A Nehari theorem for Continuous-time FIR systems

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Abstract

Explicit formulae are derived for Nehari extensions of continuous time FIR systems.

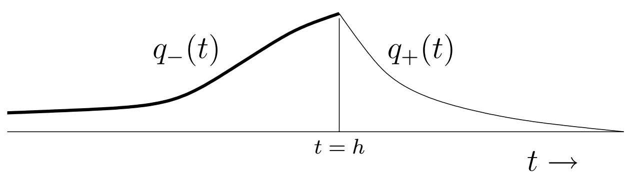

Figure 1: Nehari extension

1 Introduction

The Nehari problem is a problem in operator theory about optimal extension of functions or operators. The idea is depicted in Fig. 1. Given a function $q_-(t)$ for $t < h$ the problem is to extend $q(t)$ over $t > h$ in such a way that the convolution operator

$$u \mapsto q * u, \quad (q * u)(t) = \int_{-\infty}^{\infty} q(t-\tau)u(\tau)\,\mathrm{d}\tau \tag{1.1}$$

has smallest possible $\mathcal{L}_2(-\infty, \infty)$-induced norm

$$\|q\|_{\text{ind}} = \sup_{u \neq 0} \frac{\|q * u\|_{\mathcal{L}_2(-\infty,\infty)}}{\|u\|_{\mathcal{L}_2(-\infty,\infty)}}. \tag{1.2}$$

The standard lower bound for this induced norm is obtained by considering in the convolution (1.1) only $t < h$ and $\tau > 0$. Indeed in that case $t - \tau < h$ so that the convolution mapping (1.1) is determined by the given q_-. Restricting $t < h$ and $\tau > 0$ means that we only consider the "past" of $(q * u)(t)$ and the "future" of $u(\tau)$. A lower bound for the induced norm (1.2) hence is the induced norm $\|\Gamma_q\|$ of the operator restricted to this past and future,

$$\Gamma_q : \mathcal{L}_2(0, \infty) \to \mathcal{L}_2(-\infty, h), \qquad \Gamma_q(u) = q * u.$$

This operator is known as the Hankel operator and the famous Nehari theorem states that the lower bound $\|\Gamma_q\|$ can be attained, i.e., an extension q_+ exists such that $\|\Gamma_q\| = \|q\|_{\text{ind}}$, see (Nehari, 1957; Partington, 1988; Young, 1988).

For finite dimensional systems $q_-(t) = C\mathrm{e}^{At}B$ there is a well developed theory about Nehari extensions and the results are constructive, see e.g. (Glover, 1986; Green and Limebeer, 1995; Zhou et al., 1995). For general *infinite* dimensional systems however it is hard to come up with *computable* formulae for the optimal Nehari extension $q_+(t)$ and the suboptimal extensions $q_+(t)$ (these are extensions $q_+(t)$ for which $\|q\|_{\text{ind}} < \gamma$ for some given bound $\gamma > 0$, assuming any exist, i.e. assuming $\gamma > \|\Gamma_q\|_{\text{ind}}$).

In this note we derive explicit formulae for the suboptimal extensions $q_+(t)$ for the case that $q_-(t)$ is a matrix function of compact support of the form

$$q_-(t) = C\mathrm{e}^{-At}B\, \mathbb{1}_{[0,h]}(t) \tag{1.3}$$

with $A, B, C \in \mathbb{R}^{\cdot \times \cdot}$ of appropriate dimensions. Nehari extension problems of this type have turned up in recent results on $\mathcal{H}_\infty$ control problems for systems with delays, see Mirkin (2000). It is these results that motivated this research.

2 Preliminaries

This section introduces some notation and conventions that we use in this note.

For transfer matrices $P(s)$ we use $P^\sim(s)$ to denote its adjoint $P^\sim(s) = [P(-\bar{s})]^*$. The right conformal mapping $\mathcal{C}_\mathrm{r}(G, U)$ is defined as $\mathcal{C}_\mathrm{r}(G, U) = (G_{11}U + G_{12})(G_{21}U + G_{22})^{-1}$. From the context it will be clear what partitioning of $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right]$ is meant.

