More Progress on Tough Graphs - The Y2K Report

Doug Bauer [a,1] Hajo Broersma [b,1] Edward Schmeichel [c,2]

[a]*Department of Mathematical Sciences, Stevens Institute of Technology, Hoboken, NJ 07030, U.S.A.,* dbauer@stevens-tech.edu.

[b]*Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE, Enschede, The Netherlands,* broersma@math.utwente.nl.

[c]*Department of Mathematics and Computer Science, San Jose State University, San Jose, CA 95192, U.S.A.,* schmeich@mathcs.sjsu.edu.

Abstract

We now know that not every 2-tough graph is hamiltonian. In fact for every $\epsilon > 0$, there exists a $(9/4-\epsilon)$ - tough nontraceable graph. We continue our quadrennial survey of results that relate the toughness of a graph to its cycle structure.

Key words: toughness, t-tough graph, Hamilton cycle, hamiltonian graph, traceable graph, circumference, factor, chordal graph, complexity
1991 MSC: 05C38, 05C45, 05C35

1 Introduction

For the last three Kalamazoo conferences [10–12] we surveyed results on toughness and its relationship to cycle structure. Since the last conference in June 1996 two noteworthy events took place. On a sad note, one author of the last three "surveys", Henk Jan Veldman, passed away in October 1998. He was a very close friend of the first named author, and both a close friend and thesis advisor of the second named author. More than a year earlier, Henk Jan played an intrumental role in settling, in the negative, the longstanding conjecture that 2-tough graphs are hamiltonian. More will be said about this later.

The spirit of this survey is similar to that of the last three, namely to point the interested reader in the right direction. Consequently many details are omitted. We will

[1] Supported in part by NATO Collaborative Research Grant CRG 973085.
[2] Supported in part by the National Science Foundation under Grant DMS9206991.

review some important definitions and results; however we encourage the reader to see the earlier surveys [10–12].

We begin with the 1973 paper in which Chvátal [22] introduced the definition of toughness. From the definition (given below) it is clear that being 1-tough is a necessary condition for a graph to be hamiltonian. In [22] Chvátal conjectured that there exists a finite constant t_0 such that every t_0-tough graph is hamiltonian. For many years, however, the focus was on determining whether all 2-tough graphs are hamiltonian. One reason for this is that if all 2-tough graphs are hamiltonian, a number of important consequences [3] would follow. In addition, the results below (for $k = 2$) seemed to indicate that two might be the threshold for toughness that would imply hamiltonicity.

Theorem 1 [32]. *Let G be a k-tough graph on n vertices with $n \geq k + 1$ and kn even. Then G has a k-factor.*

Theorem 2 [32]. *Let $k \geq 1$. For every $\epsilon > 0$, there exists a $(k - \epsilon)$-tough graph G on n vertices with $n \geq k + 1$ and kn even which has no k-factor.*

However, it turns out that not all 2-tough graphs are hamiltonian, as indicated by the result below.

Theorem 3 [4]. *For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough nontraceable graph.*

The graphs in [4] are shown in Section 3.

We begin with a brief section on terminology and notation and then try to organize the work into a few self explanatory categories. These categories are the same as four years ago, except that we have substituted a section on chordal graphs for the section on triangle-free graphs. As before, many of the results fit easily into more than one category. Also, as stated in our first Kalamazoo note 12 years ago [10], our "survey" is undoubtedly not comprehensive. We can only hope we are not omitting any of our own results.

2 Terminology

Much of the background for this note can be found in [10–12]. A good reference for any undefined terms in graph theory is [19] and in complexity theory is [37]. We consider only undirected graphs with no loops or multiple edges. The definitions and terminology presented below will appear often in the sequel. Other definitions will be given later as needed.

