Inductive τ-values in cooperative transportation games under computational time constraints

Reinoud Joosten & Rogier Harmelink*
Financial Engineering, University of Twente
The Netherlands.
Email corresponding author: r.a.m.g.joosten@utwente.nl.

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Abstract

The τ-value is the efficient convex combination of the 'minimal-rights' and 'utopia' vectors in a cooperative game.The *utopia amount* is the player's marginal contribution to the grand coalition. The *remainder* for a given player in a certain coalition containing him, is the amount left after the sum of the utopia amounts of all other players is subtracted from the coalition's worth. The *minimal-rights amount* of the player is then the largest remainder taken over all coalitions containing him.

We deal with the problem that a solution for a cooperative transportation game (CTG), e.g., an allocation of profits, may be required **before** all worths of the coalitions are known. In CTGs computing the worth of a coalition may involve solving a traveling salesman or vehicle routing problem which are NP-hard, and there are exponentially many worths to determine. Finally, computing the τ-value is NP-hard itself.

An *inductive* τ-value is the efficient allocation closest to the set of all convex combinations of *approximations* of the utopia and minimal-rights vectors. First, the worth of the grand coalition is determined which (by assumption) is always possible. As long as the time constraint is not met, the worths for all coalitions with cardinality 1 are computed, then for those with cardinality 2, and so on. The utopia and minimal rights vectors of the game restricted to the cardinality at hand are determined, too. If the computation time reaches the constraint while establishing worths for coalitions with cardinality $U+1$, the approximations of the utopia and minimal rights vectors based on the completed calculations up to U are used. If the time constraint is not binding, the inductive τ-value coincides with a natural extension of the τ-value for hyperplane singular games, and with the τ-value for quasi-balanced games.

We show that the inductive τ-value satisfies efficiency, symmetry, desirability and two axioms introduced in Joosten & Lalla-Ruiz [2019]. The latter four axioms incorporate aspects of fairness guaranteed for the computations being based on restricted information.
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1 Introduction

Cooperation in real-world combinatorial optimization problems is a promising idea, especially in transportation. It may lead to considerable reductions in individual and collective costs within the cooperating entity, but also in negative externalities e.g., environmental pollution or congestion, affecting society.[1]

Applications of cooperative game theory to transportation problems suffer from a *triple curse of dimensionality*. Finding solutions[2] to cooperative games[3] may be increasingly hard if the number of agents increases, even if the primitives for the computations are given. As to obtaining such a primitive, e.g., the worth of a coalition, this may involve solving a traveling salesman, a vehicle routing or a truck scheduling problem, cf., e.g., Gansterer & Hartl [2018]. Not only are these problems notorious for being *NP*-hard (cf., e.g., Schulte *et al.* [2018]) already if restricted to the worth of one coalition only, but the number of primitives to determine increases exponentially in the cardinality of the player set.

Tijs[4] [1981,1987] introduced and characterized an interesting single-valued solution, the τ-value, restricted to a subclass of all cooperative games. This value gives each player in a cooperative game an amount in between his minimal-right and utopia amounts. The *utopia amount* is the player's marginal contribution to the grand coalition. The *remainder* for a given player in a certain coalition containing him is the amount left after the sum of the utopia amounts of all other players is subtracted from the worth of that coalition. The *minimal-right amount* of the player is then the largest remainder for that player taken over all coalitions containing him. For the minimal-right amount Tijs [1987] offers a beautiful interpretation oozing reasoning related to the notion of the core of Gillies [1953]. A player should get at least his minimal-right amount because otherwise he could entice the coalition associated with this amount, to split off from the grand coalition by offering them their utopia amounts leaving him the remainder, which would make him better off.

The τ-value satisfies efficiency, symmetry, the null-player property. On the class of quasi-balanced games, it also satisfies individual rationality, i.e., the solution gives each player at least his stand-alone worth, implying that no player is ever worse off by cooperating than by remaining alone.

Many solutions become increasingly hard to compute for games with increasing size of the player set, which is reflected implicitly by Schulte *et al.* [2018] and Guajardo & Rönnqvist [2016]: a large part of the contributions found using the Shapley value[5] focus on *CTPs* with a small player set. The τ-value is close in terms of complexity to the Shapley value (Shapley [1953b]), and the order of the latter is exponential (cf., e.g., Faigle & Kern [1992]).

[1] See e.g., Figliozzi [2010], Pradenas *et al.* [2013], Schulte *et al.* [2018]. Externalities are not considered in decision making, but may be of great relevance to society at large.

[2] For instance, the τ-value (Tijs [1981,1987]), the Shapley value (Shapley [1953b]), the (pre)nucleolus (Schmeidler [1969], Sobolev [1975]), the Banzhaf value [1965] and alternatives, e.g., Shapley [1953a], Joosten [1996,2016], or the consensus value (Ju *et al.* [2007]).

[3] See Peters [2008], Maschler *et al.* [2013], Gonzáles Díaz *et al.* [2010] for overviews.

[4] Stef Tijs, the 'godfather' of game theory in The Netherlands, was a prolific scientist, a great motivator/inspirator/teacher/mentor to those around him, as well as a great story teller. His passing away in 2023 prompted us to write this paper and dedicate it to his memory.

[5] Schulte *et al.* [2018] find 14 out of 20 contributions with at most 5 players, an additional 2 papers feature 5-10 players. Guajardo & Rönnqvist [2016] find a distribution of 34, 5 and 9 contributions with at most 5, 6-10 and 11-80 players, respectively.

Cruijssen *et al.* [2007] discuss severe impediments to horizontal cooperation: (i) the difficulty of determining benefits and savings beforehand, (ii) the complexity of ensuring a fair allocation of the shared workload in advance, and (iii) the problem of finding a fair allocation of benefits. Impediment (iii) can be taken care of theoretically by using game-theoretical concepts, provided that (i) and (ii) are covered by efficient computational procedures.

What seems behind these impediments to form larger entities of cooperating agents is the nature of their relationships. If they are not cooperating, these agents compete, sometimes fiercely. Information may yield competitive advantages, having certain connections or returning customers may be crucial, too. Sharing information or losing control may be considered dangerous (cf., e.g., Cruijssen *et al.* [2007]), even if the greater good is served.

Such impediments can be partly solved by centralized collaborative planning (cf., Gansterer & Hartl [2018]) i.e., by appointing an agent, e.g., a government agency or consultancy firm, trusted to provide a fair distribution of the gains of cooperation, and to deal diligently with information shared. As this agent may have a hard nut to crack computationally, (s)he should have ample computational means. There may be some efficiency gain in having one agent incur the necessary investments in personnel and infrastructure, instead of a large number of agents making the same investments each to compare and check the others' computations and resulting proposals for distributing the gains of cooperation.

We envision computations to be checked regularly, i.e., re-run to confirm a solution found earlier. Such a re-run, or at least a possibility of it if demanded, should foster trust regarding the validity of the method of computing. It could be detrimental to the trust in the agent computing the solution, if a re-run requested by a disgruntled agent would yield a significantly different outcome. For this reason we rule out computational methods based on randomization such as Castro *et al.* [2009] for instance, who present a rather efficient randomization-based method to approximate the Shapley value.

Gansterer & Hartl [2018], citing Guajardo & Rönnqvist [2016], mention that the vast majority of contributions in the field use one of three methods to share the benefits from cooperation: the Shapley value, proportional methods and the nucleolus. By our contribution, we promote an alternative, the τ-value, because of its, in our view, attractive motivation, as a compromise between what each player should expect and what he might hope for.

Gansterer & Hartl [2018] also distinguish two main streams of research: (1) work with focus on transportation problems, neglecting the aspect of profit sharing somewhat; (2) work with focus on profit sharing, neglecting the aspect of the transportation problem somewhat. They find only sparse work on the combination of both, e.g., Krajewska *et al.* [2008].

Our main motivation is practical, namely that computation of the τ-value in a *CTP* may be subject to contextual time constraints, similar to Joosten & Lalla-Ruiz [2019]. These constraints might imply that it is not possible to compute all worths of the coalitions, necessary as inputs for the computation of game-theoretical solutions such as the τ-value, on time. In terms of the distinction made by Gansterer & Hartl [2018], we focus on the computational difficulties arising from the **combination** of both streams of contributions mentioned, in the interest of dealing with the aspect of profit sharing. Admittedly, we slightly neglect the transportation problem aspect.

We propose the **inductive τ-value**, which entails that we use approximations of the utopia and the minimal-rights vectors based on all coalitions with at most U members. First, however, we need to determine the necessary primitives for the approximations, i.e., the worths of coalitions. We start by determining the worth of the grand coalition, which is always possible by assumption. We initialize the approximations for the utopia and the minimal-rights vectors by setting them both equal to the $|N|$-dimensional zero vector. Then, we proceed with establishing the worths of all stand-alone coalitions, i.e., coalitions with cardinality 1. If all worths can be computed for sets with cardinality one, we have obtained our first (real) approximation of both vectors namely both the utopia amount and the minimal-rights amount are equal to the stand-alone worth of each player. If however, the time-constraint is met in the midst of trying to compute the stand-alone worths, we use the approximation of the utopia and minimal-rights vectors obtained last, instead.

If the time constraint is not met yet, we continue with all 2-person coalitions, then 3-person coalitions until either the contextual time constraint is met, or all $|N|-1$-person coalitions have been determined. We find an approximation of either vector for coalitions of increasing cardinality as described, namely if the contextual time-constraint is met in the midst of computing the worths of all coalitions with cardinality $U+1$, we use the approximated utopia and minimal-rights vectors established last, i.e., after the computations for all coalitions with cardinality smaller than or equal to U have been completed.

Note that U can only be determined by actually performing the computations involved. The approximation of the utopia vector relies on determining for each player his maximum marginal contribution over all coalitions with cardinality U containing him. The approximation of the minimal-rights vector relies on computing for each player the maximum remainder for each coalition with at most U members containing him based on the approximated utopia amount.

The inductive τ-value is to be applied to the class of hyperplane-singular games, i.e., the games satisfying that whenever the approximated utopia and minimal-rights vectors are on the same hyperplane, they coincide. This is close to one of the properties quasi-balanced games to which the τ-value apply, satisfy.

We show that if the contextual time restriction is quite loose and the computations for all coalitions with cardinality smaller than or equal to $|N|$ can be completed, we obtain $I\tau = \tau$, i.e., the inductive τ-value and the τ-value coincide, for quasi-convex games. Otherwise, we get a new value in spirit rather similar to the original τ-value. If the restriction is very tight, i.e., $U = 0$, we obtain the egalitarian value as the inductive τ-value, i.e., $I\tau = \eta$, (e.g., Joosten [1996]). For $U = 1$, we obtain the center-of-the-imputation-set value, i.e., $I\tau = CIS$ (Driessen & Funaki [1991], Moulin [2003]).

We use two criteria from Joosten & Lalla-Ruiz [2019] dealing with links between the information available and the value at hand. We call a value sensitive up to cardinality U if the value is able to differentiate in attributing amounts to the players for pairs of games which are perfectly identical for coalitions with more than U members, but different for at least one coalition with at most U members. A value is called insensitive beyond cardinality U if the difference between the amounts two players receive depends only on what they contribute to coalitions with at most U members. So, a value that is sensitive up to cardinality U uses all information from the worths of coalitions up to a certain size, and if it is insensitive beyond U it does not differentiate between players on the basis of information from worths of coalitions larger than U.

Each inductive τ-value satisfies the axioms of efficiency, symmetry and desirability, as well as individual rationality for a rather large class of *CTP* games. The inductive τ-value is both insensitive beyond any cardinality U and generically sensitive up to any cardinality U. We regard this as an appropriate pair of properties for the present context involving two different notions of *fairness*. All differences in contributions players have in the part of the cooperative game for which we can compute the exact worths, do matter for the inductive τ-value. In this sense, this value can be regarded as fair, as it recognizes and acknowledges differences in players' power. On the other hand, for the part of the cooperative game for which the restriction regarding the computation time causes a lack of information about the worths of, and hence contributions of players to the latter, of coalitions with more members, the value treats each and every player equally. So, information which is simply not available (under the restriction) does not distort any (in)differences between players justified by the former part.

