



Weighted Composition Semigroups on Spaces of Continuous Functions and Their Subspaces

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Abstract

This paper is dedicated to weighted composition semigroups on spaces of continuous functions and their subspaces. We consider semigroups induced by semiflows and semicycles on Banach spaces $\mathcal{F}(\Omega)$ of continuous functions on a Hausdorff space Ω such that the norm-topology is stronger than the compact-open topology like the Hardy spaces, the weighted Bergman spaces, the Dirichlet space, the Bloch type spaces, the space of bounded Dirichlet series and weighted spaces of continuous or holomorphic functions. It was shown by Gallardo-Gutiérrez, Siskakis and Yakubovich that there are no non-trivial norm-strongly continuous weighted composition semigroups on Banach spaces $\mathcal{F}(\mathbb{D})$ of holomorphic functions on the open unit disc \mathbb{D} such that $H^\infty \subset \mathcal{F}(\mathbb{D}) \subset \mathcal{B}_1$ where H^∞ is the Hardy space of bounded holomorphic functions on \mathbb{D} and \mathcal{B}_1 the Bloch space. However, we show that there are non-trivial weighted composition semigroups on such spaces which are strongly continuous w.r.t. the mixed topology between the norm-topology and the compact-open topology. We study such weighted composition semigroups in the general setting of Banach spaces of continuous functions and derive necessary and sufficient conditions on the spaces involved, the semiflows and semicycles for strong continuity w.r.t. the mixed topology and as a byproduct for norm-strong continuity as well. Moreover, we give several characterisations of their generator and their space of norm-strong continuity.

Keywords Weighted composition semigroup · Semiflow · Semicycle · Saks space · Mixed topology

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1 Introduction

Let $(\mathcal{F}(\Omega), \|\cdot\|)$ be a Banach space of scalar-valued continuous functions on a Hausdorff space Ω such that the $\|\cdot\|$ -topology is stronger than the compact-open topology τ_{co} on $\mathcal{F}(\Omega)$. Suppose that $\varphi := (\varphi_t)_{t \geq 0}$ is a semiflow and $m := (m_t)_{t \geq 0}$ an associated semicyclope on Ω such that the induced weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ given by $C_{m,\varphi}(t)f := m_t \cdot (f \circ \varphi_t)$ for all $t \geq 0$ and $f \in \mathcal{F}(\Omega)$ is a well-defined semigroup of linear maps from $\mathcal{F}(\Omega)$ to $\mathcal{F}(\Omega)$.

In the case that $\Omega = \mathbb{D} \subset \mathbb{C}$ is the open unit disc, φ a jointly continuous holomorphic semiflow and $\mathcal{F}(\mathbb{D})$ a space of holomorphic functions on \mathbb{D} such semigroups are well-studied in the unweighted case, i.e. $m_t(z) = 1$ for all $t \geq 0$ and $z \in \mathbb{D}$, on the Hardy spaces H^p for $1 \leq p < \infty$ in [10], on the weighted Bergman spaces A_α^p for $\alpha > -1$ and $1 \leq p < \infty$ in [69], on the Dirichlet space \mathcal{D} in [70] and on more general spaces $\mathcal{F}(\mathbb{D})$ in [4, 12, 23, 38, 71]. In particular, they are always $\|\cdot\|$ -strongly continuous on H^p , A_α^p and \mathcal{D} .

The weighted case, where m is a jointly continuous holomorphic semicyclope, is more complicated and got more attention recently [5, 11, 17, 18, 39, 63, 77]. However, to the best of our knowledge from the spaces mentioned above only for the Hardy spaces H^p , $1 \leq p < \infty$, and the weighted Bergman spaces A_α^p , $1 \leq p < \infty$ and $\alpha > -1$, sufficient (non-trivial) conditions on general m are known such that the weighted composition semigroup becomes $\|\cdot\|$ -strongly continuous [47, 63, 68, 71, 77]. In [5] non-trivial sufficient conditions for $\|\cdot\|$ -strong continuity are given for Banach spaces $\mathcal{F}(\mathbb{D})$ of holomorphic functions in which the polynomials are dense in the case that $m_t = \varphi_t'$ for all $t \geq 0$.

Considering the Hardy space for $p = \infty$, it is shown in [39] that the only $\|\cdot\|$ -strongly continuous weighted composition semigroups on $\mathcal{F}(\mathbb{D})$ such that $H^\infty \subset \mathcal{F}(\mathbb{D}) \subset \mathcal{B}_1$ are the trivial ones, i.e. $\varphi_t = \text{id}$ for all $t \geq 0$. Here, \mathcal{B}_1 stands for the Bloch space. In the unweighted case this has already been observed in [3, 38]. Similarly, it is shown in [21, Theorem 7.1, p. 34] that there no non-trivial $\|\cdot\|$ -strongly continuous composition semigroups on the space \mathcal{H}^∞ of bounded Dirichlet series on the open right half-plane. Nevertheless, there are non-trivial weighted composition semigroups on such spaces $\mathcal{F}(\mathbb{D})$ resp. \mathcal{H}^∞ as well and it is said in [38, p. 494] that it would be desirable to substitute the $\|\cdot\|$ -strong continuity by a weaker property so that the description of the generator of a (weighted) composition semigroup given in [38, Main theorem, p. 490] ([39, Theorems 2.1, 3.1, pp. 68–69] in the weighted case) remains valid. This is one of the problems we solve in the present paper. We substitute the $\|\cdot\|$ -strong continuity by γ -strong continuity where $\gamma := \gamma(\|\cdot\|, \tau_{\text{co}})$ is the mixed topology between the $\|\cdot\|$ -topology and τ_{co} , which was introduced in [76] and is a Hausdorff locally convex topology.

Let us outline the content of our paper. In Sect. 2 we recall the notions of a Saks space $(X, \|\cdot\|, \tau)$, where $(X, \|\cdot\|)$ is a normed space and τ a coarser norming Hausdorff locally convex topology on X , the mixed topology $\gamma := \gamma(\|\cdot\|, \tau)$, some

background on semigroups on Hausdorff locally convex spaces and in Theorem 2.7 how γ -strongly continuous, locally γ -equicontinuous semigroups are related to the concept of a τ -bi-continuous semigroup, which was introduced in [54, 55]. We then give several examples of Saks spaces of the form $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$, which include among others the Hardy spaces H^p for $1 \leq p < \infty$, the weighted Bergman spaces A_α^p for $1 \leq p < \infty$ and $\alpha > -1$, and the Dirichlet space \mathcal{D} in Example 2.9, the v -Bloch spaces w.r.t. a continuous weight v in Example 2.10, especially the Bloch type spaces \mathcal{B}_α for $\alpha > 0$, as well as weighted spaces of continuous resp. holomorphic functions, especially, the Hardy space H^∞ and the space \mathcal{H}^∞ of bounded Dirichlet series in Examples 2.11, 2.12 and 2.13.

In Sect. 3 we recap the notions of a semiflow φ , a semicycle m for φ and introduce the notion of a co-semiflow (m, φ) . We give equivalent characterisations of their joint continuity depending on the topological properties of Ω , present several examples and generalise the concept of a generator of a semiflow, which was introduced in [10] for jointly continuous holomorphic semiflows.

In Sect. 4 we use the concepts and results of the preceding sections to prove one of our main results Proposition 4.5, which generalises [36, Proposition 2.10, p. 5] and [50, Corollary 4.3, p. 20], that the weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on a Saks space $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ is γ -strongly continuous and locally γ -equicontinuous if the semigroup is locally bounded (w.r.t. the operator norm) and the co-semiflow (m, φ) is jointly continuous. Then we derive sufficient conditions depending on (m, φ) for the local boundedness of the semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ for the Saks spaces mentioned above, in particular in the case $\mathcal{F}(\Omega) = \mathcal{H}^\infty$ in Theorem 4.16.

In Sect. 5 we turn to the generator $(A, D(A))$ of locally bounded, γ -strongly continuous weighted composition semigroups and show in Proposition 5.4 that it coincides with the Lie generator, i.e. the pointwise generator, if the Saks space $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ is sequentially complete (w.r.t. the mixed topology γ) and the co-semiflow (m, φ) is jointly continuous. This generalises [36, Proposition 2.12, p. 6] and [26, Proposition 2.4, p. 118] where $\mathcal{F}(\Omega) = \mathcal{C}_b(\Omega)$ is the space of bounded continuous functions on a completely regular Hausdorff k -space resp. Polish space Ω and m trivial. The connection to τ -bi-continuous semigroups from Sect. 2 also allows us to deduce in Proposition 5.3 that on sequentially complete Saks spaces $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ such semigroups are $\|\cdot\|$ -strongly continuous if $(\mathcal{F}(\Omega), \|\cdot\|)$ is reflexive, and to show that the space of $\|\cdot\|$ -strong continuity coincides with the $\|\cdot\|$ -closure of $D(A)$. In Theorem 5.11 we turn to the special case that $\Omega \subset \mathbb{C}$ is open, the jointly continuous co-semiflow (m, φ) continuously differentiable w.r.t. t and $\mathcal{F}(\Omega)$ for instance a space of holomorphic functions, which results in the representation

$$D(A) = \{f \in \mathcal{F}(\Omega) \mid Gf' + gf \in \mathcal{F}(\Omega)\}, \quad Af = Gf' + gf, \quad f \in D(A), \quad (1)$$

of the generator, where $G := \dot{\varphi}_0$ and $g := \dot{m}_0$ denote the derivatives w.r.t. t of φ and m in $t = 0$, respectively. In Theorem 5.10 we also handle the more complicated case where $\Omega \subset \mathbb{R}$ is open and the space $\mathcal{F}(\Omega)$ need not be a space of continuously differentiable or holomorphic functions, for example $\mathcal{F}(\Omega) = \mathcal{C}_b(\Omega)$.

Section 6 is dedicated to the converse of (1) in the sense that a γ -strongly continuous semigroup with such a generator for some holomorphic functions G and g on Ω must

be a weighted composition semigroup w.r.t. some jointly continuous holomorphic co-semiflow (m, φ) at least if $\Omega = \mathbb{D}$, $(\mathcal{F}(\mathbb{D}), \|\cdot\|, \tau_{\text{co}})$ is a sequentially complete Saks space of holomorphic functions and the embedding $\mathcal{H}(\overline{\mathbb{D}}) \hookrightarrow (\mathcal{F}(\mathbb{D}), \|\cdot\|)$ continuous where $\mathcal{H}(\overline{\mathbb{D}})$ denotes the space of holomorphic germs near $\overline{\mathbb{D}}$ with its inductive limit topology (see Theorem 6.1). In Example 6.2 we show that this embedding condition is fulfilled for the spaces like H^p for $1 \leq p \leq \infty$, A_α^p for $\alpha > -1$ and $1 \leq p < \infty$, \mathcal{D} and \mathcal{B}_α for $\alpha > 0$. Theorem 6.1 in combination with Theorem 5.11 (see also Theorem 7.4) is the counterpart of [38, Main theorem, p. 490] and [39, Theorems 2.1, 3.1, pp. 68–69] we were searching for.

In the closing Sect. 7 we generalise in Proposition 7.1 results from [25, 26] on unweighted composition semigroups on $C_b(\mathbb{R})$ to the more general setting of weighted composition semigroups on weighted spaces of continuous functions. Further, Theorem 7.2 gives us necessary and sufficient conditions for a weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on a sequentially complete Saks space $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ of holomorphic functions on an open connected set $\Omega \subset \mathbb{C}$ to be $\|\cdot\|$ -strongly continuous resp. γ -strongly continuous. In combination with Theorem 4.14 we obtain sufficient conditions on (m, φ) so that $(C_{m,\varphi}(t))_{t \geq 0}$ is $\|\cdot\|$ -strongly continuous on the Hardy spaces H^p for $1 < p < \infty$, the weighted Bergman spaces A_α^p for $\alpha > -1$ and $1 < p < \infty$ and on the Dirichlet space \mathcal{D} . For the first two spaces we get back the conditions from [63, 77] by a different proof and they improve the already known ones from [47, 68, 71]. In combination with Theorems 4.15 and 4.17 we obtain sufficient conditions on (m, φ) so that $(C_{m,\varphi}(t))_{t \geq 0}$ is γ -strongly continuous on the Hardy space H^∞ and the Bloch type spaces \mathcal{B}_α for $\alpha > 0$.

2 Background on Semigroups on Saks Spaces

In this section we recall some basic notions and results in the context of semigroups on Hausdorff locally convex spaces, bi-continuous semigroups, the mixed topology and Saks spaces to keep this work practically self-contained. We refer the interested reader for more detailed information to [22, 45, 46, 54, 79]. For a Hausdorff locally convex space (X, τ) over the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} we use the symbol $\mathcal{L}(X, \tau)$ for the space of continuous linear operators from (X, τ) to (X, τ) . If $(X, \|\cdot\|)$ is a normed space, we just write $\mathcal{L}(X) := \mathcal{L}(X, \tau_{\|\cdot\|})$ where $\tau_{\|\cdot\|}$ is the $\|\cdot\|$ -topology. First, we recall the notions of strong continuity and equicontinuity.

2.1 Definition Let (X, τ) be a Hausdorff locally convex space, I a set and $(T(t))_{t \in I}$ a family of linear maps $X \rightarrow X$.

- (a) Let I be a Hausdorff space. $(T(t))_{t \in I}$ is called τ -strongly continuous if $T(t) \in \mathcal{L}(X, \tau)$ for every $t \in I$ and the map $T_x : I \rightarrow (X, \tau)$, $T_x(t) := T(t)x$, is continuous for every $x \in X$.
- (b) Let σ be an additional Hausdorff locally convex topology on X . $(T(t))_{t \in I}$ is called σ - τ -equicontinuous if

$$\forall p \in \Gamma_\tau \exists \tilde{p} \in \Gamma_\sigma, C \geq 0 \forall t \in I, x \in X : p(T(t)x) \leq C\tilde{p}(x)$$

where Γ_τ and Γ_σ are directed systems of continuous seminorms that generate τ and σ , respectively. If $\tau = \sigma$, we just write τ -*equicontinuous* instead of τ - τ -*equicontinuous*.

In the context of semigroups of linear maps there are different degrees of equicontinuity and boundedness. In the following definition we use the symbol id for the identity map on a set X , i.e. the map $\text{id}: X \rightarrow X, \text{id}(x) := x$.

2.2 Definition Let X be a linear space and $(T(t))_{t \geq 0}$ a family of linear maps $X \rightarrow X$.

- (a) $(T(t))_{t \geq 0}$ is called a *semigroup* if $T(0) = \text{id}$ and $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$.
- (b) Let (X, τ) be a Hausdorff locally convex space. $(T(t))_{t \geq 0}$ is called *locally τ -equicontinuous* if $(T(t))_{t \in [0, t_0]}$ is τ -equicontinuous for all $t_0 \geq 0$. $(T(t))_{t \geq 0}$ is called *quasi- τ -equicontinuous* if there exists $\omega \in \mathbb{R}$ such that $(e^{-\omega t} T(t))_{t \geq 0}$ is τ -equicontinuous.
- (c) Let $(X, \|\cdot\|)$ be a normed space. $(T(t))_{t \geq 0}$ is called *locally bounded* if for all $t_0 \geq 0$ it holds that

$$\sup_{t \in [0, t_0]} \|T(t)\|_{\mathcal{L}(X)} < \infty$$

where $\|T(t)\|_{\mathcal{L}(X)} := \sup_{x \in X, \|x\| \leq 1} \|T(t)x\|$. $(T(t))_{t \geq 0}$ is called *exponentially bounded* if there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}$ for all $t \geq 0$.

Since local boundedness of semigroups of linear maps on normed spaces will be an essential condition in our work, we give the following characterisation, which carries over from the case of norm-strongly continuous semigroups on Banach spaces in [33, Chap. I, 5.5 Proposition, p. 39] and whose simple proof we omit.

2.3 Proposition Let $(X, \|\cdot\|)$ be a normed space and $(T(t))_{t \geq 0}$ a semigroup of linear maps $X \rightarrow X$. Then the following assertions are equivalent.

- (a) $(T(t))_{t \geq 0}$ is exponentially bounded.
- (b) $(T(t))_{t \geq 0}$ is locally bounded.
- (c) There exists $t_0 > 0$ such that $\sup_{t \in [0, t_0]} \|T(t)\|_{\mathcal{L}(X)} < \infty$.

If $(X, \|\cdot\|)$ is complete, then each of the preceding assertions is equivalent to:

- (d) There exists $t_0 > 0$ such that $\sup_{t \in [0, t_0]} \|T(t)x\| < \infty$ for all $x \in X$ and $T(t) \in \mathcal{L}(X)$ for all $t \in [0, t_0]$.

Let us recall the definition of the mixed topology, [76, Section 2.1], and the notion of a Saks space, [22, I.3.2 Definition, p. 27–28], which will be important for the rest of the paper.

2.4 Definition [49, 2.1 Definition, pp. 3–4] Let $(X, \|\cdot\|)$ be a normed space and τ a Hausdorff locally convex topology on X that is coarser than the $\|\cdot\|$ -topology $\tau_{\|\cdot\|}$. Then

- (a) the *mixed topology* $\gamma := \gamma(\|\cdot\|, \tau)$ is the finest linear topology on X that coincides with τ on $\|\cdot\|$ -bounded sets and such that $\tau \subset \gamma \subset \tau_{\|\cdot\|}$;
- (b) the triple $(X, \|\cdot\|, \tau)$ is called a *Saks space* if there exists a directed system of continuous seminorms Γ_τ that generates the topology τ such that

$$\|x\| = \sup_{p \in \Gamma_\tau} p(x), \quad x \in X. \quad (2)$$

The mixed topology is actually Hausdorff locally convex and the definition given above is equivalent to the one introduced by Wiweger [76, Section 2.1] due to [76, Lemmas 2.2.1, 2.2.2, p. 51].

2.5 Definition [51, Definitions 2.2, 5.7, pp. 423, 433], [49, 2.4 Definition, pp. 4–5] Let $(X, \|\cdot\|, \tau)$ be a Saks space.

- (a) We call $(X, \|\cdot\|, \tau)$ (*sequentially*) *complete* if (X, γ) is (sequentially) complete.
- (b) We call $(X, \|\cdot\|, \tau)$ *semi-Montel* if (X, γ) is a semi-Montel space.
- (c) We call $(X, \|\cdot\|, \tau)$ *C-sequential* if (X, γ) is C-sequential, i.e. every convex sequentially open subset of (X, γ) is already open (see [72, p. 273]).

A Saks space is complete if and only if the $\|\cdot\|$ -closed unit ball $B_{\|\cdot\|} := \{x \in X \mid \|x\| \leq 1\}$ is τ -complete by [22, I.1.14 Proposition, p. 11]. The definition of the mixed topology given in Definition 2.4 (a) is a bit abstract but sometimes the mixed topology coincides with a more concrete topology whose origin is [76, Theorem 3.1.1, p. 62], see for instance [22, 49, 76] for such cases. This happens for example if $(X, \|\cdot\|, \tau)$ is a semi-Montel Saks space. It is semi-Montel if and only if $B_{\|\cdot\|} := \{x \in X \mid \|x\| \leq 1\}$ is τ -compact by [22, I.1.13 Proposition, p. 11]. Since the space (X, γ) is usually not barrelled by [22, I.1.15 Proposition, p. 12], one cannot apply automatic local equicontinuity results like [46, Proposition 1.1, p. 259] to γ -strongly continuous semigroups. A way to circumvent this problem is the condition that the space is C-sequential. A sufficient condition that guarantees that $(X, \|\cdot\|, \tau)$ is C-sequential is that τ is metrisable on $B_{\|\cdot\|}$ by [53, Proposition 5.7, p. 2681–2682].

2.6 Remark If $(X, \|\cdot\|, \tau)$ is a sequentially complete Saks space, then the normed space $(X, \|\cdot\|)$ is complete because γ is a coarser topology than the norm $\|\cdot\|$ -topology and completeness of a normed space is equivalent to sequential completeness.

On sequentially complete Saks spaces $(X, \|\cdot\|, \tau)$ there is another notion of strongly continuous semigroups, namely, so-called *τ -bi-continuous semigroups* which were introduced in [54, Definition 1.3, p. 6–7] (cf. [55, Definition 3, p. 207]). Due to [52, 2.9 Remark (b)] (cf. [34, Proposition A.1.3, p. 73]) and the comments after [51, Definition 2.2, p. 423], a semigroup $(T(t))_{t \geq 0}$ of linear maps $X \rightarrow X$ is τ -bi-continuous if and only if it is locally sequentially γ -equicontinuous and the map $T_x: [0, \infty) \rightarrow (X, \gamma)$, $T_x(t) = T(t)x$, is continuous for every $x \in X$ (we note that in contrast to Definition 2.1 (a) the definition of γ -strong continuity in [52] does not include the condition that $T(t) \in \mathcal{L}(X, \gamma)$ for every $t \geq 0$). Hence every γ -strongly continuous and locally γ -equicontinuous semigroup on a sequentially complete Saks space is τ -bi-continuous and the converse is not true in general by [35, Example 4.1,

p. 320]. However, on sequentially complete C-sequential Saks spaces the converse is also true by [48, Theorem 7.4, p. 180] and [75, Theorem 7.4, p. 52], even more, every τ -bi-continuous semigroup is γ -strongly continuous and quasi- γ -equicontinuous by [50, Theorem 3.17, p. 13]. Summarising, we have the following theorem.

2.7 Theorem *Let $(X, \|\cdot\|, \tau)$ be a sequentially complete C-sequential Saks space and $(T(t))_{t \geq 0}$ a semigroup of linear maps $X \rightarrow X$. Then the following assertions are equivalent.*

- (a) $(T(t))_{t \geq 0}$ is τ -bi-continuous.
- (b) $(T(t))_{t \geq 0}$ is γ -strongly continuous and locally γ -equicontinuous.
- (c) $(T(t))_{t \geq 0}$ is γ -strongly continuous and quasi- γ -equicontinuous.

We close this section with some examples of Saks spaces and a convention we will use throughout the paper. We denote by $\mathcal{C}(\Omega)$ the space of \mathbb{K} -valued continuous functions on a Hausdorff space Ω and by τ_{co} the compact-open topology on $\mathcal{C}(\Omega)$, i.e. the topology of uniform convergence on compact subsets of Ω .

2.8 Convention Let Ω be a Hausdorff space and $(\mathcal{F}(\Omega), \|\cdot\|)$ a normed space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$. Then $\mathcal{C}(\Omega)$ induces the relative compact-open topology $\tau_{co|\mathcal{F}(\Omega)}$ on $\mathcal{F}(\Omega)$ and we get a Saks space $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co|\mathcal{F}(\Omega)})$ if and only if

- (i) for every compact set $K \subset \Omega$ there is $C \geq 0$ such that

$$\sup_{x \in K} |f(x)| \leq C \|f\|, \quad f \in \mathcal{F}(\Omega),$$

which is equivalent to the inclusion $I: (\mathcal{F}(\Omega), \|\cdot\|) \rightarrow (\mathcal{C}(\Omega), \tau_{co}), I(f) := f$, being continuous, and

- (ii) there exists a directed system of continuous seminorms $\Gamma_{\tau_{co|\mathcal{F}(\Omega)}}$ that generates the topology $\tau_{co|\mathcal{F}(\Omega)}$ such that

$$\|f\| = \sup_{p \in \Gamma_{\tau_{co|\mathcal{F}(\Omega)}}} p(f), \quad f \in \mathcal{F}(\Omega).$$

If this is fulfilled, we write that $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co}) := (\mathcal{F}(\Omega), \|\cdot\|, \tau_{co|\mathcal{F}(\Omega)})$ is a Saks space.