Let $\omega(G)$ denote the number of components of a graph G. A graph G is ***t*-tough** if

$|S| \geq t\omega(G-S)$ for every subset S of the vertex set $V(G)$ with $\omega(G-S) > 1$. The **toughness** of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). Hence if G is not complete, $\tau(G) = \min\{|S|/\omega(G-S)\}$, where the minimum is taken over all cutsets of vertices in G. In [55], Plummer defined a set $S \subseteq V(G)$ to be a **tough set** if $\tau(G) = |S|/\omega(G-S)$. We let $\alpha(G)$ denote the cardinality of a maximum set of independent vertices of G, and $c(G)$ denote the **circumference** of G, i.e., the length of a longest cycle in G. We use $\kappa(G)$ for the vertex connectivity of G and $\gamma(G)$ to denote the genus of G. A graph G is **hamiltonian** if G contains a Hamilton cycle (a cycle containing every vertex of G); G is **traceable** if G contains a Hamilton path (a path containing every vertex of G); G is **hamiltonian-connected** if for every pair of distinct vertices x and y of G there is a Hamilton path with endvertices x and y. A **k-factor** of a graph is a k-regular spanning subgraph. Of course, a Hamilton cycle is a (connected) 2-factor. We say G is **chordal** if it contains no chordless cycle of length at least four and is **k-chordal** if a longest chordless cycle in G has length at most k.
We use $d(v)$ to denote the degree of vertex v, $\delta(G)$ for the minimum degree in G, and $\sigma_k(G)$ for the minimum degree sum taken over all independent sets of k vertices of G ($k \geq 2$). If no ambiguity can arise we often omit the reference to the graph G, e.g., we use E for the edge set $E(G)$, etc.

3 Toughness and Circumference

We start this section by presenting the graphs that were used in [4] to prove Theorem 3.

In [3] a construction of a nontraceable graph from non-hamiltonian-connected building blocks was used to show that Chvátal's conjecture on the hamiltonicity of 2-tough graphs is equivalent to several other statements, some seemingly weaker, some seemingly stronger. This construction was inspired by examples of graphs of high toughness without 2-factors occurring in [9].

In [4] the same construction was used to prove Theorem 3, thereby refuting the 2-tough conjecture. We now give a brief outline of the construction of these counterexamples.

For a given graph H and $x, y \in V(H)$ we define the graph $G(H, x, y, l, m)$ as follows. Take m disjoint copies $H_1, \ldots, H_m$ of H, with x_i, y_i the vertices in H_i corresponding to the vertices x and y in H ($i = 1, \ldots, m$). Let F_m be the graph obtained from $H_1 \cup \ldots \cup H_m$ by adding all possible edges between pairs of vertices in $\{x_1, \ldots, x_m, y_1 \ldots, y_m\}$. Let $T = K_l$ and let $G(H, x, y, l, m)$ be the join $T \vee F_m$ of T and F_m.

The proof of the following theorem occurred in [4] and almost literally also in [3].

Theorem 4 *Let H be a graph and x, y two vertices of H which are not connected by a Hamilton path of H. If $m \geq 2l + 3$, then $G(H, x, y, l, m)$ is nontraceable.*

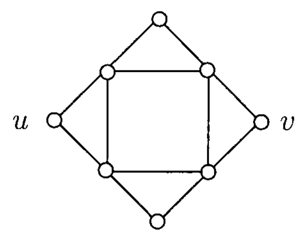

Figure 1. The graph L.

Consider the graph L of Figure 1. There is obviously no Hamilton path in L between u and v. Hence $G(L, u, v, l, m)$ is nontraceable for every $m \geq 2l + 3$. The toughness of these graphs has been established in [4].

Theorem 5 *For $l \geq 2$ and $m \geq 1$,*

$$\tau(G(L, u, v, l, m)) = \frac{l + 4m}{2m + 1}.$$

Combining Theorems 4 and 5 for sufficiently large values of m and l, one obtains the next result.

Corollary 6 [4]. *For every $\epsilon > 0$, there exists a $\left(\frac{9}{4} - \epsilon\right)$-tough nontraceable graph.*

It is easily seen from the proof in [4] that Theorem 4 remains valid if "$m \geq 2l + 3$" and "nontraceable" are replaced by "$m \geq 2l + 1$" and "nonhamiltonian", respectively. Thus the graph $G(L, u, v, 2, 5)$ is a nonhamiltonian graph, which by Theorem 5 has toughness 2. This graph is sketched in Figure 2. It follows that a smallest counterexample to the 2-tough conjecture has at most 42 vertices. Similarly, a smallest nontraceable 2-tough graph has at most $|V(G(L, u, v, 2, 7))| = 58$ vertices.