As an extra, we present a value inspired by Driessen & Tijs [1984] in which several extensions of the original τ-value are to be found. One extension, unfortunately unnamed, seems to fit the analogy τ-value versus inductive τ-value, perfectly. We call our value the inductive $\widetilde{\tau}$-value and retrospectively name the variant of the τ-value in Driessen & Tijs [1984], the $\widetilde{\tau}$-value. The difference between the τ-value and the $\widetilde{\tau}$-value is that for the latter the utopia vector is determined by taking each player's highest marginal contribution to any set, instead of simply taking the player's marginal contribution to the grand coalition. The difference between the inductive versions of both values is then that for each approximation of the utopia vector for the inductive $\widetilde{\tau}$-value relies on determining for each player his maximum marginal contribution over all coalitions with cardinality smaller than or equal to U, instead of just equal to U. A remarkable feat is that the $\widetilde{\tau}$-value is definitely harder to compute than the τ-value. This disadvantage does not hold at all for the inductive versions as differences in complexity are minimal.

2 Preliminaries

$\mathbb{R}$ denotes the set of real numbers, $Z = \{1, 2...\}$ is the set of natural numbers, representing the **set of potential players**. (Strict) inclusions are denoted by $(\subset)$ $\subseteq$. A **coalition** is a finite subset of Z. A **transferable utility game** is a pair (N, v) where $N = \{1, 2..., |N|\}$ is a coalition and $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, is a **characteristic function**. G^N is the set of all games with player set N, G denotes the set of all games.

Let $(N, v) \in G$, then:

- for $M \subseteq N$, the characteristic function of the game (M, v) is the map v restricted to 2^M;
- the **marginal contribution** of player $i \in S \subseteq N$ is given by $\Delta_i^v(S) = v(S) - v(S \backslash \{i\})$;
- player $i \in N$ is a **null-player** in (N, v) if $\Delta_i^v(S) = 0$ for all $S \subseteq N, S \ni i$; $\mathcal{N}(N, v)$ is the **set of null-players** in (N, v);
- players $i, j \in N$ are **symmetric** in (N, v), if $\Delta_i^v(S) = \Delta_j^v(S)$ for all $S \subseteq N, S \supseteq \{i, j\}$.

Let $v, w : 2^N \to \mathbb{R}$, and $\alpha, \lambda \in \mathbb{R}$, then $((\alpha v + \lambda w))(S) = \alpha v(S) + \lambda w(S)$, for all $S \subseteq N$. With this operation G^N is a linear space.

For given $\gamma \in \mathbb{R}$ we associate a hyperplane in the $|N|$-dimensional Euclidian space $H(\gamma)$ by

$$H(\gamma) = \left\{ x \in \mathbb{R}^{|N|} | \sum_{i \in N} x_i = \gamma \right\}.$$

We call $H(v(N))$ the **efficient hyperplane** of $(N, v) \in G$.

A **value** is a map ψ assigning to each game (N, v), a vector in $\mathbb{R}^{|N|}$. The interpretation is that if the value is applied to (N, v), its i-th component represents the utility attributed to player $i \in N$. Let ψ be a value, then:

- ψ is **efficient** if $\psi \in H(v(N))$ for every $(N, v) \in G$;
- ψ is **symmetric** if $\psi_i(N, v) = \psi_j(N, v)$ whenever $i, j \in N$ are symmetric players in $(N, v) \in G$;
- ψ is **linear** if $\psi(N, \lambda v + \mu w) = \lambda\psi(N, v) + \mu\psi(N, w)$ for all $\lambda, \mu \in \mathbb{R}$, and all $(N, v), (N, w) \in G$;
- ψ is **individually rational** with respect to $G' \subset G$, if $\psi_i(N, v) \geq v(\{i\})$ for every $(N, v) \in G'$;
- ψ satisfies the **null-player property** if $\psi_i(N, v) = 0$ for all $i \in \mathcal{N}(N, v)$;
- ψ satisfies **desirability** iff $\psi_i\,(N, v) \leq \psi_j\,(N, v)$ whenever for some $i, j \in S \subseteq N$ it holds that $\Delta_i^v(S) \leq \Delta_j^v(S)$ if $i, j \in S$, for any $(N, v) \in G$.

Desirability is due to Isbell [1958], Maschler & Peleg [1966]. An equivalent axiom *local monotonicity* has gained some popularity more recently (e.g., Levinský & Silársky [2004], Van den Brink et al. [2013]).

These are but a small subset of the axioms conceived in the field of cooperative game theory. We focus on a solution for the problem regarding the time restriction on computations, and refrain from characterizing our contributions. However, we mention the properties above, because they are particularly prominent, and useful in the sequel. Furthermore, we argue in the sequel why a characterization of our solution seems impossible.

The following pair of axioms deal with the (in)ability of values to distinguish between games in attributing amounts to the players.

Definition 1 *(Joosten & Lalla-Ruiz [2019]) The value φ is called*

- ***sensitive up to cardinality U**, $1 < U \leq |N|$, if for any $(N, v) \in G$ and any $s \leq U$, there exists $(N, w^s) \in G$ such that $w^s(S) \neq 0$ for some S with $|S| = s$, and $w^s(S) = 0$ whenever $|S| > U$, such that $\varphi(N, v + w^s) \neq \varphi(N, v)$;*
- ***insensitive beyond cardinality U**, $0 \leq U \leq |N|$, if $\varphi_i(N, v + w) - \varphi_j(N, v + w) = \varphi_i(N, v) - \varphi_j(N, v)$ for all $(N, v) \in G$, all $i, j \in N$ and all $(N, w) \in G$, satisfying $w(S) = 0$ for all $|S| = 1, 2, ..., U, |N|$.*

Hence, if a value is sensitive up to some cardinality U, it is responsive to changes in worths of coalitions with cardinality up to U. If the value is insensitive beyond some cardinality U, it means that if (N, v) were to change to $(N, v + w)$ as described above, gains (or losses) would be identical for any pair of players.

2.1 Values from the literature

For $(N, v) \in G$, the **egalitarian value** η (cf., e.g., Joosten [1996]) is given by

$$\eta_i(N, v) = \frac{v(N)}{\mid N \mid} \text{ for all } i \in N,$$

i.e., η distributes the worth of the grand coalition equally among the players. So, η is efficient, symmetric, linear, and it satisfies desirability, but does not satisfy the null-player property.[6]

The **center(-of-gravity) of the imputation set value**, or CIS (Driessen & Funaki [1991], Moulin [2003]) is defined by

$$CIS_i(N, v) = v(\{i\}) + \frac{v(N) - \sum_{j=1}^{n} v(\{j\})}{\mid N \mid} \text{ for all } i \in N,\ (N, v) \in G.$$

This value gives to each player i his stand alone-worth $v(\{i\})$ whereas the remaining amount $v(N) - \sum_{j=1}^{n} v(\{j\})$ is divided equally among all members of the grand coalition. So, CIS is efficient, symmetric, linear, it satisfies desirability, but it does not satisfy the null-player property.

The **proportional division value** PD (cf., Zou *et al.* [2021], Moriarity [1975], Banker [1981], Kamijo & Kongo [2015]) is defined by

$$\begin{aligned} PD_i(N, v) &= v(\{i\}) + \frac{v(\{i\})}{\sum_{j=1}^{n} v(\{j\})} \left[v(N) - \sum_{j=1}^{n} v(\{j\}) \right] \\ &= \frac{v(\{i\})}{\sum_{j=1}^{n} v(\{j\})} v(N) \text{ for all } i \in N,\ (N, v) \in G. \end{aligned}$$

For the above to make sense $\min_{i \in N} v(\{i\}) \geq 0 < \max_{i \in N} v(\{i\})$ should hold. So, each player receives his stand-alone worth and the remaining amount is divided among all members of the grand coalition proportional to their stand alone worths. PD is efficient, symmetric, and it satisfies desirability and the null-player property, but it is not linear.

Proportional methods (cf., e.g., Cruijssen *et al.* [2007], Guajardo & Rönnqvist [2016], Béal *et al.* [2016]) can be described similarly. Let $w \in \mathbb{R}^{|N|}$ be a vector of weights satisfying $w_i > 0$ for all $i \in N$, then define **proportional method** PM^w by

$$PM_i^w(N, v) = \frac{w_i}{\sum_{j=1}^{n} w_j} v(N) \text{ for all } i \in N,\ (N, v) \in G.$$

The weights might be determined outside the cooperative game, e.g., if $w_i = c > 0$ for all $i \in N$ we have $PM^w(N, v) = \eta(N, v)$. However, if the weights are taken exactly equal to the vector of stand-alone worths, i.e., $w_i = v(\{i\})$ for all $i \in N$, then $PM^w(N, v) = PD(N, v)$. Cruijssen *et al.* [2007] and Guajardo & Rönnqvist [2016] offer practical alternatives on how to establish weights w. Proportional methods may be viewed as unfair and this in turn may impede cooperation in real-world problems, cf., Cruijssen *et al.* [2007]. PM^w is efficient, linear, and it does not satisfy the null-player property, nor does it satisfy desirability.

[6] Van den Brink [2007] characterizes the egalitarian value by efficiency, linearity, symmetry and the nullifying player property.

These four values need very little information from the game. This means that they may be computed using little time. The PD and CIS simply project the vector of stand-alone worths on the efficient hyperplane by ray-projection and orthogonal projection, respectively.[7] The CIS and η use an orthogonal projection unto the efficient hyperplane of the vector of stand-alone worths respectively the origin. We reiterate that these four values are deemed unfair because they are bound to neglect differences in strength, which real-world actors would like to see expressed (cf., Cruijssen *et al.* [2007]).

A value fulfilling this demand at the price of requiring substantially more information and computational operations than the foursome discussed, is the **Shapley value** (cf., Shapley [1953b], Roth [1988]) Sh. For every $(N, v) \in G$ and every $i \in N$, Sh is given by

$$Sh_i(N, v) = \sum_{S \subseteq N : i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \Delta_i^v(S). \qquad (1)$$

The Shapley value is based on permutations of the player set. Recall that $\Delta_i^v(S)$ is player i's marginal contribution to the set S, and observe that the predecessors to i in S could have arrived in $(|S|-1)!$ different orders and the players outside of S could be arriving after i in $(|N|-|S|)!$ different orders, the total number of orders for all players is $|N|!$. The Shapley value assigns to each player his average marginal contribution taken over all permutations of the player set. As an immediate consequence, this value satisfies desirability. However, to apply (1), the worths of all coalitions should be known and the computation involves $|N|!$ steps. There is a more efficient way of computing the Shapley value of order $2^{|N|}$ based on potentials (cf., Hart & Mas-Colell [1988,1989]).

The Banzhaf value (cf., Banzhaf [1965]) is also based on the average of the players' marginal contributions but slightly differently. Here, the average is taken over all coalitions containing each player, i.e., for every $(N, v) \in G$ and every $i \in N$, the Banzhaf value BZ is given by

$$BZ_i(N, v) = \frac{1}{2^{|N|-1} - 1} \sum_{S \subseteq N : i \in S} \Delta_i^v(S). \qquad (2)$$

It is quite easy to prove that this value is symmetric, linear, that it is not efficient, and it satisfies desirability and the null-player property. To apply (2), the worths of all coalitions should be known, too. Table 1 below, summarizes.

Value	**E**	**S**	**L**	**N**	**D**
η	yes	yes	yes	no	yes
CIS	yes	yes	yes	no	yes
PD	yes	yes	no	yes	yes
PM	yes	no	no	no	no
BZ	no	yes	yes	yes	yes
Sh	yes	yes	yes	yes	yes

Table 1. Overview of values and axioms: **E,S,L,N,D** denote efficiency, symmetry, linearity, the null-player property and desirability.

As to the axioms presented in Joosten & Lalla-Ruiz [2019], clearly, η and PM are not sensitive up to cardinality U for any strictly positive U. Similarly,

[7] Following the terminology of Joosten & Roorda [2008,2011] in a different framework.

PD and CIS are sensitive up to cardinality 1; η is insensitive beyond cardinality 0, PM is not insensitive beyond cardinality 0, CIS is insensitive beyond cardinality 1 and PD is not insensitive beyond cardinality 1. The τ-value is sensitive up to cardinality $|N|$, where N is the player set. Table 2 summarizes.

Value	**Sut**	**Insb**	U
η	-	yes	0
CIS	yes	yes	1
PD	yes	no	1
PM	no	no	any
Sh	yes	-	$\|N\|$
BZ	yes	-	$\|N\|$

Table 2. Summary of connections between values and axioms: **Sut** means sensitive up to, **Insb** means insensitive beyond.