A partitioned matrix with vertical and horizontal lines separating the entries, denotes the Schur complement of that matrix with respect to its upper-left block. So $\left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right] = S - RP^{-1}Q$. This notation has proved useful. In particular we have that $\left[\begin{array}{c|c} A - sI & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D$.

Borrowing from (Mirkin, 2000) we define the *truncation* and *completion* operators τ_h and π_h. These are operators that act on causal systems. The truncation operator truncates the system's impulse response beyond a given positive time-delay h. For finite dimensional causal systems with transfer matrix $P(s) = C(sI - A)^{-1}B + D$ the truncation operator equals

$$\tau_h(P) = \left[\begin{array}{c|c} A - sI & B \\ \hline C & D \end{array}\right] - \mathrm{e}^{-sh} \left[\begin{array}{c|c} A - sI & \mathrm{e}^{Ah}B \\ \hline C & 0 \end{array}\right].$$

The *completion* operator π_h "analytically completes" the impulse response of an h-delay system to a 0-delay system. The "analytic completion" for delayed systems of the form

$\mathrm{e}^{-sh}P(s) := \mathrm{e}^{-sh}(C(sI-A)^{-1}B+D)$ is defined formally for $h>0$ as

$$\pi_h(\mathrm{e}^{-sh}P) = \left[\begin{array}{c|c} A-sI & B \\ \hline C\mathrm{e}^{-Ah} & 0 \end{array}\right] - \mathrm{e}^{-sh}\left[\begin{array}{c|c} A-sI & B \\ \hline C & D \end{array}\right].$$

For finite dimensional P, the sum of $\mathrm{e}^{-sh}P$ an its completion $\pi_h(\mathrm{e}^{-sh}P)$ is again finite dimensional with the same state dimension as that of P.

3 A Nehari theorem for FIR systems

From now on we assume that $q_-(t)$ is given by (1.3) for some given $h>0$ and matrices $A,B,C \in \mathbb{R}^{\cdot\times\cdot}$ of appropriate dimensions. We can see q_- as the truncation of the finite dimensional system $P(s) = C(sI-A)^{-1}B$. Because q_- has a finite impulse response (FIR) it follows that the Hankel norm $\|\Gamma_{q_-}\|$ equals the induced norm over the finite interval $[0,h]$,

$$\|\Gamma_{q_-}\| = \sup_{u\in\mathcal{L}_2(0,h)} \frac{\|q_- * u\|_{\mathcal{L}_2(0,h)}}{\|u\|_{\mathcal{L}_2(0,h)}},$$

in which $(q_-*u)(t) = \int_0^h q_-(t-\tau)u(\tau)\,\mathrm{d}\tau$. This norm has been studied in detail in the sampled data and dead-time literature, see e.g. Green and Limebeer (1995); Chen and Francis (1995); Gu et al. (1996) and by now there are various ways to express this norm in a more explicit form. For our purposes the following such form is important.

Theorem 3.1. *Let $q_-(t) = C\mathrm{e}^{At}B\,\mathbb{1}_{[0,h]}(t)$. Then $\|\Gamma_{q_-}\|_{ind} < 1$ if and only if $\Sigma_{22}(t)$ is nonsingular for every $t\in[0,h]$. Here $\Sigma_{22}(t)$ is the lower-right block of the symplectic matrix $\Sigma(t)$ defined as*

$$\Sigma(t) := \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} := \exp\left(\begin{bmatrix} A & BB^{\mathrm{T}} \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{bmatrix} t\right). \tag{3.4}$$

The following theorem characterizes all suboptimal Nehari extensions. This theorem is formulated in frequency domain, that is we seek $Q_+ \in \mathcal{H}_\infty$ such that $\|Q_- + \mathrm{e}^{-sh}Q_+\|_{\mathcal{H}_\infty} < 1$. Now Q_- is the truncation of the causal $P := C(sI-A)^{-1}B + D$. Therefore $\|Q_- + \mathrm{e}^{-sh}Q_+\|_{\mathcal{H}_\infty} < 1$ has a solution $Q_+\in\mathcal{H}_\infty$ iff $\|P + \mathrm{e}^{-sh}K_+\|_{\mathcal{H}_\infty} < 1$ has a causal solution K_+. We use the latter formulation.

