e^2_\eta \wedge f^1_\eta + e^1_u \wedge f^2_u + e^2_u \wedge f^1_u + e^1_v \wedge f^2_v \\ & + e^2_v \wedge f^1_v \Big) + \int_{\partial\Omega} \Big(e^1_b \wedge f^2_b + e^2_b \wedge f^1_b \Big), \end{aligned} \tag{3.25}$$

where

$$f^i = (f^i_\eta, f^i_u, f^i_v, f^i_b) \in \mathcal{F}, \quad e^i = (e^i_\eta, e^i_u, e^i_v, e^i_b) \in \mathcal{E}, \quad i = 1, 2.$$

Remark 3.3.2. *Using the facts* $H\Lambda^{n-p}(\Omega) \subset L^2\Lambda^{n-p}(\Omega)$ *and* $H^1\Lambda^{n-q}(\Omega) \subset L^2\Lambda^{n-q}(\Omega)$, *we apply the duality between* $L^2\Lambda^p(\Omega)$ *and* $L^2\Lambda^{n-p}(\Omega)$, *and between* $L^2\Lambda^q(\Omega)$ *and* $L^2\Lambda^{n-q}(\Omega)$ *on* Ω *in (3.25). Note that we use the duality of* $H^{\frac{1}{2}}\Lambda^{n-q}(\partial\Omega)$ *and* $H^{-\frac{1}{2}}\Lambda^{n-p}(\partial\Omega)$ *at* $\partial\Omega$ *in (3.25).*

Theorem 3.3.3 (Dirac structure). *Let* $\Omega \subset \mathbb{R}^n$ *be an n-dimensional oriented connected manifold with Lipschitz continuous boundary* $\partial\Omega$. *Given the function spaces* $\mathcal{F}$ *and* $\mathcal{E}$, *defined in (3.24), together with the bilinear form (3.25). Then* $D \subset \mathcal{F} \times \mathcal{E}$, *defined as*

$$
\begin{aligned}
D := \Bigg\{ &\big((f_\eta, f_u, f_v, f_b), (e_\eta, e_u, e_v, e_b)\big) \in \mathcal{F} \times \mathcal{E} \mid \\
&\begin{pmatrix} f_\eta \\ f_u \\ f_v \end{pmatrix} = \begin{pmatrix} 0 & d & 0 \\ (-1)^r d & 0 & -\frac{c}{h}* \\ 0 & \frac{c}{h}* & 0 \end{pmatrix} \begin{pmatrix} e_\eta \\ e_u \\ e_v \end{pmatrix}, \\
&\begin{pmatrix} f_b \\ e_b \end{pmatrix} = \begin{pmatrix} 0 & (-1)^q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{tr}(e_\eta) \\ \mathrm{tr}(e_u) \end{pmatrix} \Bigg\},
\end{aligned}
\tag{3.26}
$$

with given $h^{-1}, c, \frac{c}{h} \in L^2\Lambda^0(\Omega)$, $r = pq + 1$, *is a Dirac structure.*

Proof. (1) $\boldsymbol{D \subseteq D^\perp}$. Let $(f^1, e^1) \in D$ be fixed, and consider any $(f^2, e^2) \in D$. Using the integration by parts formula (3.22), we obtain

$$
\begin{aligned}
&\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle \\
=&\int_\Omega \Big(e_\eta^1 \wedge de_u^2 + e_\eta^2 \wedge de_u^1 + e_u^1 \wedge \Big((-1)^r de_\eta^2 - \frac{c}{h} * e_v^2 \Big) \\
&+ e_u^2 \wedge \Big((-1)^r de_\eta^1 - \frac{c}{h} * e_v^1 \Big) + \frac{c}{h} e_v^1 \wedge * e_u^2 + \frac{c}{h} e_v^2 \wedge * e_u^1 \Big) \\
&+ \int_{\partial\Omega} \big((-1)^q \mathrm{tr}(e_\eta^1) \wedge \mathrm{tr}(e_u^2) + (-1)^q \mathrm{tr}(e_\eta^2) \wedge \mathrm{tr}(e_u^1) \big) \\
=&(-1)^{q-1} \int_\Omega \Big(d(e_\eta^1 \wedge e_u^2) + d(e_\eta^2 \wedge e_u^1) \Big) \\
&+ (-1)^q \int_{\partial\Omega} \Big(\mathrm{tr}(e_\eta^1) \wedge \mathrm{tr}(e_u^2) + \mathrm{tr}(e_\eta^2) \wedge \mathrm{tr}(e_u^1) \Big) \\
=& \, 0.
\end{aligned}
$$

Hence, we have $(f^1, e^1) \in D^\perp$, which implies that $D \subset D^\perp$.

(2) $\boldsymbol{D^\perp \subseteq D}$. Let $(f^1, e^1) \in D^\perp \subset \mathcal{F} \times \mathcal{E}$. Then

$$
\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle = 0, \quad \forall (f^2, e^2) \in D, \tag{3.27}
$$

or equivalently for all $(f^2, e^2) \in D$ we have

$$\int_\Omega \Big(e_\eta^1 \wedge de_u^2 + e_\eta^2 \wedge f_\eta^1 + e_u^1 \wedge \Big((-1)^r de_\eta^2 - \frac{c}{h} * e_v^2 \Big) + e_u^2 \wedge f_u^1 + \frac{c}{h} e_v^1 \wedge * e_u^2 + e_v^2 \wedge f_v^1 \Big) + \int_{\partial\Omega} \big((-1)^q e_b^1 \wedge \mathrm{tr}(e_u^2) + \mathrm{tr}(e_\eta^2) \wedge f_b^1 \big) = 0. \tag{3.28}$$

Take $e_\eta^2 \in H\Lambda^{n-p}(\Omega)$ and $e_u^2 \in H^1\Lambda^{n-q}(\Omega)$ such that $\mathrm{tr}(e_\eta^2) = \mathrm{tr}(e_u^2) = 0$ at $\partial\Omega$. Then (3.28) equals

$$\int_\Omega \Big(e_\eta^1 \wedge de_u^2 + e_\eta^2 \wedge f_\eta^1 + e_u^1 \wedge \Big((-1)^r de_\eta^2 - \frac{c}{h} * e_v^2 \Big) + e_u^2 \wedge f_u^1 + \frac{c}{h} e_u^2 \wedge * e_v^1 + e_v^2 \wedge f_v^1 \Big) = 0. \tag{3.29}$$

Using the integration by parts formula (3.22) and the fact that $\mathrm{tr}(e_\eta^2) = \mathrm{tr}(e_u^2) = 0$ at $\partial\Omega$, we can rewrite (3.29) as

$$\int_\Omega \Big((-1)^{r-1} e_u^2 \wedge de_\eta^1 + e_\eta^2 \wedge f_\eta^1 - e_\eta^2 \wedge de_u^1 - e_v^2 \wedge \frac{c}{h} * e_u^1 + e_u^2 \wedge f_u^1 + e_u^2 \wedge \frac{c}{h} * e_v^1 + e_v^2 \wedge f_v^1 \Big) = 0.$$

Since $e_\eta^2 \in H\Lambda^{n-p}(\Omega)$, $e_u^2 \in H^1\Lambda^{n-q}(\Omega)$ and $e_v^2 \in L^2\Lambda^{n-q}(\Omega)$ can be chosen arbitrarily in (3.26), we have

$$\begin{aligned} f_\eta^1 &= de_u^1, \\ f_u^1 &= (-1)^r de_\eta^1 - \frac{c}{h} * e_v^1, \\ f_v^1 &= \frac{c}{h} * e_u^1. \end{aligned} \tag{3.30}$$

After introducing (3.30) into (3.28), we obtain

$$\begin{aligned} &\int_\Omega \Big(e_\eta^1 \wedge de_u^2 + e_\eta^2 \wedge de_u^1 + e_u^1 \wedge \Big((-1)^r de_\eta^2 - \frac{c}{h} * e_v^2 \Big) \\ &+ e_u^2 \wedge \Big((-1)^r de_\eta^1 - \frac{c}{h} * e_v^1 \Big) + \frac{c}{h} e_v^1 \wedge * e_u^2 + \frac{c}{h} e_v^2 \wedge * e_u^1 \Big) \\ &+ \int_{\partial\Omega} \big(e_b^1 \wedge (-1)^q \mathrm{tr}(e_u^2) + \mathrm{tr}(e_\eta^2) \wedge f_b^1 \big) \\ &= \int_{\partial\Omega} \Big((-1)^{q-1} \mathrm{tr}(e_\eta^1) \wedge \mathrm{tr}(e_u^2) + (-1)^{q-1} \mathrm{tr}(e_\eta^2) \wedge \mathrm{tr}(e_u^1) \end{aligned}$$

$$+ (-1)^q e_b^1 \wedge \mathrm{tr}(e_u^2) + \mathrm{tr}(e_\eta^2) \wedge f_b^1\Big) = 0.$$

For arbitrary $e_\eta^2 \in H\Lambda^{n-p}(\Omega)$ and $e_u^2 \in H^1\Lambda^{n-q}(\Omega)$, this integral can be zero only if

$$\begin{aligned} f_b^1 &= (-1)^q \mathrm{tr}(e_u^1), \\ e_b^1 &= \mathrm{tr}(e_\eta^1). \end{aligned} \tag{3.31}$$

Hence $D^\perp \subset D$, which together with $D \subset D^\perp$ gives that $D = D^\perp$, thus D is a Dirac structure. □

3.3.4 Port-Hamiltonian formulation

The Hamiltonian $H : L^2\Lambda^p(\Omega) \times L^2\Lambda^q(\Omega) \times L^2\Lambda^q(\Omega) \to \mathbb{R}$ for the linear dynamical system (3.6) is given by

$$H(\eta, u, v) = \frac{1}{2}\int_\Omega \big(g\eta \wedge (*\eta) + hu \wedge (*u) + hv \wedge (*v)\big). \tag{3.32}$$

Definition 3.3.5. *Given a functional $F : L^2\Lambda^k(\Omega) \to \mathbb{R}$, the functional derivative $\frac{\delta F}{\delta \mu} \in L^2\Lambda^{n-k}(\Omega)$ is defined by*

$$\int_\Omega \frac{\delta F}{\delta \mu} \wedge \partial\mu = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(F(\mu + \epsilon\partial\mu) - F(\mu)\big), \ \forall \partial\mu \in L^2\Lambda^k(\Omega). \tag{3.33}$$

Lemma 3.3.4. *The functional derivatives of the Hamiltonian (3.32) with respect to (η, u, v) are*

$$\begin{aligned} \frac{\delta H}{\delta \eta} &= (-1)^{p(n-p)} g(*\eta), \\ \frac{\delta H}{\delta u} &= (-1)^{q(n-q)} h(*u), \\ \frac{\delta H}{\delta v} &= (-1)^{q(n-q)} h(*v). \end{aligned} \tag{3.34}$$

Proof. The results directly follow from the Hamiltonian (3.32) and the definition of functional derivative (3.33). □

Define the flow variables in the Dirac structure (3.26) in terms of (η, u, v) as

$$f_\eta = -\frac{\partial \eta}{\partial t} \in L^2\Lambda^p(\Omega), \ f_u = -\frac{\partial u}{\partial t} \in L^2\Lambda^q(\Omega), \ f_v = -\frac{\partial v}{\partial t} \in L^2\Lambda^q(\Omega), \tag{3.35}$$

and assume the energy variables in terms of (η, u, v) as

$$e_\eta = \frac{\delta H}{\delta \eta} \in H\Lambda^{n-p}(\Omega),\ e_u = \frac{\delta H}{\delta u} \in H^1\Lambda^{n-q}(\Omega),\ e_v = \frac{\delta H}{\delta v} \in L^2\Lambda^{n-q}(\Omega), \tag{3.36}$$

with the Hamiltonian given by (3.32). Using Theorem 3.3.3, the port-Hamiltonian formulation can be stated in the following corollary.

Corollary 3.3.1. *Let $\Omega \subset \mathbb{R}^n$ be an n-dimensional oriented connected manifold with Lipschitz continuous boundary $\partial\Omega$. The distributed-parameter port-Hamiltonian system for the linear dynamical system (3.6) with respect to the state space variables $(\eta, u, v) \in L^2\Lambda^p(\Omega) \times L^2\Lambda^q(\Omega) \times L^2\Lambda^q(\Omega)$, Dirac structure (3.26), and Hamiltonian (3.32), is given by*

$$\begin{aligned} \begin{pmatrix} \frac{\partial \eta}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} &= \begin{pmatrix} 0 & -d & 0 \\ (-1)^{r+1} d & 0 & \frac{c}{h}* \\ 0 & -\frac{c}{h}* & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \eta} \\ \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix}, \\ \begin{pmatrix} f_b \\ e_b \end{pmatrix} &= \begin{pmatrix} 0 & (-1)^q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{tr}(\frac{\delta H}{\delta \eta}) \\ \mathrm{tr}(\frac{\delta H}{\delta u}) \end{pmatrix}, \end{aligned} \tag{3.37}$$

where $f_b \in H^{\frac{1}{2}}\Lambda^{n-q}(\partial\Omega)$ and $e_b \in H^{-\frac{1}{2}}\Lambda^{n-p}(\partial\Omega)$ are external boundary ports, which are determined by the (specific) boundary conditions.

Lemma 3.3.5 ([93])**.** *By the power-conserving property of the Dirac structure, together with the bilinear form (3.25), it immediately follows that for any $\big((f_\eta, f_u, f_v, f_b),(e_\eta, e_u, e_v, e_b)\big) \in D \subset \mathcal{F} \times \mathcal{E}$ we have*

$$\int_\Omega \big(e_\eta \wedge f_\eta + e_u \wedge f_u + e_v \wedge f_v\big) + \int_{\partial\Omega} e_b \wedge f_b = 0, \tag{3.38}$$

which implies for the Hamiltonian (3.32) that for classical solutions

$$\frac{dH}{dt}(\eta, u, v) = \int_{\partial\Omega} e_b \wedge f_b. \tag{3.39}$$

Proof. Using the chain rule together with (3.35) and (3.36) to compute $\frac{dH}{dt}$, and combined with (3.38), we have (3.39). □

3.4 Input-state-output port-Hamiltonian systems

In this section, we consider the input-state-output form of the Dirac structure (3.26) for a domain Ω that is subdivided into the tessellation $\mathcal{T}_h$. This formulation will be used in Section 3.5 to obtain a PHDG discretization for the dynamical system (3.6).

3.4.1 Broken Sobolev spaces of differential forms

Let $\Omega \subset \mathbb{R}^n$ be an n-dimensional oriented connected polyhedral manifold with boundary $\partial\Omega$. Let $\mathcal{T}_h$ denote a tessellation of Ω with shape-regular n-dimensional polyhedral elements K [35], e.g. triangles, tetrahedra or hexahedra. Let Γ denote the set of all faces in the tessellation $\mathcal{T}_h$, with Γ_i the set of interior faces and Γ_b the set of faces at the domain boundary. We first give the definitions of the broken Sobolev spaces of differential forms

$$\begin{aligned}
L^2\Lambda^k(\mathcal{T}_h) &:= \{\lambda \in L^2\Lambda^k(\Omega) \mid \lambda|_K \in L^2\Lambda^k(K), \forall K \in \mathcal{T}_h\},\\
H^1\Lambda^k(\mathcal{T}_h) &:= \{\lambda \in L^2\Lambda^k(\Omega) \mid \lambda|_K \in H^1\Lambda^k(K), \forall K \in \mathcal{T}_h\},\\
H\Lambda^k(\mathcal{T}_h) &:= \{\lambda \in L^2\Lambda^k(\Omega) \mid \lambda|_K \in H\Lambda^k(K), \forall K \in \mathcal{T}_h\},
\end{aligned}$$

and the trace of the broken Sobolev spaces

$$\begin{aligned}
H^{\frac{1}{2}}\Lambda^k(\partial\mathcal{T}_h) &:= \{\lambda \in L^2\Lambda^k(\Gamma) \mid \lambda|_{\partial K} \in H^{\frac{1}{2}}\Lambda^k(\partial K), \forall K \in \mathcal{T}_h\},\\
H^{-\frac{1}{2}}\Lambda^k(\partial\mathcal{T}_h) &:= \{\lambda \in L^2\Lambda^k(\Gamma) \mid \lambda|_{\partial K} \in H^{-\frac{1}{2}}\Lambda^k(\partial K), \forall K \in \mathcal{T}_h\}.
\end{aligned}$$

3.4.2 Interconnection Dirac structure

Before the presentation of the Dirac structure on a polyhedral tessellation, we first consider the interconnection structure defined on the interior faces Γ_i.

If there are two elements K_1 and K_2 belonging to $\mathcal{T}_h$, with $f = \partial K_1 \cap \partial K_2$, $f_1 = \partial K_1 \backslash f$ and $f_2 = \partial K_2 \backslash f$. We denote the trace at $\partial K_1 \cap f$ by using the subscript L and at $\partial K_2 \cap f$ with the subscript R. Using $\partial K_L = f \cap \partial K_1$ and $\partial K_R = f \cap \partial K_2$, we define the function spaces

$$\begin{aligned}
\mathcal{F}_\mathcal{I} &= H^{-\frac{1}{2}}\Lambda^{n-p}(\partial K_L) \times H^{\frac{1}{2}}\Lambda^{n-q}(\partial K_L) \times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial K_R)\\
&\quad \times H^{\frac{1}{2}}\Lambda^{n-q}(\partial K_R),\\
\mathcal{E}_\mathcal{I} &= H^{\frac{1}{2}}\Lambda^{n-q}(\partial K_L) \times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial K_L) \times H^{\frac{1}{2}}\Lambda^{n-q}(\partial K_R)\\
&\quad \times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial K_R),
\end{aligned} \tag{3.40}$$

with $p + q = n + 1$. The bilinear form defined on $\mathcal{F}_\mathcal{I} \times \mathcal{E}_\mathcal{I}$ is given as

$$\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle_{\partial K \cap f}$$

$$
\begin{aligned}
&= \int_{\partial K_L} (e_L^1 \wedge \tilde{f}_L^2 + e_L^2 \wedge \tilde{f}_L^1 + \tilde{e}_L^1 \wedge f_L^2 + \tilde{e}_L^2 \wedge f_L^1) \\
&+ \int_{\partial K_R} (e_R^1 \wedge \tilde{f}_R^2 + e_R^2 \wedge \tilde{f}_R^1 + \tilde{e}_R^1 \wedge f_R^2 + \tilde{e}_R^2 \wedge f_R^1), \qquad (3.41)
\end{aligned}
$$

where

$$
f^i = (\tilde{f}_L^i, f_L^i, \tilde{f}_R^i, f_R^i) \in \mathcal{F}_{\mathcal{I}}, \quad e^i = (e_L^i, \tilde{e}_L^i, e_R^i, \tilde{e}_R^i) \in \mathcal{E}_{\mathcal{I}}, \quad i = 1, 2.
$$

Remark 3.4.1. *Note that for $f \in \Gamma_i$ the bilinear form (3.41) can be rewritten as*

$$
\begin{aligned}
&\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle_{\partial K \cap f} \\
&= \int_f (e_L^1 \wedge \tilde{f}_L^2 + e_L^2 \wedge \tilde{f}_L^1 + \tilde{e}_L^1 \wedge f_L^2 + \tilde{e}_L^2 \wedge f_L^1 \\
&- e_R^1 \wedge \tilde{f}_R^2 - e_R^2 \wedge \tilde{f}_R^1 - \tilde{e}_R^1 \wedge f_R^2 - \tilde{e}_R^2 \wedge f_R^1), \qquad (3.42)
\end{aligned}
$$

where the minus sign for contributions from element K_2 results from the opposite orientation of K_2 with respect to element K_1.

Lemma 3.4.2. *(**Interconnection Dirac structure**) Let K_1 and K_2 be two elements in the tessellation $\mathcal{T}_h$ with boundaries ∂K_1 and ∂K_2, and there exists $f \in \Gamma_i$ such that $f = \partial K_1 \cap \partial K_2 \neq \emptyset$. Given the function spaces $\mathcal{F}_{\mathcal{I}}$ and $\mathcal{E}_{\mathcal{I}}$, defined in (3.40), together with the bilinear form (3.41), the structure $\mathcal{D}_{\mathcal{I}} \subset \mathcal{F}_{\mathcal{I}} \times \mathcal{E}_{\mathcal{I}}$, defined as*

$$
\begin{aligned}
D_{\mathcal{I}} := \Bigg\{ &\big((\tilde{f}_L, f_L, \tilde{f}_R, f_R), (e_L, \tilde{e}_L, e_R, \tilde{e}_R)\big) \in \mathcal{F}_{\mathcal{I}} \times \mathcal{E}_{\mathcal{I}} \mid \\
&\begin{pmatrix} \tilde{f}_L \\ f_L \\ \tilde{f}_R \\ f_R \end{pmatrix} = \begin{pmatrix} 0 & d_{12} & 0 & d_{14} \\ d_{21} & 0 & d_{23} & 0 \\ 0 & d_{32} & 0 & d_{34} \\ d_{41} & 0 & d_{43} & 0 \end{pmatrix} \begin{pmatrix} e_L \\ \tilde{e}_L \\ e_R \\ \tilde{e}_R \end{pmatrix} \Bigg\}, \qquad (3.43)
\end{aligned}
$$

where

$$
\begin{aligned}
d_{12} &= (-1)^{r+p-1}(\frac{1}{2} - \theta), & d_{14} &= (-1)^{r+p-1}(\theta - 1), \\
d_{21} &= (-1)^{q-1}(\theta - \frac{1}{2}), & d_{23} &= (-1)^q \theta, \\
d_{32} &= (-1)^{r+p}\theta, & d_{34} &= (-1)^{r+p}(\frac{1}{2} - \theta), \\
d_{41} &= (-1)^{q-1}(\theta - 1), & d_{43} &= (-1)^q(\theta - \frac{1}{2}),
\end{aligned}
$$

with $\theta \in [0, 1]$, is a Dirac structure.

Proof. (1) $\boldsymbol{\mathcal{D}_{\mathcal{I}} \subset \mathcal{D}_{\mathcal{I}}^{\perp}}$. For fixed $(f^1, e^1) \in \mathcal{D}_{\mathcal{I}}$ and any $(f^2, e^2) \in \mathcal{D}_{\mathcal{I}}$, from (3.42) we have that

$$\begin{aligned}
\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle_{\partial K \cap f} &= \int_f \Big((-1)^{r+p-1} e_L^1 \wedge \big((\frac{1}{2} - \theta)\tilde{e}_L^2 + (\theta - 1)\tilde{e}_R^2 \big) \\
&\quad + (-1)^{r+p-1} e_L^2 \wedge \big((\frac{1}{2} - \theta)\tilde{e}_L^1 + (\theta - 1)\tilde{e}_R^1 \big) \\
&\quad + (-1)^{q-1} \tilde{e}_L^1 \wedge \big((\theta - \frac{1}{2}) e_L^2 - \theta e_R^2 \big) \\
&\quad + (-1)^{q-1} \tilde{e}_L^2 \wedge \big((\theta - \frac{1}{2}) e_L^1 - \theta e_R^1 \big) \\
&\quad - (-1)^{r+p} e_R^1 \wedge \big(\theta \tilde{e}_L^2 + (\frac{1}{2} - \theta)\tilde{e}_R^2 \big) \\
&\quad - (-1)^{r+p} e_R^2 \wedge \big(\theta \tilde{e}_L^1 + (\frac{1}{2} - \theta)\tilde{e}_R^1 \big) \\
&\quad - (-1)^{q} \tilde{e}_R^1 \wedge \big((1 - \theta) e_L^2 + (\theta - \frac{1}{2}) e_R^2 \big) \\
&\quad - (-1)^{q} \tilde{e}_R^2 \wedge \big((1 - \theta) e_L^1 + (\theta - \frac{1}{2}) e_R^1 \big) \Big) \\
&= 0.
\end{aligned}$$

Hence, we have $(f^1, e^1) \in \mathcal{D}_{\mathcal{I}}^{\perp}$, which implies that $\mathcal{D}_{\mathcal{I}} \subset \mathcal{D}_{\mathcal{I}}^{\perp}$.