3 The classical τ-value

The τ-value of a game (N, v) is the efficient convex combination of two $|N|$-dimensional vectors, the so-called the minimal-rights vector $m(N, v)$, and the utopia vector $M(N, v)$. The **utopia amount** $M_i(N, v)$ for each player $i \in N$ in a game (N, v) is defined as

$$M_i(N, v) = v(N) - v(N \backslash \{i\}),$$

i.e., the player i's marginal contribution to the worth of the grand coalition. Then, the utopia vector $M(N, v)$ is simply $\left(M_1(N, v), ..., M_{|N|}(N, v)\right)$.

To determine the minimal-rights vector, we need an auxiliary concept called the **remainder of player i in coalition $S \subseteq N$ containing him**, i.e.,

$$R_i^{(N,v)}(S) = v(S) - \sum_{j \in S \backslash \{i\}} M_j(N, v).$$

Hence, $R_i^{(N,v)}(S)$ can be interpreted as the amount of money remaining from the worth of coalition S, i.e., $v(S)$, if all other players in S are paid their utopia amounts, first. The **minimal-rights amount** for player $i \in N$ is then given by

$$m_i(N, v) = \max_{S \subseteq N : S \ni i} R_i^{(N,v)}(S).$$

The **minimal-rights vector** $m(N, v)$ is $m(N, v) = \left(m_1(N, v), , ..., m_{|N|}(N, v)\right)$.

Paraphrasing Tijs [1987], the minimal-rights amount can be regarded as a lower bound to what any player should get under a solution, because if that player receives less, then he could entice all players involved in the coalition in which his remainder is equal to $m_i(v)$, to form that coalition and split off from the grand coalition, by giving them their utopia amounts and keeping the remainder to himself. We recognize similarities to the reasoning connected to the core (Gillies [1953]), in which allocations of profits should be such that no coalition has an incentive to split off from the grand coalition.

The original contributions of Tijs [1981,1987] restrict the applicability of the τ-value to those classes of cooperative games (N, v) guaranteeing

$$\begin{aligned} m_i(N, v) &\leq M_i(N, v) \text{ for all } i \in N, \text{ and} \\ \gamma_{m(N,v)} &\leq v(N) \leq \gamma_{M(N,v)}, \end{aligned}$$

here, $\gamma_x \equiv \sum_{i\in N} x_i$ for any $x \in \mathbb{R}^{|N|}$. Tijs [1981,1987] calls the class of games satisfying the two properties above **quasi-balanced games**, denoted by $Q^{|N|}$.

We are now ready to present the following definition of the classical τ-value.

Definition 2 *Let $(N,v) \in Q^{|N|}$, then the τ-value is given by*

$$\tau(N,v) = \begin{cases} m(N,v) = M(N,v) \text{ if } \gamma_{M(N,v)} = \gamma_{m(N,v)}, \\ \\ m(N,v) + \frac{v(N)-\gamma_{m(N,v)}}{\gamma_{M(N,v)}-\gamma_{m(N,v)}} \left[M(N,v) - m(N,v)\right] \text{ otherwise.} \end{cases}$$

It follows easily that the set of convex combinations of the minimal-rights and the utopia vectors, i.e.,

$$Y(N,v) = \left\{ y \in \mathbb{R}^{|N|} \mid \exists_{\lambda\in[0,1]}\ y = \lambda m(v) + (1-\lambda)\, M(v) \right\},$$

has a unique intersection point with the hyperplane $H(v(N))$, which is precisely the τ-value.

Alternatively stated, since $\tau(v) \in H(v(N))$ if $(N,v) \in Q^{|N|}$, we have

$$\tau(N,v) = \underset{x\in H(v(N))}{\arg\min}\ d_2(x, Y(v)),$$

where $d_2(\cdot,\cdot)$ denotes the Euclidean distance, moreover the distance between a point $z \in \mathbb{R}^{|N|}$ and a set $S \subset \mathbb{R}^{|N|}$ is defined in the usual manner as

$$d_2(z,S) = \inf_{y\in S} d_2(z,y).$$

3.1 A convenient short-cut in calculation

As noted quite early, the actual computations can be simplified avoiding a large number of simple and repetitive steps. The result below helps to find the minimal-rights vector, its validity is proven in Driessen & Tijs [1983].

Lemma 3 *(Driessen & Tijs [1983]) Let game $(N,v) \in Q^{|N|}$ and utopia vector $M(N,v)$ be given. Denote $b(S) = \sum_{i\in S} M_i(N,v)$, $g(S) = b(S) - v(S)$, $\mu_i = \min_{i\in S} g(S)$ for all $i \in N$, and $\mu = \left(\mu_1, ..., \mu_{|N|}\right)$, then $m(N,v) = M(N,v) - \mu$.*

The function $g : 2^{|N|} \to \mathbb{R}$ is called the **gap function**.

Example 4 *We give an example for the calculations regarding the classical τ-value. Later on, the results obtained here serve as a point of reference. For $N = \{1,2,3,4\}$, let the cooperative game (N,v) be given by*

$v(\emptyset) = v(\{1\}) = 0$	$v(\{2\}) = 1$
$v(\{3\}) = v(\{4\}) = 2$	
$v(\{1,2\}) = 4$	$v(\{1,3\}) = v(\{1,4\}) = 9$
$v(\{2,3\}) = v(\{2,4\}) = 12$	$v(\{3,4\}) = 14$
$v(\{1,2,3\}) = v(\{1,2,4\}) = 15$	$v(\{1,3,4\}) = 18$
$v(\{2,3,4\}) = 16$	$v(N) = 24$

Note that players 3 *and* 4 *are symmetric. First, we establish the utopia vector:*

$$M(N, v) = (24 - 16, 24 - 18, 24 - 15, 24 - 15) = (8, 6, 9, 9).$$

To compute $m(N, v)$ *using the short-cut computation from Lemma 3, we construct the following table omiting accolades and commas to save space.*

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$b(S)$	8	6	9	9	14	17	17	15	15	18	23	23	26	24	32
$v(S)$	0	1	2	2	4	9	9	12	12	14	15	15	18	16	24
$g(S)$	8	5	7	7	10	8	8	3	3	4	8	8	8	8	8

Confirm $\mu_1 = \min_{S:1 \in S} g(S) = 8$, *and likewise* $\mu_2 = 3$, $\mu_3 = 3$, $\mu_4 = 3$, *so*

$$\begin{aligned} \mu &= (8, 3, 3, 3) \textit{ hence} \\ m(N, v) &= M(N, v) - \mu \\ &= (8, 6, 9, 9) - (8, 3, 3, 3) = (0, 3, 6, 6). \end{aligned}$$

Since, $\gamma_{m(N,v)} = 15 < v(N) = 24 < \gamma_{M(N,v)} = 32$, *we have*

$$\begin{aligned} \tau(N, v) &= (0, 3, 6, 6) + \frac{24 - 15}{32 - 15}((8, 6, 9, 9) - (0, 3, 6, 6)) \\ &= (0, 3, 6, 6) + \frac{9}{17}(8, 3, 3, 3) = \left(4\frac{4}{17}, 4\frac{10}{17}, 7\frac{10}{17}, 7\frac{10}{17}\right). \end{aligned}$$

4 The inductive τ-value

The centralized collaborative planner in the terminology of Gansterer & Hartl [2018] to be entrusted with performing the required computations, operates under the following assumptions on $(N, v) \in G$:

A1 A finite upper bound on the computation time $CT_{\max}$ exists. However, $CT_{\max} \geq CT_{v(N)}$, i.e., $CT_{v(N)}$ the computing time for $v(N)$, the worth of the grand coalition, is small enough to compute $v(N)$.

A2 The remaining computing time $CT_{\max} - CT_{v(N)}$ results in a number U such that all worths $v(S)$ with $|S| \leq U$ can be computed. If $U = |N| - 1$, then we have established all primitives. Otherwise, at least one worth of a coalition with cardinality $U + 1$ cannot be computed.

A3 $U \geq 0$, but U need not be known before computations start.

The computational procedure performs some initialization steps, finds the worth of the grand coalition, and proceeds by trying to establish the worths of all coalitions with cardinality 1. If this succeeds and computation time remains, similar computations for coalitions with cardinality 2 are started, and so forth. If however, computing time reaches its constraint while establishing worths of coalitions with cardinality $U + 1$, approximations of the utopia and minimal-rights vectors for U, are used to determine the approximation of the τ-value.

We have, by the way, an important practical motivation for dealing with the smaller sized coalitions first. Note that there are as many coalitions with k members as there are coalitions with $|N| - k$ members, so it is not in that aspect that differences in consuming computation time might arises. Rather, we may expect that accomplishing the computations for one of the former coalitions is in general less time-consuming than for one of the latter. Furthermore, since there are $\binom{|N|}{k}$ of coalitions with cardinality k and $|N|-k$, each, the expected savings in computation times accrue. In order to obtain more information measured in the dimension of the number of coalitions for which information relevant to the computation of the value at hand is available, we work from low-cardinality to high-cardinality coalitions, instead of the other way around.

To generate an inductive version of the τ-value we adapt the utopia and the minimal-rights vectors straightforwardly. For **any strictly positive integer** U, we define the U**-restricted utopia amount** $M_i^U(N,v)$ **for each player** $i \in N$ in a game (N,v) as

$$M_i^U(N,v) = \max_{S \ni i: |S|=U} \left[v(S) - v(S \backslash \{i\}) \right].$$

So, instead of taking the marginal contribution of each player to the grand coalition we take the player's maximum marginal contribution taken over all sets of cardinality U that he belongs to. Analogously, we define the U**-restricted remainder of player** i **in coalition** $S \subseteq N$ **containing him**, by

$$R_i^{(N,v),U}(S) = v(S) - \sum_{j \in S \backslash \{i\}} M_j^U(N,v).$$

The U**-restricted minimal-rights amount for player** $i \in N$ in $(N,v) \in G$ is given by

$$\begin{aligned} m_i^U(N,v) &= \max_{S \subseteq N: |S| \leq U, S \ni i} R_i^{(N,v),U}(S), \text{ and hence} \\ m^U(N,v) &= \left(m_1^U(N,v), ..., m_{|N|}^U(N,v) \right) \end{aligned}$$

For the sake of completeness, we extend the definition to $U = 0$ by the following normalization

$$m_i^0(N,v) = M_i^0(N,v) = v(\emptyset) = 0 \text{ for all } i \in N \text{ in any } (N,v) \in G.$$

The U-restricted amounts $m_i^U(N,v)$, $M_i^U(N,v)$ for player i can be seen as the natural analogy of the minimal rights amount $m_i(N,v)$ and the utopia amount $M_i(N,v)$ to all subsets with cardinality at most U.

For $(N,v) \in G^{|N|}$ and $U = 0, 1, ..., |N|$, then let

$$Y^U(N,v) = \left\{ y \in \mathbb{R}^{|N|} \mid \exists_{\lambda \in [0,1]} \; y = \lambda m^U(N,v) + (1-\lambda) M^U(N,v) \right\}.$$

For our aims in formulating an analogy to the τ-value which allows the most general application, we need a minimal restriction with respect to the set

$$\widehat{Y}(N,v) = \left\{ Y^U(N,v) \subset \mathbb{R}^{|N|} \mid U \in \{0, 1, ..., |N|\} \right\}.$$

Definition 5 *The game* $(N,v) \in G^{|N|}$ *is hyperplane-singular if* $m^U(N,v) = M^U(N,v)$ *whenever* $\gamma_{m^U(N,v)} = \gamma_{M^U(N,v)}$.

In the generic case, $m^U(N,v), M^U(N,v)$ are on different hyperplanes. However if they are not, they must coincide, i.e., $Y^U(N,v)$ is a singleton. We denote the class of $|N|$-person games satisfying Definition 5 by $HS^{|N|}$, i.e.,

$$HS^{|N|} = \left\{ (N,v) \in G^{|N|} \mid m^U(N,v) = M^U(N,v) \text{ iff } \gamma_{m^U(N,v)} = \gamma_{M^U(N,v)} \right\}.$$

Definition 6 *For* $(N,v) \in HS^{|N|}$, ***the*** U***-restricted*** τ***-value*** *is given by*

$$\tau^U(N,v) = \underset{x \in H(v(N))}{\arg\min}\; d_2(x, Y^U(N,v)).$$

The formulation is required because we do not have any information whether $Y^U(N,v)$ and $H(v(N))$ intersect. Of course, if they do, the intersection point minimizes the distance to the efficient hyperplane.