Theorem 3.2 (All suboptimal extensions). *Let $P(s) = C(sI-A)^{-1}B$ and suppose $h>0$. There exist causal K_+ such that $\|P+\mathrm{e}^{-sh}K_+\|_{\mathcal{H}_\infty} < 1$ if and only if $\|\Gamma_{\tau_h(P)}\| < 1$. In this case all suboptimal extensions K_+ are given by*

$$K_+ = \mathcal{C}_r\left(\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right] Z_r, U\right) \tag{3.5}$$

where U satisfies $\|U\|_{\mathcal{H}_\infty} < 1$ but otherwise arbitrary. Here Δ is the FIR system defined as

$$\Delta = \pi_h(\mathrm{e}^{-sh}(P^\sim P - I)^{-1}P^\sim)$$

and Z_r is the finite dimensional system

$$Z_r = \left[\begin{array}{c|cc} A - sI & \Sigma_{22}^{-\mathrm{T}}(h)\Sigma_{12}^{\mathrm{T}}(h)C^{\mathrm{T}} & \Sigma_{22}^{-\mathrm{T}}(h)B \\ \hline -C & I & 0 \\ -B^{\mathrm{T}}\Sigma_{21}^{\mathrm{T}}(h) & 0 & I \end{array}\right].$$

A proof is given in Section 4. It is interesting to see that Z_{r} is well defined precisely if $\Sigma_{22}(h)$ is invertible. The symplectic matrix Σ also shows up in the formulae for the FIR system Δ. Indeed, from the proof it follows that

$$\Delta = \pi_h\left(\mathrm{e}^{-sh}\left[\begin{array}{cc|c} A - sI & BB^{\mathrm{T}} & 0 \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} - sI & C^{\mathrm{T}} \\ \hline 0 & B^{\mathrm{T}} & 0 \end{array}\right]\right) = \left[\begin{array}{cc|c} A - sI & BB^{\mathrm{T}} & 0 \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} - sI & C^{\mathrm{T}} \\ \hline -B^{\mathrm{T}}\Sigma_{21}^{\mathrm{T}}(h) & B^{\mathrm{T}}(\Sigma_{11}^{\mathrm{T}}(h) - \mathrm{e}^{-sh}I) & 0 \end{array}\right].$$

Example 3.1. Suppose the given part of q is the indicator function with support $[0, h]$. That is, $q_-(t) = \mathbb{1}_{[0,h]}(t)$. To find the Nehari extension we use that $q_-(t) = C\mathrm{e}^{At}B\,\mathbb{1}_{[0,h]}(t)$, with $(A, B, C) = (0, 1, 1)$. With this data the symplectic matrix defined in (3.4) becomes

$$\Sigma(t) = \exp\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t\right) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

Now $\Sigma_{22}(h) = \cos(h)$ and it follows from Thm. 3.1 that $\|\Gamma_{q_-}\| < 1$ iff $h < \pi/2$. In that case we may continue with Thm. 3.2 and we find for Δ and Z_{r},

$$\Delta(s) = \frac{\sin(h) + (\cos(h) - \mathrm{e}^{-sh})s}{s^2 + 1}$$

and

$$Z_{\mathrm{r}} = \left[\begin{array}{c|cc} -s & \tan(h) & \frac{1}{\cos(h)} \\ \hline -1 & 1 & 0 \\ \sin(h) & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 - \frac{\tan(h)}{s} & -\frac{1}{\cos(h)\,s} \\ \frac{\sin^2(h)}{\cos(h)\,s} & 1 + \frac{\tan(h)}{s} \end{bmatrix}.$$

(Note that the impulse response of Δ is $\cos(t - h)\,\mathbb{1}_{[0,h]}(t)$.) The "central extension" $K_+ = \mathcal{C}_{\mathrm{r}}(\left[\begin{smallmatrix} 1 & 0 \\ \Delta & 1 \end{smallmatrix}\right] Z_{\mathrm{r}}, 0)$ then is

$$K_+(s) = \frac{-\frac{1}{\cos(h)}\frac{1}{s}}{-\frac{1}{\cos(h)}\frac{1}{s}\Delta(s) + 1 + \tan(h)\frac{1}{s}} = -\frac{s^2 + 1}{\cos(h)s^3 + \sin(h)s^2 + s\mathrm{e}^{-sh}}. \tag{3.6}$$