(2) $\boldsymbol{\mathcal{D}_{\mathcal{I}}^{\perp} \subset \mathcal{D}_{\mathcal{I}}}$. Let $(f^1, e^1) \in \mathcal{D}_{\mathcal{I}}^{\perp}$. Then

$$\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle_{\partial K \cap f} = 0, \quad \forall (f^2, e^2) \in \mathcal{D}_{\mathcal{I}}, \tag{3.44}$$

and (3.44) gives

$$\begin{aligned}
&\int_f \Big((-1)^{r+p-1} e_L^1 \wedge \big((\frac{1}{2} - \theta)\tilde{e}_L^2 + (\theta - 1)\tilde{e}_R^2 \big) + e_L^2 \wedge \tilde{f}_L^1 \\
&+ (-1)^{q-1} \tilde{e}_L^1 \wedge \big((\theta - \frac{1}{2}) e_L^2 - \theta e_R^2 \big) + \tilde{e}_L^2 \wedge f_L^1 - (-1)^{r+p} e_R^1 \wedge \big(\theta \tilde{e}_L^2 \\
&+ (\frac{1}{2} - \theta)\tilde{e}_R^2 \big) - e_R^2 \wedge \tilde{f}_R^1 - (-1)^{q} \tilde{e}_R^1 \wedge \big((1 - \theta) e_L^2 + (\theta - \frac{1}{2}) e_R^2 \big) \\
&- \tilde{e}_R^2 \wedge f_R^1 \Big) = 0,
\end{aligned}$$

which can be written as

$$\begin{aligned}
&\int_f \Big(e_L^2 \wedge \big(\tilde{f}_L^1 + (-1)^{r+p-1} (\theta - \frac{1}{2}) \tilde{e}_L^1 + (-1)^{r+p-1} (1 - \theta) \tilde{e}_R^1 \big) \\
&\quad + \tilde{e}_L^2 \wedge \big(f_L^1 + (-1)^{q-1} (\frac{1}{2} - \theta) e_L^1 + (-1)^{q-1} \theta e_R^1 \big)
\end{aligned}$$

$$
\begin{aligned}
&-e_R^2 \wedge \big(\tilde{f}_R^1 + (-1)^{r+p-1}\theta \tilde{e}_L^1 + (-1)^{r+p}(\theta - \frac{1}{2})\tilde{e}_R^1\big) \\
&-\tilde{e}_R^2 \wedge \big(f_R^1 + (-1)^q(\theta-1)e_L^1 + (-1)^q(\frac{1}{2}-\theta)e_R^1\big)\Big) = 0. \qquad (3.45)
\end{aligned}
$$

Since $e^2 \in \mathcal{E}_\mathcal{I}$ are arbitrary, we obtain that

$$
\begin{aligned}
\tilde{f}_L^1 =& (-1)^{r+p-1}\big((\frac{1}{2}-\theta)\tilde{e}_L^1 + (\theta-1)\tilde{e}_R^1\big),\\
f_L^1 =& (-1)^{q-1}\big((\theta-\frac{1}{2})e_L^1 - \theta e_R^1\big),\\
\tilde{f}_R^1 =& (-1)^{r+p}\big(\theta \tilde{e}_L^1 + (\frac{1}{2}-\theta)\tilde{e}_R^1\big),\\
f_R^1 =& (-1)^{q-1}\big((\theta-1)e_L^1 + (\frac{1}{2}-\theta)e_R^1\big).
\end{aligned} \qquad (3.46)
$$

Hence, we have that $(f^1, e^1) \in \mathcal{D}_\mathcal{I}$, therefore $\mathcal{D}_\mathcal{I} \subset \mathcal{D}_\mathcal{I}^\perp$.

□

3.4.3 Dirac structure on a polyhedral tessellation

In this section, we derive the weak form of the Dirac structure on a polyhedral tessellation $\mathcal{T}_h$ using the interconnection Dirac structure given in (3.43). We define the following function spaces on the polyhedral tessellation $\mathcal{T}_h$

$$
\begin{aligned}
\mathcal{F}(\mathcal{T}_h) :=& H^*\Lambda^p(\mathcal{T}_h) \times H^{-1}\Lambda^q(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial\mathcal{T}_h)\\
&\times H^{\frac{1}{2}}\Lambda^{n-q}(\partial\mathcal{T}_h),\\
\mathcal{E}(\mathcal{T}_h) :=& H\Lambda^{n-p}(\mathcal{T}_h) \times H^1\Lambda^{n-q}(\mathcal{T}_h) \times L^2\Lambda^{n-q}(\mathcal{T}_h) \times H^{\frac{1}{2}}\Lambda^{n-q}(\partial\mathcal{T}_h)\\
&\times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial\mathcal{T}_h), \qquad (3.47)
\end{aligned}
$$

where $H^*\Lambda^p(\mathcal{T}_h)$ is the dual space of $H\Lambda^{n-p}(\mathcal{T}_h)$ defined in (3.16) and $p+q = n+1$. The bilinear form defined on $\mathcal{F}(\mathcal{T}_h) \times \mathcal{E}(\mathcal{T}_h)$ is defined as

$$
\begin{aligned}
\langle\!\langle (f^1,e^1),(f^2,e^2)\rangle\!\rangle_{\mathcal{T}_h} :=& \sum_{K\in\mathcal{T}_h}\int_K \big(e_\eta^1 \wedge f_\eta^2 + e_\eta^2 \wedge f_\eta^1 + e_u^1 \wedge f_u^2 + e_u^2 \wedge f_u^1\\
&+ e_v^1 \wedge f_v^2 + e_v^2 \wedge f_v^1\big) + \sum_{f\in\Gamma_b}\int_f \big(u_b^1 \wedge \tilde{y}_b^2 + u_b^2 \wedge \tilde{y}_b^1\\
&+ \tilde{u}_b^1 \wedge y_b^2 + \tilde{u}_b^2 \wedge y_b^1\big), \qquad (3.48)
\end{aligned}
$$

where

$$
f^i = (f_\eta^i, f_u^i, f_v^i, \tilde{u}_b^i, u_b^i) \in \mathcal{F}(\mathcal{T}_h),\ e^i = (e_\eta^i, e_u^i, e_v^i, y_b^i, \tilde{y}_b^i) \in \mathcal{E}(\mathcal{T}_h),\ i = 1,2.
$$

Theorem 3.4.3 (Weak form of Dirac structure on $\mathcal{T}_h$). *Let $\Omega \subset \mathbb{R}^n$ be an oriented connected polyhedral domain subdivided into $K \in \mathcal{T}_h$ non-overlapping polyhedral elements with boundary ∂K. Given the function spaces $\mathcal{F}(\mathcal{T}_h)$ and $\mathcal{E}(\mathcal{T}_h)$, defined in (3.47), together with the bilinear form (3.48). Given a face $f \in \Gamma_i$, there are two connected elements K_L and K_R such that $f = \partial K_L \cap \partial K_R$. We denote the input port variables $(u_b, \tilde{u}_b)$ at $\partial K_L \cap f$ by $(u_{b,L}, \tilde{u}_{b,L})$, and at $\partial K_R \cap f$ by $(u_{b,R}, \tilde{u}_{b,R})$, and the output port variables $(y_b, \tilde{y}_b)$ at $\partial K_L \cap f$ by $(y_{b,L}, \tilde{y}_{b,L})$, and at $\partial K_R \cap f$ by $(y_{b,R}, \tilde{y}_{b,R})$. Define the relation between the element input ports and output ports as*

$$\begin{aligned}
u_{b,L} &= (-1)^{q-1}\left(\left(\theta - \frac{1}{2}\right)y_{b,L} - \theta y_{b,R}\right), \\
u_{b,R} &= (-1)^{q}\left((1-\theta)y_{b,L} + \left(\theta - \frac{1}{2}\right)y_{b,R}\right), \\
\tilde{u}_{b,L} &= (-1)^{r+p-1}\left(\left(\frac{1}{2} - \theta\right)\tilde{y}_{b,L} + (\theta-1)\tilde{y}_{b,R}\right), \\
\tilde{u}_{b,R} &= (-1)^{r+p}\left(\theta\tilde{y}_{b,L} + \left(\frac{1}{2} - \theta\right)\tilde{y}_{b,R}\right),
\end{aligned} \tag{3.49}$$

with $\theta \in [0,1]$. Then the structure $\mathcal{D}_{\mathcal{T}} \subset \mathcal{F}(\mathcal{T}_h) \times \mathcal{E}(\mathcal{T}_h)$, stated in weak form as: if $\big((f_\eta, f_u, f_v, \tilde{u}_b, u_b), (e_\eta, e_u, e_v, y_b, \tilde{y}_b)\big) \in \mathcal{D}_{\mathcal{T}}$, for any test forms $(w_\eta, w_u, w_v) \in H\Lambda^{n-p}(\mathcal{T}_h) \times H^1\Lambda^{n-q}(\mathcal{T}_h) \times L^2\Lambda^{n-q}(\mathcal{T}_h)$, it satisfies

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} \int_K w_\eta \wedge f_\eta = &\sum_{K\in\mathcal{T}_h} \int_K w_\eta \wedge de_u \\
&+ \sum_{K\in\mathcal{T}_h} (-1)^{r+p} \int_{\partial K} \left(\frac{1}{2}\mathrm{tr}(e_u) + u_b\right) \wedge \mathrm{tr}(w_\eta),
\end{aligned} \tag{3.50a}$$

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} \int_K w_u \wedge f_u = &\sum_{K\in\mathcal{T}_h} \int_K \left((-1)^r w_u \wedge de_\eta - \frac{c}{h} w_u \wedge * e_v\right) \\
&+ \sum_{K\in\mathcal{T}_h} (-1)^{q} \int_{\partial K} \left(\frac{1}{2}\mathrm{tr}(e_\eta) + \tilde{u}_b\right) \wedge \mathrm{tr}(w_u),
\end{aligned} \tag{3.50b}$$

$$\sum_{K\in\mathcal{T}_h} \int_K w_v \wedge f_v = \sum_{K\in\mathcal{T}_h} \int_K \frac{c}{h} w_v \wedge * e_u, \tag{3.50c}$$

with output ports defined

$$\begin{pmatrix} y_b \\ \tilde{y}_b \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{q-1}\mathrm{tr} \\ (-1)^{r+p-1}\mathrm{tr} & 0 \end{pmatrix} \begin{pmatrix} e_\eta \\ e_u \end{pmatrix}, \tag{3.50d}$$

is a Dirac structure.

Proof. (1) First, we prove that (3.50) is a Dirac structure on each element $K \in \mathcal{T}_h$, which means that the function spaces $\mathcal{F}(\mathcal{T}_h)$ and $\mathcal{E}(\mathcal{T}_h)$ are

$$\begin{aligned}\mathcal{F}(K) :=& H^*\Lambda^p(K) \times H^{-1}\Lambda^q(K) \times L^2\Lambda^q(K) \times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial K)\\ &\times H^{\frac{1}{2}}\Lambda^{n-q}(\partial K),\\ \mathcal{E}(K) :=& H\Lambda^{n-p}(K) \times H^1\Lambda^{n-q}(K) \times L^2\Lambda^{n-q}(K) \times H^{\frac{1}{2}}\Lambda^{n-q}(\partial K)\\ &\times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial K),\end{aligned} \tag{3.51}$$

and the bilinear form on $\mathcal{F}(K) \times \mathcal{E}(K)$ is defined as

$$\begin{aligned}&\langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle_K\\ &= \int_K \big(e^1_\eta \wedge f^2_\eta + e^2_\eta \wedge f^1_\eta + e^1_u \wedge f^2_u + e^2_u \wedge f^1_u + e^1_v \wedge f^2_v + e^2_v \wedge f^1_v\big)\\ &+ \int_{\partial K} \big(u^1_b \wedge \tilde{y}^2_b + u^2_b \wedge \tilde{y}^1_b + \tilde{u}^1_b \wedge y^2_b + \tilde{u}^2_b \wedge y^1_b\big),\end{aligned} \tag{3.52}$$

where

$f^i = (f^i_\eta, f^i_u, f^i_v, \tilde{u}^i_b, u^i_b) \in \mathcal{F}(K), e^i = (e^i_\eta, e^i_u, e^i_v, y^i_b, \tilde{y}^i_b) \in \mathcal{E}(K), i = 1, 2.$

Furthermore, $\mathcal{D}_\mathcal{T}$ restricted to K is the structure $\mathcal{D}_K \subset \mathcal{F}(K) \times \mathcal{E}(K)$ that satisfies, for any test forms $(w_\eta, w_u, w_v) \in H\Lambda^{n-p}(K) \times H^1\Lambda^{n-q}(K) \times L^2\Lambda^{n-q}(K)$,

$$\begin{aligned}\int_K w_\eta \wedge f_\eta = \int_K w_\eta \wedge de_u&\\ + (-1)^{r+p} \int_{\partial K} \big(\frac{1}{2}\mathrm{tr}(e_u) + u_b\big) \wedge \mathrm{tr}(w_\eta),&\end{aligned} \tag{3.53a}$$

$$\begin{aligned}\int_K w_u \wedge f_u = \int_K \big((-1)^r w_u \wedge de_\eta - \frac{c}{h} w_u \wedge *e_v\big)&\\ + (-1)^q \int_{\partial K} \big(\frac{1}{2}\mathrm{tr}(e_\eta) + \tilde{u}_b\big) \wedge \mathrm{tr}(w_u),&\end{aligned} \tag{3.53b}$$

$$\int_K w_v \wedge f_v = \int_K \frac{c}{h} w_v \wedge *e_u, \tag{3.53c}$$

with output ports defined as

$$\begin{pmatrix} y_b \\ \tilde{y}_b \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{q-1}\mathrm{tr} \\ (-1)^{r+p-1}\mathrm{tr} & 0 \end{pmatrix} \begin{pmatrix} e_\eta \\ e_u \end{pmatrix}. \tag{3.54}$$

Next we prove that D_K is a Dirac structure.

(2) We begin with some useful relations. Taking $w_\eta = e^i_\eta, w_u = e^i_u, w_v = e^i_v, i = 1, 2$ in (3.53), we obtain that

$$\begin{aligned}
\int_K e^i_\eta \wedge f^j_\eta &= \int_K e^i_\eta \wedge de^j_u + (-1)^{r+p} \int_{\partial K} \big(\frac{1}{2}\mathrm{tr}(e^j_u) + u^j_b\big) \wedge \mathrm{tr}(e^i_\eta),\\
\int_K e^i_u \wedge f^j_u &= \int_K \big((-1)^r e^i_u \wedge de^j_\eta - \frac{c}{h} e^i_u \wedge * e^j_v\big)\\
&\quad + (-1)^q \int_{\partial K} \big(\frac{1}{2}\mathrm{tr}(e^j_\eta) + \tilde{u}^j_b\big) \wedge \mathrm{tr}(e^i_u),\\
\int_K e^i_v \wedge f^j_v &= \int_K \frac{c}{h} e^i_v \wedge * e^j_u,
\end{aligned}$$

with $j = 1, 2$.

(3) $\boldsymbol{\mathcal{D}_K \subseteq \mathcal{D}_K^\perp}$. From step (2), (3.53) and (3.54), for fixed $(f^1, e^1) \in \mathcal{D}_K$, and any $(f^2, e^2) \in \mathcal{D}_K$, using the integration by parts formula (3.22), we obtain that

$$\begin{aligned}
&\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle_K\\
&= \int_K \big(e^1_\eta \wedge de^2_u + e^2_\eta \wedge de^1_u + (-1)^r e^1_u \wedge de^2_\eta + (-1)^r e^2_u \wedge de^1_\eta\\
&- \frac{c}{h} e^1_u \wedge * e^2_v - \frac{c}{h} e^2_u \wedge * e^1_v + \frac{c}{h} e^1_v \wedge * e^2_u + \frac{c}{h} e^2_v \wedge * e^1_u\big)\\
&+ \int_{\partial K} \Big((-1)^{r+p}\big(\frac{1}{2}\mathrm{tr}(e^1_u) + u^1_b\big) \wedge \mathrm{tr}(e^2_\eta) + (-1)^{r+p}\big(\frac{1}{2}\mathrm{tr}(e^2_u) + u^2_b\big) \wedge \mathrm{tr}(e^1_\eta)\\
&+ (-1)^q\big(\frac{1}{2}\mathrm{tr}(e^1_\eta) + \tilde{u}^1_b\big) \wedge \mathrm{tr}(e^2_u) + (-1)^q\big(\frac{1}{2}\mathrm{tr}(e^2_\eta) + \tilde{u}^2_b\big) \wedge \mathrm{tr}(e^1_u)\\
&+ (-1)^{r+p-1} u^1_b \wedge \mathrm{tr}(e^2_\eta) + (-1)^{q-1} \tilde{u}^1_b \wedge \mathrm{tr}(e^2_u) + (-1)^{r+p-1} u^2_b \wedge \mathrm{tr}(e^1_\eta)\\
&+ (-1)^{q-1} \tilde{u}^2_b \wedge \mathrm{tr}(e^1_u)\Big)\\
&= \int_K (-1)^{q-1}\big(d(e^1_\eta \wedge e^2_u + e^2_\eta \wedge e^1_u)\big) + \int_{\partial K} (-1)^q \big(\mathrm{tr}(e^1_\eta) \wedge \mathrm{tr}(e^2_u)\\
&+ \mathrm{tr}(e^2_\eta) \wedge \mathrm{tr}(e^1_u)\big)\\
&= 0.
\end{aligned}$$

Hence, we have $(f^1, e^1) \in \mathcal{D}_K^\perp$, which implies that $\mathcal{D}_K \subset \mathcal{D}_K^\perp$.

(4) $\boldsymbol{\mathcal{D}_K^\perp \subseteq \mathcal{D}_K}$. Let $(f^1, e^1) \in \mathcal{D}_K^\perp$, then

$$\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle_K = 0, \quad \forall (f^2, e^2) \in \mathcal{D}_K,$$

which is equal to

$$\int_K (e_\eta^2 \wedge f_\eta^1 + e_\eta^1 \wedge de_u^2 + e_u^2 \wedge f_u^1 + (-1)^r e_u^1 \wedge de_\eta^2 - \frac{c}{h} e_u^1 \wedge * e_v^2 + e_v^2 \wedge f_v^1 + \frac{c}{h} e_v^1 \wedge * e_u^2) + \int_{\partial K} ((-1)^{r+p} (\frac{1}{2} \mathrm{tr}(e_u^2) + u_b^2) \wedge \mathrm{tr}(e_\eta^1) + (-1)^{r+p-1} u_b^1 \wedge \mathrm{tr}(e_\eta^2) + (-1)^q (\frac{1}{2} \mathrm{tr}(e_\eta^2) + \tilde{u}_b^2) \wedge \mathrm{tr}(e_u^1) + (-1)^{q-1} \tilde{u}_b^1 \wedge \mathrm{tr}(e_u^2) + u_b^2 \wedge \tilde{y}_b^1 + \tilde{u}_b^2 \wedge y_b^1 \big) = 0. \tag{3.55}$$

Using the integration by parts formula (3.22), we can rewrite (3.55) as

$$\int_K \Big(e_\eta^2 \wedge f_\eta^1 - e_\eta^2 \wedge de_u^1 + e_u^2 \wedge f_u^1 + (-1)^{r+1} e_u^2 \wedge de_\eta^1 + \frac{c}{h} e_u^2 \wedge * e_v^1 + e_v^2 \wedge f_v^1 - \frac{c}{h} e_v^2 \wedge * e_u^1\Big) + \int_{\partial K} \Big((-1)^{r+p-1} \big(\frac{1}{2} \mathrm{tr}(e_u^1) + u_b^1\big) \wedge \mathrm{tr}(e_\eta^2) + (-1)^{q-1} \big(\frac{1}{2} \mathrm{tr}(e_\eta^1) + \tilde{u}_b^1\big) \wedge \mathrm{tr}(e_u^2) + u_b^2 \wedge \big(\tilde{y}_b^1 + (-1)^{r+p} \mathrm{tr}(e_\eta^1)\big) + \tilde{u}_b^2 \wedge \big(y_b^1 + (-1)^q \mathrm{tr}(e_u^1)\big)\Big) = 0. \tag{3.56}$$

When additionally, assuming $\mathrm{tr}(e_\eta^2) = \mathrm{tr}(e_u^2) = u_b^2 = \tilde{u}_b^2 = 0$ at ∂K, this implies further

$$\int_K \Big(e_\eta^2 \wedge f_\eta^1 - e_\eta^2 \wedge de_u^1 + e_u^2 \wedge f_u^1 + (-1)^{r+1} e_u^2 \wedge de_\eta^1 + \frac{c}{h} e_u^2 \wedge * e_v^1 + e_v^2 \wedge f_v^1 - \frac{c}{h} e_v^2 \wedge * e_u^1\Big) = 0. \tag{3.57}$$

Equation (3.57) holds for any $(f^2, e^2) \in \mathcal{D}_K$, with e^2 having zero traces at ∂K. From (3.53)–(3.54) is clear that the pair (f^2, e^2) with $e^2 = (e_\eta^2, e_u^2, e_v^2, y_b^2, \tilde{y}_b^2) = (e_\eta^2, 0, 0, 0, 0)$, $f^2 = (f_\eta^2, f_u^2, f_v^2, \tilde{u}_b^2, u_b^2) = (0, (-1)^r de_\eta^2, 0, 0, 0)$ and $\mathrm{tr}(e_\eta^2) = 0$ is in $\mathcal{D}_K$. Thus (3.57) gives

$$\int_K e_\eta^2 \wedge f_\eta^1 = \int_K e_\eta^2 \wedge de_u^1. \tag{3.58}$$

The pair (f^2, e^2) with $e^2 = (0, e_u^2, 0, 0, 0)$, $f^2 = (de_u^2, 0, \frac{c}{h} * e_u^2, 0, 0)$ and $\mathrm{tr}(e_u^2) = 0$ belongs to $\mathcal{D}_K$, so (3.57) implies that

$$\int_K e_u^2 \wedge f_u^1 = \int_K \Big((-1)^r e_u^2 \wedge de_\eta^1 - \frac{c}{h} e_u^2 \wedge * e_v^1\Big). \tag{3.59}$$

Similarly, since the pair (f^2, e^2) with $e^2 = (0, 0, e_v^2, 0, 0)$ and $f^2 = (0, -\frac{c}{h} * e_v^2, 0, 0, 0)$ is in $\mathcal{D}_K$, it holds that

$$\int_K e_v^2 \wedge f_v^1 = \int_K \frac{c}{h} e_v^2 \wedge *e_u^1. \tag{3.60}$$