For $m^U(N,v)$, the approximation of the minimal-rights vector $m(N,v)$, we have the following result[8] analogous to Driessen & Tijs [1983].

Lemma 7 *Let game* $(N,v) \in HS^{|N|}$. *Given* U *and* $M^U(N,v)$, *let* $b^U(S) = \sum_{i \in S} M_i^U(N,v)$, $g^U(S) = b^U(S) - v(S)$, $\mu_i^U = \min_{i \in S} g^U(S)$ *for all* $i \in N$, *and* $\mu^U = \left(\mu_1^U, ..., \mu_{|N|}^U\right)$, *then* $m^U(N,v) = M^U(N,v) - \mu^U$.

The function $g^U : 2^{|N|} \to \mathbb{R}$ is called the U**-restricted gap function**.

Remark 1 *In order to avoid confusion, we emphasize the following. The inductive* τ*-value is uniquely determined for* $(N,v) \in HS^{|N|}$ *depending on the termination time of the computations, i.e.,* $U = 0,1,...,|N|$, *because the set of* ***'pre-solutions'***

$$\{\tau^u(N,v)\}_{u=0}^{|N|}$$

is uniquely determined, but ***unknown*** *at the start of the computations. Only one element out of this set is 'chosen' by the termination of the computations*

$$I\tau(N,v) = \tau^U(N,v) \in \{\tau^u(N,v)\}_{u=0}^{|N|}.$$

Remark 2 *If* $(N,v) \in HS^{|N|} \cap Q^{|N|}$, *i.e., also quasi-balanced, then* $\tau^N(N,v) = \tau(N,v)$. *Otherwise, the inductive* τ*-value yields a generalization of the* τ*-value.*

4.1 Orthogonal projections used fruitfully

For $(N,v) \in HS^{|N|}$ denote

$$B^U(N,v) = \left\{\beta \in \mathbb{R} | \exists_{x \in Y^U(N,v)} : x + \beta\,(1,...,1) \in H(v(N))\right\}.$$

The set $B^U(N,v)$ connects each element of $Y^U(N,v)$ with an orthogonal projection unto the efficient hyperplane $H(v(N))$. We show that the projection vector from x to $x + \beta\,(1,...,1)$ is perpendicular to the hyperplane $H(v(N))$, its distance is closely connected to β.

Lemma 8 *Let for* $(N,v) \in HS^{|N|}$, $x \in Y^U(N,v)$, $y = x + \beta\,(1,...,1) \in H(v(N))$ *and let* $z \in H(v(N))$, *then the vectors* $(x-y)$ *and* $(y-z)$ *are perpendicular. Moreover,* $d_2(x, H(v(N))) = |\beta| \cdot \sqrt{|N|}$.

Since both $Y^0(N,v)$ and $Y^1(N,v)$ are singletons, we have the following.

Lemma 9 *Let* $(N,v) \in HS^{|N|}$, *then* $\tau^0(N,v) = \eta(N,v)$, $\tau^1(N,v) = CIS(N,v)$.

[8] Proofs of our results are to be found in the Appendix.

So, for tightly restricted computations, the inductive τ-value boils down to η, the egalitarian value in the terminology of, e.g., Joosten [1996], or the center of gravity of the imputation set value of Driessen & Funaki [1991], Moulin [2003].

The following result helps in finding precise relations between $Y^U(N,v)$ and the set of projections of its elements unto the efficient hyperplane.

Lemma 10 *Let $(N,v) \in HS^{|N|}$. Then, for every $x \in Y^U(N,v)$ there is a unique $\beta \in B^U(N,v)$ such that $x + \beta(1,...,1) \in H(v(N))$. Moreover, for every $\beta \in B^U(N,v)$ there is a unique $x \in Y^U(N,v)$ such that $x + \beta(1,...,1) \in H(v(N))$.*

We defined $\tau^U(N,v)$ as the efficient allocation closest to the set $Y^U(N,v)$, reasoning the other way around, let

$$\pi^U(N,v) = \underset{x \in Y^U(N,v)}{\arg\min}\ d_2(x, H(v(N))).$$

We call $\pi^U(N,v) \in Y^U(N,v)$ the **nearest allocation**, as it is the allocation belonging to the set of convex combinations of the minimal rights vector $m(N,v)$ and the utopia vector $M(N,v)$ minimizing the Euclidean distance to the efficient hyperplane, which as we have seen by Lemmas 8 and 9 is equivalent to minimizing $|\beta|$.

If $Y^U(N,v)$ intersects with $H(v(N))$, $\beta = 0$. This means

$$\begin{aligned} \gamma_{\pi^U(N,v)} &= v(N) = \lambda\gamma_{M^U(N,v)} + (1-\lambda)\gamma_{m^U(N,v)} \Longrightarrow \\ \pi^U(N,v) &= m^U(N,v) + \frac{v(N) - \gamma_{m^U(N,v)}}{\gamma_{M^U(N,v)} - \gamma_{m^U(N,v)}}\left[M^U(N,v) - m^U(N,v)\right] \\ &= \overline{\tau}^U(N,v). \end{aligned}$$

As $Y^U(N,v)$ is convex and compact, if its intersection with the efficient hyperplane is empty, its element minimizing the distance to $H(v(N))$ must be obtained at an end point of the line segment, i.e., the nearest allocation $\pi^U(N,v) \in \{m^U(N,v), M^U(N,v)\}$. This in turn means that we have

$$\pi^U(N,v) = \begin{cases} \overline{\tau}^U(N,v) \text{ if } \begin{cases} \gamma_{M^U(N,v)} < v(N) < \gamma_{m^U(N,v)}, \text{ or} \\ \gamma_{M^U(N,v)} < v(N) < \gamma_{m^U(N,v)}. \end{cases} \\ m^U(N,v) \text{ if } \begin{cases} \gamma_{M^U(N,v)} \le \gamma_{m^U(N,v)} \le (N), \text{ or} \\ v(N) \le \gamma_{m^U(N,v)} \le \gamma_{M^U(N,v)}. \end{cases} \\ M^U(N,v) \text{ if } \begin{cases} \gamma_{m^U(N,v)} \le \gamma_{M^U(N,v)} \le v(N), \text{ or} \\ v(N) \le \gamma_{M^U(N,v)} \le \gamma_{m^U(N,v)}. \end{cases} \end{cases} \quad \textbf{(1)}$$

Corollary 11 *The U-restricted τ-value $\tau^U(N,v)$ is the orthogonal projection of the nearest allocation $\pi^U(N,v)$ unto $H(v(N))$.*

Example 12 *We continue with the game of Example 3. For $U = 0$, we have*

$$m^0(N,v) = M^0(N,v) = (0,0,0,0).$$

Then, $\pi^0(N,v) = (0,0,0,0)$ *and therefore*

$$\tau^0(N,v) = (6,6,6,6)\,.$$

Moreover, for $U = 1$*, we have*

$$m^1(N,v) = M^1(N,v) = (0,1,2,2).$$

Then, $\pi^1(N,v) = (0,1,2,2)$ *which in turn yields*

$$\tau^1(N,v) = \left(4\frac{3}{4}, 5\frac{3}{4}, 6\frac{3}{4}, 6\frac{3}{4}\right).$$

For $U = 2$*, we have*

$$\begin{aligned} M_1^2(N,v) &= \max\{v(\{1,2) - v(\{2\}), v(\{1,3) - v(\{3\}), v(\{1,4) - v(\{4\})\} \\ &= \max\{4-1, 9-2, 9-2\} = 7. \end{aligned}$$

Similarly we find $M_2^2(N,v) = 10, M_3^2(N,v) = 12$, $M_4^2(N,v) = 12$. *Then,*

$$M^2(N,v) = (7,10,12,12)\,.$$

We again omit accolades and commas in the notation of the coalitions.

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$b^2(S)$	7	10	12	12	17	19	19	22	22	24	29	29	31	34	41
$v^2(S)$	0	1	2	2	4	9	9	12	12	14	15	15	18	16	24
$g^2(S)$	7	9	10	10	13	10	10	10	10	20	14	14	13	18	17

As $\mu_1^2 = \min_{S:1\in S} g^2(S) = 7$, $\mu_2^2 = \min_{S:2\in S} g^2(S) = 9$, $\mu_3^2 = \min_{S:3\in S} g^2(S) = 10$, $\mu_4^2 = \min_{S:4\in S} g^2(S) = 10$, *we obtain*

$$\begin{aligned} \mu^2(N,v) &= (7,9,10,10) \text{ hence} \\ m^2(N,v) &= M^2(N,v) - \mu^2 = (7,10,12,12) - (7,9,10,10) = (0,1,2,2)\,. \end{aligned}$$

This leads to

$$\begin{aligned} \tau^2(N,v) &= (0,1,2,2) + \frac{v(N) - \gamma_{m^2(N,v)}}{\gamma_{M^2(N,v)} - \gamma_{m^2(N,v)}} \left[M^2(N,v) - m^2(N,v)\right] \\ &= (0,1,2,2) + \frac{19}{36}(7,9,10,10) = \left(3\frac{25}{36}, 5\frac{3}{4}, 7\frac{10}{36}, 7\frac{10}{36}\right). \end{aligned}$$

Recall that the τ*-value* $\tau(N,v)$ *and* $\tau^3(N,v), \tau^4(N,v)$ *coincide as* $|N| - 1 = 3$.

U	***Presolution***	***Equal to***
0	$\tau^0(N,v) = (6,6,6,6)$	$\eta(N,v)$
1	$\tau^1(N,v) = \left(4\frac{3}{4}, 5\frac{3}{4}, 6\frac{3}{4}, 6\frac{3}{4}\right)$	$CIS(N,v)$
2	$\tau^2(N,v) = \left(3\frac{25}{36}, 5\frac{3}{4}, 7\frac{10}{36}, 7\frac{10}{36}\right)$	*?*
3	$\tau^3(N,v) = \left(4\frac{4}{17}, 4\frac{10}{17}, 7\frac{10}{17}, 7\frac{10}{17}\right)$	$\tau(N,v)$
4	$\tau^4(N,v) = \left(4\frac{4}{17}, 4\frac{10}{17}, 7\frac{10}{17}, 7\frac{10}{17}\right)$	$\tau(N,v)$

Table 3. *Overview of all presolutions for the inductive* τ*-value of Examples 3 & 7.*

Note that players 3 and 4 get more as more information accrues, player 1's amounts both decrease and increase, whereas player 2's never increase.

5 On axioms

It is common in cooperative game theory to provide characterizations of values introduced, i.e., to obtain a set of axioms and show that a value is the unique one satisfying them. Here, such an ambition is likely to fail, as the inductive τ-value is picked among a set of 'presolutions' $\{\tau^u(N,v)\}_{u=0}^{|N|}$, each requiring its own characterization, not necessarily generalizable over their entire range.

The axiom of efficiency is important, especially in real-life situations, as 'no money is left on the table,' and no alternative feasible allocation makes at least one player better off while keeping all others to the same amount. Symmetry of a value is widely regarded as an important aspect of **fairness**: 'likes should be treated alike.' If players are identical with respect to what they contribute to each and every coalition they join, they should receive the same. For these two axioms we have the following positive result.

Proposition 13 *For any $(N,v) \in HS^{|N|}$ and for any $U = 0,...,|N|$, the U-restricted τ-value $\tau^U(N,v)$ satisfies efficiency and symmetry.*

Both the Shapley value and the τ-value satisfy these two axioms, the Shapley value is then characterized by adding the null-player property and additivity (cf., e.g., Peters [2008]); the τ-value is characterized by adding individual rationality, the null-player property, S-equivalence and continuity (cf., e.g., Driessen & Tijs [1983]). Additivity is useful for technical reasons, yet lacks a strong conceptual underpinning (cf., e.g., Peters [2008]). The null-player property formalizes that if a player does not change the worth of any coalition by his joining, then he should receive nothing. The null-player property is under scrutiny in the egalitarianism versus marginalism discussion (see, e.g., Van den Brink et al. [2013]). Our next result is negative and pertains to a subset of these properties.