Although Theorem 3.2 is about *sub*-optimal extensions only, it is readily seen that (3.6) remains valid for the optimal case $h = \pi/2$. In that case the above K_+ is the optimal Nehari extension, and it proves to be of an interesting form:

$$\begin{aligned} Q(s) &:= P(s) + \mathrm{e}^{-s\frac{\pi}{2}}K_+(s) = \frac{1}{s} - \mathrm{e}^{-s\frac{\pi}{2}}\frac{s^2 + 1}{s^2 + s\mathrm{e}^{-s\frac{\pi}{2}}} \\ &= \frac{1 - s\mathrm{e}^{-s\frac{\pi}{2}}}{s + \mathrm{e}^{-s\frac{\pi}{2}}} = \frac{1}{s} + \sum_{k=1}^{\infty}(-1)^k\frac{s^2 + 1}{s^{k+1}}\mathrm{e}^{-k\,s\frac{\pi}{2}}. \end{aligned} \tag{3.7}$$

Equation (3.7) shows that Q is inner (as may be expected) and it also shows that the corresponding impulse response $q(t)$ is the causal solution of the delay-differential equation $\dot{q}(t)+q(t-\pi/2) = \delta(t)-\delta^{(1)}(t-\pi/2)$. Alternatively we may determine the impulse response as the inverse Laplace transform of the last expression of Eqn. (3.7),

$$q(t) = \mathbb{1}_{(0,\infty)}(t) - \delta(t-\frac{\pi}{2}) - (t-\frac{\pi}{2})\,\mathbb{1}_{(\frac{\pi}{2},\infty)}(t) + \sum_{k=2}^{\infty}(-1)^k\Big[\frac{(t-k\frac{\pi}{2})^{k-2}}{(k-2)!} + \frac{(t-k\frac{\pi}{2})^k}{k!}\Big]\,\mathbb{1}_{(k\frac{\pi}{2},\infty)}(t).$$

The result is depicted in Fig. 2. Note that the optimal Nehari extension is smooth at all t except at multiples of $\frac{\pi}{2}$. At $t=\frac{\pi}{2}$ the function has a delta-function component, at $t=\pi$ the function is discontinuous, at $t=3\frac{\pi}{2}$ it is continuous but not differentiable, etcetera.

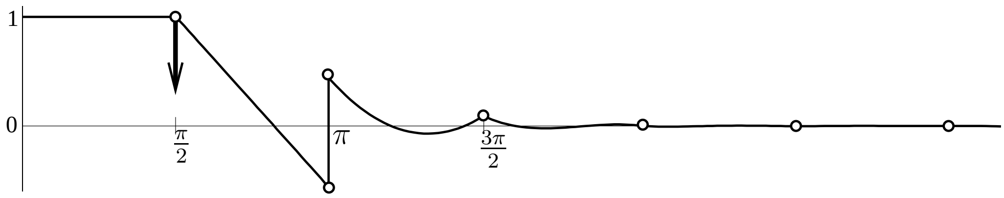

Figure 2: Optimal Nehari extension $1/s + \mathrm{e}^{-s\frac{\pi}{2}}K_+$

4 Appendix: proof

This section describes a proof of Theorem 3.2. (The technical state space formulae are collected Subsection 4.1.) The aim is to find all causal K_+ for which $Q := P + \mathrm{e}^{-sh}K_+$ is stable and contractive. First realize that Q equals

$$Q = \mathcal{C}_{\mathrm{r}}(G, K_+) \quad \text{for} \quad G := \begin{bmatrix} \mathrm{e}^{-sh}I & P \\ 0 & I \end{bmatrix}. \tag{4.8}$$