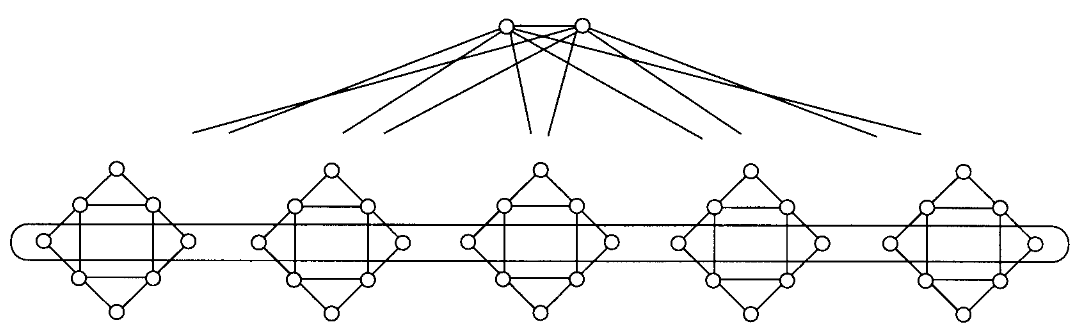

Figure 2. The graph $G(L, u, v, 2, 5)$.

A graph G is **neighborhood-connected** if the neighborhood of each vertex of G induces a connected subgraph of G. In [22] Chvátal also stated the following weaker version of the 2-tough conjecture: every 2-tough neighborhood-connected graph is hamiltonian. Since all counterexamples described above are neighborhood-connected, this weaker conjecture is also false.

Most of the ingredients used in the above counterexamples were already present in [3]. It only remained to observe that using the specific graph L as a "building block" produced a graph with toughness at least 2. We hope that other building blocks and/or smarter constructions will lead to counterexamples with a higher toughness.

3.1 Chordal graphs

Chvátal [22] obtained $\left(\frac{3}{2}-\epsilon\right)$-tough graphs without a 2-factor for arbitrary $\epsilon > 0$. These examples are all chordal. Recently it was shown in [6] that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Based on this, Kratsch [47] raised the question whether every $\frac{3}{2}$-tough chordal graph is hamiltonian. Using Theorem 4 in [4] it has been shown that this conjecture, too, is false. A key observation in this context is that the graphs $G(H, x, y, l, m)$ are chordal whenever H is chordal, as is easily shown.

Consider the graph M of Figure 3.

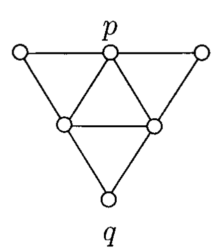

Figure 3. The graph M.

The graph M is chordal and has no Hamilton path with endvertices p and q. Hence by Theorem 4 the chordal graph $G(M, p, q, l, m)$ is nontraceable whenever $m \geq 2l + 3$. By arguments similar to those used in the proof of Theorem 5 (in [4]), the toughness of $G(M, p, q, l, m)$ is $\frac{l+3m}{2m+1}$ if $l \geq 2$. Hence for $l \geq 2$ the graph $G(M, p, q, l, 2l+3)$ is a chordal nontraceable graph with toughness $\frac{7l+9}{4l+7}$. This gives the following result.

Theorem 7 [4]. *For every $\epsilon > 0$, there exists a $\left(\frac{7}{4}-\epsilon\right)$-tough chordal nontraceable graph.*

We will come back to questions on tough chordal graphs in Section 5.

In a recent paper [29], Ellingham and Zha used the same construction to give an infinite class of graphs of relatively high toughness without a k-walk. A ***k*-walk** of a graph G is a closed spanning walk of G that visits every vertex of G at most k times. Of course a Hamilton cycle is then a 1-walk. In terms of toughness the following results on k-walks appeared in [29].