Further, we denote by $\mathcal{H}(\Omega) := \mathcal{C}_{\mathbb{C}}^1(\Omega)$ the space of holomorphic functions on an open set $\Omega \subset \mathbb{C}$ and set $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. Moreover, we write $\mathcal{C}^1(\Omega) := \mathcal{C}_{\mathbb{R}}^1(\Omega)$ for the space of continuously differentiable functions on an open set $\Omega \subset \mathbb{R}$. Due to [49, 3.5 Corollary, p. 9–10] and [81, Theorems 4.25, 9.8, p. 81, 260] we have the following result.

2.9 Example For the following spaces $(\mathcal{F}(\mathbb{D}), \|\cdot\|)$ the triples $(\mathcal{F}(\mathbb{D}), \|\cdot\|, \tau_{co})$ are complete semi-Montel C-sequential Saks spaces and $\mathcal{F}(\mathbb{D}) \subset \mathcal{H}(\mathbb{D})$:

- (a) The Hardy spaces $(H^p, \|\cdot\|_p)$ given by

$$H^p := \left\{ f \in \mathcal{H}(\mathbb{D}) \mid \|f\|_p^p := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty \right\}.$$

(b) The *weighted Bergman spaces* $(A_\alpha^p, \|\cdot\|_{\alpha,p})$ for $\alpha > -1$ and $1 \leq p < \infty$ given by

$$A_\alpha^p := \left\{ f \in \mathcal{H}(\mathbb{D}) \mid \|f\|_{\alpha,p}^p := \frac{\alpha+1}{\pi} \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dz < \infty \right\}.$$

(c) The *Dirichlet space* $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ given by

$$\mathcal{D} := \left\{ f \in \mathcal{H}(\mathbb{D}) \mid \|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 dz < \infty \right\}$$

with the inner product

$$\langle f, g \rangle := f(0)\overline{g(0)} + \frac{1}{\pi} \int_{\mathbb{D}} f'(z)\overline{g'(z)} dz, \quad f, g \in \mathcal{D}.$$

Moreover, the spaces in (a) for $1 < p < \infty$, in (b) for $\alpha > -1$ and $1 < p < \infty$, and in (c) are reflexive.

Our next example is the Bloch space w.r.t. a weight v .

2.10 Example ([49, 4.10 Corollary, p. 19]) For a continuous function $v: \mathbb{D} \rightarrow (0, \infty)$ we define the *v-Bloch space*

$$\mathcal{B}v(\mathbb{D}) := \{f \in \mathcal{H}(\mathbb{D}) \mid \|f\|_{\mathcal{B}v(\mathbb{D})} := |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)|v(z) < \infty\}.$$

Then $(\mathcal{B}v(\mathbb{D}), \|\cdot\|_{\mathcal{B}v(\mathbb{D})}, \tau_{\text{co}})$ is a complete semi-Montel C-sequential Saks space.

For $\alpha > 0$ we get with $v_\alpha(z) := (1 - |z|^2)^\alpha$, $z \in \mathbb{D}$, the usual Bloch type space $\mathcal{B}_\alpha := \mathcal{B}v_\alpha(\mathbb{D})$ (see [80, p. 1144]).

The case $p = \infty$ in Example 2.9 (a) can also be covered, namely, we have the following result for weighted H^∞ -spaces.

2.11 Example [49, 4.6 Corollary, p. 17–18] For an open set $\Omega \subset \mathbb{C}$ and a continuous function $v: \Omega \rightarrow (0, \infty)$ we set

$$\mathcal{H}v(\Omega) := \{f \in \mathcal{H}(\Omega) \mid \|f\|_v := \sup_{z \in \Omega} |f(z)|v(z) < \infty\}.$$

Then $(\mathcal{H}v(\Omega), \|\cdot\|_v, \tau_{\text{co}})$ is a complete semi-Montel C-sequential Saks space.

If $v = \mathbb{1}$, we set $H^\infty(\Omega) := \mathcal{H}v(\Omega)$ and $H^\infty := H^\infty(\mathbb{D})$, which is the Hardy space of bounded holomorphic functions on \mathbb{D} (see [81, p. 253]). Here, $\mathbb{1}$ denotes the map $\mathbb{1}: \Omega \rightarrow \mathbb{K}$, $\mathbb{1}(x) := 1$, on a set $\Omega \subset \mathbb{K}$. In our following example we consider the space of bounded Dirichlet series which is a topological subspace of $H^\infty(\mathbb{C}_+)$ where $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \text{Re } z > 0\}$ is the open right half-plane, see e.g. [21, p. 27].

2.12 Example We define the space \mathcal{H}^∞ of bounded Dirichlet series as the topological subspace of $H^\infty(\mathbb{C}_+)$ consisting of bounded holomorphic functions on \mathbb{C}_+ that can be written as a Dirichlet series on some half-plane contained in \mathbb{C}_+ . Further, we denote by $\|\cdot\|_{\mathcal{H}^\infty}$ the restriction of the norm $\|\cdot\|_{\mathbb{1}}$ of $H^\infty(\mathbb{C}_+)$ to \mathcal{H}^∞ . Then $(\mathcal{H}^\infty, \|\cdot\|_{\mathcal{H}^\infty}, \tau_{\text{co}})$ is a complete semi-Montel C-sequential Saks space.

Proof Due to [7, Lemma 18, p. 227] the unit ball $B_{\|\cdot\|_{\mathcal{H}^\infty}}$ of \mathcal{H}^∞ is τ_{co} -compact. Therefore $(\mathcal{H}^\infty, \|\cdot\|_{\mathcal{H}^\infty}, \tau_{\text{co}})$ is a complete semi-Montel Saks space by [22, I.3.1 Lemma, p. 27] and [22, I.1.14, I.1.13 Propositions, p. 11]. Moreover, the topology τ_{co} is metrisable on \mathcal{H}^∞ and thus $(\mathcal{H}^\infty, \|\cdot\|_{\mathcal{H}^\infty}, \tau_{\text{co}})$ is C-sequential by [53, Proposition 5.7, p. 2681–2682]. \square

For our last example of this section we recall that a completely regular Hausdorff space Ω is called a $k_{\mathbb{R}}$ -space if any map $f: \Omega \rightarrow \mathbb{R}$ whose restriction to each compact $K \subset \Omega$ is continuous, is already continuous on Ω (see [59, p. 487]). In particular, Polish spaces and locally compact Hausdorff spaces are $k_{\mathbb{R}}$ -spaces. Due to [49, 4.5 Corollary, p. 15–16] and [50, Remark 3.19 (a), p. 14] we have the following result.

2.13 Example For a completely regular Hausdorff space Ω and a continuous function $v: \Omega \rightarrow (0, \infty)$ we set

$$\mathcal{C}v(\Omega) := \{f \in \mathcal{C}(\Omega) \mid \|f\|_v := \sup_{x \in \Omega} |f(x)|v(x) < \infty\}.$$

Then $(\mathcal{C}v(\Omega), \|\cdot\|_v, \tau_{\text{co}})$ is a Saks space, which is complete if Ω is a $k_{\mathbb{R}}$ -space. It is C-sequential if Ω is a hemicompact Hausdorff $k_{\mathbb{R}}$ -space or a Polish space.

If $\Omega \subset \mathbb{C}$ is open, then $\mathcal{H}v(\Omega)$ from Example 2.11 is a subspace of $\mathcal{C}v(\Omega)$ and the mixed topologies are compatible, i.e.

$$\gamma(\|\cdot\|_{v|\mathcal{H}v(\Omega)}, \tau_{\text{co}|\mathcal{H}v(\Omega)}) = \gamma(\|\cdot\|_v, \tau_{\text{co}|\mathcal{C}v(\Omega)})|_{\mathcal{H}v(\Omega)}$$

by [22, I.4.6 Lemma, p. 44]. If $v = \mathbb{1}$, then we get the space of bounded continuous functions $\mathcal{C}_b(\Omega) = \mathcal{C}v(\Omega)$ on Ω and $\|\cdot\|_v = \|\cdot\|_\infty$ is the supremum norm. Further examples of Saks spaces may be found, for instance, in [22, 49, 50, 52, 54].

3 Semiflows, Semicocycles and Semicoboundaries

In this section we recall the notions and properties of semiflows, associated semicycles and of semicoboundaries, which form a special class of semicycles.

3.1 Definition Let I, Ω and Y be Hausdorff spaces and $\varphi := (\varphi_t)_{t \in I}$ a family of functions $\varphi_t: \Omega \rightarrow Y$.

- (a) We call φ *separately continuous* if φ_t and $\varphi_{(\cdot)}(x): I \rightarrow Y$ are continuous for all $t \in I$ and $x \in \Omega$.
- (b) We call φ *jointly continuous* if the map $I \times \Omega \rightarrow Y, (t, x) \mapsto \varphi_t(x)$, is continuous where $I \times \Omega$ is equipped with the product topology.

- (c) Let $I := [0, \infty)$. We say that $\varphi_{(\cdot)}(x) \in \mathcal{C}^1[0, \infty)$ for $x \in \Omega$ if $\varphi_{(\cdot)}(x)$ is continuously differentiable on $[0, \infty)$ where differentiability in $t = 0$ means right-differentiability in $t = 0$. Further, we set $\dot{\varphi}_{t_0}(x) := (\frac{\partial}{\partial t} \varphi_{(\cdot)}(x))(t_0)$ for all $t_0 \in [0, \infty)$.
- (d) Let $\Omega \subset \mathbb{K}$ be open. If $\varphi_t \in \mathcal{C}_{\mathbb{K}}^1(\Omega)$ for $t \in I$, we set $\varphi'_t(x_0) := (\frac{\partial}{\partial x} \varphi_t)(x_0)$ for all $x_0 \in \Omega$.

Let us come to semiflows.

3.2 Definition Let Ω be a Hausdorff space. A family $\varphi := (\varphi_t)_{t \geq 0}$ of continuous functions $\varphi_t: \Omega \rightarrow \Omega$ is called a *semiflow* if

- (i) $\varphi_0(x) = x$ for all $x \in \Omega$, and
(ii) $\varphi_{t+s}(x) = (\varphi_t \circ \varphi_s)(x)$ for all $t, s \geq 0$ and $x \in \Omega$.

We call a semiflow φ *trivial* and write $\varphi = \text{id}$ if $\varphi_t = \text{id}$ for all $t \geq 0$. We call a semiflow φ a C_0 -*semiflow* if $\lim_{t \rightarrow 0^+} \varphi_t(x) = x$ for all $x \in \Omega$. If $\Omega \subset \mathbb{C}$ is open, we call a semiflow φ *holomorphic* if $\varphi_t \in \mathcal{H}(\Omega)$ for all $t \geq 0$.

A lot of examples of jointly continuous holomorphic semiflows and their whole classification are given in [71, p. 4–5] for $\Omega = \mathbb{D}$, in [1, Proposition 1.4.26, p. 98] for $\Omega = \mathbb{C}$, in [1, Proposition 1.4.27, p. 98] for $\Omega = \mathbb{C} \setminus \{0\}$, in [1, Proposition 1.4.29, p. 99] for $\Omega = \{z \in \mathbb{C} \mid r < |z| < 1\}$, $0 < r < 1$, and in [1, Proposition 1.4.30, p. 99] for $\Omega = \mathbb{D} \setminus \{0\}$. Further examples may be found in [13, Chap. 8] and also in the following sections of our paper.

3.3 Proposition ([24, Theorems 2.2, 2.3, p. 692], [20, 2., p. 318–319]) *Let φ be a semiflow on a locally compact Hausdorff space Ω .*

- (a) *Let Ω be σ -compact. Then φ is jointly continuous if and only if φ is C_0 .*
(b) *Let Ω be metrisable. Then φ is jointly continuous if and only if φ is separately continuous.*

An important concept for semiflows is their generator.

3.4 Definition Let φ be a semiflow on a Hausdorff space Ω such that $\varphi_{(\cdot)}(x) \in \mathcal{C}^1[0, \infty)$ for all $x \in \Omega$. A continuous function $G: \Omega \rightarrow \Omega$ is called the *generator* of φ if $\dot{\varphi}_t(x) = G(\varphi_t(x))$ for all $t \geq 0$ and $x \in \Omega$.

3.5 Remark If existing, the generator of φ is uniquely determined because we have $G(x) = G(\varphi_0(x)) = \dot{\varphi}_0(x)$ for all $x \in \Omega$.

The generator is also called the *speed* of the semiflow (see [6, p. 210] where the symbol λ is used for G). For a separately continuous semiflow φ the existence of the generator is equivalent to right-differentiability in $t = 0$ and continuity of $\dot{\varphi}_0$.

3.6 Proposition *Let Ω be a Hausdorff space and φ a separately continuous semiflow on Ω . Then $\varphi_{(\cdot)}(x) \in \mathcal{C}^1[0, \infty)$ for all $x \in \Omega$ and $\dot{\varphi}_t \in \mathcal{C}(\Omega)$ for all $t \geq 0$ if and only if $\varphi_{(\cdot)}(x)$ is right-differentiable in $t = 0$ for all $x \in \Omega$ and $\dot{\varphi}_0 \in \mathcal{C}(\Omega)$. In this case $\dot{\varphi}_t(x) = \dot{\varphi}_0(\varphi_t(x))$ for all $t \geq 0$ and $x \in \Omega$, and $\dot{\varphi}_0$ is the generator of φ .*

Proof We only need to prove the implication \Leftarrow . Let $\varphi_{(\cdot)}(x)$ be right-differentiable in $t = 0$ for all $x \in \Omega$ and $\dot{\varphi}_0 \in \mathcal{C}(\Omega)$. For $x \in \Omega$ we claim that $\varphi_{(\cdot)}(x)$ is continuously right-differentiable on $[0, \infty)$ with right-derivative $\dot{\varphi}_0(\varphi_t(x))$ for all $t \geq 0$. Indeed, we have

$$\lim_{s \rightarrow 0^+} \frac{\varphi_{t+s}(x) - \varphi_t(x)}{s} = \lim_{s \rightarrow 0^+} \frac{\varphi_s(\varphi_t(x)) - \varphi_t(x)}{s} = \dot{\varphi}_0(\varphi_t(x))$$

for all $t \geq 0$. Thus $\varphi_{(\cdot)}(x)$ is right-differentiable on $[0, \infty)$ and the right-derivative is continuous (in t) because $\dot{\varphi}_0 \in \mathcal{C}(\Omega)$ and φ is separately continuous. It follows that the continuous function $\varphi_{(\cdot)}(x)$ is continuously differentiable on $[0, \infty)$ with $\dot{\varphi}_t(x) = \dot{\varphi}_0(\varphi_t(x))$ for all $t \geq 0$ (see e.g. [62, Chap. 2, Corollary 1.2, p. 43]). \square

Further sufficient and necessary conditions for a given continuous function $G: \mathbb{R} \rightarrow \mathbb{R}$ to be the generator of a jointly continuous flow on \mathbb{R} are contained in [6, Lemma 2.2, p. 212]. The notion of a generator in the case of a jointly continuous holomorphic semiflow was introduced in [10, p. 103]. In this case the generator always exists and is not only continuous but even holomorphic.

3.7 Theorem [10, (1.1) Theorem, p. 101–102] *Let $\Omega \subset \mathbb{C}$ be open and φ a jointly continuous holomorphic semiflow on Ω . Then $\varphi_{(\cdot)}(z) \in \mathcal{C}^1[0, \infty)$ for all $z \in \Omega$ and there is $G \in \mathcal{H}(\Omega)$ such that $\dot{\varphi}_t(z) = G(\varphi_t(z))$ for all $t \geq 0$ and $z \in \Omega$.*

We also have the following generalisation of [13, Proposition 10.1.8 (1), p. 276–277] where $\mathbb{K} = \mathbb{C}$ and $\Omega = \mathbb{D}$.

3.8 Proposition *Let φ be a semiflow on an open set $\Omega \subset \mathbb{K}$ such that $\varphi_{(\cdot)}(x) \in \mathcal{C}^1[0, \infty)$ for all $x \in \Omega$ and $\varphi_t \in \mathcal{C}_{\mathbb{K}}^1(\Omega)$ for all $t \geq 0$. Then*

$$\dot{\varphi}_0(\varphi_t(x)) = \varphi'_t(x)\dot{\varphi}_0(x), \quad t \geq 0, x \in \Omega.$$

If in addition φ has a generator G , then

$$\dot{\varphi}_t(x) = G(\varphi_t(x)) = \varphi'_t(x)G(x), \quad t \geq 0, x \in \Omega.$$

Proof For all $s, t \geq 0$ and $x \in \Omega$ we have $\varphi_s(\varphi_t(x)) = \varphi_{t+s}(x) = (\varphi_t \circ \varphi_s)(x)$. By differentiating w.r.t. s we get

$$\dot{\varphi}_s(\varphi_t(x)) = \varphi'_s(\varphi_t(x))\dot{\varphi}_s(x)$$

for all $s, t \geq 0$ and $x \in \Omega$, which yields $\dot{\varphi}_0(\varphi_t(x)) = \varphi'_t(x)\dot{\varphi}_0(x)$ for $s = 0$. The rest of the statement follows from the definition of a generator and Remark 3.5. \square

Next, let us recall the notion of a semicycle for a semiflow.

3.9 Definition Let $\varphi := (\varphi_t)_{t \geq 0}$ be a semiflow on a Hausdorff space Ω . A family $m := (m_t)_{t \geq 0}$ of continuous functions $m_t: \Omega \rightarrow \mathbb{K}$ is called a multiplicative *semicycle* for φ if

- (i) $m_0(x) = 1$ for all $x \in \Omega$, and
- (ii) $m_{t+s}(x) = m_t(x)m_s(\varphi_t(x))$ for all $t, s \geq 0$ and $x \in \Omega$.

We call a semicyclole m *trivial* and write $m = \mathbb{1}$ if $m_t = \mathbb{1}$ for all $t \geq 0$. We call a semicyclole m a C_0 -semicyclole if $\lim_{t \rightarrow 0^+} m_t(x) = 1$ for all $x \in \Omega$. If $\Omega \subset \mathbb{C}$ is open, we call a semicyclole m *holomorphic* if $m_t \in \mathcal{H}(\Omega)$ for all $t \geq 0$.

If φ is a holomorphic semiflow on an open set $\Omega \subset \mathbb{C}$, then a simple example of a holomorphic semicyclole m for φ is given by the complex derivatives $m_t := \varphi'_t$ for $t \geq 0$ by the chain rule. There is an analog of Proposition 2.3 for semicycloles due to König [47] which will be important later on. The similarity to Proposition 2.3 is not a coincidence because they use the same ideas, which can be found in the proofs of [27, VIII.1.4 Lemma, VIII.1.5 Corollary, p. 618–619].

3.10 Proposition *Let φ be a semiflow on a Hausdorff space Ω and m a semicyclole for φ . Then the following assertions are equivalent.*

- (a) *There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|m_t\|_\infty \leq Me^{\omega t}$ for all $t \geq 0$.*
- (b) *It holds that $\sup_{t \in [0, t_0]} \|m_t\|_\infty < \infty$ for all $t_0 \geq 0$.*
- (c) *There exists $t_0 > 0$ such that $\sup_{t \in [0, t_0]} \|m_t\|_\infty < \infty$.*
- (d) *It holds that $\|m_t\|_\infty < \infty$ for all $t \geq 0$.*
- (e) *It holds that $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$.*

Proof The implications (d) \Rightarrow (a) \Rightarrow (e) \Rightarrow (c) \Rightarrow (d) follow from the proof of [47, Lemma 2.1 (a), p. 472] (we note that it is not relevant for the proof that $(h_t)_{t \geq 0} := m$ in the cited lemma is assumed to be holomorphic and Ω to be equal to \mathbb{D}). Moreover, the implications (a) \Rightarrow (b) \Rightarrow (c) clearly hold. □

3.11 Definition Let φ be a semiflow on a Hausdorff space Ω and m a semicyclole for φ . We call the tuple (m, φ) a *co-semiflow* on Ω . We call a co-semiflow (m, φ) jointly continuous (separately continuous, C_0 , holomorphic) if φ and m are both jointly continuous (separately continuous, C_0 , holomorphic).

3.12 Proposition *Let (m, φ) be a co-semiflow on an open subset Ω of a metric space and φ jointly continuous. Then m is jointly continuous if and only if m is C_0 .*

Proof The implication \Rightarrow clearly holds. The other implication follows from [31, Definition 3.1, p. 1203], the proof of [31, Theorem 3.1, p. 1204] with $\mathcal{D} := \Omega$ and $\mathcal{A} := \mathbb{C}$, and the observation that the assumption that \mathcal{D} is an open connected subset of a Banach space (see [31, p. 1200]) is not needed in the proof of [31, Theorem 3.1, p. 1204]. □

Analogously to Proposition 3.6 we have the following result for semicycloles.

3.13 Proposition *Let Ω be a Hausdorff space and (m, φ) a separately continuous co-semiflow on Ω . Then $m_{(\cdot)}(x) \in C^1[0, \infty)$ for all $x \in \Omega$ and $\dot{m}_t \in \mathcal{C}(\Omega)$ for all $t \geq 0$ if and only if $m_{(\cdot)}(x)$ is right-differentiable in $t = 0$ for all $x \in \Omega$ and $\dot{m}_0 \in \mathcal{C}(\Omega)$. In this case $\dot{m}_t(x) = m_t(x)\dot{m}_0(\varphi_t(x))$ and $m_t(x) = \exp(\int_0^t \dot{m}_0(\varphi_s(x))ds)$ for all $t \geq 0$ and $x \in \Omega$.*

Proof We only need to prove the implication \Leftarrow . Let $m_{(\cdot)}(x)$ be right-differentiable in $t = 0$ for all $x \in \Omega$ and $\dot{m}_0 \in \mathcal{C}(\Omega)$. For $x \in \Omega$ we claim that $m_{(\cdot)}(x)$ is continuously right-differentiable on $[0, \infty)$ with right-derivative $m_t(x)\dot{m}_0(\varphi_t(x))$ for all $t \geq 0$. Indeed, we have

$$\lim_{s \rightarrow 0^+} \frac{m_{t+s}(x) - m_t(x)}{s} = m_t(x) \lim_{s \rightarrow 0^+} \frac{m_s(\varphi_t(x)) - 1}{s} = m_t(x)\dot{m}_0(\varphi_t(x))$$

for all $t \geq 0$. Thus $m_{(\cdot)}(x)$ is right-differentiable on $[0, \infty)$ and the right-derivative is continuous (in t) because $\dot{m}_0 \in \mathcal{C}(\Omega)$ and (m, φ) is separately continuous. It follows from [62, Chap. 2, Corollary 1.2, p. 43] that the continuous function $m_{(\cdot)}(x)$ is continuously differentiable on $[0, \infty)$ with $\dot{m}_t(x) = m_t(x)\dot{m}_0(\varphi_t(x))$ for all $t \geq 0$. Thus for $x \in \Omega$ we know that the map $t \mapsto m_t(x)$ solves the initial value problem

$$\dot{w}(t) = w(t)g(\varphi_t(x)), \quad t \geq 0, \quad w(0) = 1,$$

with $g := \dot{m}_0 \in \mathcal{C}(\Omega)$. Another solution of this initial value problem is given by the map $t \mapsto \exp(\int_0^t \dot{m}_0(\varphi_s(x))ds)$. Since the solution of this initial value problem is unique (e.g. by [42, Chap. 1, Theorem 3, p. 7]), we get that $m_t(x) = \exp(\int_0^t \dot{m}_0(\varphi_s(x))ds)$ for all $t \geq 0$. \square

We have the following construction of a semicyclope given a jointly continuous semiflow and a continuous function on a locally compact metric space.