Hence when (3.57) holds for all $(f^2, e^2) \in \mathcal{D}_K$ with e^2's having zeros traces then (f^1, e^1) satisfies (3.58)–(3.60). Now the function space for e^2 corresponds exactly to the test forms $(w_\eta, w_u, w_v) \in H\Lambda^{n-p}(K) \times H^1\Lambda^{n-q}(K) \times L^2\Lambda^{n-q}(K)$, with $\mathrm{tr}(w_\eta) = \mathrm{tr}(w_u) = 0$ at ∂K. Hence

$$\int_K w_\eta \wedge f_\eta^1 = \int_K w_\eta \wedge de_u^1, \tag{3.61a}$$

$$\int_K w_u \wedge f_u^1 = \int_K \Big((-1)^r w_u \wedge de_\eta^1 - \frac{c}{h} w_u \wedge *e_v^1\Big), \tag{3.61b}$$

$$\int_K w_v \wedge f_v^1 = \int_K \frac{c}{h} w_v \wedge *e_u^1. \tag{3.61c}$$

The third equality is (3.53c), and (3.61a) and (3.61b) equal (3.53a) and (3.53b) respectively, without the ∂K term. So it remains to show the extra terms in (3.53a) and (3.53b). For any test forms $(w_\eta, w_u) \in H\Lambda^{n-p}(K) \times H^1\Lambda^{n-q}(K)$, we have that there exist $T_1 \in H^{\frac{1}{2}}\Lambda^{n-q}(\partial K)$ and $T_2 \in H^{-\frac{1}{2}}\Lambda^{n-p}(\partial K)$, which are linear mapping of e^1, such that

$$\int_K w_\eta \wedge f_\eta^1 - \int_K w_\eta \wedge de_u^1 = \int_{\partial K} \mathrm{tr}(w_\eta) \wedge T_1, \tag{3.62a}$$

$$\int_K w_u \wedge f_u^1 - \int_K \Big((-1)^r w_u \wedge de_\eta^1 - \frac{c}{h} w_u \wedge *e_v^1\Big) = \int_{\partial K} \mathrm{tr}(w_u) \wedge T_2. \tag{3.62b}$$

Taking $w_\eta = e_\eta^2$, $w_u = e_u^2$ and $w_v = e_v^2$ in (3.61c)–(3.62), and substituting it into (3.56), we have that

$$\begin{aligned}\int_{\partial K} \Big(&\mathrm{tr}(e_\eta^2) \wedge T_1 + \mathrm{tr}(e_u^2) \wedge T_2 + (-1)^{r+p-1}\big(\frac{1}{2}\mathrm{tr}(e_u^1) + u_b^1\big) \wedge \mathrm{tr}(e_\eta^2) \\ &+ (-1)^{q-1}\big(\frac{1}{2}\mathrm{tr}(e_\eta^1) + \tilde{u}_b^1\big) \wedge \mathrm{tr}(e_u^2) + u_b^2 \wedge \big(\tilde{y}_b^1 + (-1)^{r+p}\mathrm{tr}(e_\eta^1)\big) \\ &+ \tilde{u}_b^2 \wedge \big(y_b^1 + (-1)^q \mathrm{tr}(e_u^1)\big)\Big) = 0,\end{aligned} \tag{3.63}$$

which equals

$$\int_{\partial K} \Big(\mathrm{tr}(e_\eta^2) \wedge \big(T_1 + (-1)^{q-1}\frac{1}{2}\mathrm{tr}(e_u^1) + (-1)^{q-1}u_b^1\big)$$

$$+ \operatorname{tr}(e_u^2) \wedge \big(T_2 + (-1)^{r+p-1}\frac{1}{2}\operatorname{tr}(e_\eta^1) + (-1)^{r+p-1}\tilde{u}_b^1\big)\Big) \tag{3.64}$$
$$+ u_b^2 \wedge \big(\tilde{y}_b^1 + (-1)^{r+p}\operatorname{tr}(e_\eta^1)\big) + \tilde{u}_b^2 \wedge \big(y_b^1 + (-1)^q \operatorname{tr}(e_u^1)\big)\Big) = 0.$$

Since $(e_\eta^2, e_u^2, u_b^2, \tilde{u}_b^2) \in H\Lambda^{n-p}(K) \times H^1\Lambda^{n-q}(K) \times H^{\frac{1}{2}}\Lambda^{n-q}(\partial K) \times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial K)$ are free, the integral can be zero only if

$$\begin{aligned}
T_1 =&(-1)^q\big(\frac{1}{2}\operatorname{tr}(e_u^1) + u_b^1\big),\\
T_2 =&(-1)^{r+p}\big(\frac{1}{2}\operatorname{tr}(e_\eta^1) + \tilde{u}_b^1\big),\\
y_b^1 =&(-1)^{q-1}\operatorname{tr}(e_u^1),\\
\tilde{y}_b^1 =&(-1)^{r+p-1}\operatorname{tr}(e_\eta^1).
\end{aligned} \tag{3.65}$$

Substituting (3.65) into (3.62), combining with (3.53c), we obtain that $(f^1, e^1) \in \mathcal{D}_K$, hence $\mathcal{D}_K^\perp \subseteq \mathcal{D}_K$. Combining with (3), we have that $\mathcal{D}_K = \mathcal{D}_K^\perp$, i.e., D_K is a Dirac structure.

(5) From steps (1)–(4), we know that $\mathcal{D}_K$ defined via (3.53)–(3.54) on element $K \in \mathcal{T}_h$ is a Dirac structure. After summation over each element $K \in \mathcal{T}_h$, we obtain that with the function spaces (3.47), and the bilinear form

$$\begin{aligned}
&\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle_{\mathcal{T}_h}^*\\
:=& \sum_{K \in \mathcal{T}_h} \int_K \big(e_\eta^1 \wedge f_\eta^2 + e_\eta^2 \wedge f_\eta^1 + e_u^1 \wedge f_u^2 + e_u^2 \wedge f_u^1 + e_v^1 \wedge f_v^2 + e_v^2 \wedge f_v^1\big)\\
&+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \big(u_b^1 \wedge \tilde{y}_b^2 + u_b^2 \wedge \tilde{y}_b^1 + \tilde{u}_b^1 \wedge y_b^2 + \tilde{u}_b^2 \wedge y_b^1\big),
\end{aligned} \tag{3.66}$$

(3.50) is a Dirac structure. It remain to show that under the bilinear form (3.48) and the interconnection relations (3.49), (3.50) is still a Dirac structure.

For any interior face $f \in \Gamma_i$, there are two connected elements K_L and K_R such that $f = \partial K_L \cap \partial K_R$. Define the interconnection Dirac structure between element K_L and K_R at the common boundary f by choosing

$$\begin{aligned}
u_{b,L} = f_L, \quad \tilde{u}_{b,L} = \tilde{f}_L,\\
y_{b,L} = e_L, \quad \tilde{y}_{b,L} = \tilde{e}_L,
\end{aligned} \tag{3.67}$$

and

$$\begin{aligned} u_{b,R} &= f_R, \quad \tilde{u}_{b,R} = \tilde{f}_R, \\ y_{b,R} &= e_R, \quad \tilde{y}_{b,R} = \tilde{e}_R, \end{aligned} \tag{3.68}$$

in (3.43), then we obtain (3.49). It is clear that

$$\langle u_{b,L} | \tilde{y}_{b,L} \rangle = \langle f_L | \tilde{e}_L \rangle, \quad \langle u_{b,R} | \tilde{y}_{b,R} \rangle = \langle f_R | \tilde{e}_R \rangle, \tag{3.69}$$

since we consider the integration on f from the opposite direction. Using (3.42) and Lemma 3.4.2, with (3.67) and (3.68), the power conservation at the interior face f can be expressed as

$$\begin{aligned} &\int_f \big(y^1_{b,L} \wedge \tilde{u}^2_{b,L} + y^2_{b,L} \wedge \tilde{e}^1_{b,L} + \tilde{y}^1_{b,L} \wedge u^2_{b,L} + \tilde{y}^2_{b,L} \wedge u^1_{b,L} - y^1_{b,R} \wedge \tilde{u}^2_{b,R} \\ &- y^2_{b,R} \wedge \tilde{e}^1_{b,R} - \tilde{y}^1_{b,R} \wedge u^2_{b,R} - \tilde{y}^2_{b,R} \wedge u^1_{b,R} \big) = 0. \end{aligned} \tag{3.70}$$

Hence the bilinear form (3.66) can written as

$$\begin{aligned} &\langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle^*_{\mathcal{T}_h} \\ &= \sum_{K \in \mathcal{T}_h} \int_K \big(e^1_\eta \wedge f^2_\eta + e^2_\eta \wedge f^1_\eta + e^1_u \wedge f^2_u + e^2_u \wedge f^1_u + e^1_v \wedge f^2_v + e^2_v \wedge f^1_v \big) \\ &+ \sum_{f \in \Gamma_b} \int_f \big(u^1_b \wedge \tilde{y}^2_b + u^2_b \wedge \tilde{y}^1_b + \tilde{u}^1_b \wedge y^2_b + \tilde{u}^2_b \wedge y^1_b \big) \\ &= \langle\!\langle (f^1, e^1), (f^2, e^2) \rangle\!\rangle_{\mathcal{T}_h}. \end{aligned}$$

In conclusion, due to the compositionality property of the Dirac structure [21, Theorem 3], (3.49)–(3.50) is still a Dirac structure defined on the function spaces (3.47) and the bilinear form (3.48).

□

Remark 3.4.4. *In [50], the strong form of the Dirac structure for equation (3.6) with $c = 0$ is presented for the finite-dimensional function spaces.*

Using the Hamiltonian $H : L^2\Lambda^p(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \to \mathbb{R}$ for the polyhedral domain Ω, which is given as

$$H(\eta, u, v) = \sum_{K \in \mathcal{T}_h} \frac{1}{2} \int_K \big(g\eta \wedge (*\eta) + hu \wedge (*u) + hv \wedge (*v) \big), \tag{3.71}$$

we obtain the following result.

Corollary 3.4.1. *Let $\Omega \subset \mathbb{R}^n$ be an oriented connected polyhedral domain subdivided into $K \in \mathcal{T}_h$ non-overlapping polyhedral elements with boundary ∂K. The weak form of distributed parameter port-Hamiltonian system for the linear dynamical system (3.6) with respect to the state space variables $(\eta, u, v) \in L^2\Lambda^p(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h)$, input variables $(\tilde{u}_b, u_b) \in H^{-\frac{1}{2}}\Lambda^{n-p}(\partial\mathcal{T}_h) \times H^{\frac{1}{2}}\Lambda^{n-q}(\partial\mathcal{T}_h)$ (3.49), Dirac structure (3.50), and Hamiltonian (3.71), is given as*

$$\sum_{K\in\mathcal{T}_h}\int_K w_\eta \wedge \frac{\partial \eta}{\partial t} = -\sum_{K\in\mathcal{T}_h}\int_K w_\eta \wedge d(\frac{\delta H}{\delta u}) + \sum_{K\in\mathcal{T}_h}(-1)^{r+p-1}\int_{\partial K}\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H}{\delta u}) + u_b\Big)\wedge \mathrm{tr}(w_\eta), \tag{3.72a}$$

$$\sum_{K\in\mathcal{T}_h}\int_K w_u \wedge \frac{\partial u}{\partial t} = \sum_{K\in\mathcal{T}_h}\int_K \Big((-1)^{r-1} w_u \wedge d(\frac{\delta H}{\delta \eta}) + \frac{c}{h} w_u \wedge *\frac{\delta H}{\delta v}\Big) + \sum_{K\in\mathcal{T}_h}(-1)^{q-1}\int_{\partial K}\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H}{\delta \eta}) + \tilde{u}_b\Big)\wedge \mathrm{tr}(w_u), \tag{3.72b}$$

$$\sum_{K\in\mathcal{T}_h}\int_K w_v \wedge \frac{\partial v}{\partial t} = -\sum_{K\in\mathcal{T}_h}\int_K \frac{c}{h} w_v \wedge *\frac{\delta H}{\delta u}, \tag{3.72c}$$

for any test forms $(w_\eta, w_u, w_v) \in H\Lambda^{n-p}(\mathcal{T}_h) \times H^1\Lambda^{n-q}(\mathcal{T}_h) \times L^2\Lambda^{n-q}(\mathcal{T}_h)$, together with the output variables $(y_b, \tilde{y}_b) \in H^{\frac{1}{2}}\Lambda^{n-q}(\partial\mathcal{T}_h) \times H^{-\frac{1}{2}}\Lambda^{n-p}(\partial\mathcal{T}_h)$

$$\begin{pmatrix} y_b \\ \tilde{y}_b \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{q-1}\mathrm{tr} \\ (-1)^{r+p-1}\mathrm{tr} & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta H}{\delta \eta} \\ \frac{\delta H}{\delta u} \end{pmatrix}, \tag{3.73}$$

where $(u_b, \tilde{u}_b)$ and $(y_b, \tilde{y}_b)$ are the boundary ports, which are determined by the specific boundary conditions at Γ_b.

Remark 3.4.5. *Considering the domain $\Omega = K$, and choosing the input ports as*

$$u_b = -\frac{1}{2}\mathrm{tr}(\frac{\delta H}{\delta u}), \quad \tilde{u}_b = -\frac{1}{2}\mathrm{tr}(\frac{\delta H}{\delta \eta}),$$

formulation (3.72) is exactly the weak form of (3.6) using (3.34).

3.4.4 Poisson bracket on tessellation $\mathcal{T}_h$

In this section, following the definition presented in [93, Section 4.1], we define the pseudo-Poisson bracket associated with the Dirac structure (3.50).

We prove then that the pseudo-Poisson bracket is actually a Poisson bracket which is bilinear, skew-symmetric and satisfies the Jacobi identity. First, we define the skew-symmetric bilinear form

$$\begin{aligned}[e^1, e^2] := \langle e^2 | f^1 \rangle = &\sum_{K\in\mathcal{T}_h} \int_K (e^2_\eta \wedge f^1_\eta + e^2_u \wedge f^1_u + e^2_v \wedge f^1_v) \\ &+ \sum_{f\in\Gamma_b} \int_f \big(u^2_b \wedge \tilde{y}^1_b + \tilde{u}^2_b \wedge y^1_b\big),\end{aligned} \tag{3.74}$$

where $e^1, e^2 \in \mathcal{E}_{adm} := \{e \in \mathcal{E}(\mathcal{T}_h) \mid \exists f \in \mathcal{F}(\mathcal{T}_h) \text{ such that } (f, e) \in \mathcal{D}_\mathcal{T}\}$.

For any functionals $F, G : L^2\Lambda^p(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \to \mathbb{R}$, the functional derivatives with respect to (η, u, v) are $\frac{\delta F}{\delta \eta}, \frac{\delta G}{\delta \eta} \in H\Lambda^{n-p}(\mathcal{T}_h)$, $\frac{\delta F}{\delta u}, \frac{\delta G}{\delta u} \in H^1\Lambda^{n-q}(\mathcal{T}_h)$ and $\frac{\delta F}{\delta v}, \frac{\delta G}{\delta v} \in L^2\Lambda^{n-q}(\mathcal{T}_h)$, separately. The pseudo-Poisson bracket $\{\cdot,\cdot\}(\eta, u, v)$ associated to the Dirac structure (3.50) is defined as

$$\begin{aligned}&\{F, G\} := [\delta F, \delta G] \\ &= \sum_{K\in\mathcal{T}_h} \int_K \left(\frac{\delta G}{\delta \eta} \wedge d\frac{\delta F}{\delta u} + (-1)^r \frac{\delta G}{\delta u} \wedge d\frac{\delta F}{\delta \eta} - \frac{c}{h}\frac{\delta G}{\delta u} \wedge *\frac{\delta F}{\delta v} + \frac{c}{h}\frac{\delta G}{\delta v} \wedge *\frac{\delta F}{\delta u} \right) \\ &+ \sum_{K\in\mathcal{T}_h} \int_{\partial K} \bigg((-1)^{r+p}\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta F}{\delta u}) + u^1_b\Big) \wedge \mathrm{tr}(\frac{\delta G}{\delta \eta}) + (-1)^q \Big(\frac{1}{2}\mathrm{tr}(\frac{\delta F}{\delta \eta}) + \tilde{u}^1_b\Big) \\ &\wedge \mathrm{tr}(\frac{\delta G}{\delta u}) \bigg) + \sum_{f\in\Gamma_b} \int_f \left((-1)^{r+p-1} u^2_b \wedge \mathrm{tr}(\frac{\delta F}{\delta \eta}) + (-1)^{q-1}\tilde{u}^2_b \wedge \mathrm{tr}(\frac{\delta F}{\delta u}) \right),\end{aligned} \tag{3.75}$$

with for $f \in \Gamma_i$,

$$\begin{aligned}u^j_{b,L} =& (-1)^{q-1} \left(\Big(\theta - \frac{1}{2}\Big) y^j_{b,L} - \theta y^j_{b,R} \right), \\ u^j_{b,R} =& (-1)^q \left((1-\theta) y^j_{b,L} + \Big(\theta - \frac{1}{2}\Big) y^j_{b,R} \right), \\ \tilde{u}^j_{b,L} =& (-1)^{r+p-1} \left(\Big(\frac{1}{2} - \theta\Big) \tilde{y}^j_{b,L} + (\theta - 1)\tilde{y}^j_{b,R} \right), \\ \tilde{u}^j_{b,R} =& (-1)^{r+p} \left(\theta \tilde{y}^j_{b,L} + \Big(\frac{1}{2} - \theta\Big)\tilde{y}^j_{b,R} \right),\end{aligned} \tag{3.76}$$

with $j = 1, 2$, $\theta \in [0, 1]$, and for $f \in \Gamma$,

$$y^1_b = (-1)^{q-1}\mathrm{tr}(\frac{\delta F}{\delta u}), \quad y^2_b = (-1)^{q-1}\mathrm{tr}(\frac{\delta G}{\delta u}),$$

$$\tilde{y}_b^1 = (-1)^{r+p-1}\mathrm{tr}(\frac{\delta F}{\delta \eta}), \quad \tilde{y}_b^2 = (-1)^{r+p-1}\mathrm{tr}(\frac{\delta G}{\delta \eta}). \tag{3.77}$$

Lemma 3.4.6 (Skew-symmetry). *For any functionals $F, G : L^2\Lambda^p(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \to \mathbb{R}$, the bracket $\{F, G\}(\eta, u, v)$ defined in (3.75) is skew-symmetric.*

Proof. If $(e^1, e^2) \in \mathcal{E}_{adm}$, then there exist $f^1, f^2 \in \mathcal{F}(\mathcal{T}_h)$ such that (f^1, e^1), $(f^2, e^2) \in \mathcal{D}_{\mathcal{T}}$. Since $\mathcal{D}_{\mathcal{T}} = \mathcal{D}_{\mathcal{T}}^{\perp}$, and using (3.23), we have that

$$\langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle = \langle f^1 | e^2 \rangle + \langle f^2 | e^1 \rangle = [e^2, e^1] + [e^1, e^2] = 0.$$

Hence, for any functionals $F, G : L^2\Lambda^p(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \to \mathbb{R}$, we obtain

$$[\delta F, \delta G] + [\delta G, \delta F] = 0,$$

which implies that

$$\{F, G\} = -\{G, F\}.$$

□

Lemma 3.4.7 (Jacobi identity). *For any functionals $F, G, J : L^2\Lambda^p(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \to \mathbb{R}$, the bracket $\{F, G\}(\eta, u, v)$ defined in (3.75) with $u_b^j = \tilde{u}_b^j = 0$, $j = 1, 2$ at Γ_b satisfies the Jacobi identity, i.e.*

$$\{\{F, G\}, J\} + \{\{G, J\}, F\} + \{\{J, F\}, G\} = 0.$$

Proof. The proof is given in Appendix 3.A. □

Remark 3.4.8. *Using the product rule for functional derivatives, see Appendix 2.B, it is straightforward to show that the bracket $\{F, G\}$ satisfies the Leibniz rule.*

The bilinearity, skew-symmetry and Jacobi identity immediately imply the following theorem:

Theorem 3.4.9. *For any functionals $F, G : L^2\Lambda^p(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \times L^2\Lambda^q(\mathcal{T}_h) \to \mathbb{R}$, the pseudo-Poisson bracket $\{F, G\}(\eta, u, v)$ (3.75) is a Poisson bracket.*

3.4.5 Port-Hamiltonian formulation on tessellation $\mathcal{T}_h$

In this section, we express the port-Hamiltonian formulation (3.50) on the tessellation $\mathcal{T}_h$ in terms of the Poisson bracket (3.75). Then applying the skew-symmetry of the Poisson bracket (3.75), we show that the energy exchange of this system is controlled by the ports on the domain boundary Γ_b.

Theorem 3.4.10 (**Energy exchange**). *Let $\mathcal{T}_h$ be a tessellation of the oriented connected polyhedral domain $\Omega \subset \mathbb{R}^n$ with shape-regular polyhedral elements K. Given an arbitrary functional $F(\eta, u, v) : L^2\Lambda^p(\Omega) \times L^2\Lambda^q(\Omega) \times L^2\Lambda^q(\Omega) \to \mathbb{R}$, the Hamiltonian H (3.71), Poisson bracket (3.75) and the input ports $(u_b, \tilde{u}_b)$ at Γ_i (3.49), the weak form of distributed-parameter port-Hamiltonian formulation (3.72)–(3.73) can be expressed as*

$$\frac{dF}{dt} = \{F, H\} + \sum_{f \in \Gamma_b} \int_f \big(u_b^1 \wedge \tilde{y}_b^2 + \tilde{u}_b^1 \wedge y_b^2\big). \tag{3.78}$$

Proof. Taking $w_\eta = \frac{\delta F}{\delta \eta}$, $w_u = \frac{\delta F}{\delta u}$ and $w_v = \frac{\delta F}{\delta v}$ in (3.72), we have that

$$\begin{aligned}
\sum_{K \in \mathcal{T}_h} \int_K \frac{\delta F}{\delta \eta} \wedge \frac{\partial \eta}{\partial t} =& - \sum_{K \in \mathcal{T}_h} \int_K \frac{\delta F}{\delta \eta} \wedge d\big(\frac{\delta H}{\delta u}\big) \\
&+ \sum_{K \in \mathcal{T}_h} (-1)^{r+p-1} \int_{\partial K} \Big(\frac{1}{2}\mathrm{tr}\big(\frac{\delta H}{\delta u}\big) + u_b\Big) \wedge \mathrm{tr}\big(\frac{\delta F}{\delta \eta}\big), \\
\sum_{K \in \mathcal{T}_h} \int_K \frac{\delta F}{\delta u} \wedge \frac{\partial u}{\partial t} =& \sum_{K \in \mathcal{T}_h} \int_K \Big((-1)^{r-1} \frac{\delta F}{\delta u} \wedge d\big(\frac{\delta H}{\delta \eta}\big) + \frac{c}{h} \frac{\delta F}{\delta u} \wedge * \frac{\delta H}{\delta v}\Big) \\
&+ \sum_{K \in \mathcal{T}_h} (-1)^{q-1} \int_{\partial K} \Big(\frac{1}{2}\mathrm{tr}\big(\frac{\delta H}{\delta \eta}\big) + \tilde{u}_b\Big) \wedge \mathrm{tr}\big(\frac{\delta F}{\delta u}\big), \\
\sum_{K \in \mathcal{T}_h} \int_K \frac{\delta F}{\delta v} \wedge \frac{\partial v}{\partial t} =& - \sum_{K \in \mathcal{T}_h} \int_K \frac{c}{h} \frac{\delta F}{\delta v} \wedge * \frac{\delta H}{\delta u}.
\end{aligned}$$

Combining with the chain rule

$$\frac{dF}{dt} = \sum_{K \in \mathcal{T}_h} \int_K \Big(\frac{\delta F}{\delta \eta} \wedge \frac{\partial \eta}{\partial t} + \frac{\delta F}{\delta u} \wedge \frac{\partial u}{\partial t} + \frac{\delta F}{\delta v} \wedge \frac{\partial v}{\partial t}\Big),$$

we obtain that

$$\begin{aligned}
\frac{dF}{dt} =& \sum_{K \in \mathcal{T}_h} \int_K \Big(- \frac{\delta F}{\delta \eta} \wedge d\big(\frac{\delta H}{\delta u}\big) + (-1)^{r-1} \frac{\delta F}{\delta u} \wedge d\big(\frac{\delta H}{\delta \eta}\big) + \frac{c}{h} \frac{\delta F}{\delta u} \wedge * \frac{\delta H}{\delta v} \\
&- \frac{c}{h} \frac{\delta F}{\delta v} \wedge * \frac{\delta H}{\delta u}\Big) + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \Big((-1)^{r+p-1}\Big(\frac{1}{2}\mathrm{tr}\big(\frac{\delta H}{\delta u}\big) + u_b\Big) \wedge \mathrm{tr}\big(\frac{\delta F}{\delta \eta}\big) \\
&+ (-1)^{q-1}\Big(\frac{1}{2}\mathrm{tr}\big(\frac{\delta H}{\delta \eta}\big) + \tilde{u}_b\Big) \wedge \mathrm{tr}\big(\frac{\delta F}{\delta u}\big)\Big).
\end{aligned} \tag{3.79}$$

Taking $G = H$ in the bracket $\{\cdot,\cdot\}$ stated in (3.75), and using Lemma 3.4.6, we have that

$$\begin{aligned}\frac{dF}{dt} =& -\{H,F\} + \sum_{f\in\Gamma_b}\int_f \Big((-1)^{r+p-1}u_b^1 \wedge \mathrm{tr}(\frac{\delta H}{\delta \eta})\\ &+(-1)^{q-1}\tilde{u}_b^1 \wedge \mathrm{tr}(\frac{\delta H}{\delta u})\Big).\end{aligned} \tag{3.80}$$

Using the skew-symmetry of the bracket $\{H, F\}$, together with the output ports (3.73), we obtain (3.78). □

Using the skew-symmetry of the Poisson-bracket stated in Lemma 3.4.6 we have $\{H, H\} = 0$, hence (3.78) gives

$$\frac{dH}{dt} = \sum_{f\in\Gamma_b}\int_f \big(u_b \wedge \tilde{y}_b + \tilde{u}_b \wedge y_b\big). \tag{3.81}$$

3.5 Discrete port-Hamiltonian system

In this section, we will first give the DG discretization for (3.78), which guarantees that the change in system energy only occurs at the domain boundaries Γ_b. We derive a PHDG discretization for the port-Hamiltonian formulation (3.72). Error estimates for the PHDG discretization will also be discussed.