Theorem 8 *Every 4-tough graph has a 2-walk.*

Theorem 9 *For every $\epsilon > 0$ and every $k \geq 1$, there exists a $\left(\frac{8k+1}{4k(2k-1)} - \epsilon\right)$-tough graph with no k-walk.*

To prove the latter theorem they first modified the graph L from Figure 1 and then relied on the same basic construction that was used in [4].

3.3 Long cycles through specified vertex sets

We conclude this section by presenting some results on long cycles through specified vertex sets in 1-tough graphs. Let G be a graph of order n and let $X \subseteq V(G)$. Denote by $G[X]$ the subgraph of G induced by X. Let $\alpha(X)$ be the number of vertices of a maximum independent set of $G[X]$, and $\sigma_k(X)$ the minimum degree sum in G of k independent vertices in X. A cycle C of G is called **X-longest** if no cycle of G contains more vertices of X than C, and C is called **X-dominating** if all neighbors of each vertex of $X \setminus V(C)$ are on C.

The main result of [50] is the following extension of a result by Bauer et al. [7].

Theorem 10 *If G is 1-tough and $\sigma_3(X) \geq n$, then G has an X-longest cycle C such that C is an X-dominating cycle and $|V(C) \cap X| \geq \min\{|X|, |X| + \frac{1}{3}\sigma_3(X) - \alpha(X)\}$.*

A result of the same type appeared in [49].

Theorem 11 *If G is 1-tough and $\sigma_3(X) \geq n$, then G contains a cycle containing all vertices of X of length at least $\min\{|X|, |X| + \overline{\delta}(X) - \alpha(X) + 1\}$, where $\overline{\delta}(X)$ denotes the minimum integer not less than $\frac{1}{3}\sigma_3(X)$.*

4 Toughness and Factors

We begin this section with a "hot off the press" result that gives a minimum degree condition for a 1-tough graph to have a 2-factor with a specific number of cycles. First recall the well-known theorem of Jung [43] .

Theorem 12 *Let G be a 1-tough graph on $n \geq 11$ vertices with $\sigma_2 \geq n-4$. Then G is hamiltonian.*

The degree sum condition can of course be converted to a weaker minimum degree condition.

Theorem 13 *Let G be a 1-tough graph on $n \geq 11$ vertices with $\delta \geq \frac{n-4}{2}$. Then G is hamiltonian.*

Faudree et al. [33] generalized Theorem 13 as follows.

Theorem 14 *There exists an integer n_0 such that every 1-tough graph on at least n_0 vertices with $\delta \geq \frac{n-4}{2}$ has a 2-factor with k cycles, for $1 \leq k \leq \frac{n-10}{4}$.*

We now consider minimum degree conditions that imply a t-tough graph has a regular factor, where $t > 1$. Clearly Theorem 1 implies that all 2-tough graphs with at least three vertices have 2-factors, and Theorem 2 shows that this result is best possible. When the 2-tough conjecture was still alive it was natural to ask, for $1 \leq t < 2$, how large the minimum vertex degree of a t-tough graph G can be if G does not contain a 2-factor. This problem was studied in [9] and some of the results were mentioned in [11,12]. Since Theorems 1 and 2 also show that 3-tough graphs have 3-factors, and that this result is also best possible, it is natural to seek minimum degree conditions for a t-tough graph $(1 \leq t < 3)$ to have a 3-factor, i.e., a spanning cubic subgraph. This problem was considered in [8]. The results in [8] for 3-factors are similar to the type of results in [9] for 2-factors. For $t \in [1, 4/3) \cup [2, 8/3)$, a minimum degree condition for a t-tough graph to have a 3-factor was found that is best possible to within a small additive constant. In addition, for $t \in [4/3, 2)$, a bound was obtained that is asymptotically best possible to within a factor of 1.09. A bound was also obtained for $t \in [8/3, 3)$, but the quality of this bound remains unsettled.

A number of results have recently appeared relating toughness to (r, k)-factor-critical graphs. A graph G is $\boldsymbol{(r, k)}$**-factor-critical** if $G - X$ contains an r-factor for all $X \subseteq V$ with $|X| = k$. For $r \geq 2$, these graphs were studied by Liu and Yu [51] under the name (r, k)-extendable graphs. They proved the following.