Proposition 14 *The inductive τ-value does not satisfy additivity, it does not satisfy the null-player property, and it does not satisfy S-equivalence.*

Cooperative transportation games are **superadditive** (cf., e.g. Schulte *et al.* [2018]). The worth of a coalition is the solution of a maximization problem (in terms of profits). Now, let S, T be two disjoint subsets of the grand coalition and let the solutions to the corresponding maximization problems be $v(S)$ and $v(T)$ respectively. Clearly, the maximization problem for $S \cup T$ must yield a solution that is at least equal to $v(S) + v(T)$, because it is possible to let S and T do as before by default. This means $v(S \cup T) \geq v(S) + v(T)$ for any disjoint subsets S, T of the grand coalition N and this defines a superadditive game.

A value satisfies individual rationality if every player receives at least his stand-alone value, which is instrumental to induce cooperation. It implies that cooperation is always at least as good as remaining alone. Since CTGs are superadditive, which implies for instance that the worth of the grand coalition is greater than the sum of the stand-alone worths of all individual agents, there is some 'financial room' to allow individually rational values. To establish individual rationality for our inductive τ-value we need a bit more structure on the set of games analyzed on the one hand, and in view of the proof of the previous proposition we must aviod $U = 0$. Recall the notation π^U as presented in Eq. (1). Accordingly, we define a game $(N,v) \in HS^{|N|}$ as **π-superadditive** if it is superadditive and furthermore $\max_{1\leq U\leq |N|} \gamma_{\pi^U(N,v)} \leq v(N)$. The class of π-superadditive hyperplane-singular games is therefore

$$\pi - HS^{|N|} = \Big\{(N,v) \in HS^{|N|} | \; v(S\cup T) \geq v(S) + v(T) \text{ for } S, T \subset N, \text{ such that } S\cap T = \emptyset; \text{ and } \max_{1\leq U\leq |N|} \gamma_{\pi^U(N,v)} \leq v(N)\Big\}.$$

We have the following result regarding individual rationality.

Proposition 15 *For* $(N,v) \in \pi - HS^{|N|}$*, we have* $\tau_i^U(N,v) \geq v(\{i\})$ *for* $U = 1, ..., |N|$.

Although we cannot guarantee individual rationality within this class for $U = 0$, it is extremely unlikely that computations terminate with $I\tau(N,v) = \tau^0(N,v)$.

What seems an important aspect of fairness for real-life cooperative transportation problems is that an agent contributing more than another, does not obtain less under a solution than the latter (cf., e.g., Schulte et al. [2018]). This aspect is partly covered by social acceptability (Joosten *et al.* [1994], Joosten & Lalla [2019]), however *desirability* captures this fairness requirement perfectly. We present the following result on this issue.

Proposition 16 *For* $(N,v) \in HS^{|N|}$*, the value* $\tau^U(N,v)$ *satisfies desirability for* $U = 0, 1, ..., |N|$.

The axiom of insensitivity beyond cardinality U aims to formulate that otherwise relevant (payoff) information which is not available, does not have an influence on the amounts attributed by the value. We interpret this property as incorporating some sense of fairness: as we have no knowledge as to what the worths of all other coalitions, except for the grand coalition by Assumption A1, are until we solve the corresponding optimization problems, the players are treated equally for the part which takes care of efficiency.

Proposition 17 *For* $(N,v) \in HS^{|N|}$ *and fixed* $U = 0, 1, ..., |N|$*, the value* $\tau^U(N,v)$ *is insensitive beyond cardinality* U*.*

5.1 On sensitivity up to cardinality U, $U \geq 1$.

We end this section by showing that the inductive τ-value generically satisfies sensitivity up to cardinality U, $U \geq 1$. We devote an entire subsection to this because the various proofs are involved and we need several notations which are only applied here. The theme of this subsection has been raised in Joosten & Lalla-Ruiz [2019], however the present efforts in proving relevant issues differ considerably from those in the earlier paper. Our strategy of proof here, necessitates to show validity for three different ranges of U, namely $U = 1$, $U = 2$ and $U > 2$. Even then, the property holds universally only if $U = 1$, otherwise, i.e., $U \geq 2$, it holds generically.

In the sequel we use the game $(N, w^{S^*,\zeta}) \in G^{|N|}$, defined by $w^{S^*,\zeta}(S^*) = \zeta > 0$ for $S^* \subset N$, and $w^{S^*,\zeta}(S) = 0$ whenever $S^* \neq S \subset N, S \neq S^*$. The gist is that we compare $\tau^U(N,v)$ with $\tau^U(N, v + w^{S^*,\zeta})$ for a specifice game $(N, v + w^{S^*,\zeta}) = (N,v) + (N, w^{S^*,\zeta})$.

Furthermore, we define $\underline{m}^2(N, v + w^{S^*})$ by

$$\underline{m}_k^2(N, v{+}w^{S^*}) = \begin{cases} v(\{k\}) & \text{if } k \in S^*, \\ \max_{\substack{S:|S|=2, S\ni k \\ S\setminus\{k\}\cap S^*=\emptyset}} \left[v(S) - \sum_{i \in S\setminus\{k\}} M_i(N, v) \right] & \text{if } k \in N\setminus S^*. \end{cases}$$

and $\underline{m}^U(N, v + w^{S^*})$ for $U > 2$ by

$$\underline{m}_k^U(N, v + w^{S^*}) = \max_{\substack{S:|S|=U, S\ni k \\ S\setminus\{k\}\cap S^*=\emptyset}} \left[v(S) - \sum_{i \in S\setminus\{k\}} M_i(N, v) \right] \text{ for } k \in N.$$

We start with a straightforward result pertaining to sensistivity for $U = 1$.

Lemma 18 *For any $(N, v) \in G^{|N|}$, and any $S^* \subset N, |S^*| = 1$, a $(N, w^{S^*,\zeta}) \in G^{|N|}$ exists such that $\tau^1(N, v) \neq \tau^1(N, v + w^{S^*,\zeta})$.*

So, for τ^1 no special assumptions on the game at hand are necessary.

The following helps to establish restrictions for the desired result for $U = 2$.

Lemma 19 *Let $(N, v) \in G^{|N|}$, $j_1 \in N$, and let $j_2 \in N\setminus\{j_1\}$ such that $M^2_{j_1}(N, v) = v(\{j_1, j_2\}) - v(\{j_2\})$. Then, for the utopia vector $M^2(N, v{+}w^{\{j_1,j_2\},\zeta})$ we have*

$$\begin{aligned} M^2_{j_1}(N, v + w^{\{j_1,j_2\},\zeta}) &= v(\{j_1, j_2\}) + \zeta - v(\{j_2\}), \\ M^2_{j_2}(N, v + w^{\{j_1,j_2\},\zeta}) &= v(\{j_1, j_2\}) + \zeta - v(\{j_1\}), \\ M^2_k(N, v + w^{\{j_1,j_2\},\zeta}) &= M^2_k(N, v) \text{ for } k \in N\setminus\{j_1, j_2\}. \end{aligned}$$

For the minimal-rights vector $m^2(N, v + w^{\{j_1,j_2\},\zeta})$, we have

$$\begin{aligned} m^2_k(N, v + w^{\{j_1,j_2\},\zeta}) &= v(\{k\}) \text{ for } k \in \{j_1, j_2\}, \\ m^2_k(N, v + w^{\{j_1,j_2\},\zeta}) &\leq m^2_k(N, v) \text{ for } k \in N\setminus\{j_1, j_2\}. \end{aligned}$$

$S^* \neq S \subset N, S \neq S^*$. For arbitrary $j_1 \in N$, take $j_2 \in N\setminus\{j_1\}$ such that $M^2_{j_1}(N, v) = v(\{j_1, j_2\}) - v(\{j_2\})$.

The next result for $U > 2$ fulfills a similar role as Lemma 19.

Lemma 20 *For any $(N, v) \in HS^{|N|}$ and any fixed $U > 2$, a set $S^* \subset N$, $|S^*| = U$ exists, such that $\gamma_{M^U(N,v+w^{S^*,\zeta})} \geq \gamma_{M^U(N,v)}$ for any $\zeta > 0$. Moreover for* ***large enough*** *$\zeta > 0$, it holds that $\gamma_{m^U(N,v+w^{S^*,\zeta})} \leq \gamma_{m^U(N,v)}$.*

For $U \geq 2$, as ζ increases, the number $\gamma_{M^U(N,v+w^{S^*,\zeta})}$ increases and the vector $M^U(N, v + w^{S^*,\zeta})$ increases in at least one (or more) component(s) and no component decreases, so $\gamma_{M^U(N,v+w^{S^*,\zeta})} > \gamma_{M^U(N,v)}$. For $U > 2$, the number $\gamma_{m^U(N,v+w^{S^*,\zeta})}$ and the components of $m^U(N, v + w^{S^*,\zeta})$ decrease or stay put for large enough $\zeta > 0$, so $\gamma_{m^U(N,v+w^{S^*,\zeta})} \leq \gamma_{m^U(N,v)}$ in that case. For $U = 2$, no ranking between $m^2(N, v + w^{\{j_1,j_2\},\zeta})$ and $m^2(N, v)$ is implied in general, however for **superadditive games** we have $\gamma_{M^U(N,v+w^{S^*,\zeta})} \leq \gamma_{M^U(N,v)}$, too, for sufficiently large ζ.

The functions for the utopia vectors for $(N, v + w^{S^*,\zeta})$ in Lemmas 19 and 20 are clearly continuous in ζ, which in turn implies that the functions for the minimal-rights vectors in the same lemmas, are continuous in ζ as well. Using this insight, we can show that $\lim_{\zeta\to\infty} m^U(N, v + w^{S^*,\zeta})$ is actually attained for $U \geq 1$ at the vectors $\underline{m}^U(N, v + w^{S^*})$ presented.

Lemma 21 *For any* $(N,v) \in HS^{|N|}$ *and any fixed* $U > 0$*, let* $S^* \subset N$*,* $|S^*| = U$*, then* $\lim_{\zeta\to\infty} m^U(N, v + w^{S^*,\zeta}) = \underline{m}^U(N, v + w^{S^*})$*.*

Define $\underline{\zeta}$ as the smallest number such that $\zeta \geq \underline{\zeta}$ implies $m^U(N, v + w^{S^*,\zeta}) = \underline{m}^U(N, v + w^{S^*})$.

Our primary strategy of proof is to show that $\tau^U(N,v)$ and $\tau^U(N, v+w^{S^*,\zeta})$ do not concur for some sufficiently large $\zeta > 0$. This strategy is quite rough but it allows us to show that a great deal of the technicalities involved can be solved with this rather blunt approach. What remains is a patchwork of potential degeneracies[9] of the game at hand. Future research must reveal whether a rather general approach exists dealing with the remaining degeneracies.

Recall that the value $\tau^U(N,v)$ may have three different underlying calculations as presented in Eq. (1), namely

- $\pi^U(N,v) = \overline{\tau}^U(N,v)$ if **(a)** $\gamma_{m^U(N,v)} < v(N) < \gamma_{M^U(N,v)}$ or **(b)** $\gamma_{M^U(N,v)} < v(N) < \gamma_{m^U(N,v)}$.
- $\pi^U(N,v) = m^U(N,v)$ if **(c)** $v(N) \leq \gamma_{m^U(N,v)} \leq \gamma_{M^U(N,v)}$ or **(d)** $\gamma_{M^U(N,v)} \leq \gamma_{m^U(N,v)} \leq v(N)$.
- $\pi^U(N,v) = M^U(N,v)$ if **(e)** $v(N) \leq \gamma_{M^U(N,v)} \leq \gamma_{m^U(N,v)}$ or **(f)** $\gamma_{m^U(N,v)} \leq \gamma_{M^U(N,v)} \leq v(N)$.

We can also classify the resulting cases regarding $\tau^U(N, v + w^{S^*,\zeta})$ for sufficiently large ζ, and label the analogous subcases a' to f'. Since we can make $\gamma_{M^U(N,v+w^{S^*,\zeta})}$ as large as we wish, we can avoid ending up in $(b'), (d'), (e')$ or (f'). So, we have six possible transitions resulting in either (a') or (c') in general. For instance, a transition **from** (b) **into** (a') or (c'), means that from (N,v) satisfying subcase (b) in Eq. (1), we obtain $(N, v + w^{S^*,\zeta})$ satisfying

$$\begin{array}{ll} (a') & \gamma_{\underline{m}^U(N,v+w^{S^*})} < v(N) < \gamma_{M^U(N,v+w^{S^*,\zeta})} \text{ or} \\ (c') & v(N) \leq \gamma_{\underline{m}^U(N,v+w^{S^*})} \leq \gamma_{M^U(N,v+w^{S^*,\zeta})}. \end{array}$$

Next, we present a result implying that if we end up in (a'), we can **generically** obtain a pair of different efficient convex combinations of the approximated minimal-rights and utopia vectors, implying that sensitivity for $U \geq 2$ holds.