3.14 Proposition *Let φ be a jointly continuous semiflow on a locally compact metric space Ω and $g \in \mathcal{C}(\Omega)$. Then the following assertions hold.*

- (a) *The family $m := (m_t)_{t \geq 0}$ given by $m_t(x) := \exp(\int_0^t g(\varphi_s(x))ds)$ for all $t \geq 0$ and $x \in \Omega$ is a jointly continuous semicyclope for φ . In particular, $m_{(\cdot)}(x) \in \mathcal{C}^1[0, \infty)$, $\dot{m}_t(x) = m_t(x)g(\varphi_t(x))$, $\dot{m}_0(x) = g(x)$ and $m_t(x) \neq 0$ for all $t \geq 0$ and $x \in \Omega$.*
- (b) *If $\Omega \subset \mathbb{K}$ is open and $\varphi_t, g \in \mathcal{C}_{\mathbb{K}}^1(\Omega)$ for all $t \geq 0$, then $m_t \in \mathcal{C}_{\mathbb{K}}^1(\Omega)$ for all $t \geq 0$ with m from part (a).*

Proof (a) Let $t \geq 0$. We note that $g \circ \varphi_t \in \mathcal{C}(\Omega)$ and the map $g(\varphi_{(\cdot)}(x))$ is continuous and therefore integrable on $[0, t]$. For every $x_0 \in \Omega$ there is a compact neighbourhood $U \subset \Omega$ of x_0 such that $|g(\varphi_{(\cdot)}(x))| \leq \sup\{|g(\varphi_s(w))| \mid (s, w) \in [0, t] \times U\} < \infty$ on $[0, t]$ for all $x \in U$ because Ω is locally compact, $g \in \mathcal{C}(\Omega)$ and φ jointly continuous. Setting $F_t : \Omega \rightarrow \mathbb{K}$, $F_t(x) := \int_0^t g(\varphi_s(x))ds$, we deduce that F_t is continuous on the metric space Ω by [32, 5.6 Satz, p. 147] and thus $m_t = \exp \circ F_t$ as well. From here it is easy to check that m is a C_0 -semicyclope for φ and so jointly continuous by Proposition 3.12. The rest of statement (a) follows from the integral form of $m_t(x)$ and Proposition 3.13.

(b) In the case $\mathbb{K} = \mathbb{R}$ the statement follows from [32, 5.7 Satz, p. 147–148] and in the case $\mathbb{K} = \mathbb{C}$ from [32, 5.8 Satz, p. 148]. \square

On connected proper subsets of \mathbb{C} every jointly continuous holomorphic semicyclope of a jointly continuous holomorphic semiflow is actually of the integral form in Proposition 3.14 (a).

3.15 Proposition Let $\Omega \subset \mathbb{C}$ be open and connected, and (m, φ) a jointly continuous holomorphic co-semiflow on Ω . Then it holds $m_{(\cdot)}(z) \in \mathcal{C}^1[0, \infty)$, $\dot{m}_t \in \mathcal{H}(\Omega)$ and $m_t(z) = \exp(\int_0^t \dot{m}_0(\varphi_s(z)) ds)$ for all $t \geq 0$ and $z \in \Omega$.

Proof Due to [44, Theorem 4, p. 3392] we have $m_{(\cdot)}(z) \in \mathcal{C}^1[0, \infty)$ for all $z \in \Omega$ and $\dot{m}_t \in \mathcal{H}(\Omega)$ for all $t \geq 0$. Then it follows from Proposition 3.13 that $m_t(z) = \exp(\int_0^t \dot{m}_0(\varphi_s(z)) ds)$ for all $t \geq 0$ and $z \in \Omega$. \square

Proposition 3.15 improves [44, Theorem 3, p. 3392] with $g = \dot{m}_0 \in \mathcal{H}(\Omega)$ from simply connected open $\Omega \subset \mathbb{C}$ to just connected open $\Omega \subset \mathbb{C}$. Moreover, Proposition 3.15 implies [47, Lemma 2.1 (b), p. 472]. There is another way to construct semicycles for a semiflow apart from the one in Proposition 3.14, namely, so-called (semi)coboundaries, see e.g. [43, p. 240], [47, p. 469–470] and [61, p. 513]. For that construction we need the notion of a fixed point of a semiflow.

3.16 Definition Let Ω be a Hausdorff space and φ a semiflow on Ω . We call $x \in \Omega$ a *fixed point* of φ if it is a common fixed point of all φ_t , i.e. $\varphi_t(x) = x$ for all $t \geq 0$. We denote the set of all fixed points of φ by $\text{Fix}(\varphi) := \{x \in \Omega \mid \forall t \geq 0 : \varphi_t(x) = x\}$.

Let Ω be a Hausdorff space and φ a semiflow on Ω . Let $\omega \in \mathcal{C}(\Omega)$, $\omega \neq 0$, such that its set of zeros $N_\omega := \{x \in \Omega \mid \omega(x) = 0\}$ fulfils that $N_\omega \subset \text{Fix}(\varphi)$, and that $\Omega \setminus N_\omega$ is dense in Ω . We set

$$m_t^\omega(x) := \frac{\omega(\varphi_t(x))}{\omega(x)}, \quad t \geq 0, x \in \Omega \setminus N_\omega,$$

and note that $m_t^\omega : \Omega \setminus N_\omega \rightarrow \mathbb{K}$ is continuous for all $t \geq 0$. Moreover,

$$m_0^\omega(x) = 1 \quad \text{and} \quad m_{t+s}^\omega(x) = m_t^\omega(x)m_s^\omega(\varphi_t(x)), \quad t, s \geq 0, x \in \Omega \setminus N_\omega. \quad (3)$$

If $N_\omega \neq \emptyset$, suppose additionally that m_t^ω is continuously extendable on Ω for all $t \geq 0$ and denote the (unique) extension by m_t^ω as well. Then (3) also holds for $x \in N_\omega$ by continuity and the density of $\Omega \setminus N_\omega$ in Ω . Thus $m^\omega := (m_t^\omega)_{t \geq 0}$ is a semicycle for φ under this assumption.

3.17 Definition Let Ω be a Hausdorff space and φ a semiflow on Ω . A semicycle m for φ is called a *semicoboundary* for φ if there is $\omega \in \mathcal{C}(\Omega)$, $\omega \neq 0$, such that $N_\omega \subset \text{Fix}(\varphi)$, the set $\Omega \setminus N_\omega$ is dense in Ω , and $m = m^\omega$.

3.18 Example Let $\Omega \subset \mathbb{C}$ be open and connected, φ a holomorphic semiflow on Ω and $\omega \in \mathcal{H}(\Omega)$, $\omega \neq 0$, such that $N_\omega \subset \text{Fix}(\varphi)$. Then the semicoboundary m^ω for φ satisfies

$$m_t^\omega(z) = \begin{cases} \frac{\omega(\varphi_t(z))}{\omega(z)} & , z \in \Omega \setminus N_\omega, \\ (\varphi_t'(z))^{\text{ord}_\omega(z)} & , z \in N_\omega, \end{cases}$$

and $m_t^\omega \in \mathcal{H}(\Omega)$ for all $t \geq 0$ where $\text{ord}_\omega(z) \in \mathbb{N}$ is the order of the zero $z \in N_\omega$ of ω . If φ is additionally jointly continuous, then m^ω is jointly continuous.

Proof First, we note that $\Omega \setminus N_\omega$ is dense in Ω since N_ω is discrete in Ω . For $b \in N_\omega$ there is $\psi \in \mathcal{H}(\Omega)$ such that $\psi(b) \neq 0$ and $\omega(z) = (z - b)^n \psi(z)$ for all $z \in \Omega$ with $n := \text{ord}_\omega(b)$. Let $t \geq 0$. Then we have for all $z \in \Omega \setminus N_\omega$

$$\frac{\omega(\varphi_t(z))}{\omega(z)} = \left(\frac{\varphi_t(z) - b}{z - b} \right)^n \frac{\psi(\varphi_t(z))}{\psi(z)} \xrightarrow{z \rightarrow b} (\varphi'_t(b))^n \frac{\psi(\varphi_t(b))}{\psi(b)} = (\varphi'_t(b))^n$$

because $N_\omega \subset \text{Fix}(\varphi)$. Therefore m_t^ω is continuously extendable on Ω and this extension is holomorphic on Ω by Riemann’s removable singularity theorem.

Now, if φ is additionally jointly continuous, then we have $\lim_{t \rightarrow 0^+} m_t^\omega(z) = \omega(z)/\omega(z) = 1$ for all $z \in \Omega \setminus N_\omega$. Furthermore, the map $[0, \infty) \rightarrow \mathbb{C}, t \mapsto \varphi'_t(z)$, is continuous for every $z \in \Omega$ by [43, Lemma 2.1, p. 242] with connected $G := \Omega$. Hence we have $\lim_{t \rightarrow 0^+} m_t^\omega(z) = (\varphi'_0(z))^{\text{ord}_\omega(z)} = 1^{\text{ord}_\omega(z)} = 1$ for all $z \in N_\omega$. We conclude that m_t^ω is C_0 and thus jointly continuous by Proposition 3.12. \square

For $\Omega = \mathbb{D}$ the previous example is already contained in [68, p. 361–362]. We already observed that $(\varphi'_t)_{t \geq 0}$ is a simple example of a semicocycle of a holomorphic semiflow φ on an open set $\Omega \subset \mathbb{C}$. If Ω is also connected, then it holds $\text{Fix}(\varphi) = N_G$ with the generator G of φ by [1, Proposition 1.4.13 (i), p. 89]. Thus $(\varphi'_t)_{t \geq 0}$ is even a semicoboundary by Theorem 3.7 and Proposition 3.8, which is jointly continuous by the arguments in the example above.

3.19 Example Let $\Omega \subset \mathbb{C}$ be open and connected, and φ a jointly continuous holomorphic semiflow on Ω with generator $G \neq 0$. Then $m_t^G(z) = \varphi'_t(z)$ for all $t \geq 0$ and $z \in \Omega$, and m^G is jointly continuous.

For $\Omega = \mathbb{D}$ the previous example is also contained in [71, Example 7.4, p. 247–248]. The relation between semicoboundaries and the semicocycles from Proposition 3.14 of a jointly continuous holomorphic semiflow on $\Omega = \mathbb{D}$ is given in [47, Lemma 2.2, p. 472].

4 Semigroups of Weighted Composition Operators

Before introducing weighted composition semigroups induced by a co-semiflow (m, φ) , we start this section with weighted composition families induced by a tuple (m, φ) which need not be a co-semiflow. First, we generalise a part of [41, Proposition 1, p. 307].

4.1 Proposition *Let Ω be a Hausdorff space and $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ a Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$. Let I be a set, $\varphi := (\varphi_t)_{t \in I}$ and $m := (m_t)_{t \in I}$ be families of functions $\varphi_t : \Omega \rightarrow \Omega$ and $m_t : \Omega \rightarrow \mathbb{K}$ such that*

- (i) $C_{m, \varphi}(t)f := m_t \cdot (f \circ \varphi_t) \in \mathcal{F}(\Omega)$ for all $t \in I$ and $f \in \mathcal{F}(\Omega)$, and
- (ii) $\varphi_I(K) := \bigcup_{t \in I} \varphi_t(K)$ is relatively compact in Ω and $m_I(K) := \bigcup_{t \in I} m_t(K)$ is bounded in \mathbb{K} for all compact $K \subset \Omega$.

Then $(C_{m, \varphi}(t))_{t \in I}$ is τ_{co} -equicontinuous. If in addition $\sup_{t \in I} \|C_{m, \varphi}(t)\|_{\mathcal{L}(\mathcal{F}(\Omega))} < \infty$, then $(C_{m, \varphi}(t))_{t \in I}$ is γ -equicontinuous.

Proof First, we observe that $(C_{m,\varphi}(t))_{t \in I}$ is a well-defined family of linear maps $\mathcal{F}(\Omega) \rightarrow \mathcal{F}(\Omega)$ by condition (i). Second, for compact $K \subset \Omega$ we note that

$$\sup_{x \in K} |C_{m,\varphi}(t)f(x)| = \sup_{x \in K} |m_t(x)f(\varphi_t(x))| \leq \sup_{z \in m_I(K)} |z| \sup_{x \in \varphi_I(K)} |f(x)|$$

for all $t \in I$ and $f \in \mathcal{F}(\Omega)$, implying that $(C_{m,\varphi}(t))_{t \in I}$ is τ_{co} -equicontinuous by condition (ii) and the continuity of the functions in $\mathcal{F}(\Omega)$.

Now, suppose that $\sup_{t \in I} \|C_{m,\varphi}(t)\|_{\mathcal{L}(\mathcal{F}(\Omega))} < \infty$. Since τ_{co} is a coarser topology than γ , we obtain from the part above that the family $(C_{m,\varphi}(t))_{t \in I}$ is γ - τ_{co} -equicontinuous. It follows from [50, 3.16 Proposition, (f) \Leftrightarrow (g), p. 12–13] that $(C_{m,\varphi}(t))_{t \in I}$ is even γ -equicontinuous. \square

4.2 Remark Let Ω be a Hausdorff space, I a compact Hausdorff space, $\varphi := (\varphi_t)_{t \in I}$ and $m := (m_t)_{t \in I}$ families of functions $\varphi_t: \Omega \rightarrow \Omega$ and $m_t: \Omega \rightarrow \mathbb{K}$. If φ and m are both jointly continuous, then condition (ii) of Proposition 4.1 is fulfilled since $\varphi_I(K)$ and $m_I(K)$ are compact for all compact $K \subset \Omega$.

Our next goal is to derive necessary and sufficient conditions for the weighted composition family $(C_{m,\varphi}(t))_{t \in I}$ to be γ -strongly continuous. For the necessary condition we need the following definition.

4.3 Definition Let Ω and I be Hausdorff spaces, $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ a linear space, and $\varphi := (\varphi_t)_{t \in I}$ a family of functions $\varphi_t: \Omega \rightarrow \Omega$. We say that the topology of Ω is *initial-like w.r.t. $(\varphi, \mathcal{F}(\Omega))$* if for every compact set $K \subset \Omega$ the continuity of the map $I \times K \rightarrow \mathbb{K}, (t, x) \mapsto f(\varphi_t(x))$, for all $f \in \mathcal{F}(\Omega)$ implies the continuity of the map $I \times K \rightarrow \Omega, (t, x) \mapsto \varphi_t(x)$.

4.4 Remark (a) Obviously, if Ω carries the initial topology induced by $\mathcal{F}(\Omega)$, then the topology of Ω is initial-like w.r.t. $(\varphi, \mathcal{F}(\Omega))$ for any family of functions $\varphi := (\varphi_t)_{t \in I}$ with $\varphi_t: \Omega \rightarrow \Omega$. For instance, a completely regular Hausdorff space Ω carries the initial topology induced by the space $\mathcal{C}_b(\Omega)$ of bounded continuous functions on Ω (see [2, 2.55 Theorem, p. 49] and [2, 2.56 Corollary, p. 50]).

(b) If $\Omega \subset \mathbb{K}$ and the identity $\text{id}: \Omega \rightarrow \Omega, x \mapsto x$, belongs to $\mathcal{F}(\Omega)$, then the topology of Ω is initial-like w.r.t. $(\varphi, \mathcal{F}(\Omega))$ for any family of functions $\varphi := (\varphi_t)_{t \in I}$ with $\varphi_t: \Omega \rightarrow \Omega$.

Now, we use the ideas of the proofs of [36, Proposition 2.10, p. 5] (see also [66, Theorem 4.5, p. 51–52]) and [50, Corollary 4.3 (a), (b), p. 20] where $\mathcal{F}(\Omega) = \mathcal{C}_b(\Omega)$ is the space of bounded continuous functions on a completely regular Hausdorff space $\Omega, I = [0, \infty)$ and $m = \mathbb{1}$.

4.5 Proposition Let Ω be a Hausdorff space and $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$. Let I be a metric space, $\varphi := (\varphi_t)_{t \in I}$ and $m := (m_t)_{t \in I}$ be families of continuous functions $\varphi_t: \Omega \rightarrow \Omega$ and $m_t: \Omega \rightarrow \mathbb{K}$ such that $C_{m,\varphi}(t)f := m_t \cdot (f \circ \varphi_t) \in \mathcal{F}(\Omega)$ for all $t \in I$ and $f \in \mathcal{F}(\Omega)$. Further, suppose that $\sup_{t \in I} \|C_{m,\varphi}(t)\|_{\mathcal{L}(\mathcal{F}(\Omega))} < \infty$. Then the following assertions hold.

(a) If φ and m are jointly continuous, then $(C_{m,\varphi}(t))_{t \in I}$ is γ -strongly continuous.

(b) If Ω is a $k_{\mathbb{R}}$ -space whose topology is initial-like w.r.t. $(\varphi, \mathcal{F}(\Omega))$, $\mathbb{1} \in \mathcal{F}(\Omega)$, I is locally compact, $m_t(x) \neq 0$ for all $(t, x) \in I \times \Omega$ and $(C_{m,\varphi}(t))_{t \in I}$ is γ -strongly continuous, then φ and m are jointly continuous.

Proof First, we observe that $C_{m,\varphi}(t)$ is linear and γ -continuous, thus $C_{m,\varphi}(t) \in \mathcal{L}(\mathcal{F}(\Omega), \gamma)$, for every $t \in I$ due to Proposition 4.1 and Remark 4.2 applied to the singleton $I_t := \{t\}$ and the continuity of φ_t and m_t .

Since I is a metric space and $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$, the family $(C_{m,\varphi}(t))_{t \in I}$ is γ -strongly continuous if and only if the map

$$I \rightarrow \mathcal{C}(K), t \mapsto C_{m,\varphi}(t)f|_K,$$

is continuous for every compact $K \subset \Omega$ and $f \in \mathcal{F}(\Omega)$ by [22, I.1.10 Proposition, p. 9] and the assumption $\sup_{t \in I} \|C_{m,\varphi}(t)\|_{\mathcal{L}(\mathcal{F}(\Omega))} < \infty$. It follows from [30, Lemma 4.16, p. 56] that this is equivalent to the continuity of the map

$$I \times K \rightarrow \mathbb{K}, (t, x) \mapsto m_t(x)f(\varphi_t(x)), \tag{4}$$

for every compact $K \subset \Omega$ and $f \in \mathcal{F}(\Omega)$.

(a) If φ and m are jointly continuous, then the map (4) is clearly continuous for every compact $K \subset \Omega$ and $f \in \mathcal{F}(\Omega)$.

(b) Since $\mathbb{1} \in \mathcal{F}(\Omega)$, the continuity of the map (4) implies the continuity of the map

$$I \times K \rightarrow \mathbb{K}, (t, x) \mapsto m_t(x), \tag{5}$$

for every compact $K \subset \Omega$. The continuity of the maps (4) and (5), that $m_t(x) \neq 0$ for all $(t, x) \in I \times \Omega$ and that the topology of Ω is initial-like w.r.t. $(\varphi, \mathcal{F}(\Omega))$ yield the continuity of the map

$$I \times K \rightarrow \mathbb{K}, (t, x) \mapsto \varphi_t(x), \tag{6}$$

for every compact $K \subset \Omega$. Conversely, the continuity of the maps (5) and (6) clearly implies the continuity of the map (4) for every compact $K \subset \Omega$ and $f \in \mathcal{F}(\Omega)$. Hence the continuity of the map (4) for every compact $K \subset \Omega$ and $f \in \mathcal{F}(\Omega)$ is equivalent to the continuity of the maps (5) and (6) for every compact $K \subset \Omega$. Now, if I is locally compact and Ω a $k_{\mathbb{R}}$ -space, then $I \times \Omega$ is also a $k_{\mathbb{R}}$ -space by a comment after the proof of [15, Théorème (2.1), p. 54–55]. Thus the γ -strong continuity of $(C_{m,\varphi}(t))_{t \in I}$ implies the continuity of the maps (5) and (6) for every compact $K \subset \Omega$, which then implies the joint continuity of φ and m because $I \times \Omega$ is a $k_{\mathbb{R}}$ -space. \square

4.6 Remark Looking at the proof, we see that we can drop the condition that $\mathbb{1} \in \mathcal{F}(\Omega)$ in Proposition 4.5 (b) if $m_t = \mathbb{1}$ for all $t \in I$.

From now on we restrict to the case that (m, φ) is a co-semiflow on a Hausdorff space Ω . If $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ is a linear space and $C_{m,\varphi}(t)f := m_t \cdot (f \circ \varphi_t) \in \mathcal{F}(\Omega)$ for all $t \geq 0$ and $f \in \mathcal{F}(\Omega)$, then a simple computation shows that $(C_{m,\varphi}(t))_{t \geq 0}$ is

a semigroup of linear operators on $\mathcal{F}(\Omega)$, i.e. $C_{m,\varphi}(t) : \mathcal{F}(\Omega) \rightarrow \mathcal{F}(\Omega)$ is linear and $C_{m,\varphi}(t+s) = C_{m,\varphi}(t)C_{m,\varphi}(s)$ for all $t, s \geq 0$.

4.7 Definition Let (m, φ) be a co-semiflow on a Hausdorff space Ω and $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ a linear space. The tuple (m, φ) is called a *co-semiflow for $\mathcal{F}(\Omega)$* if $C_{m,\varphi}(t)f := m_t \cdot (f \circ \varphi_t) \in \mathcal{F}(\Omega)$ for all $t \geq 0$ and $f \in \mathcal{F}(\Omega)$. In this case $(C_{m,\varphi}(t))_{t \geq 0}$ is called the *weighted composition semigroup* on $\mathcal{F}(\Omega)$ w.r.t. the co-semiflow (m, φ) .

4.8 Remark Let Ω be a Hausdorff space, $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ a linear space and (m, φ) a co-semiflow for $\mathcal{F}(\Omega)$.

- (a) If $\mathbb{1} \in \mathcal{F}(\Omega)$, then $m_t = C_{m,\varphi}(t)\mathbb{1} \in \mathcal{F}(\Omega)$ for all $t \geq 0$.
- (b) If $\text{id} \in \mathcal{F}(\Omega)$, then $m_t\varphi_t = C_{m,\varphi}(t)\text{id} \in \mathcal{F}(\Omega)$ for all $t \geq 0$.