Theorem 15 *Let G be a graph on n vertices with $\tau \geq 3$. Then G is $(2, k)$-factor-critical for every integer k such that $3 \leq k \leq \tau$ and $k \leq n-3$.*

They also made the following conjecture.

Conjecture 16 *Let G be a graph on n vertices with $\tau \geq q$ and $n \geq 2q+1$ for some integer $q \geq 1$. Then G is $(2, 2q-2)$-factor-critical.*

Note that by Theorem 2 this conjecture is false for $q = 1$. However it was shown by Cai et al. [18], and independently by Enomoto [31], that the conjecture is true for all integers $q \geq 2$.

Theorem 17 *Let G be a graph on n vertices with $\tau \geq 2$. Then G is $(2, k)$-factor-critical for every non-negative integer k with $k \leq min\{2\tau - 2, n-3\}$.*

It was also shown in [18] that the bound $2\tau - 2$ is sharp.

Progress has also been made on the relationship between toughness and (r, k)-factor-critical graphs for $r = 1$ and $r = 3$. In [34], Favaron considered $r = 1$.

Theorem 18 *Let G be a graph on n vertices and k be an integer with $2 \leq k < n$ and $n + k$ even. Then G is $(1, k)$-factor-critical if $\tau > \frac{k}{2}$.*

The value $\frac{k}{2}$ was also shown to be sharp.

In [57], Shi et al. considered $r = 3$.

Theorem 19 *Let G be a graph on n vertices with $\tau \geq 4$. Then G is $(3, k)$-factor-critical for every non-negative integer k such that $n + k$ is even, $k < 2\tau - 2$ and $k \leq n - 7$.*

This result is best possible with respect to each of the upper bounds on k.

We now return to Enomoto's work in [31]. First note that in [30], he strengthened Theorem 1 by proving Theorem 20 below.

Theorem 20 *Let k be a positive integer and G be a graph on n vertices with $n \geq k+1$ and kn even. Suppose $|S| \geq k \cdot \omega(G-S) - \frac{7k}{8}$ for all $S \subseteq V$ with $\omega(G-S) \geq 2$. Then G has a k-factor.*

In [31], Enomoto first improved Theorem 20 for $k = 1$ and $k = 2$. We need the following definition. For a graph G let

$$\begin{aligned} \tau'(G) &= \max\{t \mid |S| \geq t \cdot \omega(G-S) - t \text{ for all } S \subset V(G)\} \\ &= \min\left\{\frac{|S|}{\omega(G-S)-1} \mid \omega(G-S) \geq 2\right\} \end{aligned}$$

if G is not complete. If G is complete, set $\tau'(G) = \infty$.

Theorem 21 *Let G be a graph on n vertices, where n is even. If $\tau' \geq 1$, then G has a 1-factor.*

Theorem 22 *Let G be a graph on $n \geq 3$ vertices. If $\tau' \geq 2$, then G has a 2-factor.*

Both Theorem 21 and Theorem 22 were also shown to be sharp.

Finally, he was able to generalize Theorem 22 and strengthen Theorem 1 for graphs with a sufficiently large number of vertices.

Theorem 23 *Let k be a positive integer and G be a graph on $n \geq k^2 - 1$ vertices with kn even. If $\tau' \geq k$, then G has a k-factor.*

Four years ago we mentioned a promising line of research introduced by G. Y. Katona. In [44,45] he introduced the notion of "t-edge-toughness". We refer the reader to [44] for the precise definition. We again note that it is easy to verify that a graph is not t-edge-tough in the same way one easily verifies, given a tough set, that a graph is not t-tough. Edge-toughness is nicely related to both toughness and hamiltonicity, as the following results show.

Theorem 24 [44]. *If G is a hamiltonian graph, then G is 1-edge-tough.*

Theorem 25 [44]. *If G is a t-edge-tough graph, then G is t-tough.*

Theorem 26 [44]. *If G is a $2t$-tough graph, then G is t-edge-tough.*

We know, by Theorem 1, that 2-tough graphs have 2-factors. In light of Theorem 26, it would be interesting to know if 1-edge-tough graphs have 2-factors. This was answered by Katona in the affirmative.