Lemma 22 *Let* $(N,v) \in HS^{|N|}$*,* $S^* \subset N$*,* $|S^*| = U \geq 2$*,* $\gamma_{m^U(N,v+w^{S^*,\zeta})} < v(N) < \gamma_{M^U(N,v+w^{S^*,\zeta})}$ *for* $\zeta = \zeta_1, \zeta_2 \geq \underline{\zeta}$*, and let* $\zeta_1 \neq \zeta_2$*, then* $\tau^U(N, v + w^{S^*,\zeta_1}) = \tau^U(N, v + w^{S^*,\zeta_2})$ *if and only if*

$$\begin{aligned} M^U_k(N,v) &= \underline{m}^U_k(N, v + w^{S^*}) \text{ if } k \in N\backslash S^*, \\ v(S^*\backslash\{l\}) - v(S^*\backslash\{k\}) &= \underline{m}^U_k(N, v + w^{S^*}) - \underline{m}^U_l(N, v + w^{S^*}) \quad \text{if } k,l \in S^*. \end{aligned}$$

For $U \geq 2$, we call $(N,v) \in HS^{|N|}$ a', U**-admissible** if at least one $S^* \subset N$ with $|S^*| = U$ exists such that one of the following conditions is violated

$$\begin{aligned} \gamma_{\underline{m}^U(N,v+w^{S^*,\zeta})} &< v(N) < \gamma_{M^U(N,v+w^{S^*,\zeta})} \\ M^U_k(N,v) &= \underline{m}^U_k(N, v + w^{S^*}) \text{ if } k \in N\backslash S^*, \\ v(S^*\backslash\{l\}) - v(S^*\backslash\{k\}) &= \underline{m}^U_k(N, v + w^{S^*}) - \underline{m}^U_l(N, v + w^{S^*}) \text{ if } k,l \in S^*. \end{aligned}$$

[9] Research revealed that some degeneracies cannot exist under additional assumptions on the game (such as superadditivity), or that they can be dealt with using a specialized different strategy of proof. We refrain from going into them, because of space considerations and possible distraction from the main message of this paper.

We call $(N,v) \in HS^{|N|}$ a'**-admissible** if (N,v) is a', U-admissible for all $U \geq 2$.

The other possibly problematic parts are formed by the transitions to (c'). Below, we identify conditions such that sensitivity up to $U \geq 2$ cannot be shown using our main strategy of proof, i.e., we run into the problem that $\tau^U(N,v) = \tau^U(N, v + w^{S^*,\zeta})$ for all (large) $\zeta > 0$. To facilitate the exposure we introduce another notation, first.

We call a pair $x, y \in \mathbb{R}^{|N|}$ **aligned** if $x - y = \beta \cdot (1, ..., 1)$ for some β. It is easy to see that the orthogonal projections of any pair of coaligned vectors onto the efficient hyperplane, coincide. This leads to the following framework to establish whether $\tau^U(N,v)$ and $\tau^U(N, v + w^{S^*,\zeta})$ differ.

Lemma 23 *Let $(N,v) \in HS^{|N|}$, $U \geq 2$, let $w^{S^*,\zeta}$ for some $S^* \subset N$, $|S^*| = U$, $\zeta > \underline{\zeta}$ satisfy $v(N) \leq \gamma_{\underline{m}^U(N,v+w^{S^*})} \leq \gamma_{M^U(N,v+w^{S^*,\zeta})}$. Then $\tau^U(N,v) = \tau^U(N, v + w^{S^*,\zeta})$ if any of the following conditions holds*

$$\underline{m}^U(N, v + w^{S^*}), \overline{\tau}^U(N,v) \text{ are aligned \& } \begin{cases} \gamma_{m^U(N,v)} < v(N) < \gamma_{M^U(N,v)} \text{ or} \\ \gamma_{M^U(N,v)} < v(N) < \gamma_{m^U(N,v)}; \end{cases}$$

$$\underline{m}^U(N, v + w^{S^*}), m^U(N,v) \text{ are aligned \& } \begin{cases} v(N) \leq \gamma_{m^U(N,v)} \leq \gamma_{M^U(N,v)} \text{ or} \\ \gamma_{M^U(N,v)} \leq \gamma_{m^U(N,v)} \leq v(N); \end{cases}$$

$$\underline{m}^U(N, v + w^{S^*}), M^U(N,v) \text{ are aligned \& } \begin{cases} v(N) \leq \gamma_{M^U(N,v)} \leq \gamma_{m^U(N,v)} \text{ or} \\ \gamma_{m^U(N,v)} \leq \gamma_{M^U(N,v)} \leq v(N). \end{cases}$$

We call a game $(N,v) \in HS^{|N|}$ S^*, U**-alignment-degenerate** if for $S^* \subset N$, $|S^*| = U$ any of the conditions in Lemma 23 hold. We call a game $(N,v) \in HS^{|N|}$ **alignment-admissible** if for each $U \geq 2, U \leq |N|$, a set $S^* \subset N$, $|S^*| = U$ exists for which the game is **not** S^*, U**-alignment-degenerate**.

Proposition 24 *Let $(N,v) \in HS^{|N|}$ be a'-admissible and alignment-admissible, let $U \geq 2$ be fixed, then $\tau^U(N,v)$ is sensitive up to cardinality U.*

Value	**E**	**S**	**L**	**N**	**D**	**Sut**	**Insb**	U
η	yes	yes	yes	no	yes	-	yes	0
CIS	yes	yes	yes	no	yes	yes	yes	1
PD	yes	yes	no	yes	yes	yes	no	1
PM	yes	no	no	no	no	no	no	any
BZ	no	yes	yes	yes	yes	yes	-	$\|N\|$
Sh	yes	yes	yes	yes	yes	yes	-	$\|N\|$
ISh	yes	yes	no	no	yes	yes	yes	any
τ	yes	yes	no	yes	yes	yes	-	$\|N\|$
$I\tau$	yes[13]	yes[13]	no[14]	no[14]	yes[16]	yes[17]	yes[24]	any

Table 4. Overview of values and axioms. All abbreviations have been introduced, superscripts refer to corresponding formal results.

6 An extension of the inductive τ-value

Driessen & Tijs [1984] offer some extensions regarding the τ-value of which one seems to be ready-made for the current framework. Instead of the definition of the utopia amounts given in Section 3, Driessen & Tijs [1984] propose

$$r_i(N,v) = \max_{S \subseteq N : i \in S} [v(S) - v(S \backslash \{i\})] \text{ for all } i \in N,$$

and $r(N,v) = \left(r_1(N,v), ..., r_{|N|}(N,v)\right)$. The authors seem inspired by Milnor [1952], who defines the so-called reasonable set

$$RS(N,v) = \left\{x \in \mathbb{R}^{|N|} |\ x_i \le r_i(N,v) \text{ for all } i \in N\right\}.$$

So, this set contains all allocations such that no player receives more than his maximum marginal contribution to any coalition not containing him.

Moreover, the **remainder of player i in coalition $S \subseteq N$ containing him**, with respect to $r(N,v)$ is redefined as follows

$$\widetilde{R}_i^{(N,v)}(S) = v(S) - \sum_{j \in S \backslash \{i\}} r_j(N,v).$$

Hence, $\widetilde{R}_i^{(N,v)}(S)$ can be interpreted as the amount of money remaining from the worth of coalition S, i.e., $v(S)$, if all other players in S are paid their amounts according to $r(N,v)$ instead of $M(N,v)$ first. The **minimal-rights amount** for player $i \in N$ with respect to $r(N,v)$ is then defined as

$$\widetilde{m}_i(N,v) = \max_{S \subseteq N : S \ni i} \widetilde{R}_i^{(N,v)}(S).$$

The **minimal-rights vector** with respect to $r(N,v)$ $\widetilde{m}(N,v)$ is $\widetilde{m}(N,v) = \left(\widetilde{m}_1(N,v), , ..., \widetilde{m}_{|N|}(N,v)\right)$.

Next we need a convenient class of games on which to restrict the application of the value. Analogous to the one employed by Tijs [1981,1987], we take

$$\begin{aligned} \widetilde{Q}^{|N|} &= \{(N,v) \in G |\ \widetilde{m}_i(N,v) \le r_i(N,v) \text{ for all } i \in N, \\ &\quad \text{and } \gamma_{\widetilde{m}(N,v)} \le v(N) \le \gamma_{r(N,v)}\right\}. \end{aligned}$$

This leads to $\widetilde{\tau}(N,v)$ as straightforward modification of the τ-value, for every $(N,v) \in \widetilde{Q}^{|N|}$ given by

$$\widetilde{\tau}(N,v) = \widetilde{m}(N,v) + \frac{v(N) - \gamma_{\widetilde{m}(N,v)}}{\gamma_{r(N,v)} - \gamma_{\widetilde{m}(N,v)}} \left[r(N,v) - \widetilde{m}(N,v)\right].$$

The advantage of replacing $M(N,v)$ by $r(N,v)$ is that each amount from the latter vector is greater than the corresponding amount of the former, hence

$$\gamma_{r(N,v)} \ge \gamma_{M(N,v)}.$$

The disadvantage of this is that it is much harder to compute $r(N,v)$ than $M(N,v)$, moreover the worths of all coalitions needs to be known just to determine $r(N,v)$. So, this extension of the original τ-value is even more problematic in the context of our contextual time-restriction on computation. However, in the following we show that problems become much less stringent in the general framework of the inductive τ-value presented.

We proceed on similar lines as for the inductive τ-value with respect to an approximation of $r(N,v)$. Therefore, for **any strictly positive integer U**, we define $r_i^U(N,v)$ **for each player $i \in N$** in a game (N,v) as

$$r_i^U(N,v) = \max_{S : i \in S; |S| \le U} \left[v(S) - v(S \backslash \{i\})\right].$$

So, we take a player's maximum marginal contribution taken over all sets of cardinality **at most** U that he belongs to. Analogously, we define the **U-restricted remainder of player i in coalition $S \subseteq N$ containing him** with respect to $r^U(N,v)$ by

$$\widetilde{R}_i^{(N,v),U}(S) = v(S) - \sum_{j \in S \setminus \{i\}} r_j^U(N,v).$$

The **U-restricted minimal-rights amount for player** $i \in N$ in $(N,v) \in G$ with respect to $r^U(N,v)$ is given by

$$\begin{aligned} \widetilde{m}_i^U(N,v) &= \max_{S \subseteq N: |S| \leq U, S \ni i} \widetilde{R}_i^{(N,v),U}(S), \text{ and} \\ \widetilde{m}^U(N,v) &= \left(\widetilde{m}_1^U(N,v), ..., \widetilde{m}_{|N|}^U(N,v)\right) \end{aligned}$$

For completeness' sake, we extend the definition to $U = 0$ by a normalization

$$\widetilde{m}_i^0(N,v) = r_i^0(N,v) = v(\emptyset) = 0 \text{ for all } i \in N \text{ in any } (N,v) \in G.$$

Let us denote the set of all convex combinations of $\widetilde{m}^U(N,v)$ and $r^U(N,v)$ for any game (N,v) by $W^U(N,v)$, i.e.,

$$W^U(N,v) = \left\{ w \in \mathbb{R}^{|N|} \mid \exists_{\lambda \in [0,1]} \ y = \lambda \widetilde{m}^U(N,v) + (1-\lambda)\, r^U(N,v) \right\}.$$

For our aims, we need a minimal restriction with respect to the set

$$\widehat{W}(N,v) = \left\{ W^U(N,v) \subset \mathbb{R}^{|N|} \mid U \in \{1, ..., |N|\} \right\}.$$

Definition 25 *The game $(N,v) \in G^{|N|}$ is hyperplane-singular with respect to $r^U(N,v)$ if $\widetilde{m}^U(N,v) = r^U(N,v)$ whenever $\gamma_{\widetilde{m}^U(N,v)} = \gamma_{r^U(N,v)}$.*

In the generic case, the vectors $\widetilde{m}^U(N,v), r^U(N,v)$ are on different hyperplanes. However if they are not, they must coincide, i.e., $Y^U(N,v)$ is a singleton. We denote the class of $|N|$-person games satisfying Definition 25 by $\widetilde{HS}^{|N|}$, i.e.,

$$\widetilde{HS}^{|N|} = \left\{ (N,v) \in G^{|N|} | \ \widetilde{m}^U(N,v) = r^U(N,v) \text{ iff } \gamma_{\widetilde{m}^U(N,v)} = \gamma_{r^U(N,v)} \right\}.$$

Definition 26 *For $(N,v) \in \widetilde{HS}^{|N|}$, **the U-restricted $\widetilde{\tau}$-value** is given by*

$$\widetilde{\tau}^U(N,v) = \operatorname*{arg\,min}_{x \in H(v(N))} d_2(x, W^U(N,v)).$$

7 Conclusion and discussion

Cooperative transportation games ($CTGs$) deal with groups of agents forming a collective to increase profits, to reduce costs, wasted time, materials and energy. Benefits of cooperation may be to the participating entrepreneurs, but also to society at large, in reduced externalities from transportation-related activities. Examples of such externalities are congestion, and greenhouse gasses or particulate matter emitted. An interesting challenge then is to induce cooperation of entrepreneurs serving private and public goals.