If the semicocycle is actually a semicoboundary, then the weighted composition semigroup may have a quite simple structure.

4.9 Remark Let Ω be a Hausdorff space, $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ a linear space and (m, φ) a co-semiflow for $\mathcal{F}(\Omega)$ such that there is $\omega \in \mathcal{C}(\Omega)$ with $N_\omega = \emptyset$ and $m = m^\omega$. Then a direct calculation shows that

$$C_{m,\varphi}(t) = M_{\frac{1}{\omega}} C_{\mathbb{1},\varphi}(t) M_\omega$$

for all $t \geq 0$ where $M_\omega f := \omega f$ and $M_{\frac{1}{\omega}} f := \frac{1}{\omega} f$ for all $f \in \mathcal{F}(\Omega)$. This means that $C_{m,\varphi}(t)$ and $C_{\mathbb{1},\varphi}(t)$ are similar as linear operators on $\mathcal{F}(\Omega)$ for all $t \geq 0$ if $(\mathbb{1}, \varphi)$ is also a co-semiflow for $\mathcal{F}(\Omega)$ (cf. [39, p. 67] in the case $\Omega = \mathbb{D}$ and $\mathcal{F}(\mathbb{D})$ being a space of holomorphic functions).

Sufficient and necessary conditions for the existence of ω in Remark 4.9 are given in [44, Theorem 5, p. 3393] in the case that (m, φ) is a jointly continuous holomorphic co-semiflow on an open simply connected subset Ω of \mathbb{C} . The question which tuples (m, φ) are co-semiflows for a given space $\mathcal{F}(\Omega)$ is difficult on its own. Using Proposition 4.1 and Proposition 4.5 (a), we obtain the following generalisation of [36, Proposition 2.10, p. 5] and [50, Corollary 4.3, p. 20].

4.10 Theorem Let Ω be a Hausdorff space, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ a Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ and $(C_{m,\varphi}(t))_{t \geq 0}$ a locally bounded weighted composition semigroup on $\mathcal{F}(\Omega)$ w.r.t. a jointly continuous co-semiflow (m, φ) . Then the following assertions hold.

- (a) $(C_{m,\varphi}(t))_{t \geq 0}$ is γ -strongly continuous, locally τ_{co} -equicontinuous and locally γ -equicontinuous.
- (b) If $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ is a sequentially complete Saks space, then $(C_{m,\varphi}(t))_{t \geq 0}$ is a τ_{co} -bi-continuous semigroup on $\mathcal{F}(\Omega)$.
- (c) If $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ is a sequentially complete C-sequential Saks space, then $(C_{m,\varphi}(t))_{t \geq 0}$ is quasi- γ -equicontinuous.

Proof (a) The γ -strong continuity follows from Proposition 4.5 (a) and the local boundedness with $I := [0, t_0]$ for every $t_0 \geq 0$. The local τ_{co} -equicontinuity and local γ -equicontinuity are a consequence of Proposition 4.1, Remark 4.2 and the local boundedness with $I := [0, t_0]$ for every $t_0 \geq 0$.

(b)+(c) The remaining parts follow from Theorem 2.7 and the comments before it. □

We note that $(C_b(\mathbb{R}), \|\cdot\|_\infty, \tau_{co})$ is a sequentially complete C-sequential Saks space and the left translation semigroup on $C_b(\mathbb{R})$, i.e. the (un)weighted composition semigroup $(C_{\mathbb{1},\varphi}(t))_{t \geq 0}$ w.r.t. the jointly continuous co-semiflow $(\mathbb{1}, \varphi)$ where $\varphi_t(x) := t + x$ for all $t \geq 0, x \in \mathbb{R}$, is exponentially bounded, locally τ_{co} -equicontinuous and quasi- γ -equicontinuous by (the proof of) [50, Theorem 4.1, p. 19] and [50, Example 4.2 (a), p. 19] (or by Theorem 4.10), but not quasi- τ_{co} -equicontinuous by [56, Example 3.2, p. 549]. This shows that Theorem 4.10 is sharp in the sense that we cannot expect quasi- τ_{co} -equicontinuity of weighted composition semigroups in general.

Looking at Theorem 4.10, we see that the local boundedness of $(C_{m,\varphi}(t))_{t \geq 0}$ is a crucial ingredient. The rest of this section is dedicated to deriving sufficient conditions on (m, φ) and $\mathcal{F}(\Omega)$ such that $(C_{m,\varphi}(t))_{t \geq 0}$ becomes locally bounded. Our strategy can be described as follows. Let Ω be a Hausdorff space, (m, φ) a co-semiflow on Ω and $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$. We decompose $C_{m,\varphi} = C_{m,id}C_{\mathbb{1},\varphi}$ and see that (m, φ) is a co-semiflow for $\mathcal{F}(\Omega)$ if (m, id) and $(\mathbb{1}, \varphi)$ are co-semiflows for $\mathcal{F}(\Omega)$. In this case we have the estimate

$$\|C_{m,\varphi}(t)\|_{\mathcal{L}(\mathcal{F}(\Omega))} \leq \|C_{m,id}(t)\|_{\mathcal{L}(\mathcal{F}(\Omega))} \|C_{\mathbb{1},\varphi}(t)\|_{\mathcal{L}(\mathcal{F}(\Omega))}$$

for all $t \geq 0$. Therefore the semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ is locally bounded if (m, id) and $(\mathbb{1}, \varphi)$ are co-semiflows for $\mathcal{F}(\Omega)$, and the *multiplication semigroup* $(C_{m,id}(t))_{t \geq 0}$ as well as the *unweighted composition semigroup* $(C_{\mathbb{1},\varphi}(t))_{t \geq 0}$ are locally bounded. This strategy might not be optimal but gives rather simple, more applicable, sufficient conditions that guarantee the local boundedness of $(C_{m,\varphi}(t))_{t \geq 0}$.

- 4.11 Remark** (a) If φ is a holomorphic semiflow on \mathbb{D} and $1 \leq p < \infty$, then $(\mathbb{1}, \varphi)$ is a co-semiflow for H^p and $C_{\mathbb{1},\varphi}(t) \in \mathcal{L}(H^p)$ for all $t \geq 0$ by [28, Corollary, p. 29].
- (b) If φ is a holomorphic semiflow on \mathbb{D} , $\alpha > -1$ and $1 \leq p < \infty$, then $(\mathbb{1}, \varphi)$ is a co-semiflow for A_α^p and $C_{\mathbb{1},\varphi}(t) \in \mathcal{L}(A_\alpha^p)$ for all $t \geq 0$ by [57, 3.4 Proposition, p. 884].
- (c) If φ is a holomorphic semiflow on \mathbb{D} , then $(\mathbb{1}, \varphi)$ is a co-semiflow for \mathcal{D} and $C_{\mathbb{1},\varphi}(t) \in \mathcal{L}(\mathcal{D})$ for all $t \geq 0$ by [70, p. 166].
- (d) Let $v_0: [0, 1] \rightarrow [0, \infty)$ be a continuous non-increasing function such that $v_0(x) \neq 0$ for all $x \in [0, 1)$, $v_0(1) = 0$ and set $v: \overline{\mathbb{D}} \rightarrow [0, \infty)$, $v(z) := v_0(|z|)$ (see [60, p. 873]). Let φ be a holomorphic semiflow on \mathbb{D} . Then there is a necessary and sufficient condition such that $(\mathbb{1}, \varphi)$ is a co-semiflow for $\mathcal{B}v(\mathbb{D})$ and $C_{\mathbb{1},\varphi}(t) \in \mathcal{L}(\mathcal{B}v(\mathbb{D}))$ for all $t \geq 0$ given in [60, Theorem 2.3, p. 876].
- (e) Let Ω be a completely regular Hausdorff space, $v: \Omega \rightarrow (0, \infty)$ continuous and φ a semiflow on Ω . Suppose that for every $x \in \Omega$ there is $f \in \mathcal{C}(\Omega)$ such that

$vf \in C_0(\Omega)$ and $f(x) \neq 0$. This condition is for instance fulfilled if Ω is locally compact. Then $(\mathbb{1}, \varphi)$ is a co-semiflow for $\mathcal{C}v(\Omega)$ and $C_{\mathbb{1},\varphi}(t) \in \mathcal{L}(\mathcal{C}v(\Omega))$ for all $t \geq 0$ if and only if for every $t \geq 0$ there is $K_t \geq 0$ such that $v(x) \leq K_t v(\varphi_t(x))$ for all $x \in \Omega$ by [67, 2.2 Theorem, p. 307].

- (f) Let $\Omega \subset \mathbb{C}$ be open, $v: \Omega \rightarrow (0, \infty)$ continuous and φ a holomorphic semiflow on Ω . Then $(\mathbb{1}, \varphi)$ is a co-semiflow for $\mathcal{H}v(\Omega)$ and $C_{\mathbb{1},\varphi}(t) \in \mathcal{L}(\mathcal{H}v(\Omega))$ for all $t \geq 0$ if for every $t \geq 0$ there is $K_t \geq 0$ such that $v(z) \leq K_t v(\varphi_t(z))$ for all $z \in \Omega$ by [67, 2.2 Theorem, p. 307].

Let Ω be a Hausdorff space and $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ a linear space. We define the multiplier space $\mathcal{M}(\mathcal{F}(\Omega))$ of continuous multipliers of $\mathcal{F}(\Omega)$ by $\mathcal{M}(\mathcal{F}(\Omega)) := \{g \in \mathcal{C}(\Omega) \mid \forall f \in \mathcal{F}(\Omega) : gf \in \mathcal{F}(\Omega)\}$.¹

4.12 Proposition *Let Ω be a Hausdorff space, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ and suppose that for every $x \in \Omega$ there is $f \in \mathcal{F}(\Omega)$ such that $f(x) \neq 0$. Then we have $\mathcal{M}(\mathcal{F}(\Omega)) \subset C_b(\Omega)$. Further, $C_{m,\text{id}}(t) \in \mathcal{L}(\mathcal{F}(\Omega))$ for all $t \geq 0$ if (m, id) is a co-semiflow on Ω and $m_t \in \mathcal{M}(\mathcal{F}(\Omega))$ for all $t \geq 0$.*

Proof Due to our assumption, Remark 2.6 and Convention 2.8 $(\mathcal{F}(\Omega), \|\cdot\|)$ is a functional Banach space in the sense of [29, p. 57] and thus our statement follows from [29, Lemma 11, p. 57]. □

- 4.13 Proposition** (a) $\mathcal{M}(H^p) = \mathcal{M}(A_\alpha^p) = H^\infty$ for $\alpha > -1$ and $1 \leq p < \infty$.
- (b) $\mathcal{M}(\mathcal{B}_\alpha) = H^\infty$ for $\alpha > 1$, $\mathcal{M}(\mathcal{B}_\alpha) = \mathcal{B}_\alpha$ for $0 < \alpha < 1$ and $\mathcal{M}(\mathcal{B}_1) = \{f \in H^\infty \mid \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) \ln((1 - |z|^2)^{-1}) < \infty\}$.
- (c) $\mathcal{M}(\mathcal{C}v(\Omega)) = C_b(\Omega)$ for all locally compact Hausdorff spaces Ω and continuous $v: \Omega \rightarrow (0, \infty)$.
- (d) $\mathcal{M}(\mathcal{H}v(\Omega)) = H^\infty(\Omega)$ for all open sets $\Omega \subset \mathbb{C}$ and continuous $v: \Omega \rightarrow (0, \infty)$ if $\mathbb{1} \in \mathcal{H}v(\Omega)$.

Proof In (a) we have $H^\infty \subset \mathcal{M}(H^p)$ and $H^\infty \subset \mathcal{M}(A_\alpha^p)$, in (c) we have $C_b(\Omega) \subset \mathcal{M}(\mathcal{C}v(\Omega))$, and in (d) we have $H^\infty(\Omega) \subset \mathcal{M}(\mathcal{H}v(\Omega))$. Therefore the statements in part (a), (c) and (d) follow from Proposition 4.12 and Examples 2.9, 2.11, 2.13 since in (a) $\mathbb{1} \in H^p$, A_α^p and since in (c) for every $x \in \Omega$ there is $f \in \mathcal{F}(\Omega)$ such that $f(x) \neq 0$ because Ω is a locally compact Hausdorff space. Part (b) is [80, Theorem 27, p. 1170]. □

The multiplier space $\mathcal{M}(\mathcal{D})$ of the Dirichlet space is more complicated and its elements can be described in terms of the Carleson measure by [73, Theorems 1.1 (c), 2.3, 2.7, p. 115, 122, 125] with $\alpha = \frac{1}{2}$. From Proposition 4.12 and $\mathbb{1} \in \mathcal{D}$ it follows that $\mathcal{M}(\mathcal{D}) \subset (\mathcal{D} \cap H^\infty)$.

4.14 Theorem *Let (m, φ) be a holomorphic co-semiflow on \mathbb{D} such that φ is jointly continuous. Then (m, φ) is a co-semiflow for $\mathcal{F}(\mathbb{D})$ and the weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\mathbb{D})$ is locally bounded in each of the following cases if*

¹ The multiplier space is defined as $\mathcal{M}_0(\mathcal{F}(\Omega)) := \{g: \Omega \rightarrow \mathbb{K} \mid \forall f \in \mathcal{F}(\Omega) : gf \in \mathcal{F}(\Omega)\}$ in [14, p. 272] Thus we have $\mathcal{M}(\mathcal{F}(\Omega)) \subset \mathcal{M}_0(\mathcal{F}(\Omega))$. If $\mathbb{1} \in \mathcal{F}(\Omega)$, then we even have $\mathcal{M}(\mathcal{F}(\Omega)) = \mathcal{M}_0(\mathcal{F}(\Omega))$ because $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$. We incorporated continuity in the definition of our multiplier space since our semicycles are by definition continuous.

- (a) $\mathcal{F}(\mathbb{D}) = H^p$ for $1 \leq p < \infty$ and $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$,
- (b) $\mathcal{F}(\mathbb{D}) = A_\alpha^p$ for $\alpha > -1$ and $1 \leq p < \infty$ and $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$,
- (c) $\mathcal{F}(\mathbb{D}) = \mathcal{D}$ and $m_t \in \mathcal{M}(\mathcal{D})$ for all $t \geq 0$.

Proof The condition $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$ in (a) and (b) yields that $m_t \in H^\infty$ for all $t \geq 0$ by Proposition 3.10, which is also true in (c) because $\mathcal{M}(\mathcal{D}) \subset H^\infty$. Hence in all the cases $(\mathbb{1}, \varphi)$, (m, id) and (m, φ) are co-semiflows for $\mathcal{F}(\mathbb{D})$ by Remark 4.11 and Proposition 4.13.

(a) We have

$$\|f \circ \varphi_t\|_p^p \leq \frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \|f\|_p^p$$

for all $t \geq 0$ and $f \in H^p$ by [28, Corollary, p. 29], yielding

$$\|C_{\mathbb{1},\varphi}(t)\|_{\mathcal{L}(H^p)} \leq \left(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|}\right)^{\frac{1}{p}}$$

for all $t \geq 0$. Further, we note that $\|C_{m,\text{id}}(t)\|_{\mathcal{L}(H^p)} \leq \|m_t\|_\infty$ for all $t \geq 0$.

(b) Due to [13, Theorem 8.1.15, p. 211] and [13, Proposition 10.1.7 (1) ⇔ (2), p. 275] there is $t_0 > 0$ such that $\|\varphi_t - \text{id}\|_\infty \leq 1$ for all $t \in [0, t_0]$ since φ is jointly continuous, which implies that $\|\varphi_t\|_\infty \leq 2$ for all $t \in [0, t_0]$. Therefore we obtain

$$\|C_{\mathbb{1},\varphi}(t)\|_{\mathcal{L}(A_\alpha^p)} \leq K(\varphi_t) \left(\frac{\|\varphi_t\|_\infty + |\varphi_t(0)|}{\|\varphi_t\|_\infty - |\varphi_t(0)|}\right)^{\frac{\alpha+2}{p}}$$

for all $t \in [0, t_0]$ by [69, Lemma 1, p. 399] where $K(\varphi_t) := 1$ if $\alpha \geq 0$, and $K(\varphi_t) := (\|\varphi_t\|_\infty + |\varphi_t(0)|)^{\alpha/p} (\|\varphi_t\|_\infty + 3|\varphi_t(0)|)^{-\alpha/p}$ if $-1 < \alpha < 0$. Furthermore, we note that $\|C_{m,\text{id}}(t)\|_{\mathcal{L}(A_\alpha^p)} \leq \|m_t\|_\infty$ for all $t \geq 0$.

(c) Due to [58, Theorem 2 (a), p. 26] and [1, Proposition 1.4.6, p. 85] we have

$$\|C_{\mathbb{1},\varphi}(t)\|_{\mathcal{L}(\mathcal{D})}^2 \leq 1 + \frac{1}{2} \left(L(\varphi_t) + (L(\varphi_t)(4 + L(\varphi_t)))^{\frac{1}{2}} \right)$$

for all $t \geq 0$ where $L(\varphi_t) := -\ln(1 - |\varphi_t(0)|^2)$. By [81, Proposition 5.1, p. 101] we have $H^\infty \subset \mathcal{B}_1$ and

$$\|g\|_{\mathcal{B}_1} \leq \|g\|_\infty \tag{7}$$

for all $g \in H^\infty$. By [73, p. 114] with $\alpha = \frac{1}{2}$ there is $M > 0$ such that

$$\frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{-2} dz \leq M \|f\|_{\mathcal{D}}^2 \tag{8}$$

for all $f \in \mathcal{D}$. We deduce that

$$\begin{aligned}
 \|C_{m,\text{id}}(t)f\|_{\mathcal{D}}^2 &= |m_t(0)f(0)|^2 + \frac{1}{\pi} \int_{\mathbb{D}} |(m_t f)'(z)|^2 dz \\
 &\leq \|m_t\|_{\infty}^2 \|f\|_{\mathcal{D}}^2 + \frac{2}{\pi} \int_{\mathbb{D}} |m'_t(z)f(z)|^2 + |m_t(z)f'(z)|^2 dz \\
 &\leq 2\|m_t\|_{\infty}^2 \|f\|_{\mathcal{D}}^2 + \frac{2}{\pi} \int_{\mathbb{D}} |m'_t(z)|^2 (1 - |z|^2)^2 |f(z)|^2 (1 - |z|^2)^{-2} dz \\
 &\leq 2\|m_t\|_{\infty}^2 \|f\|_{\mathcal{D}}^2 + \frac{2}{\pi} \|m_t\|_{\mathcal{B}_1}^2 \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{-2} dz \\
 &\stackrel{(7), (8)}{\leq} 2(1 + M)\|m_t\|_{\infty}^2 \|f\|_{\mathcal{D}}^2
 \end{aligned}$$

for all $t \geq 0$ and $f \in \mathcal{D}$, which implies $\|C_{m,\text{id}}(t)\|_{\mathcal{L}(\mathcal{D})}^2 \leq 2(1 + M)\|m_t\|_{\infty}^2$ for all $t \geq 0$.

Therefore we derive in (a), (b) and (c) from the joint continuity of φ that $(C_{\mathbb{1},\varphi}(t))_{t \geq 0}$ is locally bounded. Proposition 3.10 yields that $(C_{m,\text{id}}(t))_{t \geq 0}$ is locally bounded and thus $(C_{m,\varphi}(t))_{t \geq 0}$ is locally bounded, too. \square

Any semiflow φ on \mathbb{D} given in [71, p. 4–5] is holomorphic and C_0 , thus jointly continuous by Proposition 3.3 (a). For any such φ take the semicyclope m given by $m_t : \mathbb{D} \rightarrow \mathbb{C}$, $m_t(z) := \exp(\int_0^t g(\varphi_s(z)) ds)$, for all $t \geq 0$ for some $g \in \mathcal{H}(\mathbb{D})$ (see Proposition 3.14 (a)). If $M := \sup_{z \in \mathbb{D}} \text{Re}(g(z)) < \infty$, then $\|m_t\|_{\infty} \leq e^{tM}$ for all $t \geq 0$ and so $\limsup_{t \rightarrow 0^+} \|m_t\|_{\infty} \leq 1$ (cf. [47, p. 474]). Hence $(C_{m,\varphi}(t))_{t \geq 0}$ is a locally bounded semigroup on $\mathcal{F}(\mathbb{D})$ in case (a) and (b) of Theorem 4.14.

An example in case (c) of the Dirichlet space \mathcal{D} is the jointly continuous holomorphic co-semiflow (φ', φ) on \mathbb{D} given by $\varphi_t : \mathbb{D} \rightarrow \mathbb{D}$, $\varphi_t(z) := e^{-ct}z$, for all $t \geq 0$ for some $c \in \mathbb{C}$ with $\text{Re}(c) \geq 0$ since $\varphi'_t(z) = e^{-ct}$ for all $t \geq 0$ and $z \in \mathbb{D}$, which implies $m_t := \varphi'_t \in \mathcal{D}$ for all $t \geq 0$. The same is true if we choose $\varphi_t(z) := e^{-t}z + 1 - e^{-t}$ for all $t \geq 0$ and $z \in \mathbb{D}$. Thus $(C_{\varphi',\varphi}(t))_{t \geq 0}$ is a locally bounded semigroup on \mathcal{D} by Theorem 4.14 (c) in both cases.

4.15 Theorem *Let (m, φ) be a holomorphic co-semiflow on \mathbb{D} and $\alpha > 0$. Then (m, φ) is a co-semiflow for \mathcal{B}_{α} and the weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on \mathcal{B}_{α} is locally bounded if*

$$K_{\alpha}(\varphi_t) := \sup_{z \in \mathbb{D}} |\varphi'_t(z)|(1 - |\varphi_t(z)|^2)^{-\alpha}(1 - |z|^2)^{\alpha} < \infty$$

for all $t \geq 0$, there exists $t_0 > 0$ such that $\sup_{t \in [0, t_0]} K_{\alpha}(\varphi_t) < \infty$ and

- (a) for $\alpha > 1$ if $\limsup_{t \rightarrow 0^+} \|m_t\|_{\infty} < \infty$,
- (b) for $\alpha = 1$ if $\limsup_{t \rightarrow 0^+} \|m_t\|_{\infty} < \infty$ and

$$\sup_{t \in [0, t_0]} \sup_{z \in \mathbb{D}} |m'_t(z)|(1 - |z|^2) \ln((1 - |z|^2)^{-1}) < \infty,$$

(c) for $0 < \alpha < 1$ if $m_t \in \mathcal{B}_\alpha$ for all $t \geq 0$ and $\sup_{t \in [0, t_0]} \|m_t\|_{\mathcal{B}_\alpha} < \infty$.