In Subsection 4.1 we construct a bicausal solution W of the equation $G^{\sim}JG = W^{\sim}JW$ with the properties that $\lim_{s\to\infty} W(s) = I$ and such that $M_h := GW^{-1}$ is entire. (Here J is defined as $J = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ with the partitioning compatible with that of G.) By construction we then have that $\|Q\|_{\mathcal{L}_\infty} < 1$ iff $\|U\|_{\mathcal{L}_\infty} < 1$ for U defined as

$$U := \mathcal{C}_{\mathrm{r}}(W, K_+). \tag{4.9}$$

Now this U is causal iff K_+ is causal by the fact that $\lim_{s\to\infty} W(s) = I$. Yet the set of causal operators in $\mathcal{L}_\infty$ is in fact $\mathcal{H}_\infty$, (Curtain and Zwart, 1995, A6.26.c, A6.27). So if K_+ solves Thm. 3.2 then necessarily $\|U\|_{\mathcal{H}_\infty} < 1$. This condition on U is also sufficient as we shall now see. The thing to note is that

$$M_h := GW^{-1}$$

is not only stable and J-unitary (i.e., $M_h^\sim J M_h = J$) but in fact J-lossless (meaning that in addition $M_{h,22}$ is bistable). Indeed, from $M_h^\sim J M_h = J$ it follows that $M_{h,22} M_{h,22} \geq I$, and as the M_t that we construct (see Subsection 4.1) is stable and is continuous as a function of $t \in [0, h]$, and $M_{t,22}\big|_{t=0} = I$ it follows that $M_{h,22}$ is bistable. It is well known that for J-lossless M_h we have that $Q = \mathcal{C}_\mathrm{r}(M_h, U)$ is stable for any $\|U\|_{\mathcal{H}_\infty} < 1$, see, e.g., (Meinsma and Zwart, 2000, Thm. 6.2). Hence any $\|U\|_{\mathcal{H}_\infty} < 1$ yields a solution. Now (4.9) is invertible,

$$K_+ = \mathcal{C}_\mathrm{r}(W^{-1}, U)$$

and the W constructed below is of the form $W = W_\mathrm{r}\left[\begin{smallmatrix} I & 0 \\ -\Delta & I \end{smallmatrix}\right]$ so that

$$K_+ = \mathcal{C}_\mathrm{r}(\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right] Z_\mathrm{r}, U) \quad \text{where} \quad Z_\mathrm{r} := W_\mathrm{r}^{-1}.$$

4.1 State space formulae

The rest of the subsection documents the more gory state space details.

To find a suitable W we first extract the infinite dimensional part from

$$G^\sim J G = \begin{bmatrix} I & \mathrm{e}^{sh} P \\ \mathrm{e}^{-sh} P^\sim & P^\sim P - I \end{bmatrix}.$$

To this end define the FIR system $\Delta := \pi_h(\mathrm{e}^{-sh}(P^\sim P - I)^{-1} P^\sim)$. Then Θ defined as

$$\Theta := \begin{bmatrix} I & \Delta^\sim \\ 0 & I \end{bmatrix} G^\sim J G \begin{bmatrix} I & 0 \\ \Delta & I \end{bmatrix} \tag{4.10}$$

is rational

$$\Theta = \begin{bmatrix} I - P(P^\sim P - I)P^\sim + R^\sim(P^\sim P - I)R & R(P^\sim P - I) \\ (P^\sim P - I)R & P^\sim P - I \end{bmatrix}.$$

Given a realization of $P(s) = C(sI - A)^{-1}B + D$ and with G defined in (4.8) we get the realization

$$G^\sim J G = \left[\begin{array}{cc|cc} A - sI & 0 & 0 & \mathrm{e}^{sh} B \\ -C^\mathrm{T} C & -A^\mathrm{T} - sI & -C^\mathrm{T} & 0 \\ \hline C & 0 & I & 0 \\ 0 & \mathrm{e}^{-sh} B^\mathrm{T} & 0 & -I \end{array}\right]. \tag{4.11}$$