A significant aspect is how to divide the private benefits of cooperation, as it is unlikely that an entrepreneur will join if he is worse off by cooperating than before. For *CTGs*, any coalition (a subset of the player set) can collectively make at least the aggregated amount that its members can make on their own (cf., e.g., Schulte *et al.* [2018]). So, if cooperation occurs, there is something to divide at least, and the gains culminate if all cooperate.

An important theme in cooperative game theory is the division of the benefits of cooperation. However, for real-world cooperative transportation problems (*CTPs*), determining a game-theoretical solution for dividing the gains of cooperation may be quite involved. Many well-known point-valued solutions, e.g., the Shapley value, the nucleolus or the τ-value, but also set-valued solutions such as the core, are increasingly difficult to compute as the number of agents grows. This is already a considerable problem with known primitives, i.e., the worths of all coalitions in a cooperative game. To make matters worse for *CTGs*, the very calculation of each worth of a coalition is known to be *NP*-hard as well, and there are exponentially many to be calculated. This triple curse of dimensionality forms an impediment to attribute the gains of cooperation according to any of these solutions. Existing methods based on less information or requiring fewer computations are widely regarded as lacking important aspects of fairness (cf., e.g., Cruijssen *et al.* [2007]).

We consider the situation that an agent is given the task of central collective planning (cf., Gansterer & Hartl [2018[) to propose a distribution of the gains of cooperation among all agents. This agent operates under the following explicit assumptions: a finite upper bound on the computation time exists, but it is always possible to compute the worth of the grand coalition; computations on the worths of the remaining coalitions might end prematurely; it is unknown at the start of the computations which information the computations will end.

We propose a method of obtaining an approximation of the τ-value based on inductive approximations of the minimal-rights and the utopia vectors depending on some cardinality. The final cardinality is determined endogenously, i.e., computations terminate due to the contextual time-constraints having finished all computations for the worths of all coalitions with that cardinality, but not for coalitions with cardinality one higher. The approximation of the utopia vector consists of an amount for each player equal to his largest marginal contribution determined over all coalitions of a certain cardinality, instead of his marginal contribution to the grand coalition. The approximation of the remainder to a specific player in a certain coalition is the amount remaining after all other players have received their approximated utopia amounts. Then, the approximated minimal-rights vector consists for each player of the amount equal to the largest of all remainders determined over all coalitions with at most the cardinality at hand, instead of over the coalitions up to the cardinality of the grand coalition. This is close to the intentions expressed in Tijs [1981,1987].

The second parallell to the original work of Tijs [1981,1987] is the manner in which the approximation of the τ-value is determined once the approximated minimal-rights and utopia vectors have been established. We take the efficient allocation closest, measured in the Euclidean distance, to the set of all convex combinations of the approximated minimal-rights and utopia vectors. For the τ-value, precautions are taken that the set mentioned intersects with the efficient hyperplane. The unique efficient element in the set of convex combination of the (unrestricted) minimal-rights and the utopia vectors is then the τ-value, i.e., the τ-value is then that sets element closest to the efficient hyperplane. However, in the framework considered, we had to deal with the possibility that the set of convex combinations of the approximated minimal-rights and utopica vectors does **not** intersect with the efficient hyperplane. Taking then the efficient allocation closest to this set is the appropriate parallell in our opinion. To guarantee that there is indeed a unique closest allocation to the efficient hyperplane, we introduced a condition that if the set mentioned is a subset of some hyperplane, then the set must be a singleton.

In order to justify and select among the many possible solutions, cooperative game theory has come up with criteria. A well-known set is efficiency, symmetry, individual rationality and desirability. Efficiency means that 'no money is left on the table', and symmetry means that 'equals are treated equally', informally speaking. Efficiency is hard to argue against, since if some money were left on the table any or all could be made better off without anyone being worse off. Symmetry is widely regarded as an axiom covering an aspect of fairness. Individual rationality formalizes 'by cooperation each gets at least the amount (s)he can get on his (her) own.' Desirability means that 'anyone contributing at least the amount of another to the worth of any coalition by his joining, should receive at least the same amount the other gets.' For *CTGs*, the inductive τ-value satisfies these four axioms.

Clearly, these four axioms deal with some aspects of *fairness,* other such aspects, incorporated by the inductive τ-value are the pair introduced in Joosten & Lalla Ruiz [2019]: sensitivity up to and insensitivity beyond cardinality U (for all games and for general U). As to the latter two, the inductive τ-value uses the relevant information that can be gathered under the restrictions specified and bases the allocation of the gains from cooperation on this information exclusively. Hence, the allocation does not depend on information that cannot be obtained. We see this combination as highly relevant fairness properties in the present context of contextual time-constraints on computating necessary.

We also present the inductive $\widetilde{\tau}$-value, an extension of our own inductive τ-value inspired by an extension of the τ-value by Driessen & Tijs [1984]. Whereas the original τ-value uses the marginal contribution of each player to the grand coalition in order to establish the utopia vector, the **unnamed** extension of Driessen & Tijs [1984] uses the maximum marginal contribution of each player taken over all sets not just the grand coalition, in order to establish an extension of the utopia vector. The extension of Driessen & Tijs [1984] is conceptually quite close to the original, but involves considerably more computations to be completed. For the inductive $\widetilde{\tau}$-value, our extension of the inductive τ-value in the spirit of Driessen & Tijs [1984], the closeness in concepts is definitely there, too, but precisely the objection does not hold.

8 Appendix

Proof of Lemma 7 Let $k \in N$, then

$$\begin{aligned} & m_k^U(N,v) \\ = & \max_{S\subseteq N:|S|\leq U, S\ni k} R_k^{(N,v),U}(S) = \max_{S\subseteq N:|S|\leq U, S\ni k} [v(S) - \sum_{j\in S\setminus\{k\}} M_j^U(N,v)] \end{aligned}$$

$$
\begin{aligned}
&= \max_{S\subseteq N:|S|\leq U, S\ni k}[v(S) - b^U(S) + M_k^U(N,v)] \\
&= M_k^U(N,v) + \max_{S\subseteq N:|S|\leq U, S\ni k}\left\{v(S) - b^U(S)\right\} \\
&= M_k^U(N,v) + \max_{S\subseteq N:|S|\leq U, S\ni k}\left\{-g^U(S)\right\} \\
&= M_k^U(N,v) - \min_{S\subseteq N:|S|\leq U, S\ni k} g^U(S) = M_k^U(N,v) - \mu_k^U.
\end{aligned}
$$

Since $k \in N$ was arbitrarily chosen, this completes the proof. ■

Proof of Lemma 8. First, $(x-y) = (x - x - \beta\,(1,...,1)) = -\beta\,(1,...,1)$. So,

$$
\begin{aligned}
(x-y)\cdot(y-z) &= -\beta\,(1,...,1)\cdot(y-z) = -\beta\,(1,...,1)\cdot y + \beta\,(1,...,1)\cdot z \\
&= -\beta\sum_{i=1}^{|N|} y_i + \beta\sum_{i=1}^{|N|} z_i = -\beta v(N) + \beta v(N) = 0.
\end{aligned}
$$

This proves the first part of the statement of the lemma.
As, vectors $(x-y)$ and $(y-z)$ are perpendicular, by the Pythagorean theorem

$$
d_2^2(x,z) = d_2^2(x,y) + d_2^2(y,z),
$$

which implies that $d_2^2(x,y) \leq d_2^2(x,z)$. Because $z \in H(v(N))$ was chosen arbitrarily, we may conclude $y = \underset{x\in H(v(N))}{\arg\min}\ d_2(x, Y^U(N,v))$. Clearly,

$$
d_2^2(x,y) = ||x-y||^2 = ||\,(x - x - \beta\,(1,...,1))\,||^2 = \beta^2|N|.
$$

So, $d_2(x,y) = \sqrt{\beta^2|N|} = |\beta|\cdot\sqrt{|N|}$. ■

Proof of Lemma 9. Recall that for all $(N,v) \in HS^{|N|}$

$$
Y^0(N,v) = \{(0,...,0)\} \text{ and } Y^1(N,v) = \{(v(\{1\}),...,v(\{|N|\}))\}.
$$

So, $\tau^0\,(N,v)$ is the orthogonal projection of $\pi^0(N,v) = (0,...,0)$ unto $H(v(N))$, i.e.,

$$
\tau^0\,(N,v) = (0,...,0) + \frac{v(N)-0}{|N|}\,(1,...,1) = \left(\frac{v(N)}{|N|},...,\frac{v(N)}{|N|}\right) = \eta(N,v).
$$

Similarly, $\tau^1(N,v)$ is the orthogonal projection of $(v(\{1\}),...,v(\{|N|\}))$ unto the same hyperplane, i.e.,

$$
\begin{aligned}
&\tau^1(N,v) \\
&= (v(\{1\}),...,v(\{|N|\})) + \frac{v(N)-\sum_{i=1}^{|N|} v(\{i\})}{|N|}\,(1,...,1) \\
&= \left(v(\{1\}) + \frac{v(N)-\sum_{i=1}^{|N|} v(\{i\})}{|N|},...,v(\{|N|\}) + \frac{v(N)-\sum_{i=1}^{|N|} v(\{i\})}{|N|}\right) \\
&= CIS(N,v).
\end{aligned}
$$

■

Proof of Lemma 10. Let $x \in Y^U(N,v)$ then $x = \lambda m^U(N,v) + (1-\lambda)M^U(N,v)$, hence

$$\begin{array}{ll} \lambda m^U(N,v) + (1-\lambda)M^U(N,v) + \beta\,(1,...,1) \in H(v(N)) & \Rightarrow \\ \lambda\gamma_{m^U(N,v)} + (1-\lambda)\gamma_{M^U(N,v)} + \beta\sum_{i=1}^{|N|} 1 = v(N) & \Rightarrow \\ \beta = \frac{v(N)-\lambda\gamma_{m^U(N,v)}-(1-\lambda)\gamma_{M^U(N,v)}}{|N|}. & \end{array}$$

As β is unique we have proven the first part of the lemma.
Now, let $\beta \in B^U(N,v)$ and assume $x, x' \in Y^U(N,v)$ exist satisfying

$$x + \beta\,(1,...,1)\,,\ x' + \beta\,(1,...,1) \in H(v(N)).$$

This implies

$$\begin{aligned} 0 &= v(N) - v(N) = \sum_{i=1}^{|N|} x_i + |N|\beta - \sum_{i=1}^{|N|} x'_i - |N|\beta \\ &= \lambda_x\gamma_{m^U(N,v)} + (1-\lambda_x)\gamma_{M^U(N,v)} - \lambda_{x'}\gamma_{m^U(N,v)} - (1-\lambda_{x'})\gamma_{M^U(N,v)} \\ &= (\lambda_x - \lambda_{x'}) \cdot \Big(\gamma_{m^U(N,v)} - \gamma_{M^U(N,v)}\Big)\,. \end{aligned}$$

Then $(\lambda_x - \lambda_{x'}) = 0$, or $\Big(\gamma_{m^U(N,v)} - \gamma_{M^U(N,v)}\Big) = 0$. Both imply $m^U(N,v) = M^U(N,v)$ because $(N,v) \in HS^{|N|}$. ■