Proof By the proof of [78, Theorem 2.2 (i), p. 115] we have

$$\|C_{\mathbb{1}, \varphi}(t)f\| \leq \|f\|_{\mathcal{B}_\alpha} \sup_{z \in \mathbb{D}} |\varphi'_t(z)|(1 - |\varphi_t(z)|^2)^{-\alpha} (1 - |z|^2)^\alpha = K_\alpha(\varphi_t)\|f\|_{\mathcal{B}_\alpha}$$

for all $t \geq 0$ and $f \in \mathcal{B}_\alpha$, yielding $f \circ \varphi_t \in \mathcal{B}_\alpha$ and that $(\mathbb{1}, \varphi)$ is a co-semiflow for \mathcal{B}_α . In addition, the existence of $t_0 > 0$ such that $\sup_{t \in [0, t_0]} K_\alpha(\varphi_t) < \infty$ gives that $(C_{\mathbb{1}, \varphi}(t))_{t \geq 0}$ is a locally bounded semigroup.

Moreover, the conditions in (a), (b) and (c) guarantee that $m_t \in \mathcal{M}(\mathcal{B}_\alpha)$ for all $t \geq 0$ by Proposition 3.10 and Proposition 4.13 (b). Hence in all the cases (m, id) and so (m, φ) are co-semiflows for \mathcal{B}_α .

(a) For $\alpha > 1$ we have by the proof of [80, Proposition 7, p. 1147] that there is $L_\alpha > 0$ such that

$$|f(z) - f(0)| \leq L_\alpha \|f\|_{\mathcal{B}_\alpha} (1 - |z|^2)^{-(\alpha-1)} \tag{9}$$

for all $z \in \mathbb{D}$ and $f \in \mathcal{B}_\alpha$. It follows that

$$\begin{aligned} \|m_t f\|_{\mathcal{B}_\alpha} &\leq |m_t(0)f(0)| + \sup_{z \in \mathbb{D}} |m'_t(z)f(z)|(1 - |z|^2)^\alpha + \sup_{z \in \mathbb{D}} |m_t(z)f'(z)|(1 - |z|^2)^\alpha \\ &\leq \sup_{z \in \mathbb{D}} |m'_t(z)|(1 - |z|^2) \sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2)^{\alpha-1} + 2\|m_t\|_\infty \|f\|_{\mathcal{B}_\alpha} \\ &\leq \|m_t\|_{\mathcal{B}_1} \sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2)^{\alpha-1} + 2\|m_t\|_\infty \|f\|_{\mathcal{B}_\alpha} \\ &\stackrel{(9)}{\leq} \|m_t\|_{\mathcal{B}_1} \sup_{z \in \mathbb{D}} (L_\alpha \|f\|_{\mathcal{B}_\alpha} + |f(0)|(1 - |z|^2)^{\alpha-1}) + 2\|m_t\|_\infty \|f\|_{\mathcal{B}_\alpha} \\ &\stackrel{(7)}{\leq} \|m_t\|_\infty (L_\alpha \|f\|_{\mathcal{B}_\alpha} + |f(0)|) + 2\|m_t\|_\infty \|f\|_{\mathcal{B}_\alpha} \\ &\leq (3 + L_\alpha)\|m_t\|_\infty \|f\|_{\mathcal{B}_\alpha} \end{aligned}$$

for all $t \geq 0$ and $f \in \mathcal{B}_\alpha$.

(b) For $\alpha = 1$ we get as in the proof of [80, Proposition 7, p. 1147], using [80, Proposition 7 3), p. 1146] with $t = \alpha - 1 = 0$ and $s = 0$, that there is $L_1 > 0$ such that

$$|f(z) - f(0)| \leq L_1 \|f\|_{\mathcal{B}_1} \ln((1 - |z|^2)^{-1}) \tag{10}$$

for all $z \in \mathbb{D}$ and $f \in \mathcal{B}_1$. It follows that

$$\begin{aligned} \|m_t f\|_{\mathcal{B}_1} &\leq \sup_{z \in \mathbb{D}} |m'_t(z)|(1 - |z|^2)(|f(z) - f(0)| + |f(0)|) + 2\|m_t\|_\infty \|f\|_{\mathcal{B}_1} \\ &\stackrel{(10)}{\leq} L_1 \|f\|_{\mathcal{B}_1} \sup_{z \in \mathbb{D}} |m'_t(z)|(1 - |z|^2) \ln((1 - |z|^2)^{-1}) \\ &\quad + |f(0)| \sup_{z \in \mathbb{D}} |m'_t(z)|(1 - |z|^2) + 2\|m_t\|_\infty \|f\|_{\mathcal{B}_1} \\ &\stackrel{(7)}{\leq} (3\|m_t\|_\infty + L_1 \sup_{z \in \mathbb{D}} |m'_t(z)|(1 - |z|^2) \ln((1 - |z|^2)^{-1})) \|f\|_{\mathcal{B}_1} \end{aligned}$$

for all $t \geq 0$ and $f \in \mathcal{B}_1$.

(c) Again, for $0 < \alpha < 1$ we get as in the proof of [80, Proposition 7, p. 1147], using [80, Proposition 7 1), p. 1146] with $t = \alpha - 1 < 0$ and $s = 0$, that there is $L_\alpha > 0$ such that

$$|f(z) - f(0)| \leq L_\alpha \|f\|_{\mathcal{B}_\alpha} \tag{11}$$

for all $z \in \mathbb{D}$ and $f \in \mathcal{B}_\alpha$. This implies $\mathcal{B}_\alpha \subset H^\infty$ and $\|g\|_\infty \leq (1 + L_\alpha)\|g\|_{\mathcal{B}_\alpha}$ for all $g \in \mathcal{B}_\alpha$ and $0 < \alpha < 1$. It follows that

$$\begin{aligned} \|m_t f\|_{\mathcal{B}_\alpha} &\leq \sup_{z \in \mathbb{D}} |m'_t(z)| (1 - |z|^2)^\alpha (|f(z) - f(0)| + |f(0)|) + 2\|m_t\|_\infty \|f\|_{\mathcal{B}_\alpha} \\ &\stackrel{(11)}{\leq} (1 + L_\alpha) \|f\|_{\mathcal{B}_\alpha} \sup_{z \in \mathbb{D}} |m'_t(z)| (1 - |z|^2)^\alpha + 2\|m_t\|_\infty \|f\|_{\mathcal{B}_\alpha} \\ &\leq (2\|m_t\|_\infty + (1 + L_\alpha)\|m_t\|_{\mathcal{B}_\alpha}) \|f\|_{\mathcal{B}_\alpha} \leq 3(1 + L_\alpha) \|m_t\|_{\mathcal{B}_\alpha} \|f\|_{\mathcal{B}_\alpha} \end{aligned}$$

for all $t \geq 0$ and $f \in \mathcal{B}_\alpha$.

Hence our conditions in (a) and (b) combined with Proposition 3.10, and in (c) guarantee that $(C_{m,\text{id}}(t))_{t \geq 0}$ is a locally bounded semigroup and thus $(C_{m,\varphi}(t))_{t \geq 0}$ as well. \square

The jointly continuous holomorphic co-semiflow (φ', φ) on \mathbb{D} given by $\varphi_t : \mathbb{D} \rightarrow \mathbb{D}$, $\varphi_t(z) := e^{-ct}z$, for all $t \geq 0$ for some $c \in \mathbb{C}$ with $\text{Re}(c) \geq 0$, fulfils $m_t(z) := \varphi'_t(z) = e^{-ct}$,

$$K_\alpha(\varphi_t) = \sup_{z \in \mathbb{D}} e^{-\text{Re}(c)t} (1 - e^{-2\text{Re}(c)t} |z|^2)^{-\alpha} (1 - |z|^2)^\alpha \leq e^{-\text{Re}(c)t}$$

and $\|\varphi'_t\|_\infty = e^{-\text{Re}(c)t}$ as well as $m'_t(z) = \varphi''_t(z) = 0$ and $\|m_t\|_{\mathcal{B}_\alpha} = |\varphi'_t(0)| = e^{-\text{Re}(c)t}$ for all $t \geq 0$ and $z \in \mathbb{D}$. Thus $(C_{\varphi',\varphi}(t))_{t \geq 0}$ is a locally bounded semigroup on \mathcal{B}_α for all $\alpha > 0$ by Theorem 4.15.

Let us turn to the space of bounded Dirichlet series. We say that a holomorphic function $\varphi : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ belongs to the class \mathcal{G}_∞ if there exist $c_\varphi \in \mathbb{N}_0$ and a Dirichlet series ρ_φ which converges on some open half-plane and extends holomorphically to \mathbb{C}_+ such that $\varphi(z) = c_\varphi z + \rho_\varphi(z)$ for all $z \in \mathbb{C}_+$ (see [21, Definition 2.6, p. 9]).

4.16 Theorem *Let (m, φ) be a holomorphic co-semiflow on \mathbb{C}_+ . Then (m, φ) is a co-semiflow for \mathcal{H}^∞ and the weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on \mathcal{H}^∞ is locally bounded if $\varphi_t \in \mathcal{G}_\infty$, $m_t \in \mathcal{H}^\infty$ for all $t \geq 0$ and $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$. The converse is true if $\inf_{z \in \mathbb{C}_+} |m_t(z)| > 0$ for all $t \geq 0$.*

Proof By [8, Proposition 2, p. 219] $(\mathbb{1}, \varphi)$ is a co-semiflow for \mathcal{H}^∞ if and only if $\varphi_t \in \mathcal{G}_\infty$ for all $t \geq 0$. Further, we observe that $\|C_{\mathbb{1},\varphi}(t)f\|_{\mathcal{H}^\infty} \leq \|f\|_{\mathcal{H}^\infty}$ for all $f \in \mathcal{H}^\infty$ if $\varphi_t \in \mathcal{G}_\infty$ for $t \geq 0$. Due to [74, Theorem 7 a), p. 9–10] we have $\mathcal{M}(\mathcal{H}^\infty) = \mathcal{H}^\infty$, which implies that (m, id) is a co-semiflow for \mathcal{H}^∞ if and only if $m_t \in \mathcal{H}^\infty$ for all $t \geq 0$. We note that $\|C_{m,\text{id}}(t)f\|_{\mathcal{H}^\infty} \leq \|m_t\|_\infty \|f\|_{\mathcal{H}^\infty}$ for all $f \in \mathcal{H}^\infty$ if $m_t \in \mathcal{H}^\infty$ for $t \geq 0$.

Now, the first implication follows from our considerations above and Proposition 3.10. Let us consider the converse implication. The properties $m_t \in \mathcal{H}^\infty$ for all $t \geq 0$ and $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$ follow from the local boundedness of the weighted composition semigroup and the observation that $\mathbb{1} \in \mathcal{H}^\infty$ since $\|m_t\|_\infty = \|m_t\|_{\mathcal{H}^\infty} = \|C_{m,\varphi}(t)\mathbb{1}\|_{\mathcal{H}^\infty}$ for all $t \geq 0$. The property $\varphi_t \in \mathcal{G}_\infty$ for all $t \geq 0$ follows from our first observation of the proof, writing $C_{\mathbb{1},\varphi}(t) = (1/m_t)C_{m,\varphi}(t)$ and noting that $1/m_t$ belongs to the Banach algebra \mathcal{H}^∞ by [64, Theorem 6.2.1, p. 147] if and only if $\inf_{z \in \mathbb{C}_+} |m_t(z)| > 0$. \square

The jointly continuous holomorphic co-semiflow $(\mathbb{1}, \varphi)$ on \mathbb{C}_+ given by $\varphi_t: \mathbb{C}_+ \rightarrow \mathbb{C}_+, \varphi_t(z) := z + t$, for all $t \geq 0$ fulfils with $m_t(z) := 1$ for all $t \geq 0$ and $z \in \mathbb{C}_+$ that $\varphi_t \in \mathcal{G}_\infty, m_t \in \mathcal{H}^\infty$ for all $t \geq 0$ and $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty = 1$. Thus $(C_{\mathbb{1},\varphi}(t))_{t \geq 0}$ is a locally bounded semigroup on \mathcal{H}^∞ by Theorem 4.16.

4.17 Theorem *Let $\Omega \subset \mathbb{C}$ be open, $v: \Omega \rightarrow (0, \infty)$ continuous, $\mathbb{1} \in \mathcal{H}v(\Omega)$ and (m, φ) a holomorphic co-semiflow on Ω . If $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$,*

$$K(\varphi_t) := \sup_{z \in \Omega} \frac{v(z)}{v(\varphi_t(z))} < \infty$$

for all $t \geq 0$ and there exists $t_0 > 0$ such that $\sup_{t \in [0, t_0]} K(\varphi_t) < \infty$, then (m, φ) is a co-semiflow for $\mathcal{H}v(\Omega)$ and the weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on $\mathcal{H}v(\Omega)$ is locally bounded. If $v = \mathbb{1}$, then the converse holds as well.

Proof We deduce from Remark 4.11 (f) and Proposition 4.13 (d) that $(m, \text{id}), (\mathbb{1}, \varphi)$ and so (m, φ) are co-semiflows for $\mathcal{H}v(\Omega)$. We observe that

$$\|C_{\mathbb{1},\varphi}(t)f\|_v = \sup_{z \in \Omega} |f(\varphi_t(z))|v(\varphi_t(z)) \frac{v(z)}{v(\varphi_t(z))} \leq K(\varphi_t)\|f\|_v$$

for all $t \geq 0$ and $f \in \mathcal{H}v(\Omega)$, yielding $\|C_{\mathbb{1},\varphi}(t)\|_{\mathcal{L}(\mathcal{H}v(\Omega))} \leq K(\varphi_t)$.

Moreover, we note that

$$\|C_{m,\text{id}}(t)f\|_v = \sup_{z \in \Omega} |m_t(z)f(z)|v(z) \leq \|m_t\|_\infty \|f\|_v$$

for all $t \geq 0$ and $f \in \mathcal{H}v(\Omega)$, yielding $\|C_{m,\text{id}}(t)\|_{\mathcal{L}(\mathcal{H}v(\Omega))} \leq \|m_t\|_\infty$. Therefore $(C_{m,\text{id}}(t))_{t \geq 0}$ is locally bounded by Proposition 3.10. The same is true for the semigroup $(C_{\mathbb{1},\varphi}(t))_{t \geq 0}$ by the existence of $t_0 > 0$ and so $(C_{m,\varphi}(t))_{t \geq 0}$ is locally bounded as well.

If $v = \mathbb{1}$, then the converse holds as well by Proposition 3.10 since then $K(\varphi_t) = 1$ and $\|m_t\|_\infty = \|m_t\|_v = \|C_{m,\varphi}(t)\mathbb{1}\|_v$ for all $t \geq 0$. \square

Analogously we obtain the corresponding result for $\mathcal{C}v(\Omega)$ by using Remark 4.11 (e) and Proposition 4.13 (c).

4.18 Theorem *Let Ω be a locally compact Hausdorff space, $v: \Omega \rightarrow (0, \infty)$ continuous and (m, φ) a co-semiflow on Ω . If $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$,*

$$K(\varphi_t) := \sup_{x \in \Omega} \frac{v(x)}{v(\varphi_t(x))} < \infty$$

for all $t \geq 0$ and there exists $t_0 > 0$ such that $\sup_{t \in [0, t_0]} K(\varphi_t) < \infty$, then (m, φ) is a co-semiflow for $\mathcal{C}v(\Omega)$ and the weighted composition semigroup $(C_{m, \varphi}(t))_{t \geq 0}$ on $\mathcal{C}v(\Omega)$ is locally bounded. If $\mathbb{1} \in \mathcal{C}v(\Omega)$ and $v = \mathbb{1}$, then the converse holds as well.

Apart from the weight $v = \mathbb{1}$ there are other weights that fulfil the conditions of Theorem 4.17 and Theorem 4.18. For instance, take the left-translation semiflow φ given by $\varphi_t: \mathbb{K} \rightarrow \mathbb{K}$, $\varphi_t(x) := x + t$, for $t \geq 0$ and set $v: \mathbb{K} \rightarrow (0, \infty)$, $v(x) := e^{-|x|}$. Then $K(\varphi_t) \leq e^t$ for all $t \geq 0$ and we can choose any $t_0 > 0$ (cf. [67, (i), p. 310]). Taking now any (holomorphic) semicycle m for φ such that $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$, we get a locally bounded semigroup $(C_{m, \varphi}(t))_{t \geq 0}$ on $\mathcal{H}v(\mathbb{C})$ (if $\mathbb{K} = \mathbb{C}$ and $\mathbb{1} \in \mathcal{H}v(\mathbb{C})$) resp. $\mathcal{C}v(\mathbb{K})$ by Theorem 4.17 resp. Theorem 4.18.

5 Generators of Weighted Composition Semigroups

In this section we give several characterisations of the generator of a γ -strongly continuous weighted composition semigroup on a Saks space. Let $(X, \|\cdot\|, \tau)$ be a Saks space and $(T(t))_{t \geq 0}$ a γ -strongly continuous semigroup on X . We define the generator $(A, D(A))$ of $(T(t))_{t \geq 0}$ according to [46, p. 260] by

$$D(A) := \left\{ x \in X \mid \gamma\text{-}\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}$$

and

$$Ax := \gamma\text{-}\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}, \quad x \in D(A).$$

If $(X, \|\cdot\|, \tau)$ is sequentially complete, then $D(A)$ is γ -dense in X by [46, Proposition 1.3, p. 261]. The bi-generator $(A_{\|\cdot\|, \tau}, D(A_{\|\cdot\|, \tau}))$ of $(T(t))_{t \geq 0}$ is given by

$$D(A_{\|\cdot\|, \tau}) := \left\{ x \in X \mid \tau\text{-}\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists in } X, \sup_{0 < t \leq 1} \frac{\|T(t)x - x\|}{t} < \infty \right\}$$

and

$$A_{\|\cdot\|, \tau}x := \tau\text{-}\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}, \quad x \in D(A_{\|\cdot\|, \tau}).$$

In the context of τ -bi-continuous semigroups their generators are actually defined as bi-generators (see [34, Definition 1.2.6, p. 7]). The notion of the bi-generator was originally introduced in [54, 55] (and corrected in [34]).

5.1 Proposition *Let $(X, \|\cdot\|, \tau)$ be a Saks space and $(T(t))_{t \geq 0}$ a γ -strongly continuous, locally bounded semigroup on X . Then we have*

$$D(A) = D(A_{\|\cdot\|, \tau}) \text{ and } A = A_{\|\cdot\|, \tau}.$$

Proof The inclusion $D(A_{\|\cdot\|, \tau}) \subset D(A)$ follows from [22, I.1.10 Proposition, p. 9], which says that a sequence in X is γ -convergent if and only if it is τ -convergent and $\|\cdot\|$ -bounded. Further, $Af = A_{\|\cdot\|, \tau}f$ for $f \in D(A)$ because τ is coarser than γ .

Conversely, suppose that there is $x \in D(A)$ such that $\sup_{t \in (0, 1]} \frac{\|T(t)x - x\|}{t} = \infty$. Due to the local boundedness of $(T(t))_{t \geq 0}$ this implies that there is a sequence $(t_n)_{n \in \mathbb{N}}$ in $(0, 1]$ such that $t_n \rightarrow 0+$ and $\sup_{n \in \mathbb{N}} \frac{\|T(t_n)x - x\|}{t_n} = \infty$. Since γ - $\lim_{n \rightarrow \infty} \frac{T(t_n)x - x}{t_n}$ exists in X , this is a contradiction because γ -convergent sequences are $\|\cdot\|$ -bounded by [22, I.1.10 Proposition, p. 9]. \square

Due to Theorem 4.10 (a) and (b) we directly get the following corollary of Proposition 5.1.

5.2 Corollary *Let Ω be a Hausdorff space, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ and $(A, D(A))$ the generator of a locally bounded weighted composition semigroup $(C_{m, \varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ w.r.t. a jointly continuous co-semiflow (m, φ) . Then we have*

$$D(A) = D(A_{\|\cdot\|, \tau}) \text{ and } A = A_{\|\cdot\|, \tau}.$$

For our next observation we recall the definition of the generator of a norm-strongly continuous semigroup on a Banach space. Let $(X, \|\cdot\|)$ be a Banach space and $(T(t))_{t \geq 0}$ a $\|\cdot\|$ -strongly continuous semigroup on X . We define the *norm-generator* $(A_{\|\cdot\|}, D(A_{\|\cdot\|}))$ of $(T(t))_{t \geq 0}$ according to [33, Chap. 2, 1.2 Definition, p. 49] by

$$D(A_{\|\cdot\|}) := \left\{ x \in X \mid \|\cdot\| \text{-} \lim_{t \rightarrow 0+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}$$

and

$$A_{\|\cdot\|}x := \|\cdot\| \text{-} \lim_{t \rightarrow 0+} \frac{T(t)x - x}{t}, \quad x \in D(A_{\|\cdot\|}).$$

5.3 Proposition *Let Ω be a Hausdorff space, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ and $(A, D(A))$ the generator of the weighted composition semigroup $(C_{m, \varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ w.r.t. a jointly continuous co-semiflow (m, φ) . Then the following assertions hold.*

(a) *If $(C_{m, \varphi}(t))_{t \geq 0}$ is $\|\cdot\|$ -strongly continuous, then*

$$D(A) = D(A_{\|\cdot\|}) \text{ and } A = A_{\|\cdot\|}.$$

(b) *If $(\mathcal{F}(\Omega), \|\cdot\|)$ is reflexive and $(C_{m, \varphi}(t))_{t \geq 0}$ locally bounded, then the semigroup $(C_{m, \varphi}(t))_{t \geq 0}$ is $\|\cdot\|$ -strongly continuous.*

(c) Let $[(m, \varphi), \mathcal{F}(\Omega)]$ denote the space of $\|\cdot\|$ -strong continuity of $(C_{m,\varphi}(t))_{t \geq 0}$, i.e.

$$[(m, \varphi), \mathcal{F}(\Omega)] := \{f \in \mathcal{F}(\Omega) \mid \|\cdot\| \text{-} \lim_{t \rightarrow 0^+} C_{m,\varphi}(t)f = f\}.$$

If $(C_{m,\varphi}(t))_{t \geq 0}$ is locally bounded, then we have

$$[(m, \varphi), \mathcal{F}(\Omega)] = \overline{D(A)}^{\|\cdot\|}$$

where $\overline{D(A)}^{\|\cdot\|}$ denotes the closure of $D(A)$ w.r.t. the $\|\cdot\|$ -topology.

Proof Due to [33, Chap. I, 5.5 Proposition, p. 39] a $\|\cdot\|$ -strongly continuous semigroup is exponentially bounded and thus locally bounded. Hence part (a) follows from Theorem 4.10 (b) and [53, Lemma 5.15, p. 2684]. Parts (b) and (c) are a consequence of Theorem 4.10 (b) and [54, Corollary 1.25, p. 26] resp. [16, Theorem 5.6, p. 340]. \square

Let Ω be a Hausdorff space, $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ a linear space and (m, φ) a co-semiflow for $\mathcal{F}(\Omega)$. Then the Lie generator of the co-semiflow (m, φ) is given by

$$D(A_{m,\varphi}) := \left\{ f \in \mathcal{F}(\Omega) \mid \forall x \in \Omega : g(x) := \lim_{t \rightarrow 0^+} \frac{m_t(x)f(\varphi_t(x)) - f(x)}{t} \text{ exists in } \mathbb{K} \right. \\ \left. \text{and } g \in \mathcal{F}(\Omega) \right\}$$

and

$$A_{m,\varphi}f(x) := \lim_{t \rightarrow 0^+} \frac{m_t(x)f(\varphi_t(x)) - f(x)}{t}, \quad f \in D(A_{m,\varphi}), x \in \Omega.$$

In other words, the Lie generator is the generator of the weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ w.r.t. the topology of pointwise convergence. The Lie generator was introduced in [26, p. 115] for the space $\mathcal{F}(\Omega) = \mathcal{C}_b(\Omega)$ of bounded continuous functions on a Polish space Ω , equipped with the supremum norm $\|\cdot\| = \|\cdot\|_\infty$, and a jointly continuous co-semiflow $(\mathbb{1}, \varphi)$. The following proposition generalises [36, Proposition 2.12, p. 6] and [26, Proposition 2.4, p. 118] where $\mathcal{F}(\Omega) = \mathcal{C}_b(\Omega)$ and Ω a completely regular Hausdorff k -space resp. Polish space.