The construction of a realization of Θ requires several steps. A first step is to associate with $G^\sim J G$ the equation $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] = G^\sim J G \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right]$. This equation may be rearranged as

$$\begin{bmatrix} y_1 \\ -u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} I - P(P^\sim P - I)^{-1} P^\sim & \mathrm{e}^{sh} P(P^\sim P - I)^{-1} \\ \mathrm{e}^{-sh}(P^\sim P - I)^{-1} P^\sim & -(P^\sim P - I)^{-1} \end{bmatrix}}_{\Omega} \begin{bmatrix} u_1 \\ y_2 \end{bmatrix}.$$

This defines Ω. Rearranging the realization of $G^{\sim}JG$ similarly gives a realization of Ω:

$$\Omega = \left[\begin{array}{cc|cc} A-sI & BB^{\mathrm{T}} & 0 & -\mathrm{e}^{sh}B \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}}-sI & -C^{\mathrm{T}} & 0 \\ \hline C & 0 & I & 0 \\ 0 & -\mathrm{e}^{-sh}B^{\mathrm{T}} & 0 & I \end{array}\right].$$

Looking at the lower left block of Ω we see that

$$\Delta := \pi_h(\mathrm{e}^{-sh}(P^{\sim}P-I)^{-1}P^{\sim}) = \pi_h\,\Omega_{21} = \pi_h\left(\mathrm{e}^{-sh}\left[\begin{array}{cc|c} A-sI & BB^{\mathrm{T}} & 0 \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}}-sI & -C^{\mathrm{T}} \\ \hline 0 & -B^{\mathrm{T}} & 0 \end{array}\right]\right)$$

Consequently

$$R := \Delta + \mathrm{e}^{-sh}(P^{\sim}P-I)^{-1}P^{\sim} = \left[\begin{array}{cc|c} A-sI & BB^{\mathrm{T}} & \Sigma^{-1}\begin{bmatrix} 0 \\ -C^{\mathrm{T}} \end{bmatrix} \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}}-sI & \\ \hline 0 & -B^{\mathrm{T}} & 0 \end{array}\right]$$

Based on this we now combine the various blocks and obtain the realization

$$\begin{bmatrix} I-P(P^{\sim}P-I)P^{\sim} & R^{\sim} \\ R & -(P^{\sim}P-I)^{-1} \end{bmatrix} = \left[\begin{array}{cc|cc} A-sI & BB^{\mathrm{T}} & \Sigma^{-1}\begin{bmatrix} 0 \\ -C^{\mathrm{T}} \end{bmatrix} & \begin{matrix} -B \\ 0 \end{matrix} \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}}-sI & & \\ \hline \multicolumn{2}{c|}{\begin{bmatrix} C & 0 \end{bmatrix}\Sigma} & I & 0 \\ 0 & -B^{\mathrm{T}} & 0 & I \end{array}\right]$$

As a final step we associate with this the equation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I-P(P^{\sim}P-I)P^{\sim} & R^{\sim} \\ R & -(P^{\sim}P-I)^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and we rewrite it as

$$\begin{bmatrix} y_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} I-P(P^{\sim}P-I)P^{\sim}+R^{\sim}(P^{\sim}P-I)R & R(P^{\sim}P-I) \\ (P^{\sim}P-I)R & P^{\sim}P-I \end{bmatrix}}_{\Theta} \begin{bmatrix} u_1 \\ -y_2 \end{bmatrix}.$$

Here we recognize Θ. In terms of state space manipulations we similarly obtain

$$\Theta = \left[\begin{array}{cc|cc} A-sI & 0 & \Sigma^{-1}\begin{bmatrix} 0 \\ -C^{\mathrm{T}} \end{bmatrix} & \begin{matrix} B \\ 0 \end{matrix} \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}}-sI & & \\ \hline \multicolumn{2}{c|}{\begin{bmatrix} C & 0 \end{bmatrix}\Sigma} & I & 0 \\ 0 & B^{\mathrm{T}} & 0 & -I \end{array}\right]. \tag{4.12}$$

Then

$$\Theta^{-1} = \left[\begin{array}{c|c} \Sigma^{-1}\begin{bmatrix} A-sI & BB^{\mathrm{T}} \\ 0 & -A^{\mathrm{T}}-sI \end{bmatrix}\Sigma & ? \\ \hline ? & ? \end{array}\right]$$