Theorem 27 [45]. *Let G be a 1-edge-tough graph on $n \geq 3$ vertices. Then G has a 2-factor.*

Conjecture 28 [45]. *Let t be a positive integer and G be a t-edge-tough graph on $n \geq 2t + 1$ vertices. Then G has a $2t$-factor.*

A final result concerning toughness and factors will be given in the next section on chordal graphs.

5 Toughness and Chordal Graphs

During the last four years a problem that has received much attention is that of determining the minimum level of toughness to ensure that a chordal graph is hamiltonian. We have seen in Section 3 an infinite class of chordal graphs with toughness close to $\frac{7}{4}$ having no Hamilton path. Hence 1-tough chordal graphs need not be hamiltonian. However for other classes of perfect graphs (for definitions, see [16]), being 1-tough will ensure hamiltonicity. For example, in [46] it was shown (implicitly) that 1-tough interval graphs are hamiltonian, and in [27] it was shown that 1-tough cocomparability graphs are hamiltonian. However in [15] it was proven that for chordal planar graphs, 1-toughness does not ensure hamiltonicity. The following result was established, however.

Theorem 29 *Let G be a chordal, planar graph with $\tau > 1$. Then G is hamiltonian.*

To see that being 1-tough will not suffice, we must first define the "shortness exponent" of a class of graphs. This concept was first introduced in [41] as a way of measuring the size of longest cycles in polyhedral, i.e., 3-connected planar graphs.

Let Σ be a class of graphs. The **shortness exponent** of the class Σ is given by

$$\sigma(\Sigma) = \liminf_{H \in \Sigma} \frac{\log\ c(H_n)}{\log\ |V(H_n)|}.$$

The lim inf is taken over all sequences of graphs H_n in Σ such that $|V(H_n)| \longrightarrow \infty$ as $n \longrightarrow \infty$.

In [15], it was also shown that the shortness exponent of the class of all 1-tough chordal planar graphs is at most $\frac{\log 8}{\log 9}$. Hence there exists a sequence $G_1, G_2, \ldots$ of 1-tough chordal planar graphs with $\frac{c(G_i)}{|V(G_i)|} \longrightarrow 0$ as $i \longrightarrow \infty$. On the other hand, all 1-tough $K_{1,3}$-free chordal graphs are hamiltonian. This follows from the well-known result of Matthews and Sumner [52] relating toughness and vertex connectivity in $K_{1,3}$-free graphs, and a result of Balakrishnan and Paulraja [1] showing that 2-connected $K_{1,3}$-free chordal graphs are hamiltonian.

Let us now consider $\frac{3}{2}$-tough chordal graphs. We have already seen that such graphs need not be hamiltonian. However for a certain subclass of chordal graphs, namely split graphs, we have a different result. A graph G is called a **split graph** if $V(G)$ can be partitioned into an independent set and a clique. We have the following.

Theorem 30 [48]. *Every $\frac{3}{2}$-tough split graph is hamiltonian.*

Theorem 31 [48]. *There is a sequence $\{G_n\}_{n=1}^{\infty}$ of non-2-factorable split graphs with $\tau(G_n) \longrightarrow \frac{3}{2}$.*

Even though $\frac{3}{2}$-tough chordal graphs need not be hamiltonian, it was shown in [6] that they will have a 2-factor. In fact, we can say a bit more.

Theorem 32 *Let G be a $\frac{3}{2}$-tough 5-chordal graph. Then G has a 2-factor.*

Theorem 32 is best possible in two ways. Chvátal's examples in [22] show it is best possible with respect to toughness and examples in [9] contain 6-chordal graphs without a 2-factor whose toughness approaches 2 from below.

The previous results on tough chordal graphs lead to a very natural question. This question was answered by Chen et al. in the title of their paper "Tough enough chordal graphs are hamiltonian" [20]. Using an algorithmic proof they were able to prove the result below.

Theorem 33 *Every* 18*-tough chordal graph is hamiltonian.*

The authors did not claim that 18 is best possible. The natural question, in light of the disproof of the 2-tough conjecture for general graphs, is what level of toughness will ensure that a chordal graph is hamiltonian. More specifically, are 2-tough chordal graphs hamiltonian?