3.5.1 Notations

Let $P_r(K) = \oplus_{k=0}^r \mathcal{H}_k(K)$ be the space of polynomials of total degree at most r on $K \in \mathcal{T}_h$ with K a simplicial element, where $\mathcal{H}_k(K)$ denotes the space of homogeneous polynomials of degree k on K. The spaces of polynomial differential forms $P_r\Lambda^k(K)$ and $\mathcal{H}_r\Lambda^k(K)$ are the spaces whose coefficient functions belong to $P_r(K)$ and $\mathcal{H}_r(K)$, respectively.

Using the polynomial differential form spaces $P_r\Lambda^k(K)$, $\mathcal{H}_r\Lambda^k(K)$ and the Koszul differential operator κ [4, 7], the polynomial spaces on cubical meshes are defined as [6, 9]

$$S_r\Lambda^k(K) := P_r\Lambda^k(K) + \mathcal{J}_r\Lambda^k(K) + d\mathcal{J}_{r+1}\Lambda^{k-1}(K),$$

where the space of shape functions $\mathcal{J}_r\Lambda^k$ is given as

$$\mathcal{J}_r\Lambda^k(K) = \sum_{l\geq 1}\kappa\mathcal{H}_{r+l-1}\Lambda^{k+1}(K).$$

The polynomial spaces of differential forms $P_r\Lambda^k(K)$ and $S_r\Lambda^k(K)$ are all de Rham complexes [4, 9].

The broken polynomial differential form spaces are denoted by

$$P_r\Lambda^k(\mathcal{T}_h) := \{\lambda_h \in L^2\Lambda^k(\Omega) \mid \lambda_h|_K \in P_r\Lambda^k(K), \forall K \in \mathcal{T}_h, K \text{ simplex}\},$$
$$S_r\Lambda^k(\mathcal{T}_h) := \{\lambda_h \in L^2\Lambda^k(\Omega) \mid \lambda_h|_K \in S_r\Lambda^k(K), \forall K \in \mathcal{T}_h, K \text{ hypercube}\}.$$

The broken polynomial differential form spaces at element faces $\partial\mathcal{T}_h$ are denoted by

$$P_r\Lambda^k(\partial\mathcal{T}_h) :=\{\lambda_h \in L^2\Lambda^k(\Gamma) \mid \lambda_h|_{\partial K} \in P_r\Lambda^k(\partial K), \forall K \in \mathcal{T}_h, K \text{ simplex}\},$$
$$S_r\Lambda^k(\partial\mathcal{T}_h) :=\{\lambda_h \in L^2\Lambda^k(\Gamma) \mid \lambda_h|_{\partial K} \in S_r\Lambda^k(\partial K), \forall K \in \mathcal{T}_h, K \text{ hypercube}\},$$

with Γ the set of all faces in $\mathcal{T}_h$. The space of broken polynomial differential forms $P_r\Lambda^k(\mathcal{T}_h)$ is used for simplicial elements and $S_r\Lambda^k(\mathcal{T}_h)$ for hypercube elements. We will use $V^k_{h,r}$ to represent, depending on the element type, either $P_r\Lambda^k(\mathcal{T}_h)$ or $S_r\Lambda^k(\mathcal{T}_h)$, the dual space of $V^k_{h,r}$ is denoted by $\widehat{V}^{n-k}_{h,r}$ [9, 4, 50]. We use $W^k_{h,r}$ to represent $P_r\Lambda^k(\partial\mathcal{T}_h)$ or $S_r\Lambda^k(\partial\mathcal{T}_h)$ and $\widehat{W}^{n-k-1}_{h,r}$ the dual space of $W^k_{h,r}$.

In the remainder, we will use that a polyhedral element can be subdivided into simplicial elements [35].

3.5.2 Discrete Poisson bracket and port-Hamiltonian formulation

Using the discrete approximation $(\eta_h, u_h, v_h) \in V^p_{h,r} \times \widehat{V}^q_{h,r} \times V^q_{h,r}$ to the variables $(\eta, u, v) \in L^2\Lambda^p(\Omega) \times L^2\Lambda^q(\Omega) \times L^2\Lambda^q(\Omega)$. The Hamiltonian H (3.32) is approximated as $H_h(\eta_h, u_h, v_h) : V^p_{h,r} \times \widehat{V}^q_{h,r} \times V^q_{h,r}$ with

$$H_h(\eta_h, u_h, v_h) = \sum_{K\in\mathcal{T}_h} \frac{1}{2}\int_K \big(g\eta_h\wedge(*\eta_h)+hu_h\wedge(*u_h)+hv_h\wedge(*v_h)\big). \quad (3.82)$$

Since $V^p_{h,r} \subset H\Lambda^p(\mathcal{T}_h)$ and $\widehat{V}^q_{h,r} \subset H^1\Lambda^q(\mathcal{T}_h)$, for arbitrary discrete functionals $F_h : V^p_{h,r} \times \widehat{V}^q_{h,r} \times V^q_{h,r} \to \mathbb{R}$, the discrete weak formulation of the port-Hamiltonian formulation (3.79) is given by

$$\frac{dF_h}{dt} = \sum_{K\in\mathcal{T}_h} \int_K \Big(-\frac{\delta F_h}{\delta\eta_h} \wedge d\frac{\delta H_h}{\delta u_h} + (-1)^{r-1}\frac{\delta F_h}{\delta u_h} \wedge d\frac{\delta H_h}{\delta\eta_h} + \frac{c}{h}\frac{\delta F_h}{\delta u_h} \wedge *\frac{\delta H_h}{\delta v_h}$$
$$- \frac{c}{h}\frac{\delta F_h}{\delta v_h} \wedge *\frac{\delta H_h}{\delta u_h}\Big) + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \Big((-1)^{r+p-1}\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta u_h}) + u^h_b\Big) \wedge \mathrm{tr}(\frac{\delta F_h}{\delta\eta_h})$$

$$+ (-1)^{q-1}\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta \eta_h}) + \tilde{u}_b^h\Big) \wedge \mathrm{tr}(\frac{\delta F_h}{\delta u_h})\Big), \tag{3.83}$$

where $(u_b^h, \tilde{u}_b^h) \in W_{h,r}^{n-q} \times \widehat{W}_{h,r}^{n-p}$ are the discrete variables that approximate $(u_b, \tilde{u}_b)$.

Remark 3.5.1. *Given any two polyhedral elements K_1 and K_2 with $\partial K_1 \cap \partial K_2 = f$ with $f \in \Gamma_i$. We note that for any k-form $\lambda \in V_{h,r}^k$ or $\widehat{V}_{h,r}^k$, the trace of λ on element K_1 connected to face f is in general not equal to the trace on element K_2 connected to the opposite side of f due to the discontinuity of λ at f. We denote these variables, respectively, $\mathrm{tr}(\lambda)|_{f,L}$ and $\mathrm{tr}(\lambda)|_{f,R}$.*

For arbitrary functionals $F_h, G_h : V_{h,r}^p \times \widehat{V}_{h,r}^q \times V_{h,r}^q \to \mathbb{R}$ the discrete Poisson bracket $\{\cdot,\cdot\}_p : V_{h,r}^p \times \widehat{V}_{h,r}^q \times V_{h,r}^q \to \mathbb{R}$ on $\mathcal{T}_h$ is obtained from the Poisson bracket $\{\cdot,\cdot\}$ defined on $\mathcal{T}_h$ (3.75),

$$\begin{aligned}
&\{F_h, G_h\}_p(\eta_h, u_h, v_h) \\
&= \sum_{K\in\mathcal{T}_h} \int_K \Big(\frac{\delta G_h}{\delta \eta_h} \wedge d\frac{\delta F_h}{\delta u_h} + (-1)^r \frac{\delta G_h}{\delta u_h} \wedge d\frac{\delta F_h}{\delta \eta_h} - \frac{c}{h}\frac{\delta G_h}{\delta u_h} \wedge *\frac{\delta F_h}{\delta v_h} \\
&+ \frac{c}{h}\frac{\delta G_h}{\delta v_h} \wedge *\frac{\delta F_h}{\delta u_h}\Big) + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \Big((-1)^{r+p}\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta F_h}{\delta u_h}) + u_b^{h,1}\Big) \wedge \mathrm{tr}(\frac{\delta G_h}{\delta \eta_h}) \\
&+ (-1)^q\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta F_h}{\delta \eta_h}) + \tilde{u}_b^{h,1}\Big) \wedge \mathrm{tr}(\frac{\delta G_h}{\delta u_h})\Big) + \sum_{f\in\Gamma_b} \int_f \Big((-1)^{r+p-1} u_b^{h,2} \wedge \mathrm{tr}(\frac{\delta F_h}{\delta \eta_h}) \\
&+ (-1)^{q-1}\tilde{u}_b^{h,2} \wedge \mathrm{tr}(\frac{\delta F_h}{\delta u_h})\Big),
\end{aligned} \tag{3.84}$$

where given a face $f \in \Gamma_i$, there are two connected elements K_L and K_R such that $f = \partial K_L \cap \partial K_R$. We denote the trace of $(u_b^h, \tilde{u}_b^h)$ at $\partial K_L \cap f$ and $\partial K_R \cap f$ by $(u_{b,L}^h, \tilde{u}_{b,L}^h)$ and $(u_{b,R}^h, \tilde{u}_{b,R}^h)$, respectively. At interior faces Γ_i, we choose the discrete element input ports as

$$\begin{aligned}
u_{b,L}^{h,j} &= (-1)^{q-1}\left(\Big(\theta - \frac{1}{2}\Big) y_{b,L}^{h,j} - \theta y_{b,R}^{h,j}\right), \\
u_{b,R}^{h,j} &= (-1)^{q}\left((1-\theta) y_{b,L}^{h,j} + \Big(\theta - \frac{1}{2}\Big) y_{b,R}^{h,j}\right), \\
\tilde{u}_{b,L}^{h,j} &= (-1)^{r+p-1}\left(\Big(\frac{1}{2} - \theta\Big) \tilde{y}_{b,L}^{h,j} + (\theta - 1)\tilde{y}_{b,R}^{h,j}\right), \\
\tilde{u}_{b,R}^{h,j} &= (-1)^{r+p}\left(\theta \tilde{y}_{b,L}^{h,j} + \Big(\frac{1}{2} - \theta\Big) \tilde{y}_{b,R}^{h,j}\right),
\end{aligned} \tag{3.85}$$

with $j = 1, 2$ and $\theta \in [0, 1]$. At faces $f \in \Gamma$, the discrete element output ports are given as

$$
\begin{aligned}
y_b^{h,1} &= (-1)^{q-1}\mathrm{tr}(\frac{\delta F_h}{\delta u_h}), \quad y_b^{h,2} = (-1)^{q-1}\mathrm{tr}(\frac{\delta G_h}{\delta u_h}),\\
\tilde{y}_b^{h,1} &= (-1)^{r+p-1}\mathrm{tr}(\frac{\delta F_h}{\delta \eta_h}), \quad \tilde{y}_b^{h,2} = (-1)^{r+p-1}\mathrm{tr}(\frac{\delta G_h}{\delta \eta_h}).
\end{aligned} \tag{3.86}
$$

Lemma 3.5.2. *Let $\mathcal{T}_h$ be a tessellation of the oriented connected polyhedral domain $\Omega \subset \mathbb{R}^n$ with shape-regular polyhedral elements K. Then with choices (3.85) and (3.86), for any discrete functional $F_h, G_h : V_{h,r}^p \times \widehat{V}_{h,r}^q \times V_{h,r}^q \to \mathbb{R}$, the Poisson bracket $\{F_h, G_h\}_p(\eta_h, u_h, v_h)$ defined in (3.84) is skew-symmetric.*

Proof. The proof follows directly from Lemma 3.4.6 since $V_{h,r}^p \subset H\Lambda^p(\mathcal{T}_h)$ and $\widehat{V}_{h,r}^q \subset H^1\Lambda^q(\mathcal{T}_h)$. □

Theorem 3.5.3 (Discrete energy exchange). *Let $\mathcal{T}_h$ be a tessellation of the oriented polyhedral domain $\Omega \subset \mathbb{R}^n$ with shape-regular polyhedral elements K. Given the discrete element input ports (3.85), and the discrete element output ports (3.86), with the discrete Hamiltonian H_h (3.82). For any discrete functionals $F_h(\eta_h, u_h, v_h) : V_{h,r}^p \times \widehat{V}_{h,r}^q \times V_{h,r}^q \to \mathbb{R}$, we have the relation*

$$
\frac{dF_h}{dt} = \{F_h, H_h\}_p + \sum_{f \in \Gamma_b} \int_f \Big(u_b^{h,1} \wedge \tilde{y}_b^{h,2} + \tilde{u}_b^{h,1} \wedge y_b^{h,2} \Big). \tag{3.87}
$$

Proof. The proof is similar to Theorem 3.4.10. □

Corollary 3.5.1. *The rate of change of the discrete Hamiltonian (3.82) is equal to*

$$
\frac{dH_h}{dt} = \sum_{f \in \Gamma_b} \int_f \Big(u_b^h \wedge \tilde{y}_b^h + \tilde{u}_b^h \wedge y_b^h \Big), \tag{3.88}
$$

where at $f \in \Gamma_b$

$$
y_b^h = (-1)^{q-1}\mathrm{tr}(\frac{\delta H_h}{\delta u}), \quad \tilde{y}_b^h = (-1)^{r+p-1}\mathrm{tr}(\frac{\delta H_h}{\delta \eta}). \tag{3.89}
$$

Proof. Since the discrete bracket $\{F_h, G_h\}_p$ is skew-symmetric, we have $\{H_h, H_h\}_p = 0$. Combining this with (3.87), we obtain (3.88). □

3.5.3 PHDG formulation

In this section, we derive the PHDG formulation for the port-Hamiltonian system (3.72). We first transform the port-Hamiltonian formulation (3.87) into a form which is more suitable for a DG discretization.

Lemma 3.5.4. *For any discrete functional $F_h : V^p_{h,r} \times \widehat{V}^q_{h,r} \times V^q_{h,r} \to \mathbb{R}$, the rate of change of the discrete functional F_h is given by*

$$\begin{aligned}\frac{dF_h}{dt} = &\sum_{K\in\mathcal{T}_h}\int_K \Big(\frac{\delta H_h}{\delta\eta_h}\wedge d\frac{\delta F_h}{\delta u_h} + (-1)^r\frac{\delta H_h}{\delta u_h}\wedge d\frac{\delta F_h}{\delta\eta_h} - \frac{c}{h}\frac{\delta H_h}{\delta u_h}\wedge *\frac{\delta F_h}{\delta v_h}\\ &+\frac{c}{h}\frac{\delta H_h}{\delta v_h}\wedge *\frac{\delta F_h}{\delta u_h}\Big) + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\Big((-1)^{r+p}\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta u_h}) - u^h_b\Big)\wedge\mathrm{tr}(\frac{\delta F_h}{\delta\eta_h})\\ &+(-1)^q\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h}) - \tilde{u}^h_b\Big)\wedge\mathrm{tr}(\frac{\delta F_h}{\delta u_h})\Big).\end{aligned} \tag{3.90}$$

Proof. Applying integration by parts formula (3.22) to (3.83), we obtain (3.90). □

Using Lemma 3.5.4, we obtain the weak form of the distributed-parameter port-Hamiltonian system for the linear dynamical system (3.6), which is stated in the following theorem.

Theorem 3.5.5 (PHDG formulation). *Let $\mathcal{T}_h$ be a tessellation of the oriented connected polyhedral domain $\Omega\subset\mathbb{R}^n$ with shape-regular polyhedral elements K. For the discrete Hamiltonian H_h (3.82), the PHDG formulation of the linear dynamical system (3.6) on the polyhedral domain Ω is represented as: Find $(\eta_h, u_h, v_h)\in V^p_{h,r}\times\widehat{V}^q_{h,r}\times V^q_{h,r}$ such that*

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\int_K v^p\wedge\frac{\partial\eta_h}{\partial t} = &\sum_{K\in\mathcal{T}_h}(-1)^r\int_K\frac{\delta H_h}{\delta u_h}\wedge dv^p\\ &+\sum_{f\in\Gamma_b}(-1)^{r+p}\int_f \widehat{\mathrm{tr}(\frac{\delta H_h}{\delta u_h})}\wedge\mathrm{tr}(v^p) \qquad (3.91a)\\ &+\sum_{f\in\Gamma_i}(-1)^{r+p}\int_f \widehat{\mathrm{tr}(\frac{\delta H_h}{\delta u_h})}\wedge\Big(\mathrm{tr}(v^p)\Big|_L - \mathrm{tr}(v^p)\Big|_R\Big),\end{aligned}$$

$$\sum_{K\in\mathcal{T}_h}\int_K v^q\wedge\frac{\partial u_h}{\partial t} = \sum_{K\in\mathcal{T}_h}\int_K\Big(\frac{\delta H_h}{\delta\eta_h}\wedge dv^q + \frac{c}{h}\frac{\delta H_h}{\delta v_h}\wedge *v^q\Big)$$

$$+\sum_{f\in\Gamma_b}(-1)^q\int_f \widehat{\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})}\wedge \mathrm{tr}(v^q) \tag{3.91b}$$

$$+\sum_{f\in\Gamma_i}(-1)^q\int_f \widehat{\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})}\wedge \Big(\mathrm{tr}(v^q)\Big|_L-\mathrm{tr}(v^q)\Big|_R\Big),$$

$$\sum_{K\in\mathcal{T}}\int_K v^r\wedge\frac{\partial v_h}{\partial t}=-\sum_{K\in\mathcal{T}_h}\int_K\frac{c}{h}\frac{\delta H_h}{\delta u_h}\wedge *v^r, \tag{3.91c}$$

for all test functions $(v^p,v^q,v^r)\in\widehat{V}_{h,r}^{n-p}\times V_{h,r}^{n-q}\times\widehat{V}_{h,r}^{n-q}$*. The Hodge star operators are computed by (3.4), and the numerical fluxes are given by*

$$\begin{aligned}\widehat{\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})}&=\theta\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})\Big|_L+(1-\theta)\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})\Big|_R,\\ \widehat{\mathrm{tr}(\frac{\delta H_h}{\delta u_h})}&=(1-\theta)\mathrm{tr}(\frac{\delta H_h}{\delta u_h})\Big|_L+\theta\mathrm{tr}(\frac{\delta H_h}{\delta u_h})\Big|_R,\end{aligned} \tag{3.92}$$

with $\theta\in[0,1]$ *when* $f\in\Gamma_i$*, and*

$$\begin{aligned}\widehat{\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})}&=\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})-\tilde{u}_b^h,\\ \widehat{\mathrm{tr}(\frac{\delta H_h}{\delta u_h})}&=\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta u_h})-u_b^h,\end{aligned} \tag{3.93}$$

when $f\in\Gamma_b$*, with* $u_b^h,\tilde{u}_b^h$ *the given control input imposed at the domain boundary* Γ_b*.*

Proof. Choose the functional $F_h: V_{h,r}^p\times\widehat{V}_{h,r}^q\times V_{h,r}^q\to\mathbb{R}$

$$F_h(\eta_h,u_h,v_h)=\sum_{K\in\mathcal{T}_h}\int_K\big(v^p\wedge\eta_h+v^q\wedge u_h+v^r\wedge v_h\big), \tag{3.94}$$

then

$$\frac{\delta F_h}{\delta\eta_h}=v^p,\quad \frac{\delta F_h}{\delta u_h}=v^q,\frac{\delta F_h}{\delta v_h}=v^r,$$

with the test functions $v^p\in\widehat{V}_{h,r}^{n-p}$, $v^q\in V_{h,r}^{n-q}$ and $v^r\in\widehat{V}_{h,r}^{n-q}$. Combining with the chain rule

$$\frac{dF_h}{dt}=\sum_{K\in\mathcal{T}_h}\int_K\left(\frac{\delta F_h}{\delta\eta_h}\wedge\frac{\partial\eta_h}{\partial t}+\frac{\delta F_h}{\delta u_h}\wedge\frac{\partial u_h}{\partial t}+\frac{\delta F_h}{\delta v_h}\wedge\frac{\partial v_h}{\partial t}\right), \tag{3.95}$$

with (3.90), and after transforming the summation over element boundaries into a summation over faces, we obtain the following discrete formulation for the dynamical system (3.6)

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h}\int_K v^p\wedge\frac{\partial\eta_h}{\partial t} &= \sum_{K\in\mathcal{T}_h}(-1)^r\int_K\frac{\delta H_h}{\delta u_h}\wedge dv^p\\
&+\sum_{f\in\Gamma_b}(-1)^{r+p}\int_f\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta u_h})-u_b^h\Big)\wedge\mathrm{tr}(v^p)\\
&+\sum_{f\in\Gamma_i}\int_f(-1)^{r+p}\bigg(\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta u_h})\Big|_L-u_{b,L}^h\Big)\wedge\mathrm{tr}(v^p)\Big|_L\\
&-\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta u_h})\Big|_R-u_{b,R}^h\Big)\wedge\mathrm{tr}(v^p)\Big|_R\bigg),\\
\sum_{K\in\mathcal{T}_h}\int_K v^q\wedge\frac{\partial u_h}{\partial t} &= \sum_{K\in\mathcal{T}_h}\int_K\Big(\frac{\delta H_h}{\delta\eta_h}\wedge dv^q+\frac{c}{h}\frac{\delta H_h}{\delta v_h}\wedge *v^q\Big)\\
&+\sum_{f\in\Gamma_b}(-1)^q\int_f\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})-\tilde{u}_b^h\Big)\wedge\mathrm{tr}(v^q)\\
&+\sum_{f\in\Gamma_i}(-1)^q\int_f\bigg(\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})\Big|_L-\tilde{u}_{b,L}^h\Big)\wedge\mathrm{tr}(v^q)\Big|_L\\
&-\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta H_h}{\delta\eta_h})\Big|_R-\tilde{u}_{b,R}^h\Big)\wedge\mathrm{tr}(v^q)\Big|_R\bigg),\\
\sum_{K\in\mathcal{T}_h}\int_K v^r\wedge\frac{\partial v_h}{\partial t} &= -\sum_{K\in\mathcal{T}_h}\int_K\frac{c}{h}\frac{\delta H_h}{\delta u_h}\wedge *v^r. \qquad (3.96)
\end{aligned}
$$

Substituting the formulations of u_b and $\tilde{u}_b$ on the interior faces into (3.96), taking $G=H$ in (3.85) and (3.86) and ignoring the index j, we obtain the PHDG formulation (3.91)–(3.93). □

Theorem 3.5.6. *The PHDG discretization (3.91)–(3.93) is Hamiltonian.*

Proof. Using the formulation (3.87) and (3.95), we have that

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K\Big(v^p\wedge\frac{\partial\eta_h}{\partial t}+v^q\wedge\frac{\partial u_h}{\partial t}+v^r\wedge\frac{\partial v_h}{\partial t}\Big)\\
&=\{F_h,H_h\}_p+\sum_{f\in\Gamma_b}\int_f\Big(u_b^{h,1}\wedge\tilde{y}_b^{h,2}+\tilde{u}_b^{h,1}\wedge y_b^{h,2}\Big),
\end{aligned}
$$

where F_h given in (3.94). Using Theorem 3.4.9, which states that the bracket (3.84) is a Poisson bracket and the fact that $V^p_{h,r} \subset H\Lambda^p(\mathcal{T}_h)$ and $\widehat{V}^q_{h,r} \subset H^1\Lambda^q(\mathcal{T}_h)$, we obtain the result. $\square$

In each element K, there are linear discrete constitutive relations between the co-energy variables and the discrete Hamiltonian functional [19, 50]

$$\begin{aligned}
\frac{\delta H_h}{\delta \eta_h} &:= e^h_\eta \neq (-1)^{p(n-p)} g(*\eta_h),\\
\frac{\delta H_h}{\delta u_h} &:= e^h_u \neq (-1)^{q(n-q)} h(*u_h), \qquad (3.97)\\
\frac{\delta H_h}{\delta v_h} &:= e^h_v \neq (-1)^{q(n-q)} h(*v_h).
\end{aligned}$$

In contrast to the infinite dimensional case, the functional derivatives are, however, not equal in the strong sense, but only in a weak sense. Then

$$\eta_h = \frac{1}{g}(*e^h_\eta), \quad u_h = \frac{1}{h}(*e^h_u), \quad v_h = \frac{1}{h}(*e^h_v), \qquad (3.98)$$

holds in the weak form.