With Θ and Θ^{-1} in this form there is a standard procedure to find a factorization $W_\mathrm{r}^\sim J W_\mathrm{r} = \Theta$: Let X be any solution of the Riccati equation

$$\begin{bmatrix} -X & I \end{bmatrix} \Sigma^{-1} \begin{bmatrix} A-sI & BB^\mathrm{T} \\ 0 & -A^\mathrm{T}-sI \end{bmatrix} \Sigma \begin{bmatrix} I \\ X \end{bmatrix} = 0.$$

Then

$$W_\mathrm{r} = \left[\begin{array}{c|c} A-sI & \begin{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} \Sigma^{-1} \begin{bmatrix} 0 \\ -C^\mathrm{T} \end{bmatrix} & B \end{bmatrix} \\ \hline J \begin{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \Sigma \\ \begin{bmatrix} 0 & B^\mathrm{T} \end{bmatrix} \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{array}\right]$$

does the job. For any X, the poles of W_r are the eigenvalues of A. The freedom in choice of X may be used to choose the zeros W_r. If we choose $X = -\Sigma_{21}^\mathrm{T}\Sigma_{22}^{-\mathrm{T}}$, that is, if

$$\begin{bmatrix} I \\ X \end{bmatrix} = \Sigma^{-1} \begin{bmatrix} \Sigma_{22}^{-\mathrm{T}} \\ 0 \end{bmatrix} = \begin{bmatrix} \Sigma_{22}^\mathrm{T} & -\Sigma_{12}^\mathrm{T} \\ -\Sigma_{21}^\mathrm{T} & \Sigma_{11}^\mathrm{T} \end{bmatrix} \begin{bmatrix} \Sigma_{22}^{-\mathrm{T}} \\ 0 \end{bmatrix}$$

then

$$\Sigma^{-1} \begin{bmatrix} A-sI & BB^\mathrm{T} \\ 0 & -A^\mathrm{T}-sI \end{bmatrix} \Sigma \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \Sigma_{22}^\mathrm{T} A \Sigma_{22}^{-\mathrm{T}}$$

Therefore the zeros of W_r are the eigenvalues of $\Sigma_{22}^\mathrm{T} A \Sigma_{22}^{-\mathrm{T}}$ i.e., of A. The formulae for W_r and its inverse W_r^{-1} may be simplified to

$$W_\mathrm{r} = \left[\begin{array}{c|cc} A-sI & \Sigma_{12}^\mathrm{T} C^\mathrm{T} & B \\ \hline C\Sigma_{22}^{-\mathrm{T}} & I & 0 \\ B^\mathrm{T}\Sigma_{21}^\mathrm{T}\Sigma_{22}^{-\mathrm{T}} & 0 & I \end{array}\right]$$

and then $Z_\mathrm{r} := W_\mathrm{r}^{-1}$ is as in Thm. 3.2.

Now G and $W := W_\mathrm{r}\left[\begin{smallmatrix} I & 0 \\ -\Delta & I \end{smallmatrix}\right]$ have the same zeros and poles, and because $\left[\begin{smallmatrix} I & \Delta^\sim \\ 0 & I \end{smallmatrix}\right] G^\sim J G \left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right] = W_\mathrm{r}^\sim J W_\mathrm{r}$ also the directions of these zeros and poles are the same. It therefore follows that all zeros and poles are canceled in $GW^{-1} = G\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right] W_\mathrm{r}^{-1}$. Indeed it may be shown (via some not very enlightening manipulations) that $M_h := G\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right] W_\mathrm{r}^{-1}$ is entire, in fact it is a truncation:

$$M = \begin{bmatrix} \mathrm{e}^{-sh} I & 0 \\ 0 & I \end{bmatrix} + \tau_h \left(\left[\begin{array}{cc|cc} A-sI & BB^\mathrm{T} & 0 & B \\ -C^\mathrm{T}C & -A^\mathrm{T}-sI & \Sigma_{22}^{-1}C^\mathrm{T} & -\Sigma_{22}^{-1}\Sigma_{21}B \\ \hline C & 0 & 0 & 0 \\ 0 & B^\mathrm{T} & 0 & 0 \end{array}\right] \right).$$

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