6 Complexity

The problem of determining the complexity of recognizing t-tough graphs was first raised by Chvátal [21] and later appeared in [58] and [[23], p. 429]. We refer the reader to [37] for the basic ideas of complexity theory.

Consider the following decision problem, where t is any positive rational number.

t-TOUGH
INSTANCE : Graph G.
QUESTION : Is $\tau(G) \geq t$?

The following was established in [5].

Theorem 34 *For any positive rational number t, t-TOUGH is NP-hard.*

It is natural to inquire whether the problem of recognizing t-tough graphs remains NP-hard for various subclasses of graphs. One class of graphs for which this is not true is the class of split graphs. Recall that a graph G is called a **split graph** if $V(G)$ can be partitioned into an independent set and a clique. Determining whether a split graph is hamiltonian was shown to be NP-complete in [25]. On the other hand, noting that submodular functions can be minimized in polynomial time [26,40], Woeginger [61] gave a short proof of the following result.

Theorem 35 *For any rational number* $t \geq 0$*, the class of t-tough split graphs can be recognized in polynomial time.*

For many interesting subclasses of graphs, however, it is NP-hard to recognize

t-tough graphs in the subclass. One such subclass that has received considerable attention is the class of r-regular graphs. Of course, the maximum possible toughness of an r-regular graph is $r/2$, since if $r < 2t$, an r-regular graph is trivially not t-tough. The case $r = 2t$ is already interesting. Jackson and Katerinis [42] gave a characterization of $3/2$-tough cubic graphs which allowed such graphs to be recognized in polynomial time, and Goddard and Swart [39] have conjectured a characterization of $\frac{r}{2}$-tough, r-regular graphs that would allow such graphs to be recognized in polynomial time, for all $r \geq 1$. In the opposite direction, it was established in [13] that 1-tough cubic graphs are NP-hard to recognize, and recently this was generalized in [14] as follows.

Theorem 36 *For any integer $t \geq 1$ and any fixed $r \geq 3t$, it is NP-hard to recognize t-tough, r-regular graphs.*

The complexity of recognizing t-tough, r-regular graphs remains completely open when $2t < r < 3t$, and the complexity when $r = 2t + 1$ seems an especially interesting open case.

There remain many interesting subclasses of graphs for which the complexity of recognizing t-tough graphs is unknown. A number of these classes were given in [13]. In particular, we do not yet know the complexity of recognizing 1-tough planar graphs.

There are several results in hamiltonian graph theory of the form $\mathcal{P}_1$ implies $\mathcal{P}_2$, where $\mathcal{P}_1$ is an NP-hard property of graphs and $\mathcal{P}_2$ is an NP-hard cycle structure property, and one might wonder about the practical value of such theorems.

Two such theorems are the well-known theorems of Chvátal and Erdös [24] and Jung [43].

In [23], Chvátal gave a proof of the Chvátal - Erdös Theorem [24] which constructs in polynomial time either a Hamilton cycle in a graph G or an independent set of more than κ vertices in G. In [2], the authors provided a similar type of polynomial time constructive proof for Jung's Theorem [43] on graphs with at least 16 vertices.

Theorem 37 *Let G be a graph on $n \geq 16$ vertices with $\sigma_2 \geq n - 4$. Then we can construct in polynomial time either a Hamilton cycle in G or a set $X \subseteq V(G)$ with $\omega(G - X) > |X|$.*

It is possible that other theorems in graph theory with an NP-hard hypothesis and an NP-hard conclusion also have polynomial time constructive proofs.

7 Other Toughness Results

Our last section contains some results on toughness that seem to fall outside of the previous sections. We hope the reader does not infer that these results are not important. The fact that this section was **accidentally omitted** from our 1992 survey should in no way indicate a value judgement on our part!

In [38], Goddard et al. considered bounds on the toughness of a graph G in terms of the graph's connectivity and genus. They made use of the following result of Schmeichel and Bloom [56].