Corollary 3.5.2. *The PHDG formulation of (3.6) can be stated as: Find* $(e^h_\eta, e^h_u, e^h_v) \in \widehat{V}^{n-p}_{h,r} \times V^{n-q}_{h,r} \times \widehat{V}^{n-q}_{h,r}$ *such that*

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} \Big\langle \frac{1}{g}\frac{\partial e^h_\eta}{\partial t}, v^p \Big\rangle_{L^2\Lambda^{n-p}(K)} &= \sum_{K\in\mathcal{T}_h} (-1)^r \int_K e^h_u \wedge dv^p \\
&+ \sum_{f\in\Gamma_i} (-1)^{r+p} \int_f \widehat{\mathrm{tr}(e^h_u)} \wedge \big(\mathrm{tr}(v^p)|_L - \mathrm{tr}(v^p)|_R\big)\\
&+ \sum_{f\in\Gamma_b} (-1)^{r+p} \int_f \widehat{\mathrm{tr}(e^h_u)} \wedge \mathrm{tr}(v^p),\\
\sum_{K\in\mathcal{T}_h} \Big\langle \frac{1}{h}\frac{\partial e^h_u}{\partial t}, v^q \Big\rangle_{L^2\Lambda^{n-q}(K)} &= \sum_{K\in\mathcal{T}_h} \int_K \Big(e^h_\eta \wedge dv^q + \frac{c}{h} e^h_v \wedge *v^q\Big)\\
&+ \sum_{f\in\Gamma_i} (-1)^q \int_f \widehat{\mathrm{tr}(e^h_\eta)} \wedge \big(\mathrm{tr}(v^q)|_L - \mathrm{tr}(v^q)|_R\big)\\
&+ \sum_{f\in\Gamma_b} (-1)^q \int_f \widehat{\mathrm{tr}(e^h_\eta)} \wedge \mathrm{tr}(v^q),
\end{aligned}$$

$$\sum_{K\in\mathcal{T}} \langle \frac{1}{h}\frac{\partial e_v^h}{\partial t}, v^r \rangle_{L^2\Lambda^{n-q}(K)} = -\sum_{K\in\mathcal{T}_h} \int_K \frac{c}{h} e_u^h \wedge *v^r, \tag{3.99}$$

for all $(v^p, v^q, v^r) \in \widehat{V}_{h,r}^{n-p} \times V_{h,r}^{n-q} \times \widehat{V}_{h,r}^{n-q}$*. The numerical fluxes are given by*

$$\begin{aligned} \widehat{\mathrm{tr}(e_\eta^h)} &= \theta \mathrm{tr}(e_\eta^h)|_L + (1-\theta)\mathrm{tr}(e_\eta^h)|_R, \\ \widehat{\mathrm{tr}(e_u^h)} &= (1-\theta)\mathrm{tr}(e_u^h)|_L + \theta \mathrm{tr}(e_u^h)|_R, \end{aligned} \tag{3.100}$$

with $\theta \in [0,1]$*, when* $f \in \Gamma_i$*, and*

$$\begin{aligned} \widehat{\mathrm{tr}(e_\eta^h)} &= \frac{1}{2}\mathrm{tr}(e_\eta^h) - \tilde{u}_b^h, \\ \widehat{\mathrm{tr}(e_u^h)} &= \frac{1}{2}\mathrm{tr}(e_u^h) - u_b^h, \end{aligned} \tag{3.101}$$

when $f \in \Gamma_b$*, with* $u_b^h, \tilde{u}_b^h$ *the given control input imposed at the domain boundary* Γ_b*.*

3.5.4 Error estimates

In this section, we discuss error estimates for the PHDG discretization presented in Section 5.3. From (3.99), the DG scheme can be written in the following form: Find $(e_\eta^h, e_u^h, e_v^h) \in \widehat{V}_{h,r}^{n-p} \times V_{h,r}^{n-q} \times \widehat{V}_{h,r}^{n-q}$ such that

$$\langle \frac{1}{g}\frac{\partial e_\eta^h}{\partial t}, v^p \rangle + \langle \frac{1}{h}\frac{\partial e_u^h}{\partial t}, v^q \rangle + \langle \frac{1}{h}\frac{\partial e_v^h}{\partial t}, v^r \rangle + a(e_\eta^h, e_u^h, e_v^h; v^p, v^q, v^r) = 0, \tag{3.102}$$

for any $(v^p, v^q, v^r) \in \widehat{V}_{h,r}^{n-p} \times V_{h,r}^{n-q} \times \widehat{V}_{h,r}^{n-q}$, with the bilinear form $a : (\widehat{V}_{h,r}^{n-p}, V_{h,r}^{n-q}, \widehat{V}_{h,r}^{n-q}) \times (\widehat{V}_{h,r}^{n-p} \times V_{h,r}^{n-q} \times \widehat{V}_{h,r}^{n-q}) \to \mathbb{R}$ defined as

$$\begin{aligned} &a(e_\eta^h, e_u^h, e_v^h; v^p, v^q, v^r) \\ &= \sum_{K\in\mathcal{T}_h} \int_K \Big((-1)^{r-1} e_u^h \wedge dv^p - e_\eta^h \wedge dv^q - \frac{c}{h} e_v^h \wedge *v^q + \frac{c}{h} e_u^h \wedge *v^r \Big) \\ &+ \sum_{f\in\Gamma_i} \int_f \Big((-1)^{r+p-1} \widehat{\mathrm{tr}(e_u^h)} \wedge \Big(\mathrm{tr}(v^p)\Big|_L - \mathrm{tr}(v^p)\Big|_R \Big) \\ &+ (-1)^{q-1} \widehat{\mathrm{tr}(e_\eta^h)} \wedge \Big(\mathrm{tr}(v^q)\Big|_L - \mathrm{tr}(v^q)\Big|_R \Big)\Big) \\ &+ \sum_{f\in\Gamma_b} \int_f \Big((-1)^{r+p-1} \widehat{\mathrm{tr}(e_u^h)} \wedge \mathrm{tr}(v^p) + (-1)^{q-1} \widehat{\mathrm{tr}(e_\eta^h)} \wedge \mathrm{tr}(v^q) \Big), \end{aligned} \tag{3.103}$$

with

$$\langle \cdot, \cdot \rangle := \sum_{K \in \mathcal{T}_h} \langle \cdot, \cdot \rangle_{L^2\Lambda^k(K)}, \qquad \| \cdot \|^2 = \langle \cdot, \cdot \rangle. \tag{3.104}$$

Lemma 3.5.7. *The bilinear form* $a : (\widehat{V}_{h,r}^{n-p} \times V_{h,r}^{n-q} \times \widehat{V}_{h,r}^{n-q}) \times (\widehat{V}_{h,r}^{n-p} \times V_{h,r}^{n-q} \times \widehat{V}_{h,r}^{n-q}) \to \mathbb{R}$ *satisfies the following property*

$$a(e_\eta^h, e_u^h, e_v^h; e_\eta^h, e_u^h, e_v^h) = -\sum_{f \in \Gamma_b} \left(u_b^h \wedge \tilde{y}_b^h + \tilde{u}_b^h \wedge y_b^h \right), \tag{3.105}$$

with the input ports $(u_b^h, \tilde{u}_b^h) \in W_{h,r}^{n-q} \times \widehat{W}_{h,r}^{n-p}$ *determined by the specific boundary condition on the domain boundary* Γ_b *and the output ports* $(y_b^h, \tilde{y}_b^h)$ *given by (3.89).*

Proof. Using (3.97) the proof directly follows from Corollary 3.5.1. □

Definition 3.5.1 (**Canonical Projection Operator**)**.** *[7, 50]] Let* $\mathcal{T}_h$ *be a tessellation of the oriented connected polyhedral domain* $\Omega \subset \mathbb{R}^n$*, and* $K \in \mathcal{T}_h$ *be a simplex with boundary* ∂K*. For* $r \in \mathbb{N}, r \geq 1$*, the projection* $\pi_h^k : \Lambda^k(K) \to V_{h,r}^k(K)$ *is defined as*

$$\begin{aligned} \int_K (w - \pi_h^k w) \wedge \mu = 0, \quad \forall \mu \in \widehat{V}_{h,r}^{n-k}(K), \\ \int_{\partial K} \mathrm{tr}(w - \pi_h^k w) \wedge \mathrm{tr}(\mu) = 0, \quad \forall \mu \in \widehat{V}_{h,r}^{n-k-1}(K). \end{aligned} \tag{3.106}$$

Or the projection the projection $\pi_h^k : \Lambda^k(K) \to \widehat{V}_{h,r}^k(K)$ *is defined as*

$$\begin{aligned} \int_K (w - \pi_h^k w) \wedge \mu = 0, \quad \forall \mu \in V_{h,r}^{n-k}(K), \\ \int_{\partial K} \mathrm{tr}(w - \pi_h^k w) \wedge \mathrm{tr}(\mu) = 0, \quad \forall \mu \in V_{h,r}^{n-k-1}(K), \end{aligned} \tag{3.107}$$

where $C^0\Lambda^k(K)$ *are the spaces of k-th order differential forms whose coefficient functions belong to* C^0*.*

Lemma 3.5.8 ([7, 50])**.** *Let* $\pi_h^k : L^2\Lambda^k(\Omega) \to V_r\Lambda^k$ *be the projection operator defined in Definition 3.5.1. Then*

$$\|w - \pi_h^k w\| \leq c h_K^s \|w\|_{H^s\Lambda^k(\Omega)}, \quad \forall w \in H^s\Lambda^k(\Omega). \tag{3.108}$$

Theorem 3.5.9 (Error estimates). *Given the PHDG formulation (3.99)–(3.101) on the tessellation $\mathcal{T}_h$ of the oriented connected polyhedral domain $\Omega \subset \mathbb{R}^n$. Given $g, h \in L^2\Lambda^0(\Omega) \cap L^\infty\Lambda^0(\Omega)$ with $g(x) \geq g_0 > 0$ and $h(x) \geq h_0 > 0$ for all $x \in \Omega$. Assume that the exact solutions $e_\eta \in H\Lambda^{n-p}(\Omega)$, $e_u \in H^1\Lambda^{n-q}(\Omega)$ and $e_v \in H^1\Lambda^{n-q}(\Omega)$ with $t \in [0,T]$ are sufficiently smooth and the boundary conditions are applied exactly at $\partial\Omega$, then we have the following error estimate*

$$\begin{aligned}
&\Big\|\frac{1}{\sqrt{g}}(e_\eta - e_\eta^h)\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}(e_u - e_u^h)\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}(e_v - e_v^h)\Big\|^2 \\
&\leq \exp(T)\left(\Big\|\frac{1}{\sqrt{g}}(\pi_h^{n-p}e_\eta - e_\eta^h)\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}(\pi_h^{n-q}e_u - e_u^h)\Big\|^2\right. \\
&\left.+\Big\|\frac{1}{\sqrt{h}}(\pi_h^{n-q}e_v - e_v^h)\Big\|^2\right)\Big|_{t=0} + C\exp(T)h_K^{2(r+1)},
\end{aligned}$$

where h_K denotes the maximal mesh size of $K \in \mathcal{T}_h$, C is a constant independent of h_K, but depends on $(r+1)$-th Sobolev norms of $e_\eta, e_u, e_v, \frac{\partial e_\eta}{\partial t}, \frac{\partial e_u}{\partial t}, \frac{\partial e_v}{\partial t}$, h and g, and $\|\cdot\|$ is the L^2–norm defined in (3.13).

Proof. Decompose the error as

$$\begin{aligned}
\lambda_\eta &= e_\eta - e_\eta^h = e_\eta - \pi_h^{n-p}e_\eta + \pi_h^{n-p}\lambda_\eta, \\
\lambda_u &= e_u - e_u^h = e_u - \pi_h^{n-q}e_u + \pi_h^{n-q}\lambda_u, \\
\lambda_v &= e_v - e_v^h = e_v - \pi_h^{n-q}e_v + \pi_h^{n-q}\lambda_v,
\end{aligned} \tag{3.109}$$

using the fact that $\pi_h^{n-p}e_\eta^h = e_\eta^h$, $\pi_h^{n-q}e_u^h = e_u^h$ and $\pi_h^{n-q}e_v^h = e_v^h$. For any $e_\eta \in H\Lambda^{n-p}(\Omega)$, $e_u \in H^1\Lambda^{n-q}(\Omega)$ and $e_v \in H^1\Lambda^{n-q}(\Omega)$, it holds that

$$\begin{aligned}
&\Big\langle\frac{1}{g}\frac{\partial e_\eta}{\partial t}, v^p\Big\rangle + \Big\langle\frac{1}{h}\frac{\partial e_u}{\partial t}, v^q\Big\rangle + \Big\langle\frac{1}{h}\frac{\partial e_v}{\partial t}, v^r\Big\rangle \\
&+ a(e_\eta, e_u, e_v; v^p, v^q, v^r) = 0,
\end{aligned} \tag{3.110}$$

which gives the orthogonality condition

$$\begin{aligned}
&\Big\langle\frac{1}{g}\frac{\partial \lambda_\eta}{\partial t}, v^p\Big\rangle + \Big\langle\frac{1}{h}\frac{\partial \lambda_u}{\partial t}, v^q\Big\rangle + \Big\langle\frac{1}{h}\frac{\partial \lambda_v}{\partial t}, v^r\Big\rangle \\
&+ a(\lambda_\eta, \lambda_u, \lambda_v; v^p, v^q, v^r) = 0,
\end{aligned} \tag{3.111}$$

for any $v^p \in \widehat{V}_{h,r}^{n-p}$, $v^q \in V_{h,r}^{n-q}$ and $v^r \in \widehat{V}_{h,r}^{n-q}$. From the definitions of π_h^k in (3.106), we have

$$a(e_\eta - \pi_h^{n-p}e_\eta, e_u - \pi_h^{n-q}e_u, e_v - \pi_h^{n-q}e_v; \pi_h^{n-p}\lambda_\eta, \pi_h^{n-q}\lambda_u, \pi_h^{n-q}\lambda_v) = 0. \tag{3.112}$$

Using the assumption that the boundary terms are exact, with Lemma 3.5.7 we have

$$a(\pi_h^{n-p}\lambda_\eta, \pi_h^{n-q}\lambda_u, \pi_h^{n-q}\lambda_v; \pi_h^{n-p}\lambda_\eta, \pi_h^{n-q}\lambda_u, \pi_h^{n-q}\lambda_v) = 0, \tag{3.113}$$

with the equality

$$\mathrm{tr}(\pi_h^{n-p}\lambda_\eta) = 0, \quad \mathrm{tr}(\pi_h^{n-q}\lambda_u) = 0, \quad \mathrm{tr}(\pi_h^{n-q}\lambda_v) = 0. \tag{3.114}$$

Substituting (3.112) and (3.113) into (3.111), taking $v^p = \pi_h^{n-p}\lambda_\eta$, $v^q = \pi_h^{n-q}\lambda_u$ and $v^r = \pi_h^{n-q}\lambda_v$ in (3.111), we obtain

$$\begin{aligned}
&\Big\langle \frac{1}{g}\frac{\partial \pi_h^{n-p}\lambda_\eta}{\partial t}, \pi_h^{n-p}\lambda_\eta \Big\rangle + \Big\langle \frac{1}{h}\frac{\partial \pi_h^{n-q}\lambda_u}{\partial t}, \pi_h^{n-q}\lambda_u \Big\rangle + \Big\langle \frac{1}{h}\frac{\partial \pi_h^{n-q}\lambda_v}{\partial t}, \pi_h^{n-q}\lambda_v \Big\rangle \\
&= -\Big\langle \frac{1}{g}\frac{\partial (e_\eta - \pi_h^{n-p}e_\eta)}{\partial t}, \pi_h^{n-p}\lambda_\eta \Big\rangle - \Big\langle \frac{1}{h}\frac{\partial (e_u - \pi_h^{n-q}e_u)}{\partial t}, \pi_h^{n-q}\lambda_u \Big\rangle \\
&\quad - \Big\langle \frac{1}{h}\frac{\partial (e_v - \pi_h^{n-q}e_v)}{\partial t}, \pi_h^{n-q}\lambda_v \Big\rangle.
\end{aligned} \tag{3.115}$$

Applying the Cauchy-Schwarz and Young inequalities yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\Big\|\frac{1}{\sqrt{g}}\pi_h^{n-p}\lambda_\eta\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_u\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_v\Big\|^2\right) \\
&\leq \frac{1}{2}\left(\Big\|\frac{1}{\sqrt{g}}\frac{\partial (e_\eta - \pi_h^{n-p}e_\eta)}{\partial t}\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}\frac{\partial (e_u - \pi_h^{n-q}e_u)}{\partial t}\Big\|^2\right. \\
&+ \Big\|\frac{1}{\sqrt{h}}\frac{\partial (e_v - \pi_h^{n-q}e_v)}{\partial t}\Big\|^2 + \Big\|\frac{1}{\sqrt{g}}\pi_h^{n-p}\lambda_\eta\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_u\Big\|^2 \\
&\left. + \Big\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_v\Big\|^2\right).
\end{aligned}$$

Lemma 3.5.8 gives

$$\begin{aligned}
&\frac{d}{dt}\left(\Big\|\frac{1}{\sqrt{g}}\pi_h^{n-p}\lambda_\eta\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_u\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_v\Big\|^2\right) \\
&\leq \Big\|\frac{1}{\sqrt{g}}\pi_h^{n-p}\lambda_\eta\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_u\Big\|^2 + \Big\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_v\Big\|^2 \\
&+ Ch_K^{2(r+1)}\left(\Big\|\frac{1}{\sqrt{g}}\frac{\partial e_\eta}{\partial t}\Big\|^2_{H^{r+1}\Lambda^{n-p}(\Omega)} + \Big\|\frac{1}{\sqrt{h}}\frac{\partial e_u}{\partial t}\Big\|^2_{H^{r+1}\Lambda^{n-q}(\Omega)}\right. \\
&\left. + \Big\|\frac{1}{\sqrt{h}}\frac{\partial e_\eta}{\partial t}\Big\|^2_{H^{r+1}\Lambda^{n-q}(\Omega)}\right),
\end{aligned}$$

with C a constant independent of g, h, e_η and e_u. Next, we apply Gronwall's inequality [68] to obtain

$$\begin{aligned}
&\left(\left\|\frac{1}{\sqrt{g}}\pi_h^{n-p}\lambda_\eta\right\|^2 + \left\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_u\right\|^2 + \left\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_v\right\|^2\right)|_{t=T} \\
&\leq \exp(T)\left(\left\|\frac{1}{\sqrt{g}}\pi_h^{n-p}\lambda_\eta\right\|^2 + \left\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_u\right\|^2 + \left\|\frac{1}{\sqrt{h}}\pi_h^{n-q}\lambda_v\right\|^2\right)|_{t=0} \\
&+ C(\exp(T)-1)h_K^{2(r+1)} \sup_{t\in[0,T]}\left(\left\|\frac{1}{\sqrt{g}}\frac{\partial e_\eta}{\partial t}\right\|^2_{H^{r+1}\Lambda^{n-p}(\Omega)}\right. \\
&\left.+\left\|\frac{1}{\sqrt{h}}\frac{\partial e_u}{\partial t}\right\|^2_{H^{r+1}\Lambda^{n-q}(\Omega)} + \left\|\frac{1}{\sqrt{h}}\frac{\partial e_\eta}{\partial t}\right\|^2_{H^{r+1}\Lambda^{n-q}(\Omega)}\right).
\end{aligned}$$

The proof is completed by applying the triangle inequality and using Lemma 3.5.8 again. □

3.6 Numerical test cases

In this section, we provide four numerical examples to illustrate the capability and accuracy of the PHDG discretization (3.99)–(3.101). In all examples, we apply the 3rd order symplectic partitioned Runge-Kutta (SPRK) method [45] for the time discretization.

3.6.1 Two-dimensional linear shallow water equations

We consider a harmonic wave type solution of the two-dimensional linear shallow water equations (3.7) with $f = 0$ on a domain $[0, L] \times [0, L]$ with periodic boundary conditions with the following harmonic wave type solutions

$$\begin{aligned}
\eta &= A\sin(lx + ly + \omega t)dx + A\sin(lx + ly + \omega t))dy, \\
u &= -\frac{Agl}{\omega}\sin(lx + ly + \omega t)dx \wedge dy,
\end{aligned} \tag{3.116}$$

where A is the amplitude, $l = \frac{2\pi m}{L}$ the wave number, ω the frequency and $a^2 = gh_l$, $\omega^2 = a^2 l^2$, with h_l the mean free surface depth. The height of the water level is $h = h_l + \eta$. We have initialized the solution (3.116) with parameters $L = 1, g = h_l = m = 1$.