5.4 Proposition *Let Ω be a Hausdorff space, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ and $(A, D(A))$ the generator of a locally bounded weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ w.r.t. a jointly continuous co-semiflow (m, φ) . Then we have*

$$D(A) = D(A_{m,\varphi}) \quad \text{and} \quad A = A_{m,\varphi}.$$

Proof Since γ is stronger than τ_{co} and thus stronger than the topology of pointwise convergence, we only need to prove the inclusion $D(A_{m,\varphi}) \subset D(A)$. Let $f \in D(A_{m,\varphi})$ and set $g := A_{m,\varphi}f$. Since $g \in \mathcal{F}(\Omega)$ and $(C_{m,\varphi}(t))_{t \geq 0}$ is γ -strongly continuous by Theorem 4.10 (a), the γ -Riemann integral $\int_0^t C_{m,\varphi}(s)g ds$ exists in $\mathcal{F}(\Omega)$ for every

$t \geq 0$ by [45, Proposition 1.1, p. 232] because $(\mathcal{F}(\Omega), \gamma)$ is sequentially complete. Now, the function $f_x : [0, \infty) \rightarrow \mathbb{K}, s \mapsto m_s(x)f(\varphi_s(x))$, is right-differentiable with right-derivative $g_x : [0, \infty) \rightarrow \mathbb{K}, s \mapsto m_s(x)g(\varphi_s(x))$, because $f \in D(A_{m,\varphi})$ and

$$\begin{aligned} & \lim_{t \rightarrow 0^+} \frac{m_{s+t}(x)f(\varphi_{s+t}(x)) - m_s(x)f(\varphi_s(x))}{t} \\ &= m_s(x) \lim_{t \rightarrow 0^+} \frac{m_t(\varphi_s(x))f(\varphi_t(\varphi_s(x))) - f(\varphi_s(x))}{t} = m_s(x)g(\varphi_s(x)) = g_x(s) \end{aligned}$$

for every $s \geq 0$ and $x \in \Omega$. The right-derivative g_x is continuous for every $x \in \Omega$ as $g \in \mathcal{F}(\Omega)$ and (m, φ) is jointly continuous. Hence $f_x \in C^1[0, \infty)$ with derivative g_x by [62, Chap. 2, Corollary 1.2, p. 43] and

$$\begin{aligned} C_{m,\varphi}(t)f(x) - f(x) &= m_t(x)f(\varphi_t(x)) - f(x) = \int_0^t g_x(s)ds = \int_0^t m_s(x)g(\varphi_s(x))ds \\ &= \int_0^t C_{m,\varphi}(s)g(x)ds \end{aligned}$$

for every $t \geq 0$ and $x \in \Omega$ by the fundamental theorem of calculus. In combination with the existence of the γ -Riemann integral $\int_0^t C_{m,\varphi}(s)gds$ in $\mathcal{F}(\Omega)$ for every $t \geq 0$ this yields

$$C_{m,\varphi}(t)f - f = \int_0^t C_{m,\varphi}(s)gds$$

for every $t \geq 0$, implying our statement by [46, Proposition 1.2 (2), p. 260]. □

5.5 Remark Let Ω be a Hausdorff space, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ and $(A, D(A))$ the generator of a locally bounded weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ w.r.t. a jointly continuous co-semiflow (m, φ) . We may also define the τ_{co} -generator $(A_{\tau_{co}}, D(A_{\tau_{co}}))$ by

$$D(A_{\tau_{co}}) := \left\{ f \in \mathcal{F}(\Omega) \mid \tau_{co}\text{-}\lim_{t \rightarrow 0^+} \frac{C_{m,\varphi}(t)f - f}{t} \text{ exists in } \mathcal{F}(\Omega) \right\}$$

and

$$A_{\tau_{co}}f := \tau_{co}\text{-}\lim_{t \rightarrow 0^+} \frac{C_{m,\varphi}(t)f - f}{t}, \quad f \in D(A_{\tau_{co}}).$$

Then it follows from τ_{co} being coarser than γ and Proposition 5.4 that

$$D(A) = D(A_{\tau_{co}}) \quad \text{and} \quad A = A_{\tau_{co}}.$$

If we have more information on the co-semiflow than just joint continuity, then we may give a simpler characterisation of the generator of a weighted composition semigroup.

5.6 Proposition *Let Ω be a Hausdorff space, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ and $(A, D(A))$ the generator of a locally bounded weighted composition semigroup $(C_{m, \text{id}}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ w.r.t. a jointly continuous co-semiflow (m, id) . If $m_{(\cdot)}(x)$ is right-differentiable in $t = 0$ for all $x \in \Omega$, then*

$$D(A) = \{f \in \mathcal{F}(\Omega) \mid \dot{m}_0 f \in \mathcal{F}(\Omega)\} \text{ and } Af = \dot{m}_0 f, \quad f \in D(A).$$

Proof For $f \in \mathcal{F}(\Omega)$ we have

$$\lim_{t \rightarrow 0^+} \frac{m_t(x)f(x) - f(x)}{t} = \lim_{t \rightarrow 0^+} \frac{m_t(x) - 1}{t} f(x) = \dot{m}_0(x)f(x)$$

for all $x \in \Omega$, yielding our statement by Proposition 5.4. □

5.7 Proposition *Let $\Omega \subset \mathbb{K}$ be open, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ and $(A, D(A))$ the generator of a locally bounded weighted composition semigroup $(C_{m, \varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ w.r.t. a jointly continuous co-semiflow (m, φ) .*

(a) *If*

(i) $m_{(\cdot)}(x) \in C^1[0, \infty)$ and $\varphi_{(\cdot)}(x) \in C^1[0, \infty)$ for all $x \in \Omega$,

then

$$D_0 := \{f \in C^1_{\mathbb{K}}(\Omega) \cap \mathcal{F}(\Omega) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)\} \subset D(A)$$

and $Af = \dot{\varphi}_0 f' + \dot{m}_0 f$ for all $f \in D_0$.

(b) *Let $\omega \subset \Omega$ be open. If condition (i) is fulfilled and*

(ii) φ has a generator G , i.e. there is a function $G \in \mathcal{C}(\Omega)$ such that $\dot{\varphi}_t(x) = (G \circ \varphi_t)(x)$ for all $t \geq 0$ and $x \in \Omega$, and

(iii) $t_x := \inf\{t > 0 \mid \exists y \in \Omega \setminus \omega, y \neq x : \varphi_t(x) = y\} > 0$ for all $x \in \Omega \setminus \omega$,

then

$$D_1 := \{f \in C^1_{\mathbb{K}}(\omega) \cap \mathcal{F}(\Omega) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)\} \subset D(A)$$

and $Af = \dot{\varphi}_0 f' + \dot{m}_0 f$ for all $f \in D_1$, where $\dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)$ in the definition of D_1 means that there is an extension $g \in \mathcal{F}(\Omega)$ of the map $\dot{\varphi}_0 f' + \dot{m}_0 f : \omega \rightarrow \mathbb{K}$.

Proof (a) Let $f \in C^1_{\mathbb{K}}(\Omega) \cap \mathcal{F}(\Omega)$ and $x \in \Omega$. Fix $\tilde{t} > 0$. We note that the map $h_x : [0, \tilde{t}] \rightarrow \mathbb{K}, h_x(s) := m_s(x)\dot{\varphi}_s(x)f'(\varphi_s(x)) + \dot{m}_s(x)f(\varphi_s(x))$, is continuous by condition (i), the continuity of f' and the joint continuity of (m, φ) . Then we have

$$\begin{aligned} \frac{m_t(x)f(\varphi_t(x)) - f(x)}{t} &= \frac{1}{t} \int_0^t m_s(x)\dot{\varphi}_s(x)f'(\varphi_s(x)) + \dot{m}_s(x)f(\varphi_s(x)) \, ds \\ &= \frac{1}{t} \int_0^t h_x(s) \, ds \end{aligned}$$

for every $0 < t \leq \tilde{t}$ by (i) and the fundamental theorem of calculus. This implies

$$\begin{aligned} & \frac{m_t(x)f(\varphi_t(x)) - f(x)}{t} - (\dot{\varphi}_0(x)f'(x) + \dot{m}_0(x)f(x)) \\ &= \frac{1}{t} \int_0^t h_x(s) - \dot{\varphi}_0(x)f'(x) - \dot{m}_0(x)f(x) ds \end{aligned} \tag{12}$$

for every $0 < t \leq \tilde{t}$. Further, we observe that

$$\lim_{s \rightarrow 0^+} h_x(s) = m_0(x)\dot{\varphi}_0(x)f'(\varphi_0(x)) + \dot{m}_0(x)f(\varphi_0(x)) = \dot{\varphi}_0(x)f'(x) + \dot{m}_0(x)f(x),$$

by the continuity of h_x . Due to the Heine–Cantor theorem the continuous function h_x is even uniformly continuous on the compact interval $[0, \tilde{t}]$. Thus for every $\varepsilon > 0$ there is $0 < \delta \leq \tilde{t}$ such that for all $s \geq 0$ with $|s| = |s - 0| < \delta$ we have

$$\left| \frac{m_t(x)f(\varphi_t(x)) - f(x)}{t} - (\dot{\varphi}_0(x)f'(x) + \dot{m}_0(x)f(x)) \right| \stackrel{(12)}{<} \frac{1}{t} \int_0^t \varepsilon ds = \varepsilon$$

for all $0 < t < \delta$. We deduce that

$$\lim_{t \rightarrow 0^+} \frac{m_t(x)f(\varphi_t(x)) - f(x)}{t} = \dot{\varphi}_0(x)f'(x) + \dot{m}_0(x)f(x). \tag{13}$$

The rest of the statement follows from (13) and Proposition 5.4.

(b) Let $f \in C^1_{\mathbb{K}}(\omega) \cap \mathcal{F}(\Omega)$ and $x \in \Omega$. First, we consider the case that $x \in \omega$. Since ω is open, $\varphi_0(x) = x \in \omega$, and $\varphi_{(\cdot)}(x)$ is continuous, there is $\delta_x > 0$ such that $\varphi_t(x) \in \omega$ for all $t \in [0, \delta_x]$. It follows that the map $h_x : [0, \tilde{t}] \rightarrow \mathbb{K}$ from part (a) is still a well-defined continuous function for the choice $\tilde{t} := \delta_x$ and the rest of the proof carries over.

Let us turn to the case $x \in \Omega \setminus \omega$. Now, we need the restriction that $\dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)$. We set $p(x) := \inf\{t > 0 \mid \varphi_t(x) = x\}$. If $p(x) = 0$, then x is a fixed point of φ , and thus

$$\lim_{t \rightarrow 0^+} \frac{m_t(x)f(\varphi_t(x)) - f(x)}{t} = \lim_{t \rightarrow 0^+} \frac{m_t(x) - 1}{t} f(x) = \dot{m}_0(x)f(x)$$

Suppose that $p(x) > 0$. Setting $t(x) := \min\{p(x), t_x\}$, we observe that $t(x) > 0$ by condition (iii). Hence the map $h_x : (0, \tilde{t}] \rightarrow \mathbb{K}$ from part (a) is still a well-defined continuous function for the choice $\tilde{t} := t(x)$. Next, we show that h_x is continuously extendable in $s = 0$. We denote by $g \in \mathcal{F}(\Omega)$ the extension of $\dot{\varphi}_0 f' + \dot{m}_0 f$ and note that g as an element of $\mathcal{F}(\Omega)$ is continuous on Ω . Then $\tilde{g} := g - \dot{m}_0 f$ is a continuous extension of $\dot{\varphi}_0 f'$ on Ω . For $0 < s < t(x)$ we have by condition (ii) and Remark 3.5 that

$$\dot{\varphi}_s(x) = G(\varphi_s(x)) = \dot{\varphi}_0(\varphi_s(x))$$

and thus

$$\begin{aligned}
 & m_s(x)\dot{\varphi}_s(x)f'(\varphi_s(x)) - \tilde{g}(x) \\
 &= m_s(x)\dot{\varphi}_s(x)f'(\varphi_s(x)) - \dot{\varphi}_0(\varphi_s(x))f'(\varphi_s(x)) + \dot{\varphi}_0(\varphi_s(x))f'(\varphi_s(x)) - \tilde{g}(x) \\
 &= (m_s(x) - 1)\dot{\varphi}_0(\varphi_s(x))f'(\varphi_s(x)) + \dot{\varphi}_0(\varphi_s(x))f'(\varphi_s(x)) - \tilde{g}(x) \\
 &= (m_s(x) - 1)\tilde{g}(\varphi_s(x)) + \tilde{g}(\varphi_s(x)) - \tilde{g}(x).
 \end{aligned}$$

We derive that

$$\lim_{s \rightarrow 0^+} m_s(x)\dot{\varphi}_s(x)f'(\varphi_s(x)) = \tilde{g}(x)$$

since \tilde{g} is continuous in x and (m, φ) is a C_0 -co-semiflow. Hence h_x is continuously extendable in $s = 0$ by setting $h_x(0) := \tilde{g}(x) + \dot{m}_0(x)f(x) = g(x)$. From here the rest of the proof of part (a) carries over with $\dot{\varphi}_0(x)f'(x) + \dot{m}_0(x)f(x)$ replaced by $g(x)$. \square

The expression $p(x) = \inf\{t > 0 \mid \varphi_t(x) = x\}$ in the proof of part (b) is also called the *period* of $x \in \Omega$ w.r.t. φ (see [65, p. 660]). For the proof of the converse inclusion in Proposition 5.7 in the case that $\mathcal{F}(\Omega)$ is not a subspace of $C_{\mathbb{K}}^1(\Omega)$ we need to know what happens with φ_t and m_t to the left of $t = 0$, meaning we consider flows and cocycles instead of just semiflows and semicocycles.

5.8 Definition Let Ω be a Hausdorff space. A family $\varphi := (\varphi_t)_{t \in \mathbb{R}}$ of continuous functions $\varphi_t: \Omega \rightarrow \Omega$ is called a *flow* if

- (i) $\varphi_0(x) = x$ for all $x \in \Omega$, and
- (ii) $\varphi_{t+s}(x) = (\varphi_t \circ \varphi_s)(x)$ for all $t, s \in \mathbb{R}$ and $x \in \Omega$.

We call a flow *trivial* and write $\varphi = \text{id}$ if $\varphi_t = \text{id}$ for all $t \in \mathbb{R}$. We call a flow φ a C_0 -*flow* if $\lim_{t \rightarrow 0} \varphi_t(x) = x$ for all $x \in \Omega$. A family $m := (m_t)_{t \in \mathbb{R}}$ of continuous functions $m_t: \Omega \rightarrow \mathbb{K}$ is called a multiplicative *cocycle* for a flow φ if

- (i) $m_0(x) = 1$ for all $x \in \Omega$, and
- (ii) $m_{t+s}(x) = m_t(x)m_s(\varphi_t(x))$ for all $t, s \in \mathbb{R}$ and $x \in \Omega$.

We call a cocycle m *trivial* and write $m = \mathbb{1}$ if $m_t = \mathbb{1}$ for all $t \in \mathbb{R}$. We call a cocycle m a C_0 -*cocycle* if $\lim_{t \rightarrow 0} m_t(x) = 1$ for all $x \in \Omega$. We call the tuple (m, φ) a *co-flow* on Ω . We call a co-flow (m, φ) jointly continuous (separately continuous, C_0) if φ and m are both jointly continuous (separately continuous, C_0).

We have the following characterisation of joint continuity of flows and cocycles on certain Hausdorff spaces Ω .

5.9 Proposition Let (m, φ) be a co-flow on a Hausdorff space Ω .

- (a) Let Ω be locally compact and σ -compact. Then φ is jointly continuous if and only if φ is C_0 .
- (b) Let Ω be an open subset of a metric space and φ jointly continuous. Then m is jointly continuous if and only if m is C_0 .

Proof (a) We only need to prove the implication \Leftarrow . We define $\psi := (\psi_t)_{t \geq 0}$ by $\psi_t(x) := \varphi_{-t}(x)$ for all $t \geq 0$ and $x \in \Omega$. Then it is easily checked that ψ is a semiflow. Further, we have

$$\lim_{t \rightarrow 0^+} \psi_t(x) = \lim_{t \rightarrow 0^+} \varphi_{-t}(x) = \lim_{t \rightarrow 0^-} \varphi_t(x) = x$$

for all $x \in \Omega$. It follows from Proposition 3.3 that ψ is jointly continuous and $(\varphi_t)_{t \geq 0}$ as well. Since $\psi_0(x) = x = \varphi_0(x)$ for all $x \in \Omega$, we get that φ is jointly continuous. The proof of part (b) is analogous and we only need to use Proposition 3.12 instead of Proposition 3.3. \square

5.10 Theorem *Let $\Omega \subset \mathbb{R}$ be open, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}(\Omega)$ and $(A, D(A))$ the generator of a locally bounded weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ w.r.t. a C_0 -co-flow (m, φ) such that $m_{(\cdot)}(x) \in \mathcal{C}^1(\mathbb{R})$ and $\varphi_{(\cdot)}(x) \in \mathcal{C}^1(\mathbb{R})$ for all $x \in \Omega$, and $m_t(x) \neq 0$ for all $(t, x) \in \mathbb{R} \times \Omega$.*

(a) *If the map $V_\varphi \rightarrow \Omega, (t, x) \mapsto \varphi_t(x)$, is surjective, where $V_\varphi := \{(t, x) \in \mathbb{R} \times \Omega \mid \dot{\varphi}_t(x) \neq 0\}$, then*

$$D(A) = \{f \in \mathcal{C}^1(\Omega) \cap \mathcal{F}(\Omega) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)\}$$

and $Af = \dot{\varphi}_0 f' + \dot{m}_0 f$ for all $f \in D(A)$.

(b) *If*

(i) $(\varphi_t)_{t \geq 0}$ has a generator G , and

(ii) $t_x := \inf\{t > 0 \mid \exists y \in N_G, y \neq x : \varphi_t(x) = y\} > 0$ for all $x \in N_G$, where $N_G = \{z \in \Omega \mid G(z) = 0\}$,

then

$$D(A) = \{f \in \mathcal{C}^1(\Omega \setminus N_G) \cap \mathcal{F}(\Omega) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)\}$$

and $Af = \dot{\varphi}_0 f' + \dot{m}_0 f$ for all $f \in D(A)$.

Proof (a) Due to Proposition 5.7 (a) we only need to show that $D(A) \subset D_0$. Let $f \in D(A)$ and $x \in \Omega$. By assumption there is $(t_0, x_0) \in V_\varphi$ such that $x = \varphi_{t_0}(x_0)$. The arguments in the proof of Proposition 5.4 in combination with Proposition 5.9 applied to the C_0 -co-flow (m, φ) show that $f_{x_0} : \mathbb{R} \rightarrow \mathbb{R}, s \mapsto m_s(x_0) f(\varphi_s(x_0))$, is continuously differentiable. We deduce that $f \circ \varphi_{(\cdot)}(x_0)$ is continuously differentiable on \mathbb{R} since

$$f(\varphi_t(x_0)) = \frac{f_{x_0}(t)}{m_t(x_0)}$$

for all $t \in \mathbb{R}, m_{(\cdot)}(x_0) \in \mathcal{C}^1(\mathbb{R})$ and $f_{x_0} \in \mathcal{C}^1(\mathbb{R})$. By assumption we know that $\varphi_{(\cdot)}(x_0) \in \mathcal{C}^1(\mathbb{R})$ with $\dot{\varphi}_{t_0}(x_0) \neq 0$. By the inverse function theorem there is an open neighbourhood $U := U(t_0) \subset \mathbb{R}$ of t_0 such that $\varphi_{(\cdot)}(x_0)$ is invertible on U and the

inverse is continuously differentiable on the open neighbourhood $W := \varphi_U(x_0) \subset \Omega$ of $x = \varphi_{t_0}(x_0)$. Noting that

$$f(y) = \left((f \circ \varphi_{(\cdot)}(x_0)) \circ [\varphi_{(\cdot)}(x_0)|_U]^{-1} \right)(y)$$

for all $y \in W$, we conclude that f is continuously differentiable in $x = \varphi_{t_0}(x_0) \in W$, yielding $f \in C^1(\Omega) \cap \mathcal{F}(\Omega)$. Using (13) and Proposition 5.4, finishes the proof of part (a).

(b) Due to Proposition 5.7 (b) with $\omega := \Omega \setminus N_G$ we only need to show that $D(A) \subset D_1$. Let $f \in D(A)$. Since $\varphi_0(x) = x$ and

$$\dot{\varphi}_0(x) = G(\varphi_0(x)) = G(x) \neq 0$$

for every $x \in \Omega \setminus N_G$, we obtain that the map $V_\varphi \rightarrow \Omega \setminus N_G, (t, x) \mapsto \varphi_t(x)$, is surjective. Hence the proof of part (a) shows that f is continuously differentiable in every $x \in \Omega \setminus N_G$. The first part of the proof of Proposition 5.7 (b) yields that

$$g(x) := \lim_{t \rightarrow 0^+} \frac{m_t(x)f(\varphi_t(x)) - f(x)}{t} = \dot{\varphi}_0(x)f'(x) + \dot{m}_0(x)f(x)$$

holds for all $x \in \Omega \setminus N_G$. The left-hand side g of this equation belongs to $\mathcal{F}(\Omega)$ as $f \in D(A)$, yielding $f \in D_1$. □

Looking at the proof of Theorem 5.10 (b), we see that part (a) is a special case of (b) if there is a generator G such that $N_G = \emptyset$. If $(\varphi_t)_{t \geq 0}$ is the restriction of a jointly continuous holomorphic semiflow ψ on an open set $\tilde{\Omega} \subset \mathbb{C}$, i.e. $\varphi_t = \psi_t$ on $\Omega := \tilde{\Omega} \cap \mathbb{R}$ for all $t \geq 0$, then the generator G of $(\varphi_t)_{t \geq 0}$ exists by Theorem 3.7, namely, it is the restriction of the generator of ψ to Ω . So condition (i) of Theorem 5.10 (b) is fulfilled in this case. If in addition ψ is non-trivial, $\tilde{\Omega}$ simply connected and $\tilde{\Omega} \neq \mathbb{C}$, then $|N_G| \leq |\text{Fix}(\psi)| \leq 1$ by [13, Remark 10.1.6, p. 275] combined with the Riemann mapping theorem and thus $t_x = \inf \emptyset = \infty$ for all $x \in N_G$, yielding that condition (ii) of Theorem 5.10 (b) is also fulfilled.