Theorem 38 *Let G be a graph with $\kappa \geq 3$. Then*

$$\omega(G - X) \leq \frac{2}{\kappa - 2}(|X| - 2 + 2\gamma)$$

for all $X \subseteq V$ with $|X| \geq \kappa$.

After simplifying the proof of Theorem 38 they then used the result to obtain lower bounds on $\tau(G)$.

Theorem 39 *Let G be a connected graph. Then*

(1) $\tau(G) > \frac{\kappa}{2} - 1$, *if* $\gamma = 0$, *and*

(2) $\tau(G) \geq \frac{\kappa(\kappa - 2)}{2(\kappa - 2 + 2\gamma)}$, *if* $\gamma \geq 1$.

They also discussed the quality of the bounds, as well as investigated upper bounds on $\tau(G)$. In particular, they showed that Theorem 39(1) is sharp for $2 \leq \kappa \leq 5$ and that the bound in Theorem 39(2) is attained by an infinite class of graphs, all of girth four.

In [55], Plummer defined a **tough component** to be any component of $G - S$, where S is a tough set, i.e. $\tau(G) = |S|/\omega(G - S)$. He then investigated the toughness of tough components. In particular, if G is not complete and C is a tough component of G, then if $t(G) \geq 1$, $t(C) \geq \frac{\lceil t(G) \rceil}{2}$.

In [35], Ferland continued his investigation of the toughness of generalized Petersen graphs. These graphs were first defined by Watkins in [60]. For each $n \geq 3$ and $0 < k < n$, the generalized Petersen graph $G(n, k)$ has vertex set $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $E = \{(u_i, u_{i+1}) \mid 1 \leq i \leq n\} \cup \{(u_i, v_i) \mid 1 \leq i \leq n\} \cup \{(v_i, v_{i+k}) \mid 1 \leq i \leq n\}$, where all indices are modulo n. Of course, the Petersen graph is $G(5, 2)$.

In [35], as well as in his earlier paper [36], Ferland was interested in finding bounds for $\tau(G(n,k))$. He was also interested in asymptotic bounds. He called b an **asymptotic upper bound** for $\tau(G(n,k))$ if $\lim_{n\to\infty}\tau(G(n,k)) \leq b$. Asymptotic lower bounds were defined similarly. In [54], $\tau(G(n,1))$ was completely determined, and it was found that 1 is an asymptotic upper bound. In [36], it was found that for $\tau(G(n,2))$, a lower bound and an asymptotic upper bound is $\frac{5}{4}$. For $n \geq 3$ and $0 < k < n$, upper and lower (asymptotic) bounds for $\tau(G(n,k))$ were given in [35].

In [17], Broersma, Engbers and Trommel studied the relationship between the toughness of a graph and the toughness of its spanning subgraphs. In particular they proved the following.

Theorem 40 *Let G be a graph on $n \geq 4$ vertices with $\tau(G) > 1$. Then there exists a spanning subgraph H of G with $\tau(H) = 1$.*

They also defined a graph G to be **minimally t-tough** if $\tau(G) = t$ and $\tau(H) < t$ for every proper spanning subgraph H of G, and discussed conditions under which the square of a graph will either be, or not be, minimally 2-tough.

A number of recent results have concerned the existence of tough maximal planar graphs. A **maximal planar graph** is a planar graph in which every face is bounded by a triangle. Let $\Gamma(t_0)$ denote the class of all t_0-tough maximal planar graphs. In [53] , Nishizeki produced a nonhamiltonian graph on 19 vertices in $\Gamma(1)$, thus answering a question of Chvátal concerning the existence of such a graph. In [28], Dillencourt found such a graph with 15 vertices. Finally, Tkáč [59] was able to find a nonhamiltonian graph in $\Gamma(1)$ with 13 vertices, and to show that any such graph can not have fewer vertices.

Other results have considered the shortness exponent, $\sigma(\Sigma)$, of a class of graphs Σ. This concept was defined in Section 5. In [28], Dillencourt showed that $\sigma(\Gamma(1)) \leq \log_7 6$. In [59], Tkáč improved this by showing $\sigma(\Gamma(1)) \leq \log_6 5$.

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