The discretization errors in the L^2-norm and the numerical orders of accuracy with $\theta = 0, \frac{1}{2}$ and 1 in the numerical fluxes (3.100) at time $t = 1$ are given in Tables 3.1–3.2. We can observe from Table 3.1 that the order of accuracy is optimal for $P_r\Lambda^k(\mathcal{T}_h)$ basis functions, namely order $r+1$ for r-th order polynomials. The results in Table 3.2 show that the order of accuracy for $\theta = 0, 1$ is also optimal for $S_r\Lambda^k(\mathcal{T}_h)$ basis functions, but for the central flux with $\theta = \frac{1}{2}$ and $r = 1$ the order of accuracy is close to 1 instead of 2, which is suboptimal.

Figures 3.1(a)–(c) show the discrete Hamiltonian in the time interval $t \in [97, 100]$ using the $S_r\Lambda^k(\mathcal{T}_h)$ finite element spaces for $r = 0, 1, 2$ on a 32×32 mesh. These results show that the error in the discrete Hamiltonian is contained in a small bounded interval, which rapidly decreases for basis functions with increasing polynomial order. The error in the Hamiltonian $|H(t) - H(0)|$ is also calculated using the 3rd order symplectic partitioned Runge-Kutta (SPRK) method and the 3rd order TVD Runge-Kutta (RK) method [81] with $P_1\Lambda^k(\mathcal{T}_h)$ elements on a uniform 20×20 mesh consisting of triangles, and is shown in Fig. 3.1(d). These results show that the symplectic partitioned Runge-Kutta method is much better in energy conservation than the 3rd order TVD Runge-Kutta method.

Table 3.1: Accuracy test for two-dimensional linear shallow water equations for the exact solutions (3.116). $P_r\Lambda^k(\mathcal{T}_h)$ finite element spaces, numerical fluxes (3.100) with $\theta = 0, \frac{1}{2}, 1$ at time $t = 1$.

		$P_r\Lambda^k(\mathcal{T}_h)$					
		$\theta = 0$		$\theta = \frac{1}{2}$		$\theta = 1$	
r	N	L^2 error	order	L^2 error	order	L^2 error	order
	8	2.2958E-01	-	2.9582E-01	-	2.2960E-01	-
	16	1.1966E-01	0.9400	1.2329E-01	1.2627	1.1730E-01	0.9689
0	32	6.9452E-02	0.7849	5.7974E-02	1.0885	6.4303E-02	0.8673
	64	3.0876E-02	1.1695	2.8504E-02	1.0242	3.3179E-02	0.9546
	8	4.2582E-02	-	8.3696E-02	-	4.2268E-02	-
	16	1.2728E-02	1.7422	1.4978E-02	2.4822	1.2657E-02	1.7396
1	32	2.7920E-03	2.1887	3.3848E-03	2.1457	2.9398E-03	2.1062
	64	7.2197E-03	1.9513	9.2085E-04	1.8780	7.4174E-04	1.9867
	8	1.1903E-02	-	1.6220E-02	-	1.0689E-02	-
	16	1.6508E-03	2.8501	1.3358E-03	3.6020	1.2542E-03	3.0913
2	32	2.1447E-04	2.9443	1.8371E-04	2.8622	2.3538E-04	2.4138
	64	1.9860E-05	3.4328	1.2528E-05	3.8742	1.6616E-05	3.8244

Table 3.2: Accuracy test for two-dimensional linear shallow water equations for the exact solutions (3.116). $S_r\Lambda^k(\mathcal{T}_h)$ finite element spaces, numerical fluxes (3.100) with $\theta = 0, \frac{1}{2}, 1$ at time $t = 1$.

$S_r\Lambda^k(\mathcal{T}_h)$							
		$\theta = 0$		$\theta = \frac{1}{2}$		$\theta = 1$	
r	N	L^2 error	order	L^2 error	order	L^2 error	order
0	8	4.2737E-01	-	8.1290E-01	-	4.2571E-01	-
	16	2.0122E-01	1.0867	2.7490E-01	1.5642	2.0119E-01	1.0813
	32	9.8804E-02	1.0261	1.0968E-01	1.3256	9.8803E-02	1.0259
	64	4.9167E-02	1.0069	5.0599E-02	1.1161	4.9167E-02	1.0069
1	8	1.1384E-01	-	1.1446E-01	-	1.0641E-01	-
	16	2.1108E-02	2.4312	5.4229E-02	1.0776	2.2788E-02	2.2233
	32	6.3641E-03	1.7298	2.6292E-02	1.0444	7.2068E-03	1.6609
	64	1.5511E-03	2.0367	1.3013E-02	1.0146	1.3844E-03	2.3801
2	8	1.7877E-02	-	2.1371E-02	-	1.4871E-02	-
	16	1.4299E-03	3.6441	1.8393E-03	3.5382	1.3871E-03	3.4228
	32	3.0679E-04	2.2206	1.7270E-04	3.4128	2.8450E-04	2.2856
	64	2.1198E-05	3.8552	1.9235E-05	3.1664	2.0590E-05	3.7884

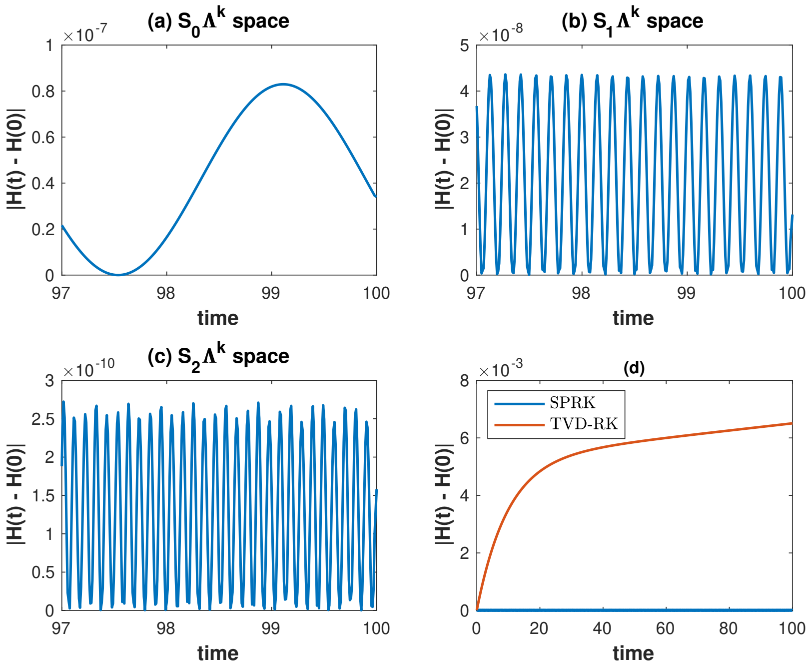

Figure 3.1: (a)(b)(c): Error in discrete Hamiltonian for the 2-D linear shallow water equations using the finite element spaces $S_r\Lambda^k(\mathcal{T}_h)$ for $r = 0, 1, 2$ in the time interval $t \in [97, 100]$. (d): Error in the discrete Hamiltonian using the $P_1\Lambda^k(\mathcal{T}_h)$ finite element space for the SPRK and TVD-RK time integration methods in the time interval $t \in [0, 100]$.

3.6.2 Two-dimensional Klein–Gordon equation

For the second numerical experiment, we consider the two-dimensional Klein–Gordon equation (3.10) with $g = \frac{1}{2}$ and $m = 1$ in the domain $[0, 2\pi] \times [0, 2\pi]$ with periodic boundary conditions. The exact solutions are given as

$$\begin{aligned} \eta &= \cos(x + y + \sqrt{2}t)dx + \cos(x + y + \sqrt{2}t)dy, \\ u &= -\sqrt{2}\cos(x + y + \sqrt{2}t)dx \wedge dy, \\ v &= \sin(x + y + \sqrt{2}t)dx \wedge dy. \end{aligned} \tag{3.117}$$

The discretization errors in the L^2-norm and the numerical orders of accuracy with θ=0, $\frac{1}{2}$ and 1 at time $t = 1$ are given in Tables 3.3 and 3.4. We can also observe that for this test case, the convergence order is not optimal for the central flux. In Fig. 3.2, we compare the phase of the discrete solution (η_h, u_h, v_h) with the exact solution (3.117) of the two-dimensional Klein–Gordon equations. These results show that the phase error is very small, with the numerical solution slightly lagging the exact solution.

Table 3.3: Accuracy test for two-dimensional Klein-Gordon equation for the exact solutions (3.117). $P_r\Lambda^k(\mathcal{T}_h)$ finite element spaces, numerical fluxes (3.100) with $\theta = 0, \frac{1}{2}, 1$ at time $t = 1$.

		$P_r\Lambda^k(\mathcal{T}_h)$					
		$\theta = 0$		$\theta = \frac{1}{2}$		$\theta = 1$	
r	N	L^2 error	order	L^2 error	order	L^2 error	order
	8	2.4793	-	2.0799	-	2.5082	-
	16	1.2888	0.9544	1.0455	0.9923	1.2628	0.9900
0	32	6.3836E-01	1.0136	5.2346E-01	0.9981	6.4053E-01	0.9793
	64	3.1961E-01	0.9980	2.6182E-01	0.9995	3.2105E-01	0.9965
	8	3.1289E-01	-	9.0978E-01	-	3.1456E-01	-
	16	7.8172E-01	2.0009	4.2046E-01	1.1135	7.3705E-02	2.0935
1	32	1.9438E-02	2.0078	2.0773E-01	1.0173	1.8357E-02	2.0055
	64	4.8578E-02	2.0005	1.0361E-01	1.0035	4.8722E-03	1.9137
	8	5.0631E-02	-	7.2338E-02	-	5.0634E-02	-
	16	6.2287E-03	3.0230	7.4796E-03	3.2737	6.2439E-03	3.0196
2	32	7.5164E-04	3.0508	8.8930E-04	3.0722	7.6205E-04	3.0345
	64	8.9345E-05	3.0726	1.0903E-04	3.0280	9.2431E-05	3.0434

3.6.3 Three-dimensional wave equation

The three-dimensional wave equation corresponds to the case $n = 3, g = 1$ in (3.7). We compute the numerical solutions $\eta_h \in V^1_{h,r}$ and $u_h \in V^3_{h,r}$ in the domain $[0,1] \times [0,1] \times [0,1]$ with periodic boundary conditions. The exact solutions are given as

$$\begin{aligned} \eta &= \cos(2\pi t)(\sin(2\pi x)dx + \sin(2\pi y)dy + \sin(2\pi z)dz), \\ u &= \sin(2\pi t)(\cos(2\pi x) + \cos(2\pi y) + \cos(2\pi z))dx \wedge dy \wedge dz. \end{aligned} \tag{3.118}$$

Table 3.5 lists the numerical error in the L^2-norm and the orders of accuracy for the $S_r\Lambda^k(\mathcal{T}_h)$ finite element spaces with $r = 0, 1, 2$. In Fig. 3.3, we compare the contour lines of the numerical solutions and the exact solutions

Table 3.4: Accuracy test for two-dimensional Klein-Gordon equation for the exact solutions (3.117). $S_r\Lambda^k(\mathcal{T}_h)$ finite element spaces, numerical fluxes (3.100) with $\theta = 0, \frac{1}{2}, 1$ at time $t = 1$.

		$S_r\Lambda^k(\mathcal{T}_h)$					
		$\theta = 0$		$\theta = \frac{1}{2}$		$\theta = 1$	
r	N	L^2 error	order	L^2 error	order	L^2 error	order
	8	4.2405	-	3.2376	-	4.1719	-
	16	2.1778	0.9613	1.6011	1.0159	2.1590	0.9503
0	32	1.0950	0.9920	7.9746E-01	1.0056	1.0902	0.9858
	64	5.4795E-01	0.9988	3.9832E-01	1.0015	5.4674E-01	0.9956
	8	9.6115E-01	-	1.2815	-	9.3027E-01	-
	16	2.4280E-01	1.9850	6.0245E-01	1.0889	2.4617E-01	1.9177
1	32	6.0127E-02	2.0137	2.9629E-01	1.0238	5.8131E-02	2.0823
	64	1.4839E-02	2.0186	1.4753E-01	1.0061	1.4378E-02	2.0154
	8	1.2262E-01	-	1.8723E-01	-	1.2709E-01	-
	16	1.7980E-02	2.7698	4.4297E-02	2.0796	1.6570E-02	2.9392
2	32	2.2340E-03	3.0087	1.0899E-02	2.0231	2.1375E-03	2.9546
	64	2.7273E-04	3.0341	2.2713E-03	2.0060	2.8498E-04	2.9070

of (3.118) at time $T = 0.5$ in the cross sections at $z = 0.0125, 0.4875, 0.9875$ for meshes with $40 \times 40 \times 40$ elements, which results compare very well.

3.6.4 Three-dimensional Maxwell equations

We consider the Maxwell equations (3.9) with $\epsilon = \mu = 1$ in a three-dimensional domain ($n = 3$). The propagative solutions are given by

$$\begin{aligned} B &= B_x dz \wedge dx + B_y dx \wedge dz + B_z dx \wedge dy, \\ D &= D_x dz \wedge dx + D_y dx \wedge dz + D_z dx \wedge dy, \end{aligned} \tag{3.119}$$

where, with $\alpha = \cos(0.3 * \pi)$ and $\beta = \sin(0.3 * \pi)$, we have

$$\begin{aligned} B_x &= 0, \\ B_y &= 0, \\ B_z &= \exp(\cos(\alpha * x + \beta * y + t)), \\ D_x &= -\beta * \exp(\cos(\alpha * x + \beta * y + t)), \\ D_y &= \alpha * \exp(\cos(\alpha * x + \beta * y + t)), \\ D_z &= 0. \end{aligned}$$

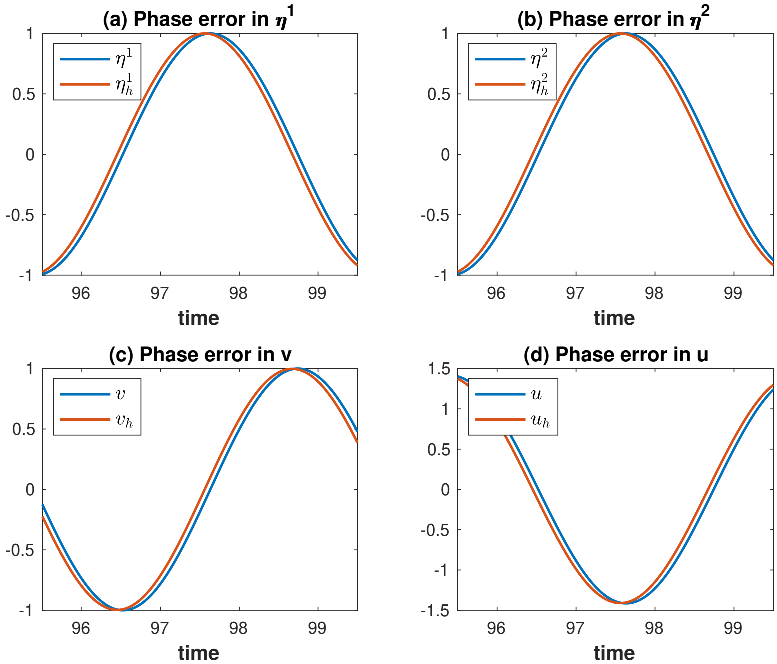

Figure 3.2: Phase error in the discrete solutions (η_h, u_h, v_h) of the 2-D Klein–Gordon equation using $P_1\Lambda^k(\mathcal{T}_h)$ finite element spaces with $N_x = N_y = 40$ in the time interval [95.5,99.5].

For the solutions (3.119), the L^2 errors and the numerical orders of accuracy for D_x on a cube with a uniform mesh using periodic boundary conditions are presented in Table 3.6. For $\theta = 0, 1$ the order of accuracy is optimal, but for $\theta = \frac{1}{2}$ and finite element spaces $S_r\Lambda^k(\mathcal{T}_h)$ the order of accuracy is suboptimal. Fig. 3.4 shows the contour of lines of numerical solutions for B_z at time $T = 20, 50, 80, 100$, which presents the behavior of numerical solutions over a long time period.

Table 3.5: Accuracy test for the three-dimensional wave equation for the exact solutions (3.119). $S_r\Lambda^k(\mathcal{T}_h)$ finite element spaces, numerical fluxes (3.100) with $\theta = 0, \frac{1}{2}, 1$ at time $t = 1.2$.

		$S_r\Lambda^k(\mathcal{T}_h)$					
		$\theta = 0$		$\theta = \frac{1}{2}$		$\theta = 1$	
r	N	L^2 error	order	L^2 error	order	L^2 error	order
0	4	1.1256	-	2.2257	-	1.1256	-
	8	5.5534E-01	1.0192	9.1817E-01	1.2774	5.5535E-01	1.0192
	16	2.7064E-01	1.0370	2.7161E-01	1.7573	2.7064E-01	1.0370
	32	1.3418E-01	1.0122	9.1149E-02	1.5752	1.3418E-01	1.0122
1	4	1.6681	-	4.4840E-01	-	1.6681E-01	-
	8	5.7360E-01	1.5401	1.7402E-01	1.3655	5.7360E-02	1.5401
	16	1.2978E-01	2.1239	5.7973E-02	1.5858	1.2978E-02	2.1439
	32	4.0867E-02	1.6671	2.2869E-02	1.3420	4.0867E-03	1.6671
2	4	3.0524E-02	-	2.2341E-02	-	3.0524E-02	-
	8	4.0721E-03	2.9061	2.0581E-03	3.5078	4.0720E-03	2.9061
	16	3.9403E-04	3.3694	2.4926E-04	3.0456	3.9403E-04	3.3694
	32	5.8742E-05	2.7458	2.9393E-05	3.0841	5.8742E-05	2.7458

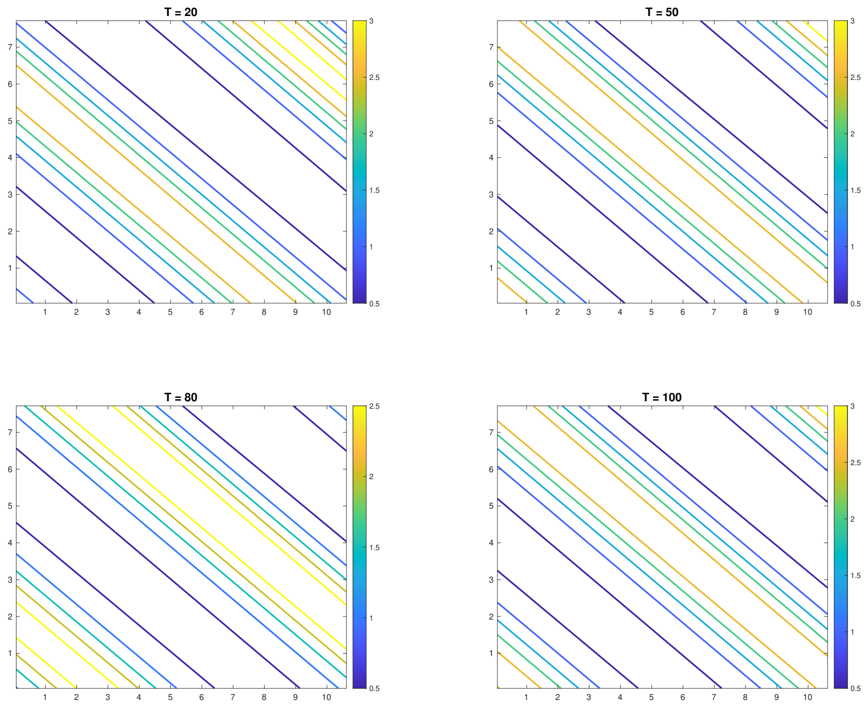

Figure 3.4: The contour plot for B_z of the solutions (3.119) of the three-dimensional Maxwell equations (3.118) at $z = 0$ at time $T = 20, 50, 80, 100$.

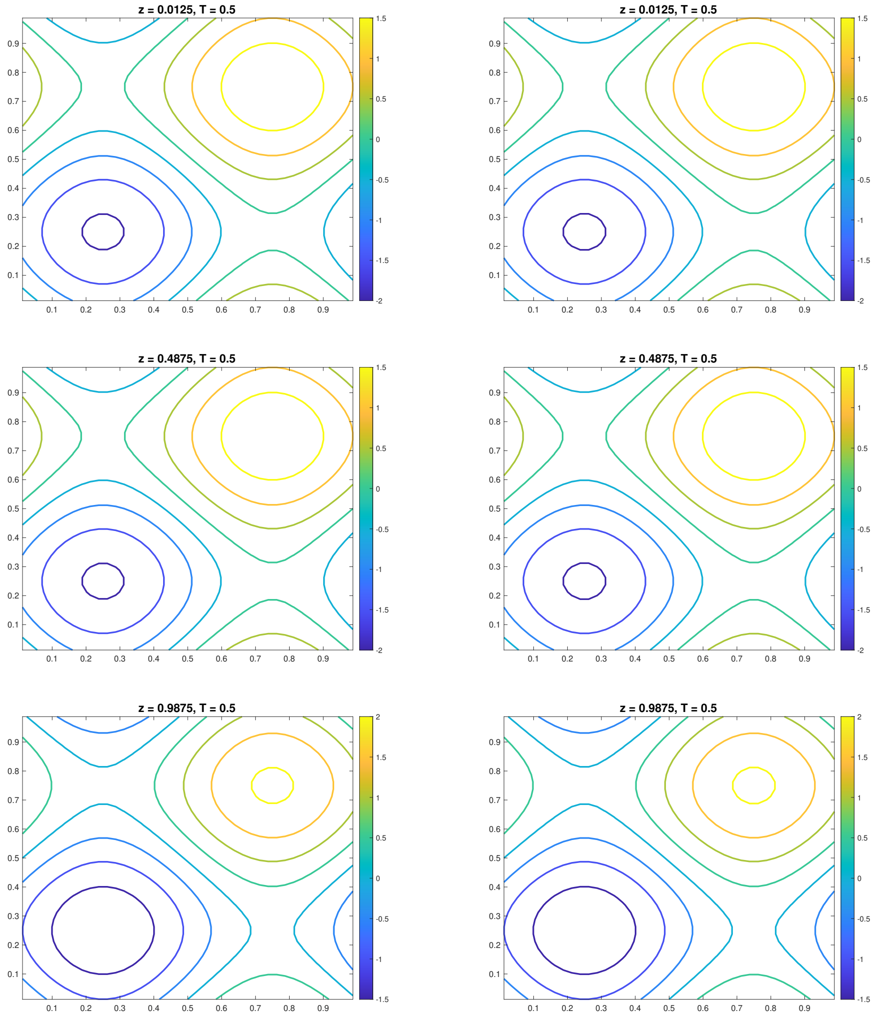

Figure 3.3: Contours of cross sections of the numerical solutions of the three-dimensional wave equations (3.118) at $z = 0.0125, 0.4875, 0.9875$ for $T = 0.5$. The figures on the left side are the graphs of the numerical solutions for η using the DG finite element space $S_1\Lambda^1(\mathcal{T}_h)$ on a uniform grid with $40\times 40\times 40$ elements. The figures on the right side are the graphs of the exact solutions for η.

Table 3.6: Accuracy test for the three-dimensional Maxwell equations for the exact solutions (3.119). $S_r\Lambda^k(\mathcal{T}_h)$ finite element spaces, numerical fluxes (3.100) with $\theta = 0, \frac{1}{2}, 1$ at time $t = 1$.