Now, we will see that the proof of the converse inclusion in Proposition 5.7 is much simpler if $\mathcal{F}(\Omega)$ is a subspace of $\mathcal{C}_{\mathbb{K}}^1(\Omega)$.

5.11 Theorem *Let $\Omega \subset \mathbb{K}$ be open, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{C}_{\mathbb{K}}^1(\Omega)$ and $(A, D(A))$ the generator of a locally bounded weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ w.r.t. a jointly continuous co-semiflow (m, φ) . If $m_{(\cdot)}(x) \in C^1[0, \infty)$ and $\varphi_{(\cdot)}(x) \in C^1[0, \infty)$ for all $x \in \Omega$, then*

$$D(A) = \{f \in \mathcal{F}(\Omega) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)\}$$

and $Af = \dot{\varphi}_0 f' + \dot{m}_0 f$ for all $f \in D(A)$.

Proof By Proposition 5.7 (a) and the assumption $\mathcal{F}(\Omega) \subset C^1_{\mathbb{K}}(\Omega)$, we have

$$D_0 = \{f \in \mathcal{F}(\Omega) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)\} \subset D(A)$$

and $Af = \dot{\varphi}_0 f' + \dot{m}_0 f$ for all $f \in D_0$. The converse inclusion holds by $\mathcal{F}(\Omega) \subset C^1_{\mathbb{K}}(\Omega)$, (13) and Proposition 5.4. \square

Now, we may use the theory on weighted composition semigroups developed so far to show (in combination with Proposition 4.5) that the condition that $m_{(\cdot)}(x) \in C^1[0, \infty)$ for all $x \in \Omega$ is quite often a necessary condition for γ -strong continuity of the induced weighted composition semigroup. The underlying idea of the proof comes from the proof of [47, Theorem 1, p. 470].

5.12 Proposition *Let $\Omega \subset \mathbb{K}$ be open, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset C^1_{\mathbb{K}}(\Omega)$ and*

$$\forall x \in \Omega \exists F \in \mathcal{F}(\Omega) : F(x) \neq 0. \tag{14}$$

If (m, φ) is a jointly continuous co-semiflow for $\mathcal{F}(\Omega)$, $(C_{m,\varphi}(t))_{t \geq 0}$ locally bounded and φ has a generator G , then $m_{(\cdot)}(x) \in C^1[0, \infty)$, $\dot{m}_0 \in \mathcal{C}(\Omega)$ and $m_t(x) = \exp(\int_0^t \dot{m}_0(\varphi_s(x)) ds)$ for all $t \geq 0$ and $x \in \Omega$. If in addition $\mathbb{K} = \mathbb{C}$ and $G \in \mathcal{H}(\Omega)$, then $\dot{m}_0 \in \mathcal{H}(\Omega)$.

Proof Let $(A, D(A))$ be the generator of the γ -strongly continuous weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$. We fix $x \in \Omega$. By (14) there is $F \in \mathcal{F}(\Omega)$ such that $F(x) \neq 0$. Since F is continuous on Ω , there is a compact neighbourhood $U \subset \Omega$ of x such that $F(z) \neq 0$ for all $z \in U$. Due to Theorem 4.10 (a) $(C_{m,\varphi}(t))_{t \geq 0}$ is γ -strongly continuous on the sequentially complete space (X, γ) , implying that $D(A)$ is γ -dense in $\mathcal{F}(\Omega)$ by [46, Proposition 1.3, p. 261]. Thus there is $f \in D(A)$ such that $f(z) \neq 0$ for all $z \in U$ since γ is stronger than the topology τ_{co} and U compact. Using that $\varphi_0(x) = x$ and the joint continuity of φ , we deduce that there are $t_0 > 0$ and a neighbourhood U_0 of x such that $\varphi_s(\zeta) \in U$ for all $s \in [0, t_0]$ and $\zeta \in U_0$. In particular, $f(\varphi_s(\zeta)) \neq 0$ for all $s \in [0, t_0]$ and $\zeta \in U_0$. Further, we have

$$\begin{aligned} \frac{m_s(\zeta) - 1}{s} &= \frac{1}{f(\varphi_s(\zeta))} \frac{m_s(\zeta)f(\varphi_s(\zeta)) - f(\varphi_s(\zeta))}{s} \\ &= \frac{1}{f(\varphi_s(\zeta))} \frac{C_{m,\varphi}(s)f(\zeta) - f(\varphi_s(\zeta))}{s} \end{aligned}$$

for all $0 < s \leq t_0$ and

$$\begin{aligned} \lim_{s \rightarrow 0^+} \frac{m_s(\zeta) - 1}{s} &= \lim_{s \rightarrow 0^+} \frac{1}{f(\varphi_s(\zeta))} \left(\frac{C_{m,\varphi}(s)f(\zeta) - f(\zeta)}{s} - \frac{f(\varphi_s(\zeta)) - f(\zeta)}{s} \right) \\ &= \frac{1}{f(\zeta)} (Af(\zeta) - f'(\zeta)\dot{\varphi}_0(\zeta)) = \frac{1}{f(\zeta)} (Af(\zeta) - f'(\zeta)G(\zeta)) \end{aligned}$$

for all $\zeta \in U_0$. Therefore $\dot{m}_0(\zeta) = \frac{1}{f(\zeta)}(Af(\zeta) - f'(\zeta)G(\zeta))$ for all $\zeta \in U_0$, yielding that \dot{m}_0 is continuous on Ω because x is arbitrary. If in addition $\mathbb{K} = \mathbb{C}$ and $G \in \mathcal{H}(\Omega)$, then we also get $\dot{m}_0 \in \mathcal{H}(\Omega)$. The continuous differentiability of $m_{(\cdot)}(x)$ on $[0, \infty)$ and $m_t(x) = \exp(\int_0^t \dot{m}_0(\varphi_s(x))ds)$ for all $t \geq 0$ and $x \in \Omega$ follow from Proposition 3.13. \square

We note that we may replace the condition $\mathcal{F}(\Omega) \subset C^1_{\mathbb{K}}(\Omega)$ by the condition $D(A) \subset C^1_{\mathbb{K}}(\Omega)$ because we only need it for the function $f \in D(A)$ in the proof of Proposition 5.12. Condition (14) is for example fulfilled if $\mathbb{1} \in \mathcal{F}(\Omega)$.

5.13 Corollary *Let $\Omega \subset \mathbb{C}$ be open, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{H}(\Omega)$ and $\mathbb{1} \in \mathcal{F}(\Omega)$. If (m, φ) is a jointly continuous holomorphic co-semiflow for $\mathcal{F}(\Omega)$ and $(C_{m,\varphi}(t))_{t \geq 0}$ locally bounded, then $m_{(\cdot)}(z) \in C^1[0, \infty)$, $\dot{m}_0 \in \mathcal{H}(\Omega)$ and $m_t(z) = \exp(\int_0^t \dot{m}_0(\varphi_s(z))ds)$ for all $t \geq 0$ and $z \in \Omega$.*

Proof Our statement follows from Proposition 5.12 since (14) is fulfilled as $\mathbb{1} \in \mathcal{F}(\Omega)$ and φ has a generator $G \in \mathcal{H}(\Omega)$ by Theorem 3.7. \square

Due to [33, Chap. I, 5.5 Proposition, p. 39] and Corollary 5.13 we get back [47, Theorem 1, p. 470] as a special case where $\Omega = \mathbb{D}$ and $\mathcal{F}(\mathbb{D}) = H^p$, $1 \leq p < \infty$, is the Hardy space on \mathbb{D} . Moreover, in contrast to Proposition 3.15, where $\Omega \subset \mathbb{C}$ is connected, Corollary 5.13 allows arbitrary open sets Ω but for the cost of more assumptions on (m, φ) . For example the condition $\mathbb{1} \in \mathcal{F}(\Omega)$ in Corollary 5.13 implies that $m_t \in \mathcal{F}(\Omega)$ for all $t \geq 0$ (see Remark 4.8).

6 Converse of the Holomorphic Generation Theorem

Let $\Omega \subset \mathbb{C}$ be open, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{H}(\Omega)$. Suppose that Ω is connected or that $\mathbb{1} \in \mathcal{F}(\Omega)$. Due to Theorem 3.7, Theorem 5.11, and Proposition 3.15 or Corollary 5.13 we know that the generator $(A, D(A))$ of a locally bounded weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ w.r.t. a jointly continuous holomorphic co-semiflow (m, φ) is given by

$$D(A) = \{f \in \mathcal{F}(\Omega) \mid Gf' + gf \in \mathcal{F}(\Omega)\}, \quad Af = Gf' + gf, \quad f \in D(A), \quad (15)$$

where $G \in \mathcal{H}(\Omega)$ is the generator of φ and $g := \dot{m}_0 \in \mathcal{H}(\Omega)$.

In this section we want to prove the converse statement, namely, if we know that the domain of a γ -strongly continuous semigroup $(T(t))_{t \geq 0}$ on $\mathcal{F}(\Omega)$ is given by (15), we want to show, under suitable conditions, that this semigroup is a weighted composition semigroup whose semiflow has G as a generator and whose semicocycle m is given by $m_t(z) := \exp(\int_0^t g(\varphi_s(z))ds)$ for all $t \geq 0$ and $z \in \Omega$.

Our main result in this direction is an analogon of [38, Main theorem, p. 490] ($g = 0$) and [39, Theorem 3.1, p. 69] where $\Omega = \mathbb{D}$ and $(T(t))_{t \geq 0}$ is a $\|\cdot\|$ -strongly continuous semigroup. We define the space of holomorphic germs near the closed unit

disc $\overline{\mathbb{D}}$ by the inductive limit

$$\mathcal{H}(\overline{\mathbb{D}}) := \varinjlim_{\omega \subset \mathbb{C} \text{ open, } \overline{\mathbb{D}} \subset \omega} (\mathcal{H}(\omega), \tau_{\text{co}})$$

equipped with its inductive limit topology (see e.g. [9, p. 81–82]). For $n \in \mathbb{N}_0$ set $e_n: \mathbb{C} \rightarrow \mathbb{C}, e_n(z) := z^n$. Then $e_n \in \mathcal{H}(\overline{\mathbb{D}})$ for all $n \in \mathbb{N}_0$. Hence the assumption that $\mathcal{H}(\overline{\mathbb{D}}) \hookrightarrow (\mathcal{F}(\mathbb{D}), \|\cdot\|)$ embeds continuously implies that $e_n \in \mathcal{F}(\mathbb{D})$ for all $n \in \mathbb{N}_0$.

6.1 Theorem *Let $(\mathcal{F}(\mathbb{D}), \|\cdot\|, \tau_{\text{co}})$ be a sequentially complete Saks space such that $\mathcal{F}(\mathbb{D}) \subset \mathcal{H}(\overline{\mathbb{D}})$, and $\mathcal{H}(\overline{\mathbb{D}}) \hookrightarrow (\mathcal{F}(\mathbb{D}), \|\cdot\|)$ embeds continuously. If $(T(t))_{t \geq 0}$ is a γ -strongly continuous semigroup on $\mathcal{F}(\mathbb{D})$ with generator $(A, D(A))$ of the form*

$$D(A) = \{f \in \mathcal{F}(\mathbb{D}) \mid Gf' + gf \in \mathcal{F}(\mathbb{D})\}, \quad Af = Gf' + gf, \quad f \in D(A),$$

for some $G, g \in \mathcal{H}(\overline{\mathbb{D}})$, then there is a jointly continuous holomorphic co-semiflow (m, φ) for $\mathcal{F}(\mathbb{D})$ such that G is the generator of φ , $m_t(z) = \exp(\int_0^t g(\varphi_s(z)) ds)$ for all $t \geq 0, z \in \mathbb{D}$, and $(T(t))_{t \geq 0} = (C_{m, \varphi}(t))_{t \geq 0}$.

Proof The proof of [39, Theorem 3.1, p. 69] carries over to our setting. We only have to adjust the proof in three instances. First, we have to use that $Af = \frac{d}{dt} T(t)f$ for all $t \geq 0$ and $f \in D(A)$ by [46, Proposition 1.2 (1), p. 260] in the proof of [39, Claim 1, p. 70]. Second, $D(A)$ is γ -dense in $\mathcal{F}(\mathbb{D})$ by [46, Proposition 1.3, p. 261]. Thus for every $f \in \mathcal{F}(\mathbb{D})$ there is a net $(f_i)_{i \in I}, I$ a directed set, which is γ -convergent to f . Since $T(t)$ is γ -continuous for every $t \geq 0$, this implies that $(T(t)f_i)_{i \in I}$ is γ -convergent to $T(t)f$ for every $t \geq 0$. This proves the validity of [39, Eq. (3.6), p. 71] because γ is finer than τ_{co} . Third, we note that $\delta_z \in (\mathcal{F}(\mathbb{D}), \gamma)'$ and $\delta_z \in (\mathcal{F}(\mathbb{D}), \|\cdot\|)'$ for all $z \in \mathbb{D}$, where $\delta_z(f) := f(z)$ for all $f \in \mathcal{F}(\mathbb{D})$, because $\delta_z \in (\mathcal{F}(\mathbb{D}), \tau_{\text{co}})'$ and γ and the $\|\cdot\|$ -topology are finer than τ_{co} . We now get by [39, Eq. (3.10), p. 72] that

$$\varphi_t(z)^n = \frac{1}{m_t(z)} T(t)(e_n)(z)$$

for all $z \in \mathbb{D}, n \in \mathbb{N}$ and $0 \leq t < t_0$ with $t_0 > 0$ from [39, p. 69]. Let Γ_γ denote a directed system of continuous seminorms that generates γ . Since $\delta_z \in (\mathcal{F}(\mathbb{D}), \gamma)'$, $T(t) \in \mathcal{L}(\mathcal{F}(\mathbb{D}), \gamma)$ and the $\|\cdot\|$ -topology is finer than γ , there are $p = p_{z,t} \in \Gamma_\gamma, K_1 = K_1(z, t) \geq 0$ and $K_2 = K_2(p) \geq 0$ such that

$$|\varphi_t(z)^n| = \left| \frac{1}{m_t(z)} T(t)(e_n)(z) \right| \leq K_1 \frac{1}{m_t(z)} p(e_n) \leq K_1 K_2 \frac{1}{m_t(z)} \|e_n\|$$

for all $z \in \mathbb{D}, n \in \mathbb{N}$ and $0 \leq t < t_0$. This implies that there is $0 < t_1 \leq t_0$ such that $|\varphi_t(z)| < 1$ for all $z \in \mathbb{D}$ and $0 \leq t < t_1$ like in the proof of [38, Claim 2, p. 492]. This is all we have to change in the proof of [39, Theorem 3.1, p. 69]. Moreover, we note that the joint continuity of φ follows from [39, p. 65] and Proposition 3.3 (b), and the joint continuity of m from Proposition 3.14 (a). \square

We note that Theorem 6.1 implies [39, Theorem 3.1, p. 69] if $(T(t))_{t \geq 0}$ is $\|\cdot\|$ -strongly continuous and $T(t) \in \mathcal{L}(X, \gamma)$ for all $t \geq 0$ because then $(T(t))_{t \geq 0}$ is also γ -strongly continuous. The semicyclope m in Theorem 6.1 is given by $m_t = T(t)\mathbb{1} = T(t)e_0$ and the semiflow φ by $\varphi_t = \frac{1}{m_t}T(t)\text{id} = \frac{1}{m_t}T(t)e_1$ for all $t \geq 0$. Further, it is shown in [11, p. 176–177] that the assumption that $\mathcal{H}(\overline{\mathbb{D}}) \hookrightarrow (\mathcal{F}(\mathbb{D}), \|\cdot\|)$ embeds continuously is equivalent to $\limsup_{n \rightarrow \infty} \|e_n\|_n^{\frac{1}{n}} \leq 1$.

6.2 Example For the following spaces $(\mathcal{F}(\mathbb{D}), \|\cdot\|)$ the embedding $\mathcal{H}(\overline{\mathbb{D}}) \hookrightarrow (\mathcal{F}(\mathbb{D}), \|\cdot\|)$ is continuous:

- (a) The Hardy spaces $(H^p, \|\cdot\|_p)$ for $1 \leq p \leq \infty$ since $\|e_n\|_p = 1$ for all $n \in \mathbb{N}$.
- (b) The Bergman spaces $(A_{\alpha}^p, \|\cdot\|_{\alpha,p})$ for $\alpha > -1$ and $1 \leq p < \infty$ since

$$\begin{aligned} \|e_n\|_{\alpha,p}^p &= \frac{\alpha + 1}{\pi} \int_{\mathbb{D}} |z|^{np} (1 - |z|^2)^{\alpha} dz = \frac{\alpha + 1}{\pi} \int_0^{2\pi} \int_0^1 r^{np+1} (1 - r^2)^{\alpha} dr d\theta \\ &\leq 2(\alpha + 1) \int_0^1 r^{p+1} (1 - r^2)^{\alpha} dr = (\alpha + 1)B\left(\frac{p}{2} + 1, \alpha + 1\right) \end{aligned}$$

for all $n \in \mathbb{N}$ by [40, Eq. 3251-1, p. 327] with $\mu := p + 2, \lambda := 2$ and $\nu := \alpha + 1$, where B denotes the Beta function, and thus $\limsup_{n \rightarrow \infty} \|e_n\|_{\alpha,p}^{\frac{1}{n}} \leq 1$.

- (c) The Dirichlet space $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ since

$$\|e_n\|_{\mathcal{D}}^2 = 0 + \frac{1}{\pi} \int_{\mathbb{D}} n^2 |z|^{2n-2} dz = \frac{n^2}{\pi} \int_0^{2\pi} \int_0^1 r^{2n-1} dr d\theta = \frac{2n^2}{2n} = n$$

for all $n \in \mathbb{N}$ and thus $\lim_{n \rightarrow \infty} \|e_n\|_{\mathcal{D}}^{\frac{1}{n}} = 1$.

- (d) The ν -Bloch spaces $(\mathcal{B}_{\nu}(\mathbb{D}), \|\cdot\|_{\mathcal{B}_{\nu}(\mathbb{D})})$ for bounded continuous $\nu: \mathbb{D} \rightarrow (0, \infty)$ since

$$\|e_n\|_{\mathcal{B}_{\nu}(\mathbb{D})} = 0 + \sup_{z \in \mathbb{D}} n|z|^{n-1} \nu(z) \leq n\|v\|_{\infty}$$

for all $n \in \mathbb{N}$ and thus $\limsup_{n \rightarrow \infty} \|e_n\|_{\mathcal{B}_{\nu}(\mathbb{D})}^{\frac{1}{n}} \leq 1$.

- (e) The spaces $(\mathcal{H}_{\nu}(\mathbb{D}), \|\cdot\|_{\nu})$ of weighted holomorphic functions on \mathbb{D} for bounded continuous $\nu: \mathbb{D} \rightarrow (0, \infty)$ since

$$\|e_n\|_{\nu} = \sup_{z \in \mathbb{D}} |z|^n \nu(z) \leq \|v\|_{\infty}$$

for all $n \in \mathbb{N}$ and thus $\limsup_{n \rightarrow \infty} \|e_n\|_{\nu}^{\frac{1}{n}} \leq 1$.

In particular, the embedding $\mathcal{H}(\overline{\mathbb{D}}) \hookrightarrow (\mathcal{B}_{\alpha}, \|\cdot\|_{\mathcal{B}_{\alpha}})$ for $\alpha > 0$ is continuous by Example 6.2 (d) because $\nu_{\alpha}(z) = (1 - |z|^2)^{\alpha} \leq 1$ for all $z \in \mathbb{D}$. Further, we note that Theorem 4.14, Theorem 4.15 and Theorem 4.17 in combination with Example 6.2 answer the question in [39, Remark 3.2, p. 72] for several spaces, namely, to give

a sufficient condition such that $C_{m,\varphi}(t) \in \mathcal{F}(\mathbb{D})$ for all $t \geq 0$ where $(\mathcal{F}(\mathbb{D}), \|\cdot\|)$ is a space whose embedding $\mathcal{H}(\overline{\mathbb{D}}) \hookrightarrow (\mathcal{F}(\mathbb{D}), \|\cdot\|)$ is continuous.

We remark that Theorem 6.1 is not restricted to the case $\Omega = \mathbb{D}$ (cf. [4, p. 177–178] ($g = 0$) and [39, Remark 3.3, p. 72–73] in the case of $\|\cdot\|$ -strongly continuous semigroups).

6.3 Remark Let $\Omega \subsetneq \mathbb{C}$ be open and simply connected, and $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{H}(\Omega)$. Suppose that $(S(t))_{t \geq 0}$ is a γ -strongly continuous semigroup on $\mathcal{F}(\Omega)$ with generator $(A, D(A))$ of the form

$$D(A) = \{f \in \mathcal{F}(\Omega) \mid Gf' + gf \in \mathcal{F}(\Omega)\}, \quad Af = Gf' + gf, \quad f \in D(A),$$

for some $G, g \in \mathcal{H}(\Omega)$. Choose a biholomorphic map $h: \mathbb{D} \rightarrow \Omega$ by the Riemann mapping theorem and define the space $\mathcal{F}_h(\mathbb{D}) := \{f \circ h \mid f \in \mathcal{F}(\Omega)\}$, which becomes a Banach space when equipped with the norm $\|f \circ h\|_{\mathcal{F}_h(\mathbb{D})} := \|f\|$ for $f \in \mathcal{F}(\Omega)$. It is easy to check that the composition operator $C_h: \mathcal{F}(\Omega) \rightarrow \mathcal{F}_h(\mathbb{D})$, $C_h(f) := f \circ h$, is an isometric isomorphism w.r.t. the norms, an isomorphism $(\mathcal{F}(\Omega), \tau_{co}) \rightarrow (\mathcal{F}_h(\mathbb{D}), \tau_{co})$, and $C_h^{-1} = C_{h^{-1}}$. Then it follows as in the proof of Proposition 4.1 that $C_h: (\mathcal{F}(\Omega), \gamma(\|\cdot\|, \tau_{co})) \rightarrow (\mathcal{F}_h(\mathbb{D}), \gamma(\|\cdot\|_{\mathcal{F}_h(\mathbb{D})}, \tau_{co}))$ is an isomorphism as well. Therefore $(\mathcal{F}_h(\mathbb{D}), \|\cdot\|_{\mathcal{F}_h(\mathbb{D})}, \tau_{co})$ is a sequentially complete Saks space such that $\mathcal{F}_h(\mathbb{D}) \subset \mathcal{H}(\mathbb{D})$ and the semigroup $(T(t))_{t \geq 0}$ on $\mathcal{F}_h(\mathbb{D})$ given by

$$T(t) := C_h \circ S(t) \circ C_{h^{-1}}, \quad t \geq 0,$$

is $\gamma(\|\cdot\|_{\mathcal{F}_h(\mathbb{D})}, \tau_{co})$ -strongly continuous. Assume that $\mathcal{H}(\overline{\mathbb{D}}) \hookrightarrow (\mathcal{F}_h(\mathbb{D}), \|\cdot\|_{\mathcal{F}_h(\mathbb{D})})$ embeds continuously. Like in [39, Remark 3.3, p. 73] it follows that the generator $(B, D(B))$ of $(T(t))_{t \geq 0}$ fulfils

$$D(B) = \{f \in \mathcal{F}_h(\mathbb{D}) \mid G_1 f' + g_1 f \in \mathcal{F}_h(\mathbb{D})\}, \quad Bf = G_1 f' + g_1 f, \quad f \in D(B),$$

where $G_1(z) := \frac{1}{h'(z)}G(h(z))$ and $g_1(z) := g(h(z))$ for all $z \in \mathbb{D}$. Hence we may apply Theorem 6.1 and obtain that there is a jointly continuous holomorphic co-semiflow (m, φ) for $\mathcal{F}_h(\mathbb{D})$ such that G_1 is the generator of φ , $m_t(z) = \exp(\int_0^t g_1(\varphi_s(z))ds)$ for all $t \geq 0$, $z \in \mathbb{D}$, and $T(t) = C_{m,\varphi}(t)$ for all $t \geq 0$. Like in [39, Remark 3.3, p. 73] we get that $S(t) = C_{\mu,\psi}(t)$ for all $t \geq 0$ with the jointly continuous holomorphic semiflow ψ on Ω given by $\psi_t := h \circ \varphi_t \circ h^{-1}$ and its jointly continuous holomorphic semicycle μ given by $\mu_t(z) := (m_t \circ h^{-1})(z) = \exp(\int_0^t g(\psi_s(z))ds)$ for all $t \geq 0$ and $z \in \Omega$.