		$S_r\Lambda^k(\mathcal{T}_h)$					
		$\theta = 0$		$\theta = \frac{1}{2}$		$\theta = 1$	
r	N	L^2 error	order	L^2 error	order	L^2 error	order
0	4	6.0324	-	7.0384	-	4.9203	-
	8	2.8956	1.0589	3.3954	1.0516	2.6032	0.9185
	16	1.4120	1.0362	1.6524	1.0390	1.3381	0.9601
	32	6.9675E-01	1.0190	8.8194E-01	1.0127	6.7832E-01	0.9803
1	4	2.1380	-	2.1380	-	1.7607	-
	8	5.7576E-01	2.1380	1.0122	1.2604	5.1611E-01	1.7704
	16	1.3968E-01	2.0431	4.7994E-01	1.0765	1.3327E-01	1.9533
	32	3.4791E-02	2.0053	2.3655E-01	1.0207	3.3501E-02	1.9921
2	4	5.7119E-01	-	5.0795E-01	-	5.8608E-01	-
	8	6.8785E-02	3.0537	1.0593E-01	2.2616	7.3704E-02	2.9913
	16	9.8889E-03	2.7982	2.7366E-02	1.9527	8.0736E-03	3.0981
	32	1.1587E-03	3.0933	6.9165E-03	1.9843	1.1227E-03	2.9386

3.7 Conclusions

In this chapter, we present a port-Hamiltonian discontinuous Galerkin (PHDG) discretization for a class of linear hyperbolic partial differential equations (3.6). A key element in the development of the PHDG discretization is the proof of a weak Dirac structure in the input-state-output form in broken Sobolev spaces for the dynamical system (3.6) on a tessellation of polyhedral elements. The pseudo-Poisson bracket related to this Dirac structure is proven to be a Poisson bracket since it also satisfies the Jacobi identity. The resulting PHDG discretization is therefore energy-conservative and Hamiltonian. In addition, we present an a priori error estimate of the PHDG method, which is supported by several numerical examples that demonstrate the accuracy and capabilities of the PHDG method.

3.A Proof of Lemma 3.4.7

In order to prove that the bracket (3.75) satisfies the Jacobi identity, we first define the dual of the trace operator.

Definition 3.A.1. *For the trace operator* $\mathrm{tr} : H^1\Lambda^k(\Omega) \to H^{\frac{1}{2}}\Lambda^k(\partial\Omega)$*, the dual operator* $\mathrm{tr}^* : H^{-\frac{1}{2}}\Lambda^{n-k-1}(\partial\Omega) \to H^{-1}\Lambda^{n-k}(\Omega)$ *is defined as*

$$\int_\Omega \lambda \wedge \mathrm{tr}^*(\mu) = \int_{\partial\Omega} \mathrm{tr}(\lambda) \wedge \mu, \ \forall \lambda \in H^1\Lambda^k(\Omega), \ \forall \mu \in H^{-\frac{1}{2}}\Lambda^{n-k-1}(\partial\Omega).$$

Similarly, for the trace operator $\mathrm{tr} : H\Lambda^k(\Omega) \to H^{-\frac{1}{2}}\Lambda^k(\partial\Omega)$*, the dual operator* $\mathrm{tr}^* : H^{\frac{1}{2}}\Lambda^{n-k-1}(\partial\Omega) \to H^*\Lambda^{n-k}(\Omega)$ *is defined as*

$$\int_\Omega \lambda \wedge \mathrm{tr}^*(\mu) = \int_{\partial\Omega} \mathrm{tr}(\lambda) \wedge \mu, \ \forall \lambda \in H\Lambda^k(\Omega), \ \forall \mu \in H^{\frac{1}{2}}\Lambda^{n-k-1}(\partial\Omega).$$

For any $f \in \Gamma_i$, there exist $K_L \in \mathcal{T}_h$ and $K_R \in \mathcal{T}_h$ such that $\partial K_L \cap \partial K_R = f$, and

$$\begin{aligned} \int_{f\cap\partial K_L} \mathrm{tr}(\lambda) \wedge \mu &= \int_f \mathrm{tr}(\lambda)|_L \wedge \mu|_L = \int_{K_L} \lambda \wedge \mathrm{tr}_L^*(\mu), \\ \int_{f\cap\partial K_R} \mathrm{tr}(\lambda) \wedge \mu &= -\int_f \mathrm{tr}(\lambda)|_R \wedge \mu|_R = -\int_{K_R} \lambda \wedge \mathrm{tr}_R^*(\mu), \end{aligned}$$

where the minus sign for contributions from element K_R results from the opposite orientation of K_R with respect to element K_L. Next, we give the definition of the second functional derivative.

Definition 3.A.2. *For arbitrary functionals* $F : H^2\Lambda^k(\Omega) \to \mathbb{R}$*, the second variation is defined as*

$$\begin{aligned} \delta^2 F[\lambda; \partial\lambda, \partial\hat{\lambda}] &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big(\delta F[\lambda + \epsilon\partial\hat{\lambda}; \partial\lambda] - \delta F[\lambda; \partial\lambda]\Big) \\ &= \int_\Omega \partial\lambda \wedge \frac{\delta^2 F}{\delta\lambda\delta\hat{\lambda}} \wedge *(\partial\hat{\lambda}), \end{aligned} \tag{3.120}$$

with variations $\partial\lambda, \partial\hat{\lambda} \in \Lambda^k(\Omega)$*, and the second functional derivative* $\frac{\delta^2 F}{\delta\lambda\delta\hat{\lambda}} \in \Lambda^0(\Omega)$*.*

Given the bracket (3.75), which we repeat here for clarity,

$$\{F, G\} = \sum_{K\in\mathcal{T}_h} \int_K \Big(\frac{\delta G}{\delta\eta} \wedge d\frac{\delta F}{\delta u} + (-1)^r \frac{\delta G}{\delta u} \wedge d\frac{\delta F}{\delta\eta} - \frac{c}{h}\frac{\delta G}{\delta u} \wedge *\frac{\delta F}{\delta v}$$

$$+\frac{c}{h}\frac{\delta G}{\delta v}\wedge *\frac{\delta F}{\delta u}\Big)+\sum_{K\in\mathcal{T}_h}\int_{\partial K}\Big((-1)^{r+p}\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta F}{\delta u})+u_b^1\Big)\wedge \mathrm{tr}(\frac{\delta G}{\delta \eta})$$
$$+(-1)^q\Big(\frac{1}{2}\mathrm{tr}(\frac{\delta F}{\delta \eta})+\tilde{u}_b^1\Big)\wedge \mathrm{tr}(\frac{\delta G}{\delta u})\Big)$$
$$+\sum_{f\in\Gamma_b}\int_f\Big((-1)^{r+p-1}u_b^2\wedge \mathrm{tr}(\frac{\delta F}{\delta \eta})+(-1)^{q-1}\tilde{u}_b^2\wedge \mathrm{tr}(\frac{\delta F}{\delta u})\Big),$$

we can prove now the Jacobi identity.

Lemma 3.A.1 (Jacobi identity). *For any functionals $F,G,H : L^2\Lambda^p(\mathcal{T}_h)\times L^2\Lambda^q(\mathcal{T}_h)\times L^2\Lambda^q(\mathcal{T}_h)\to\mathbb{R}$, the bracket $\{F,G\}(\eta,u,v)$ defined in (3.75) with $u_b^j=\tilde{u}_b^j=0$, $j=1,2$ at Γ_b satisfies the Jacobi identity, i.e.*

$$\{\{F,G\},H\}+\{\{G,H\},F\}+\{\{H,F\},G\}=0.$$

Proof. Firstly, we rewrite the bracket (3.75) in the following form:

$$\{F,G\}=\{F,G\}_1+\{F,G\}_2+\{F,G\}_3+\{F,G\}_4,$$

with

$$\{F,G\}_1=\sum_{K\in\mathcal{T}_h}\int_K\Big(\frac{\delta G}{\delta \eta}\wedge d(\frac{\delta F}{\delta u})+(-1)^r\frac{\delta G}{\delta u}\wedge d(\frac{\delta F}{\delta \eta})\Big)$$
$$+\sum_{K\in\mathcal{T}_h}\int_{\partial K}(-1)^q\mathrm{tr}(\frac{\delta G}{\delta \eta})\wedge \mathrm{tr}(\frac{\delta F}{\delta u}),$$
$$\{F,G\}_2=\sum_{K\in\mathcal{T}_h}\int_K\Big(-\frac{c}{h}\frac{\delta G}{\delta u}\wedge *\frac{\delta F}{\delta v}+\frac{c}{h}\frac{\delta G}{\delta v}\wedge *\frac{\delta F}{\delta u}\Big),$$
$$\{F,G\}_3=\sum_{K\in\mathcal{T}_h}\int_{\partial K}(-1)^q\frac{1}{2}\Big(\mathrm{tr}(\frac{\delta F}{\delta \eta})\wedge \mathrm{tr}(\frac{\delta G}{\delta u})-\mathrm{tr}(\frac{\delta G}{\delta \eta})\wedge \mathrm{tr}(\frac{\delta F}{\delta u})\Big),$$
$$\{F,G\}_4=\sum_{K\in\mathcal{T}_h}\int_{\partial K}(-1)^q\Big(\tilde{u}_b^1\wedge \mathrm{tr}(\frac{\delta G}{\delta u})+\mathrm{tr}(\frac{\delta G}{\delta \eta})\wedge u_b^1\Big)$$
$$+\sum_{f\in\Gamma_b}\int_f(-1)^{q-1}\Big(\mathrm{tr}(\frac{\delta F}{\delta \eta})\wedge u_b^2+\tilde{u}_b^2\wedge \mathrm{tr}(\frac{\delta F}{\delta u})\Big).$$

Note that the functionals F,G in this case are separable, i.e., F,G can be written in the form

$$F(\eta,u,v)=F_1(\eta)+F_2(u)+F_3(v),$$

$$G(\eta, u, v) = G_1(\eta) + G_2(u) + G_3(v).$$

Using the relations

$$\frac{\delta G}{\delta \eta} \wedge d(\frac{\delta F}{\delta u}) = (-1)^{q-1} d\big(\frac{\delta G}{\delta \eta} \wedge \frac{\delta F}{\delta u}\big) - (-1)^r \frac{\delta F}{\delta u} \wedge d(\frac{\delta G}{\delta \eta}),$$
$$(-1)^r \frac{\delta G}{\delta u} \wedge d(\frac{\delta F}{\delta \eta}) = (-1)^{r+p-1} d\big(\frac{\delta G}{\delta u} \wedge \frac{\delta F}{\delta \eta}\big) - \frac{\delta F}{\delta \eta} \wedge d(\frac{\delta G}{\delta u}),$$

we have

$$\begin{aligned}\{F, G\}_1 = &\sum_{K \in \mathcal{T}_h} \int_K \Big((-1)^{r+1} \frac{\delta F}{\delta u} \wedge d\big(\frac{\delta G}{\delta \eta}\big) - \frac{\delta F}{\delta \eta} \wedge d\big(\frac{\delta G}{\delta u}\big)\Big) \\ &+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} (-1)^{r+p-1} \mathrm{tr}\big(\frac{\delta G}{\delta u}\big) \wedge \mathrm{tr}\big(\frac{\delta F}{\delta \eta}\big),\end{aligned}$$

which gives that

$$\begin{aligned}\{\{F, G\}_1, H\}_1 = &\sum_{K \in \mathcal{T}_h} \int_K \Big((-1)^{r+1} \frac{\delta \{F, G\}_1}{\delta u} \wedge d\big(\frac{\delta H}{\delta \eta}\big) - \frac{\delta \{F, G\}_1}{\delta \eta} \wedge d\big(\frac{\delta H}{\delta u}\big)\Big) \\ &+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} (-1)^{r+p-1} \mathrm{tr}\big(\frac{\delta H}{\delta u}\big) \wedge \mathrm{tr}\big(\frac{\delta \{F, G\}_1}{\delta \eta}\big). \qquad (3.121)\end{aligned}$$

Next, we compute the functional derivatives $\frac{\delta\{F,G\}_1}{\delta\eta}$ and $\frac{\delta\{F,G\}_1}{\delta u}$. Using the definition of the functional derivative (3.33), and the relations

$$\frac{\delta F}{\delta \eta}(\eta + \epsilon \partial \eta) = \frac{\delta \mathcal{F}}{\delta \eta}(\eta) + \frac{\delta^2 \mathcal{F}}{\delta \eta^2} \wedge *(\epsilon \partial \eta) + o(\epsilon^2),$$
$$\frac{\delta G}{\delta \eta}(\eta + \epsilon \partial \eta) = \frac{\delta \mathcal{G}}{\delta \eta}(\eta) + \frac{\delta^2 \mathcal{G}}{\delta \eta^2} \wedge *(\epsilon \partial \eta) + o(\epsilon^2),$$

we have that

$$\begin{aligned}&\sum_{K \in \mathcal{T}_h} \int_K \frac{\delta \{F, G\}_1}{\delta \eta} \wedge \partial \eta = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \big(\{F, G\}_1(\eta + \epsilon \partial \eta) - \{F, G\}_1(\eta)\big) \\ &= \sum_{K \in \mathcal{T}_h} \int_K \big(\frac{\delta^2 G}{\delta \eta^2} \wedge *\partial \eta \wedge d(\frac{\delta F}{\delta u}) - \frac{\delta^2 F}{\delta \eta^2} \wedge *\partial \eta \wedge d(\frac{\delta G}{\delta u})\big)\end{aligned}$$

$$+(-1)^{r+p}\sum_{K\in\mathcal{T}_h}\int_{\partial K}\Big(\mathrm{tr}(\frac{\delta F}{\delta u})\wedge\mathrm{tr}(\frac{\delta^2 G}{\delta\eta^2})-\mathrm{tr}(\frac{\delta G}{\delta u})\wedge\mathrm{tr}(\frac{\delta^2 F}{\delta\eta^2})\Big)\wedge\mathrm{tr}(*\partial\eta)$$
$$=\sum_{K\in\mathcal{T}_h}\int_K\Big(\frac{\delta^2 G}{\delta\eta^2}\wedge *d(\frac{\delta F}{\delta u})-\frac{\delta^2 F}{\delta\eta^2}\wedge *d(\frac{\delta G}{\delta u})\Big)\wedge\partial\eta$$
$$-\sum_{K\in\mathcal{T}_h}\int_K *\mathrm{tr}^*\big(\mathrm{tr}(\frac{\delta^2 G}{\delta\eta^2}\wedge\frac{\delta F}{\delta u}-\frac{\delta^2 F}{\delta\eta^2}\wedge\frac{\delta G}{\delta u})\big)\wedge\partial\eta,$$

which implies that in each element $K\in\mathcal{T}_h$

$$\frac{\delta\{F,G\}_1}{\delta\eta}=\big(\frac{\delta^2 G}{\delta\eta^2}\wedge *d(\frac{\delta F}{\delta u})-\frac{\delta^2 F}{\delta\eta^2}\wedge *d(\frac{\delta G}{\delta u})\big)$$
$$-*\,\mathrm{tr}^*\big(\mathrm{tr}(\frac{\delta^2 G}{\delta\eta^2}\wedge\frac{\delta F}{\delta u}-\frac{\delta^2 F}{\delta\eta^2}\wedge\frac{\delta G}{\delta u})\big). \tag{3.122}$$

Similarly, we can obtain that

$$\frac{\delta\{F,G\}_1}{\delta u}=(-1)^r\big(\frac{\delta^2 G}{\delta u^2}\wedge *d(\frac{\delta F}{\delta\eta})-\frac{\delta^2 F}{\delta u^2}\wedge *d(\frac{\delta G}{\delta\eta})\big). \tag{3.123}$$

Substituting (3.122) and (3.123) into (3.121), we have that

$$\{\{F,G\}_1,H\}_1 \tag{3.124}$$
$$=\sum_{K\in\mathcal{T}_h}\int_K\Big(-\big(\frac{\delta^2 G}{\delta u^2}\wedge *d(\frac{\delta F}{\delta\eta})-\frac{\delta^2 F}{\delta u^2}\wedge *d(\frac{\delta G}{\delta\eta})\big)\wedge d(\frac{\delta H}{\delta\eta})$$
$$-\big(\frac{\delta^2 G}{\delta\eta^2}\wedge *d(\frac{\delta F}{\delta u})-\frac{\delta^2 F}{\delta\eta^2}\wedge *d(\frac{\delta G}{\delta u})\big)\wedge d(\frac{\delta H}{\delta u})$$
$$+*\,\mathrm{tr}^*\big(\mathrm{tr}(\frac{\delta^2 G}{\delta\eta^2}\wedge\frac{\delta F}{\delta u}-\frac{\delta^2 F}{\delta\eta^2}\wedge\frac{\delta G}{\delta u})\big)\wedge d(\frac{\delta H}{\delta u})\Big)$$
$$+(-1)^{r+p-1}\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathrm{tr}(\frac{\delta H}{\delta u})\wedge\mathrm{tr}\big(\frac{\delta^2 G}{\delta\eta^2}\wedge *d(\frac{\delta F}{\delta u})-\frac{\delta^2 F}{\delta\eta^2}\wedge *d(\frac{\delta G}{\delta u})\big)$$
$$+(-1)^q\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathrm{tr}\Big(*\,\mathrm{tr}^*\big(\mathrm{tr}(\frac{\delta^2 G}{\delta\eta^2}\wedge\frac{\delta F}{\delta u}-\frac{\delta^2 F}{\delta\eta^2}\wedge\frac{\delta G}{\delta u})\big)\Big)\wedge\mathrm{tr}(\frac{\delta H}{\delta u}).$$

We can obtain the relations for $\{\{G,H\}_1,F\}_1$ and $\{\{H,F\}_1,G\}_1$ using a cyclic permutation of F,G and H in (3.124). Summing over all terms, it is straightforward to check that

$$\{\{F,G\}_1,H\}_1+\{\{G,H\}_1,F\}_1+\{\{H,F\}_1,G\}_1=0.$$

Following the same approach as used above, we compute the functional derivatives for the brackets $\{\cdot,\cdot\}_2$ and $\{\cdot,\cdot\}_3$

$$\frac{\delta\{F,G\}_2}{\delta u} =(-1)^{q(n-q)}\frac{c}{h}\Big(\frac{\delta^2 F}{\delta u^2}\wedge\frac{\delta G}{\delta v}-\frac{\delta^2 G}{\delta u^2}\wedge\frac{\delta F}{\delta v}\Big), \tag{3.125a}$$

$$\frac{\delta\{F,G\}_2}{\delta v} =(-1)^{q(n-q)}\frac{c}{h}\Big(\frac{\delta^2 G}{\delta v^2}\wedge\frac{\delta F}{\delta u}-\frac{\delta^2 F}{\delta v^2}\wedge\frac{\delta G}{\delta u}\Big), \tag{3.125b}$$

$$\begin{aligned}\frac{\delta\{F,G\}_3}{\delta \eta} =&(-1)^{q}\frac{1}{2}\Big(\frac{\delta^2 F}{\delta \eta^2}\wedge *\mathrm{tr}^*(\mathrm{tr}(\frac{\delta G}{\delta u}))\\ &-\frac{\delta^2 G}{\delta \eta^2}\wedge *\mathrm{tr}^*\big(\mathrm{tr}(\frac{\delta F}{\delta u})\big)\Big),\end{aligned} \tag{3.125c}$$

$$\begin{aligned}\frac{\delta\{F,G\}_3}{\delta u} =&(-1)^{r+1}\frac{1}{2}\Big(\frac{\delta^2 G}{\delta u^2}\wedge *\mathrm{tr}^*(\mathrm{tr}(\frac{\delta F}{\delta \eta}))\\ &-\frac{\delta^2 F}{\delta u^2}\wedge *\mathrm{tr}^*\big(\mathrm{tr}(\frac{\delta G}{\delta \eta})\big)\Big).\end{aligned} \tag{3.125d}$$

Using the functional derivatives (3.122), (3.123) and (3.125), it is now straightforward to check that for each $i \in \{1,2,3\}$ and $j \in \{1,2,3\}$ we have

$$\begin{aligned}&\{\{F,G\}_i,H\}_j+\{\{F,G\}_j,H\}_i+\{\{G,H\}_i,F\}_j+\{\{G,H\}_j,F\}_i\\ &\{\{H,F\}_i,G\}_j+\{\{H,F\}_j,G\}_i=0.\end{aligned} \tag{3.126}$$

Finally, we consider the bracket $\{F,G\}_4$. Using the condition $u_b^i=\tilde{u}_b^i=0, i\in\{1,2\}$ for $f\in\Gamma_b$, and the relations for $u_b^1,\tilde{u}_b^1$ at Γ_i, stated in (3.76)–(3.77), the bracket $\{F,G\}_4$ can be divided into four parts

$$\{F,G\}_4=\{F,G\}_{4,1}+\{F,G\}_{4,2}+\{F,G\}_{4,3}+\{F,G\}_{4,4},$$

where

$$\begin{aligned}\{F,G\}_{4,1}=&\sum_{f\in\Gamma_i}(-1)^q\Big(\frac{1}{2}-\theta\Big)\int_f\Big(\mathrm{tr}(\frac{\delta F}{\delta\eta})\Big|_L\wedge\mathrm{tr}(\frac{\delta G}{\delta u})\Big|_L\\ &-\mathrm{tr}(\frac{\delta G}{\delta\eta})\Big|_L\wedge\mathrm{tr}(\frac{\delta F}{\delta u})\Big|_L\Big),\\ \{F,G\}_{4,2}=&\sum_{f\in\Gamma_i}(-1)^q\Big(\frac{1}{2}-\theta\Big)\int_f\Big(\mathrm{tr}(\frac{\delta F}{\delta\eta})\Big|_R\wedge\mathrm{tr}(\frac{\delta G}{\delta u})\Big|_R\\ &-\mathrm{tr}(\frac{\delta G}{\delta\eta})\Big|_R\wedge\mathrm{tr}(\frac{\delta F}{\delta u})\Big|_R\Big),\end{aligned}$$

$$
\begin{aligned}
\{F,G\}_{4,3} = &\sum_{f\in\Gamma_i}(-1)^q\Big(\theta-1\Big)\int_f\Big(\mathrm{tr}(\frac{\delta F}{\delta\eta})\Big|_R\wedge\mathrm{tr}(\frac{\delta G}{\delta u})\Big|_L\\
&-\mathrm{tr}(\frac{\delta G}{\delta\eta})\Big|_R\wedge\mathrm{tr}(\frac{\delta F}{\delta u})\Big|_L\Big),\\
\{F,G\}_{4,4} = &\sum_{f\in\Gamma_i}(-1)^q\theta\int_f\Big(\mathrm{tr}(\frac{\delta F}{\delta\eta})\Big|_L\wedge\mathrm{tr}(\frac{\delta G}{\delta u})\Big|_R\\
&-\mathrm{tr}(\frac{\delta G}{\delta\eta})\Big|_L\wedge\mathrm{tr}(\frac{\delta F}{\delta u})\Big|_R\Big).
\end{aligned}
$$

For every $f\in\Gamma_i$ the corresponding functional derivatives of these brackets are

$$
\begin{aligned}
\frac{\delta\{F,G\}_{4,1}}{\delta\eta}|_L =&(-1)^q(\frac{1}{2}-\theta)\Big(\frac{\delta^2F}{\delta\eta^2}\wedge *\mathrm{tr}_L^*\big(\mathrm{tr}(\frac{\delta G}{\delta u})|_L\big)\\
&-\frac{\delta^2G}{\delta\eta^2}\wedge *\mathrm{tr}_L^*\big(\mathrm{tr}(\frac{\delta F}{\delta u})|_L\big)\Big),\\
\frac{\delta\{F,G\}_{4,1}}{\delta\eta}|_R =&0,\\
\frac{\delta\{F,G\}_{4,1}}{\delta u}|_L =&(-1)^{r+1}(\frac{1}{2}-\theta)\Big(\frac{\delta^2G}{\delta u^2}\wedge *\mathrm{tr}_L^*\big(\mathrm{tr}(\frac{\delta F}{\delta\eta})|_L\big)\\
&-\frac{\delta^2F}{\delta u^2}\wedge *\mathrm{tr}_L^*\big(\mathrm{tr}(\frac{\delta G}{\delta\eta})|_L\big)\Big),\\
\frac{\delta\{F,G\}_{4,1}}{\delta u}|_R =&0;
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta\{F,G\}_{4,2}}{\delta\eta}|_L =&0,\\
\frac{\delta\{F,G\}_{4,2}}{\delta\eta}|_R =&(-1)^q(\theta-\frac{1}{2})\Big(\frac{\delta^2F}{\delta\eta^2}\wedge *\mathrm{tr}_R^*\big(\mathrm{tr}(\frac{\delta G}{\delta u})|_R\big)\\
&-\frac{\delta^2G}{\delta\eta^2}\wedge *\mathrm{tr}_R^*\big(\mathrm{tr}(\frac{\delta F}{\delta u})|_R\big)\Big),\\
\frac{\delta\{F,G\}_{4,2}}{\delta u}|_L =&0,\\
\frac{\delta\{F,G\}_{4,2}}{\delta u}|_R =&(-1)^{r+1}(\theta-\frac{1}{2})\Big(\frac{\delta^2G}{\delta u^2}\wedge *\mathrm{tr}_R^*\big(\mathrm{tr}(\frac{\delta F}{\delta\eta})|_R\big)\\
&-\frac{\delta^2F}{\delta u^2}\wedge *\mathrm{tr}_R^*\big(\mathrm{tr}(\frac{\delta G}{\delta\eta})|_R\big)\Big);
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta\{F,G\}_{4,3}}{\delta\eta}|_L =&0,\\
\frac{\delta\{F,G\}_{4,3}}{\delta\eta}|_R =&(-1)^q(1-\theta)\Big(\frac{\delta^2 F}{\delta\eta^2}\wedge *\mathrm{tr}_R^*\big(\mathrm{tr}(\frac{\delta G}{\delta u})|_L\big)\\
&-\frac{\delta^2 G}{\delta\eta^2}\wedge *\mathrm{tr}_R^*\big(\mathrm{tr}(\frac{\delta F}{\delta u})|_L\big)\Big),\\
\frac{\delta\{F,G\}_{4,3}}{\delta u}|_L =&(-1)^{r+1}(\theta-1)\Big(\frac{\delta^2 G}{\delta u^2}\wedge *\mathrm{tr}_L^*\big(\mathrm{tr}(\frac{\delta F}{\delta\eta})|_R\big)\\
&-\frac{\delta^2 F}{\delta u^2}\wedge *\mathrm{tr}_L^*\big(\mathrm{tr}(\frac{\delta G}{\delta\eta})|_R\big)\Big),\\
\frac{\delta\{F,G\}_{4,3}}{\delta u}|_R =&0;
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta\{F,G\}_{4,4}}{\delta\eta}|_L =&(-1)^q\theta\Big(\frac{\delta^2 F}{\delta\eta^2}\wedge *\mathrm{tr}_L^*\big(\mathrm{tr}(\frac{\delta G}{\delta u})|_R\big)\\
&-\frac{\delta^2 G}{\delta\eta^2}\wedge *\mathrm{tr}_L^*\big(\mathrm{tr}(\frac{\delta F}{\delta u})|_R\big)\Big),\\
\frac{\delta\{F,G\}_{4,4}}{\delta\eta}|_R =&0,\\
\frac{\delta\{F,G\}_{4,4}}{\delta u}|_L =&0,\\
\frac{\delta\{F,G\}_{4,4}}{\delta u}|_R =&(-1)^r\theta\Big(\frac{\delta^2 G}{\delta u^2}\wedge *\mathrm{tr}_R^*\big(\mathrm{tr}(\frac{\delta F}{\delta\eta})|_L\big)\\
&-\frac{\delta^2 F}{\delta u^2}\wedge *\mathrm{tr}_R^*\big(\mathrm{tr}(\frac{\delta G}{\delta\eta})|_L\big)\Big).
\end{aligned}
$$

Combined with the functional derivatives (3.122), (3.123) and (3.125), we obtain after a lengthy but straightforward calculation that for $i \in \{1,2,3,4\}$, $j \in \{1,2,3,4\}$ and $l \in \{1,2,3\}$ we have the relations

$$
\begin{aligned}
&\{\{F,G\}_{4,i},H\}_{4,j}+\{\{F,G\}_{4,j},H\}_{4,i}+\{\{G,H\}_{4,i},F\}_{4,j}+\{\{G,H\}_{4,j},F\}_{4,i}\\
&+\{\{H,F\}_{4,i},G\}_{4,j}+\{\{H,F\}_{4,j},G\}_{4,i}=0, \qquad (3.127)
\end{aligned}
$$

$$
\begin{aligned}
&\{\{F,G\}_{4,i},H\}_{l}+\{\{F,G\}_{l},H\}_{4,i}+\{\{G,H\}_{4,i},F\}_{l}+\{\{G,H\}_{l},F\}_{4,i}\\
&+\{\{H,F\}_{4,i},G\}_{l}+\{\{H,F\}_{l},G\}_{4,i}=0. \qquad (3.128)
\end{aligned}
$$

Using (3.126), (3.127) and (3.128), we obtain that the bracket (3.75) satisfies the Jacobi identity. □

Chapter 4

Conclusions and Outlook

4.1 Conclusions

In this dissertation, we study port-Hamiltonian formulations for several hyperbolic partial differential equations, including the incompressible Euler equations with a free surface and a class of linear hyperbolic partial differential equations. Furthermore, we present a general framework for port-Hamiltonian discontinuous Galerkin discretizations of linear hyperbolic partial differential equations. The main conclusions are as follows:

- Based on the traditional Hamiltonian formulations of the incompressible Euler equations with a free surface in vector form, generalized Hamiltonian formulations of these equations expressed in differential forms were derived. Three sets of variables were applied to represent the generalized Hamiltonian formulations: (i) (v, Σ); (ii) $(\eta, \phi_\partial, \Sigma)$; (iii) $(\omega, \phi_\partial, \Sigma)$, with v the velocity, η the solenoidal velocity, ϕ_∂ the potential function, ω the vorticity and Σ the free surface.

- Port-Hamiltonian formulations, including state dependent Dirac structures, of the incompressible Euler equations for a domain with a boundary consisting of a free surface and a fixed surface with inhomogeneous boundary conditions were derived based on the generalized Hamiltonian formulations in, the above mentioned, three sets of variables. The pseudo-Poisson bracket related to the port-Hamiltonian systems has been proven to be indeed a Poisson bracket.

- Port-Hamiltonian discontinuous Galerkin(PHDG) discretizations were constructed for a class of linear hyperbolic partial differential equations. This method was proven to be power-conserving and Hamiltonian. Error estimates for PHDG discretizations were also proven and several numerical results demonstrate the accuracy and capabilities of the PHDG finite element methods.

4.2 Outlook

There are several topics that are interesting for further research:

- In Theorem 3.4.3, we presented the weak form of the Dirac structure for a class of linear hyperbolic equations. Similar to the delta distribution defined in the 1-dimensional case, if we can define the dual of the trace operator (3.17), we can present the strong form of the Dirac structure in the input-state-output form for a class of linear hyperbolic partial differential equations. Then the structure of this system will be more clear. However, defining the dual operator of the trace operator requires a deeper understanding and study of the Sobolev spaces of differential forms and their properties.

- In Chapter 3, we construct a port-Hamiltonian discontinuous Galerkin discretization based on the linear constitutive relation (3.97). However for many physical systems, for example the nonlinear shallow water equations, there are only nonlinear constitutive relations instead of linear relations, which makes the computation much more complicated. How to deal with nonlinear constitutive relations between the variables and the corresponding functional derivatives in the numerical discretization is the main challenge for this topic.

- The Dirac structure of the incompressible Euler equations with a free surface (2.104), or the simpler case, the incompressible Euler equations without a free surface presented in [75], have solution dependent Dirac structures, which are different from the Dirac structure (3.26) discussed in Chapter 3. How to construct port-Hamiltonian discontinuous Galerkin discretizations for these systems will be one of our future goals.

Bibliography

[1] R. Abraham and J. Marsden. *Foundations of Mechanics*. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978. Second edition.

[2] R. Abraham, J. Marsden, and T. Ratiu. *Manifolds, Tensor analysis, and Applications*, volume 75 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1988.

[3] J. Anderson. *Fundamentals of Aerodynamics*. McGraw-Hill, 1991.

[4] D. Arnold. Spaces of finite element differential forms. In *Analysis and numerics of partial differential equations*, pages 117–140. Springer, 2013.

[5] D. Arnold. *Finite Element Exterior Calculus*, volume 93 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2018.

[6] D. Arnold and G. Awanou. Finite element differential forms on cubical meshes. *Math Comput.*, 83(288):1551–1570, 2014.

[7] D. Arnold, R. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, 15:1–155, 2006.

[8] D. Arnold, R. Falk, and R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bull. Amer. Math. Soc. (N.S.)*, 47(2):281–354, 2010.

[9] D. N. Arnold, D. Boffi, and F. Bonizzoni. Finite element differential forms on curvilinear cubic meshes and their approximation properties. *Numerische Mathematik*, 129(1):1–20, 2015.

[10] V. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, 16(1):319–361, 1966.

[11] V. Arnold and B. Khesin. *Topological Methods in Hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[12] J. Barlow, W. Rae, and A. Pope. *Low-Speed Wind Tunnel Testing*. John Wiley & Sons, 1999.

[13] G. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge university press, 1967.

[14] A. Beris and B. Edwards. Poisson bracket formulation of incompressible flow equations in continuum mechanics. *J. Rheol.*, 34(1):55–78, 1990.

[15] P. Bernhard. Chain differentials with an application to the mathematical fear operator. *Nonlinear Analysis: Theory, Methods & Applications*, 62(7):1225–1233, 2005.

[16] R. Bird, W. Stewart, and E. Lightfoot. *Transport Phenomena*. John Wiley & Sons, 2007.

[17] A. Brugnoli, R. Rashad, and S. Stramigioli. Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus. *J. Comput. Phys.*, 471:Paper No. 111601, 37, 2022.

[18] R. Camassa, G. Falqui, G. Ortenzi, and M. Pedroni. On variational formulations and conservation laws for incompressible 2d Euler fluids. In *J. Phys.: Conf. Ser.*, volume 482, page 012006. IOP Publishing, 2014.

[19] F. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. A partitioned finite element method for power-preserving discretization of open systems of conservation laws. *IMA J. Math. Control Inform.*, 38(2):493–533, 2021.

[20] F. Cardoso-Ribeiro, D. Matignon, and V. Pommier-Budinger. Port-Hamiltonian model of two-dimensional shallow water equations in moving containers. *IMA J. Math. Control. Inf.*, 37(4):1348–1366, 2020.

[21] J. Cervera, A. van der Schaft, and A. Baños. Interconnection of port-Hamiltonian systems and composition of dirac structures. *Automatica*, 43(2):212–225, 2007.

[22] M. Cessenat. *Mathematical Methods in Electromagnetism*, volume 41 of *Series on Advances in Mathematics for Applied Sciences*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. Linear theory and applications.

[23] S. Chaturantabut, C. Beattie, and S. Gugercin. Structure-preserving model reduction for nonlinear port-Hamiltonian systems. *SIAM J. Sci. Comput.*, 38(5):B837–B865, 2016.

[24] J.-Y. Chemin. *Perfect Incompressible Fluids*. Number 14. Oxford University Press, 1998.

[25] A. Chorin and J. Marsden. *A Mathematical Introduction to Fluid Mechanics*, volume 4 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1990.

[26] V. Chow, D. Maidment, and L. Mays. *Applied Hydrology*. McGraw-Hill, 1988.

[27] B. Cockburn, G. Karniadakis, and C.-W. Shu. *Discontinuous Galerkin methods: theory, computation and applications*, volume 11. Springer Science & Business Media, 2012.

[28] B. Cockburn and C.-W. Shu. TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. II. General framework. *Math. Comp.*, 52(186):411–435, 1989.

[29] B. Cockburn and C.-W. Shu. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.*, 35(6):2440–2463, 1998.

[30] B. Cockburn and C.-W. Shu. Runge-Kutta discontinuous Galerkin methods for convection-dominated problems. *J. Sci. Comput.*, 16(3):173–261, 2001.

[31] R. Courant and K. Friedrichs. *Supersonic Flow and Shock Waves*, volume 21. Springer Science & Business Media, 1999.

[32] T. Courant. Dirac manifolds. *Trans. Am. Math. Soc*, 319(2):631–661, 1990.

[33] G. Csató, B. Dacorogna, and O. Kneuss. *The Pullback Equation for Differential Forms*, volume 83. Springer Science & Business Media, 2011.

[34] J. Davis. *Wave Propagation in Solids and Fluids.* Springer Science & Business Media, 2012.

[35] D. Di Pietro and A. Ern. *Mathematical Aspects of Discontinuous Galerkin Methods*, volume 69. Springer Science & Business Media, 2011.

[36] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx. *Modeling and Control of Complex Physical Systems: the Port-Hamiltonian Approach.* Springer Science & Business Media, 2009.

[37] R. Easton. Introduction to Hamiltonian dynamical systems and the N-body problem. *SIAM Review*, 35(4):659–659, 1993.

[38] G. Ebin and J. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math. (2)*, 92:102–163, 1970.

[39] T. Frankel. *The Geometry of Physics: An Introduction.* Cambridge University Press, Cambridge, third edition, 2012.

[40] A. Gill. *Atmosphere-Ocean Dynamics*, volume 30. Academic press, 1982.

[41] P. Gross and P. Kotiuga. *Electromagnetic Theory and Computation: A Topological Approach*, volume 48 of *Mathematical Sciences Research Institute Publications.* Cambridge University Press, Cambridge, 2004.

[42] G. Hsiao and W. Wendland. *Boundary Integral Equations.* Springer, 2008.

[43] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-Dimensional Spaces*, volume 223 of *Operator Theory: Advances and Applications.* Birkhäuser/Springer Basel AG, Basel, 2012. Linear Operators and Linear Systems.

[44] J. John. *Gas Dynamics.* Allyn and Bacon Inc., Newton, MA, 1984.

[45] Z. Kalogiratou, T. Monovasilis, and T. E. Simos. Symplectic partitioned Runge-Kutta methods for the numerical integration of periodic and oscillatory problems. In *Recent advances in computational and applied mathematics*, pages 169–208. Springer, Dordrecht, 2011.

[46] J. Katz and A. Plotkin. *Low-Speed Aerodynamics: From Wing Theory to Panel Methods.* McGraw-Hill Education, 1991.

[47] B. Kolev. Poisson brackets in hydrodynamics. *Discrete Contin. Dyn. Syst.*, 19(3):555–574, 2007.

[48] P. Kotyczka. *Numerical Methods for Distributed Parameter Port-Hamiltonian Systems.* 2019.

[49] P. Kotyczka, B. Maschke, and L. Lefèvre. Weak form of Stokes-Dirac structures and geometric discretization of port-Hamiltonian systems. *J. Comput. Phys.*, 361:442–476, 2018.

[50] N. Kumar, J. van der Vegt, and H. Zwart. Port-Hamiltonian discontinuous Galerkin finite element methods. *arXiv preprint arXiv:2212.07041*, 2022.

[51] L. Landau and E. Lifshitz. *Fluid Mechanics: Landau and Lifshitz: Course of Theoretical Physics, Volume 6*, volume 6. Elsevier, 2013.

[52] D. Lewis, J. Marsden, R. Montgomery, and T. Ratiu. The Hamiltonian structure for dynamic free boundary problems. *Phys. D*, 18(1-3):391–404, 1986.

[53] J. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications. Vol. I.* Die Grundlehren der Mathematischen Wissenschaften, Band 183. Springer-Verlag, New York-Heidelberg, 1973. Translated from the French by P. Kenneth.

[54] A. Macchelli and C. Melchiorri. Modeling and control of the Timoshenko beam. The distributed port-Hamiltonian approach. *SIAM J. Control Optim.*, 43(2):743–767, 2004.

[55] A. Majda and A. Bertozzi. *Vorticity and Incompressible Flow.* Cambridge University Press, 2001.

[56] J. Marsden and T. Ratiu. *Introduction to Mechanics and Symmetry*, volume 17 of *Texts in Applied Mathematics.* Springer-Verlag, New York, second edition, 1999.

[57] J. Marsden and A. Weinstein. Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. *Phys. D*, 7(1-3):305–323, 1983.

[58] B. Maschke and A. van der Schaft. Interconnected mechanical systems, part II: the dynamics of spatial mechanical networks. In *Modelling and control of mechanical systems*, pages 17–30. World Scientific, 1997.

[59] A. Mazer and T. Ratiu. Hamiltonian formulation of adiabatic free boundary Euler flows. *J. Geom. Phys.*, 6(2):271–291, 1989.

[60] S. Meacham, P. Morrison, and G. Flierl. Hamiltonian moment reduction for describing vortices in shear. *Phys. Fluids*, 9(8):2310–2328, 1997.

[61] J. Miles. On Hamilton's principle for surface waves. *J. Fluid Mech.*, 83(1):153–158, 1977.

[62] P. Monk. *Finite Element Methods for Maxwell's Equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[63] L. Mora, Y. Gorrec, D. Matignon, H. Ramirez, and J. Yuz. About dissipative and pseudo port-Hamiltonian formulations of irreversible newtonian compressible flows. *IFAC-PapersOnLine*, 53(2):11521–11526, 2020.

[64] S. Morita. *Geometry of Differential Forms*. Number 201. American Mathematical Soc., 2001.

[65] P. Morrison. Hamiltonian description of the ideal fluid. *Rev. Modern Phys.*, 70(2):467–521, 1998.

[66] P. Morrison and J. Greene. Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics. *Phys. Review Letters*, 45(10):790–793, 1980.

[67] P. Olver. A nonlinear Hamiltonian structure for the Euler equations. *J. Math. Anal. Appl.*, 89(1):233–250, 1982.

[68] B. Pachpatte. *Inequalities for Differential and Integral Equations*. Academic Press, New York and London, 1998.

[69] R. Pasumarthy, V. R. Ambati, and A. van der Schaft. Port-Hamiltonian formulation of shallow water equations with coriolis force and topography. In *Proceedings of the 18th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2008)*, 2008.

[70] R. Pasumarthy and A. van der Schaft. A port-Hamiltonian approach to modeling and interconnections of canal systems. In *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems, 2006*. University of Groningen, Johann Bernoulli Institute for Mathematics, 2006.

[71] G. Payen, D. Matignon, and G. Haine. Modelling and structure-preserving discretization of Maxwell's equations as port-Hamiltonian system. *IFAC-PapersOnLine*, 53(2):7581–7586, 2020.

[72] M. Polner and J. van der Vegt. A Hamiltonian vorticity-dilatation formulation of the compressible Euler equations. *Nonlinear Anal.*, 109:113–135, 2014.

[73] S. Pope. *Turbulent Flows*. Cambridge Univ. Press, Cambridge, 2011.

[74] R. Rashad, F. Califano, F. Schuller, and S. Stramigioli. Port-Hamiltonian modeling of ideal fluid flow: Part I. Foundations and kinetic energy. *J. Geom. Phys.*, 164:Paper No. 104201, 21, 2021.

[75] R. Rashad, F. Califano, F. Schuller, and S. Stramigioli. Port-Hamiltonian modeling of ideal fluid flow: Part II. Compressible and incompressible flow. *J. Geom. Phys.*, 164:Paper No. 104199, 15, 2021.

[76] R. Rashad, F. Califano, A. van der Schaft, and S. Stramigioli. Twenty years of distributed port-Hamiltonian systems: a literature review. *IMA J. Math. Control. Inf.*, 37(4):1400–1422, 2020.

[77] J. Retherford. *Hilbert Space: Compact Operators and The Trace Theorem*, volume 27 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

[78] G. Schwarz. *Hodge Decomposition—A Method for Solving Boundary Value Problems*, volume 1607 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

[79] M. Seslija, A. van der Schaft, and J. Scherpen. Discrete exterior geometry approach to structure-preserving discretization of distributed-parameter port-Hamiltonian systems. *J. Geom. Phys.*, 62(6):1509–1531, 2012.

[80] A. Shapiro. *The Dynamics and Thermodynamics of Compressible Fluid Flow vol. 1*. John Wiley & Sons, New York, 1953.

[81] C.-W. Shu. Total-variation-diminishing time discretizations. *SIAM J. Sci. Stat. Comput.*, 9(6):1073–1084, 1988.

[82] M. Struwe. Semi-linear wave equations. *Bull. Amer. Math. Soc.*, 26(1):53–85, 1992.

[83] G. Sutton and O. Biblarz. *Rocket Propulsion Elements*. John Wiley & Sons, 2016.

[84] V. Talasila, G. Golo, and A. van der Schaft. The wave equation as a port-Hamiltonian system and a finite dimensional approximation. In *Proceedings of 15th international symposium mathematical theory of networks and systems (MTNS), South Bend*, 2002.

[85] T. Thoma and P. Kotyczka. Structure preserving discontinuous Galerkin approximation of one-dimensional port-Hamiltonian systems. *IFAC-PapersOnLine*, 56(2):6783–6788, 2023.

[86] V. Trenchant, W. Hu, H. Ramirez, and Y. Le Gorrec. Structure preserving finite differences in polar coordinates for heat and wave equations. *IFAC-PapersOnLine*, 51(2):571–576, 2018.

[87] V. Trenchant, H. Ramirez, Y. Le Gorrec, and P. Kotyczka. Finite differences on staggered grids preserving the port-Hamiltonian structure with application to an acoustic duct. *J. Comput. Phys.*, 373:673–697, 2018.

[88] G. Vallis. *Atmospheric and Oceanic Fluid Dynamics*. Cambridge University Press, 2017.

[89] A. van der Schaft. Port-Hamiltonian differential-algebraic systems. In *Surveys in differential-algebraic equations. I*, Differ.-Algebr. Equ. Forum, pages 173–226. Springer, Heidelberg, 2013.

[90] A. van der Schaft and D. Jeltsema. Port-Hamiltonian systems theory: An introductory overview. *Found. Trends. Syst. Co.*, 1(2-3):173–378, 2014.

[91] A. van der Schaft and B. Maschke. Interconnected mechanical systems, part I: geometry of interconnection and implicit Hamiltonian systems. In *Model. Mech. Syst.*, pages 1–15. World Scientific, 1997.

[92] A. van der Schaft and B. Maschke. Fluid dynamical systems as Hamiltonian boundary control systems. In *Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No. 01CH37228)*, volume 5, pages 4497–4502. IEEE, 2001.

[93] A. van der Schaft and B. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *J. Geom. Phys.*, 42(1-2):166–194, 2002.

[94] A. van der Schaft and B. Maschke. Dirac and Lagrange algebraic constraints in nonlinear port-Hamiltonian systems. *Vietnam J. Math.*, 48(4):929–939, 2020.

[95] S. Walker. *The Shapes of Things: A Practical Guide to Differential Geometry and The Shape Derivative*. SIAM, 2015.

[96] J. Wehausen and E. Laitone. *Surface Waves*. Handbuch der Physik, Springer, Germany, 1960.

[97] Y. Xu, J. van der Vegt, and O. Bokhove. Discontinuous hamiltonian finite element method for linear hyperbolic systems. *J. Sci. Comput*, 35:241–265, 2008.

[98] Y. Zel'Dovich and Y. Raizer. *Physics of Shock Waves and High-temperature Hydrodynamic Phenomena*. Courier Corporation, 2002.

[99] H. Zwart and B. Jacob. Distributed-parameter port-Hamiltonian systems. *CIMPA*, 2009.

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