7 Applications

In this short section we apply our results from the preceding sections. In the following proposition $\mathcal{C}v(\mathbb{R})$ denotes the weighted space of continuous functions from Example 2.13 for $\mathbb{K} = \mathbb{R}$.

7.1 Proposition *Let $v: \mathbb{R} \rightarrow (0, \infty)$ be continuous, $\varphi_t(x) := \left(x^{\frac{1}{3}} + \frac{t}{3}\right)^3$, $t, x \in \mathbb{R}$, and $m := (m_t)_{t \in \mathbb{R}}$ be a cocycle for $\varphi := (\varphi_t)_{t \in \mathbb{R}}$ such that $\lim_{s \rightarrow 0} m_s(x) = 1$ for all $x \in \mathbb{R}$, $m_t \in C_b(\mathbb{R})$ for all $t \in \mathbb{R}$, $m_{(\cdot)}(x) \in C^1(\mathbb{R})$, $m_t(x) \neq 0$ for all $(t, x) \in \mathbb{R}^2$,*

$$K(\varphi_t) := \sup_{x \in \mathbb{R}} \frac{v(x)}{v(\varphi_t(x))} < \infty \tag{16}$$

for all $t \in \mathbb{R}$ and there exists $t_0 > 0$ such that $\sup_{t \in [0, t_0]} K(\varphi_t) < \infty$. Then the following assertions hold.

- (a) *The weighted composition semigroup $(C_{m, \varphi}(t))_{t \geq 0}$ on $Cv(\mathbb{R})$ w.r.t. the co-flow (m, φ) is γ -strongly continuous, τ_{co} -bi-continuous, locally τ_{co} -equicontinuous and quasi- γ -equicontinuous.*
- (b) *The generator $(A, D(A))$ of $(C_{m, \varphi}(t))_{t \geq 0}$ fulfils*

$$D(A) = \{f \in C^1(\mathbb{R} \setminus \{0\}) \cap Cv(\mathbb{R}) \mid [x \mapsto x^{\frac{2}{3}} f'(x) + \dot{m}_0(x) f(x)] \in Cv(\mathbb{R})\}$$

and $Af(x) = x^{\frac{2}{3}} f'(x) + \dot{m}_0(x) f(x)$, $x \neq 0$, for all $f \in D(A)$. If in addition $\dot{m}_0 \in C_b(\mathbb{R})$, then $D(A) = \{f \in C^1(\mathbb{R} \setminus \{0\}) \cap Cv(\mathbb{R}) \mid [x \mapsto x^{\frac{2}{3}} f'(x)] \in Cv(\mathbb{R})\}$.

Proof (a) The flow φ is clearly C_0 . The assumption $\lim_{s \rightarrow 0} m_s(x) = 1$ for all $x \in \mathbb{R}$ means that its cocycle m is also C_0 and so the co-flow (m, φ) is jointly continuous by Proposition 5.9. Further, $C_{m, \varphi}(t)f = m_t \cdot (f \circ \varphi_t)$ is continuous on \mathbb{R} and

$$\|C_{m, \varphi}(t)f\|_v = \sup_{x \in \mathbb{R}} |m_t(x) f(\varphi_t(x))| v(\varphi_t(x)) \frac{v(x)}{v(\varphi_t(x))} \leq K(\varphi_t) \|m_t\|_\infty \|f\|_v$$

for all $t \in \mathbb{R}$ and $f \in Cv(\mathbb{R})$ by (16). Hence (m, φ) is a jointly continuous co-flow for $Cv(\mathbb{R})$ and $(C_{m, \varphi}(t))_{t \geq 0}$ locally bounded by Proposition 2.3 since $\sup_{t \in [0, t_0]} K(\varphi_t) < \infty$ and $\sup_{t \in [0, t_1]} \|m_t\|_\infty < \infty$ for some $t_1 > 0$ by Proposition 3.10. We conclude that statement (a) is valid by Example 2.13 and Theorem 4.10.

(b) Setting $G: \mathbb{R} \rightarrow \mathbb{R}$, $G(x) := x^{\frac{2}{3}}$, we note that

$$\dot{\varphi}_t(x) = \left(x^{\frac{1}{3}} + \frac{t}{3}\right)^2 = (G \circ \varphi_t)(x)$$

for all $(t, x) \in \mathbb{R}^2$. Thus $N_G = \{0\}$ and $t_0 = \inf \emptyset = \infty$. We deduce that the first part of (b) follows from Theorem 5.10 (b). If additionally $\dot{m}_0 \in C_b(\mathbb{R})$, then we have $\dot{m}_0 f \in Cv(\mathbb{R})$ and thus $D(A) = \{f \in C^1(\mathbb{R} \setminus \{0\}) \cap Cv(\mathbb{R}) \mid [x \mapsto x^{\frac{2}{3}} f'(x)] \in Cv(\mathbb{R})\}$. □

Proposition 7.1 is interesting because it illustrates the phenomenon of non-differentiability of the elements in the domain of A at the zeros of G which cannot happen in the case of weighted composition semigroups on spaces of holomorphic functions (see Corollary 5.13). For $m = v = \mathbb{1}$ the statement of Proposition 7.1 (a) is

known due to e.g. [25, Theorems 2.1, 2.2, 2.3, p. 5]. Proposition 7.1 (b) generalises [26, Example 4.2, p. 124–125] where $m = v = \mathbb{1}$.

7.2 Theorem *Let $\Omega \subset \mathbb{C}$ be open and connected, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{H}(\Omega)$ and $\{\mathbb{1}, \text{id}\} \subset \mathcal{F}(\Omega)$, and $(C_{m,\varphi}(t))_{t \geq 0}$ the weighted composition semigroup on $\mathcal{F}(\Omega)$ w.r.t. a holomorphic co-semiflow (m, φ) . Then the following assertions hold.*

- (a) $(C_{m,\varphi}(t))_{t \geq 0}$ is γ -strongly continuous and locally γ -equicontinuous if and only if (m, φ) is a C_0 -co-semiflow and $(C_{m,\varphi}(t))_{t \geq 0}$ locally bounded.
- (b) Suppose that $(\mathcal{F}(\Omega), \|\cdot\|)$ is reflexive. Then $(C_{m,\varphi}(t))_{t \geq 0}$ is $\|\cdot\|$ -strongly continuous if and only if (m, φ) is a C_0 -co-semiflow and $(C_{m,\varphi}(t))_{t \geq 0}$ locally bounded.
- (c) If $(C_{m,\varphi}(t))_{t \geq 0}$ is $\|\cdot\|$ -strongly continuous with generator $(A_{\|\cdot\|}, D(A_{\|\cdot\|}))$, then we have $\dot{\varphi}_0, \dot{m}_0 \in \mathcal{H}(\Omega)$ and

$$D(A_{\|\cdot\|}) = \{f \in \mathcal{F}(\Omega) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)\}$$

and $A_{\|\cdot\|} f = \dot{\varphi}_0 f' + \dot{m}_0 f$ for all $f \in D(A_{\|\cdot\|})$.

Proof First, we note that the open set $\Omega \subset \mathbb{C}$ is a locally compact, σ -compact space w.r.t. the relative topology induced by the metric space \mathbb{C} . In particular, Ω is a Hausdorff $k_{\mathbb{R}}$ -space, and the co-semiflow (m, φ) is jointly continuous if and only if it is C_0 by Proposition 3.3 and Proposition 3.12. Second, if φ is C_0 , so jointly continuous, then $\varphi_{(\cdot)}(z) \in \mathcal{C}^1[0, \infty)$ for all $z \in \Omega$ and $\dot{\varphi}_0 \in \mathcal{H}(\Omega)$ by Theorem 3.7. Third, $m_{(\cdot)}(z) \in \mathcal{C}^1[0, \infty)$, $\dot{m}_0 \in \mathcal{H}(\Omega)$ and $m_t(z) \neq 0$ for all $t \geq 0$ and $z \in \Omega$ by Proposition 3.14 (a) and Proposition 3.15 since $\Omega \subset \mathbb{C}$ is connected.

(a) Due to [37, Proposition 3.6 (ii), p. 1137] and [22, I.1.10 Proposition, p. 10] a γ -strongly continuous, locally γ -equicontinuous semigroup of linear operators is locally bounded. Thus implication \Rightarrow follows from Remark 4.4 (b) and Proposition 4.5 (b). The converse implication \Leftarrow follows from Theorem 4.10 (a).

(b) Let $(C_{m,\varphi}(t))_{t \geq 0}$ be $\|\cdot\|$ -strongly continuous. Then $(C_{m,\varphi}(t))_{t \geq 0}$ is locally bounded by [33, Chap. I, 5.5 Proposition, p. 39] and γ -strongly continuous by Proposition 4.1 with $I := \{t\}$ for all $t \geq 0$ and since γ is coarser than the $\|\cdot\|$ -topology. Due to the assumption $\{\mathbb{1}, \text{id}\} \subset \mathcal{F}(\Omega)$ the topology of Ω is initial-like w.r.t. $(\varphi, \mathcal{F}(\Omega))$ by Remark 4.4 (b), and thus the co-semiflow (m, φ) jointly continuous by Proposition 4.5 (b).

Let us turn to the converse. Suppose that (m, φ) is a C_0 -co-semiflow and $(C_{m,\varphi}(t))_{t \geq 0}$ locally bounded. Then (m, φ) is jointly continuous and $(C_{m,\varphi}(t))_{t \geq 0}$ γ -strongly continuous by Proposition 4.5 (a). We deduce from Proposition 5.3 (b) that $(C_{m,\varphi}(t))_{t \geq 0}$ is $\|\cdot\|$ -strongly continuous since $(\mathcal{F}(\Omega), \|\cdot\|)$ is reflexive.

(c) By the the first part of the proof of (b) we get that $(C_{m,\varphi}(t))_{t \geq 0}$ is locally bounded and (m, φ) a C_0 -co-semiflow. Applying Proposition 5.3 (a) and Theorem 5.11, this finishes the proof of part (c). □

Theorem 7.2 (b) implies [47, Lemma 3.1, p. 474] and [68, Theorem 1, p. 362] for the reflexive Hardy spaces H^p , $1 < p < \infty$, by [47, Definition 1 (i), p. 469], Example 2.9 (a), Example 3.18 and Theorem 4.14 (a). It also answers the questions

in [71, Example 7.4, p. 247–248] because it implies the $\|\cdot\|$ -strong continuity of the weighted composition semigroup induced by the semicoboundary (φ', φ) for a jointly continuous holomorphic semiflow φ on \mathbb{D} (see Example 3.19) on reflexive spaces $(\mathcal{F}(\mathbb{D}), \|\cdot\|)$ such that $\mathcal{F}(\mathbb{D}) \subset \mathcal{H}(\mathbb{D})$ such as the Hardy spaces H^p for $1 < p < \infty$, the weighted Bergman spaces A_α^p for $\alpha > -1$ and $1 < p < \infty$, and the Dirichlet space \mathcal{D} due to Example 2.9 and Theorem 4.14 (and the computations thereafter). Further, Theorem 7.2 (b) in combination with Theorem 4.14 (a) and (b) gives back [63, Lemmas 2.13, 2.14, p. 828] and [77, Corollaries 3, 4, p. 8–9] for $1 < p < \infty$ by a different proof.² It also implies in combination with Theorem 4.14 (c) [70, Theorem 1, p. 167] where $\mathcal{F}(\mathbb{D}) = \mathcal{D}$ and $m = \mathbb{1}$.

If $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ is a sequentially complete Saks such that $\mathcal{F}(\Omega) \subset \mathcal{H}(\Omega)$ and $\{\mathbb{1}, \text{id}\} \subset \mathcal{F}(\Omega)$, then Theorem 7.2 (c) implies [12, Theorem 2, p. 72] where $\Omega = \mathbb{D}$ and $m = \mathbb{1}$ but it is only assumed that $(\mathcal{F}(\Omega), \|\cdot\|)$ is complete and τ_{co} coarser than the $\|\cdot\|$ -topology (see [12, p. 67]). Moreover, Theorem 7.2 (c) generalises [68, Theorem 2, p. 364] (where $m = m^\omega$ is a semicoboundary for $\omega \in \mathcal{H}(\mathbb{D})$) and the first part of [47, Theorem 2 (b), p. 471] by Example 2.9 (a) and Example 3.18 where $\mathcal{F}(\mathbb{D}) = H^p$ is the Hardy space for $1 \leq p < \infty$. Theorem 7.2 (c) also yields [69, Theorem 1 (ii), p. 400–401] by Example 2.9 (b) where $\mathcal{F}(\mathbb{D}) = A_\alpha^p$ is the Bergman space for $\alpha > -1$ and $1 \leq p < \infty$ and $m = \mathbb{1}$. Theorem 7.2 (b) and (c) combined with Theorem 4.14 (c) also imply the $\|\cdot\|$ -strong continuity of the weighted composition semigroup on \mathcal{D} w.r.t. the holomorphic C_0 -co-semiflow (m, φ) in [70, Corollary 2, p. 170] and the form of its generator where $\text{Fix}(\varphi) = \{0\}$ and $m = m^\omega$ with $\omega(z) := z^p$ for all $z \in \mathbb{D}$ for some $p \in \mathbb{N}$. If $(\mathcal{F}(\Omega), \|\cdot\|)$ is a Banach space such that τ_{co} is coarser than the $\|\cdot\|$ -topology and $\mathcal{F}(\Omega) \subset \mathcal{H}(\Omega)$, then the assumptions in [39, Theorem 2.1 (ii), p. 68] that $\Omega = \mathbb{D}$ and the continuity of $\mathcal{H}(\overline{\mathbb{D}}) \leftrightarrow (\mathcal{F}(\Omega), \|\cdot\|)$ are stronger than in Theorem 7.2 (c) whereas vice versa the latter theorem has the stronger assumption that $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ is a sequentially complete Saks space.

We may also apply Theorem 7.2 (a) to weighted composition semigroups on the Hardy space H^1 by Theorem 4.14 (a), the Bergman space A_α^1 for $\alpha > -1$ by Theorem 4.14 (b), the Bloch type space \mathcal{B}_α for $\alpha > 0$ by Theorem 4.15, the weighted spaces $\mathcal{H}v(\Omega)$ and $\mathcal{C}v(\Omega)$ of holomorphic resp. continuous functions on Ω by Theorem 4.17 resp. Theorem 4.18 if $\{\mathbb{1}, \text{id}\} \subset \mathcal{H}v(\Omega)$ resp. $\{\mathbb{1}, \text{id}\} \subset \mathcal{C}v(\Omega)$. For instance, we have the following result for the Hardy space H^∞ .

7.3 Corollary *Let (m, φ) be a holomorphic co-semiflow on \mathbb{D} . The weighted composition semigroup $(C_{m,\varphi}(t))_{t \geq 0}$ is γ -strongly continuous and locally γ -equicontinuous on H^∞ if and only if (m, φ) is a C_0 -co-semiflow and $\limsup_{t \rightarrow 0^+} \|m_t\|_\infty < \infty$.*

Proof This statement follows from Example 2.11, Theorem 4.17 with $v := \mathbb{1}$ and Theorem 7.2 (a). □

7.4 Theorem *Let $\Omega \subset \mathbb{C}$ be open, $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{\text{co}})$ a sequentially complete Saks space such that $\mathcal{F}(\Omega) \subset \mathcal{H}(\Omega)$, and $(C_{m,\varphi}(t))_{t \geq 0}$ a locally bounded weighted composition*

² We note that the necessary condition that m has to be jointly continuous is missing in [77, Corollaries 3, 4, p. 8–9] (see [77, p. 2]) even though the cited references therein concerning semicycles like [43, 47] have this incorporated in their definition of a semicycle.

semigroup on $\mathcal{F}(\Omega)$ w.r.t. a holomorphic C_0 -co-semiflow (m, φ) . Then the following assertions hold.

- (a) $(C_{m,\varphi}(t))_{t \geq 0}$ is γ -strongly continuous, τ_{co} -bi-continuous, locally τ_{co} -equicontinuous and locally γ -equicontinuous.
- (b) If $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$ is a C -sequential Saks space, then $(C_{m,\varphi}(t))_{t \geq 0}$ is quasi- γ -equicontinuous.
- (c) If $m_{(\cdot)}(z) \in \mathcal{C}^1[0, \infty)$ for all $z \in \Omega$, then $\dot{\varphi}_0 \in \mathcal{H}(\Omega)$ and the generator $(A, D(A))$ of $(C_{m,\varphi}(t))_{t \geq 0}$ fulfils

$$D(A) = \{f \in \mathcal{F}(\Omega) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)\}$$

and $Af = \dot{\varphi}_0 f' + \dot{m}_0 f$ for all $f \in D(A)$.

- (d) If $m_{(\cdot)}(z) \in \mathcal{C}^1[0, \infty)$ for all $z \in \Omega$, then $\dot{\varphi}_0 \in \mathcal{H}(\Omega)$ and

$$[(m, \varphi), \mathcal{F}(\Omega)] = \overline{\{f \in \mathcal{F}(\Omega) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\Omega)\}}^{\|\cdot\|}$$

where $[(m, \varphi), \mathcal{F}(\Omega)]$ is the space of $\|\cdot\|$ -strong continuity of $(C_{m,\varphi}(t))_{t \geq 0}$.

Proof By the first part of the proof of Theorem 7.2 we know that the C_0 -co-semiflow (m, φ) is jointly continuous, $\varphi_{(\cdot)}(z) \in \mathcal{C}^1[0, \infty)$ for all $z \in \Omega$ and $\dot{\varphi}_0 \in \mathcal{H}(\Omega)$.

The parts (a), (b) and (c) follow directly from Theorem 4.10 and Theorem 5.11. Part (d) is a consequence of Proposition 5.3 (c) and Theorem 5.11. \square

Comparing Theorem 7.4 (d) with [5, Theorem 10, p. 9] resp. [12, Theorem 1, p. 71] where $\Omega = \mathbb{D}$ and $m_t = \varphi_t'$ for all $t \geq 0$ resp. $m = \mathbb{1}$ we see that the former theorem is more general w.r.t. the semicycles and does not need the assumption $\mathbb{1} \in \mathcal{F}(\Omega)$, and less general w.r.t. the tuples $(\mathcal{F}(\Omega), \|\cdot\|, \tau_{co})$. In the latter theorem it is only assumed that $(\mathcal{F}(\Omega), \|\cdot\|)$ is complete and τ_{co} coarser than the $\|\cdot\|$ -topology (see [12, p. 67]). However, both theorems have the assumption on joint continuity of φ (see [12, (3'), p. 68]) and local boundedness of $(C_{m,\varphi}(t))_{t \geq 0}$ (see Proposition 2.3).

For $H^\infty \subset \mathcal{F}(\mathbb{D}) \subset \mathcal{B}_1$, $(\mathcal{F}(\mathbb{D}), \|\cdot\|)$ Banach, it holds $[(\mathbb{1}, \varphi), \mathcal{F}(\mathbb{D})] \subsetneq \mathcal{F}(\mathbb{D})$ by [3, Theorem 1.1, p. 844] for any non-trivial jointly continuous holomorphic semicycle $(\mathbb{1}, \varphi)$ for $\mathcal{F}(\Omega)$ such that $C_{\mathbb{1},\varphi}(t) \in \mathcal{L}(\mathcal{F}(\mathbb{D}))$ for all $t \geq 0$. The weighted version is given in [39, Theorem 4.1, p. 74].

In the case $\mathcal{F}(\mathbb{D}) = H^\infty$ it holds $\mathcal{A} \subset [(\mathbb{1}, \varphi), H^\infty]$ by [3, Corollary 1.4, p. 844] for any jointly continuous holomorphic φ , and $\mathcal{A} \neq [(\mathbb{1}, \varphi), H^\infty]$ for some φ by [3, Proposition 4.3, p. 852] where \mathcal{A} is the *disc-algebra* of holomorphic functions on \mathbb{D} that extend continuously to $\overline{\mathbb{D}}$. If φ consists of rotations or dilations, then $\mathcal{A} = [(\mathbb{1}, \varphi), H^\infty]$ by [3, Proposition 4.1, p. 850].

Further, we have $[(\mathbb{1}, \varphi), \mathcal{B}_\alpha] \subsetneq \mathcal{B}_\alpha$ for $\alpha > 0$ and any non-trivial jointly continuous holomorphic semicycle $(\mathbb{1}, \varphi)$ for \mathcal{B}_α such that $C_{\mathbb{1},\varphi}(t) \in \mathcal{L}(\mathcal{B}_\alpha)$ for all $t \geq 0$ by [12, Theorem 3, p. 73]. In the case $\alpha = 1$ it holds $\mathcal{B}_0 \subset [(\mathbb{1}, \varphi), \mathcal{B}_1]$ for any φ by [12, p. 73], and $\mathcal{B}_0 = [(\mathbb{1}, \varphi), \mathcal{B}_1]$ if and only if the resolvent operator $R(\lambda, A) \in \mathcal{L}(\mathcal{B}_1)$ is weakly compact on \mathcal{B}_0 by [12, Corollary 1, p. 76] where \mathcal{B}_0 is the *little Bloch space* of \mathcal{B}_1 , i.e. the space consisting of all $f \in \mathcal{B}_1$ such that $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)|f'(z)| = 0$.

The assertion $\mathcal{B}_0 = [(\mathbb{1}, \varphi), \mathcal{B}_1]$ is also equivalent to φ being elliptic and its generator G fulfilling the logarithmic vanishing Bloch condition by [19, Theorem 1.1, p. 4].

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Declarations

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