Proof of Proposition 13. Every $\tau^U(N,v) \in H(v(N))$, $U = 0,...,|N|$ by design. To prove the second part of the statement, let $(N,v) \in HS^{|N|}$ and take $U = 0,...,|N|$ arbitrarily. For $U = 0$ all players get the same amount which includes any pair of symmetric players. So, $\tau^0(N,v)$ is symmetric. For $U = 1$, let i, j be symmetric players then

$$\begin{aligned} & \tau_i^1(N,v) \\ = \ & v(\{i\}) + \frac{v(N) - \sum_{k=1}^{|N|} v(\{k\})}{|N|} = v(\{i\}) - v(\emptyset) + \frac{v(N) - \sum_{k=1}^{|N|} v(\{k\})}{|N|} \\ = \ & v(\{j\}) - v(\emptyset) + \frac{v(N) - \sum_{k=1}^{|N|} v(\{k\})}{|N|} = v(\{j\}) + \frac{v(N) - \sum_{k=1}^{|N|} v(\{k\})}{|N|} \\ = \ & \tau_j^1(N,v). \end{aligned}$$

Clearly, $v(S) - v(S\backslash\{i\}) = v(S) - v(S\backslash\{j\})$ for any $S \ni i,j$, which implies $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for any S with $S \cap \{i,j\} = \emptyset$. So, we concentrate on all coalitions S with cardinality U, i.e., $|S| = U$, then

$$\begin{aligned} & M_i^U(N,v) \\ = \ & \max_{S \ni i: |S|=U} v(S) - v(S\backslash\{i\}) \\ = \ & \max\left\{ \max_{S \ni i,j: |S|=U} v(S) - v(S\backslash\{i\}), \max_{j \notin S \ni i, S: |S|=U} v(S) - v(S\backslash\{i\}) \right\} \\ = \ & \max\left\{ \max_{S \ni i,j: |S|=U} v(S) - v(S\backslash\{j\}), \max_{i,j \notin S, S: |S|=U-1} v(S \cup \{i\}) - v(S) \right\} \end{aligned}$$

$$\begin{aligned} &= \max\left\{\max_{S\ni i,j:|S|=U} v(S)-v(S\backslash\{j\}), \max_{i,j\notin S,S:|S|=U-1} v(S\cup\{j\})-v(S)\right\} \\ &= \max_{S\ni j:|S|=U} v(S)-v(S\backslash\{j\}) = M_j^U(N,v). \quad \blacksquare \end{aligned}$$

Proof of Proposition 14. Let $(N,v),(N,v),(N,\lambda v+\mu w)\in HS^{|N|}\cap Q^{|N|}$, then $\tau^{|N|}(N,v)=\tau(N,v)$, $\tau^{|N|}(N,w)=\tau(N,w)$ and $\tau^{|N|}(N,w)=\tau(N,\lambda v+\mu w)$. Since the τ-value does not satisfy additivity in general, a pair game exists for which

$$\begin{aligned} \tau^{|N|}(N,\lambda v+\mu w) &= \tau(N,\lambda v+\mu w) \\ &\neq \lambda\tau(N,v)+\mu\tau(N,w)=\lambda\tau^{|N|}(N,v)+\mu\tau^{|N|}(N,w). \end{aligned}$$

This proves the first part of the proposition. To prove the second, note that s for $j\in\mathcal{N}(N,v)$:

$$\tau_j^0(N,v)=\frac{v(N)}{|N|} \text{ and } \tau_j^1(N,v)=\frac{v(N)-\sum_{k\in N}v(\{k\})}{|N|}.$$

So, the inductive τ-value need not satisfy the null-player property.
S-equivalence implies that $\psi(N,kv+d)=k\psi(N,v)+d$ for all $k\in\mathbb{R}$ and $d\in\mathbb{R}^{|N|}$ where the $|N|$-dimensional vector d defines an $|N|$-person game d which is additive, i.e., $d(S)=\sum_{k\in S}d_k$ for all $S\subseteq N$. In that respect, we easily obtain $\tau^0(N,0\cdot v+d)=\tau^0(N,d)=\frac{\gamma_d}{|N|}\neq 0\cdot\psi(N,v)+d=d$. This proves the third part. ■

Proof of Proposition 15. Let $(N,v)\in\pi-HS^{|N|}$ and $U\in\{1,...,|N|\}$. Clearly, by definition we have for arbitrary $j\in N$,

$$m_j^U(N,v)=\max_{S\subseteq N:|S|\leq U,S\ni j} R_j^{(N,v),U}(S)\geq v(\{j\})-v(\emptyset)=v(\{j\}).$$

By superadditivity we have

$$v(S)\geq v(S\backslash\{j\})+v(\{j\})\Longrightarrow v(S)-v(S\backslash\{j\})\geq v(\{j\}) \text{ for all } S\subseteq N,\ S\ni j,$$

and this in turn implies

$$M_j^U(N,v)=\max_{S:j\in S,|S|=U}[v(S)-v(S\backslash\{j\})]\geq v(\{j\}).$$

With respect to $\pi^U(N,v)$, Eq. (1) implies that if $\pi^U(N,v)$ is the efficient convex combination of $m^U(N,v)$ and $M^U(N,v)$, then $\gamma_{\pi^U(N,v)}=v(N)$ and some $\lambda\in[0,1]$ exists such that

$$\begin{aligned} \pi^U(N,v) &= \lambda m^U(N,v)+(1-\lambda)M^U(N,v)\Longrightarrow \\ \pi_j^U(N,v) &= \lambda m_j^U(N,v)+(1-\lambda)M_j^U(N,v)=v(\{j\}). \end{aligned}$$

So, $\pi^U(N,v)=\tau^U(N,v)$ and $\tau_j^U(N,v)\geq v(\{j\})$.
Otherwise, $\pi^U(N,v)\in\left\{m^U(N,v),M^U(N,v)\right\}$ and

$$\begin{aligned} \tau^U(N,v) &= \pi^U(N,v)+\frac{v(N)-\gamma_{\pi^U(N,v)}}{|N|}(1,...,1)\Longrightarrow \\ \tau_j^U(N,v) &= \pi_j^U(N,v)+v(N)-\gamma_{\pi^U(N,v)} \\ &\geq v(\{j\})+v(N)-\gamma_{\pi^U(N,v)}\geq v(\{j\}). \end{aligned}$$

This final inequality follows from the fact that $\max_{1\leq U'\leq|N|}\gamma_{\pi^{U'}(N,v)} \leq v(N)$. This completes the proof. ■

Proof of Proposition 16. Take $(N,v) \in HS^{|N|}$ and $U = 0,1,...,|N|$, let $v(S) - v(S\backslash\{i\}) \leq v(S) - v(S\backslash\{j\}\}$ for all $S \subset N$ containing i,j. Choose S^* such that $|S^*| = U$ and

$$v(S^*) - v(S^*\backslash\{i\}) = \max_{S:i\in S,|S|=U}\left[v(S) - v(S\backslash\{i\})\right].$$

Clearly, if $j \in S^*$, we have $v(S^*)-v(S^*\backslash\{j\}\} \geq v(S^*)-v(S^*\backslash\{i\})$, which implies

$$\begin{aligned} M_j^U(N,v) &= \max_{S:i\in S,|S|=U}[v(S) - v(S\backslash\{j\})] \geq v(S^*) - v(S^*\backslash\{j\}\} \\ &\geq v(S^*) - v(S^*\backslash\{i\}) = \max_{S:i\in S,|S|=U}[v(S) - v(S\backslash\{i\})] = M_i^U(N,v). \end{aligned}$$

Otherwise, $j \notin S^*$ and then

$$\begin{aligned} & v(S^*\cup\{j\}) - v(S^*) + v(S^*) - v(S^*\backslash\{i\}) \\ = \;& v(S^*\cup\{j\}) - v(S^*\cup\{j\}\backslash\{i\}) + v(S^*\cup\{j\}\backslash\{i\}) - v(S^*\backslash\{i\}). \end{aligned}$$

For clarity's sake we define $a = v(S^*) - v(S^*\backslash\{i\})$, $b = v(S^*\cup\{j\}) - v(S^*)$, $c = v(S^*\cup\{j\}\backslash\{i\}) - v(S^*\backslash\{i\})$ and $d = v(S^*\cup\{j\}) - v(S^*\cup\{j\}\backslash\{i\})$. Then, we claim

$$\begin{aligned} a+b &= c+d, \\ b \geq d,\;\; a &\geq d, \end{aligned}$$

because $v(S^*\cup\{j\})-v(S^*) \leq v(S^*\cup\{j\})-v(S^*\cup\{j\}\backslash\{i\})$ by our assumption, and we took S^* such that $v(S^*)-v(S^*\backslash\{i\}) = \max_{S:i\in S,|S|=U}\left[v(S) - v(S\backslash\{i\})\right]$. Hence,

$$a+b-d = c \Longrightarrow a \leq c.$$

So,

$$\begin{aligned} M_j^U(N,v) &= \max_{S:i\in S,|S|=U}[v(S) - v(S\backslash\{j\})] \geq v(S^*\cup\{j\}\backslash\{i\}) - v(S^*\backslash\{i\}) \\ &\geq v(S^*) - v(S^*\backslash\{i\}) = \max_{S:i\in S,|S|=U}[v(S) - v(S\backslash\{i\})] = M_i^U(N,v). \end{aligned}$$

Hence, $M_j^U(N,v) \geq M_i^U(N,v)$.

Next, take S^{**} such that $|S^{**}| = U$ and

$$v(S^{**}) - \sum_{k\in S^{**}\backslash\{i\}} M_k^U(N,v) = \max_{S:i\in S,|S|=U}\left[v(S) - \sum_{k\in S\backslash\{i\}} M_k^U(N,v)\right].$$

If $j \in S^{**}$, then

$$\begin{aligned} v(S^{**}) - \sum_{k\in S^{**}\backslash\{j\}} M_k^U(N,v) &= v(S^{**}) - \sum_{k\in S^{**}\backslash\{i,j\}} M_k^U(N,v) - M_i^U(N,v) \\ &\geq v(S^{**}) - \sum_{k\in S^{**}\backslash\{i,j\}} M_k^U(N,v) - M_j^U(N,v) \\ &= m_i^U(N,v). \end{aligned}$$

Clearly, $m_j^U(N,v) \geq m_i^U(N,v)$.

If $j \notin S^{**}$ then

$$\begin{aligned} m_j^U(N,v) &= \max_{S:j\in S,|S|=U} \left[v(S) - \sum_{k\in S\backslash\{j\}} M_k^U(N,v) \right] \\ &\geq v(S^{**}\backslash\{i\}\cup\{j\}) - \sum_{k\in S^{**}\backslash\{i\}} M_k^U(N,v) \\ &\geq v(S^{**}) - \sum_{k\in S^{**}\backslash\{i\}} M_k^U(N,v) = m_i^U(N,v). \end{aligned}$$

The second greater than sign holds because we can apply the same reasoning to S^{**} as applied above for S^* which implies $v(S^{**}\backslash\{i\}\cup\{j\}) \geq v(S^{**})$.

If $\pi^U(N,v)$ is the efficient convex combination of $m^U(N,v)$ and $M^U(N,v)$, then $\lambda \in [0,1]$ exists such that

$$\begin{aligned} \tau_j^U(N,v) &= \pi_j^U(N,v) = \lambda m_j^U(N,v) + (1-\lambda)\, M_j^U(N,v) \\ &\geq \lambda m_i^U(N,v) + (1-\lambda)\, M_i^U(N,v) = \pi_i^U(N,v) = \tau_i^U(N,v). \end{aligned}$$

Otherwise, $\pi^U(N,v) \in \left\{ m^U(N,v), M^U(N,v) \right\}$ and some β exists such that

$$\begin{aligned} \tau_j^U(N,v) - \tau_i^U(N,v) &= \pi_j^U(N,v) + \beta - \left(\pi_i^U(N,v) + \beta\right) \\ &= \pi_j^U(N,v) - \pi_i^U(N,v) \geq 0. \end{aligned}$$

This completes the proof. ∎

Proof of Proposition 17. The amounts attributed by the inductive τ-value for $U = 0,1,2,...,|N|$ depend partly on $\pi^U(N,v)$ as $\tau^U(N,v) = \pi^U(N,v) + \beta(1,...1)$. Any difference in the utilities attributed by $\tau^U(N,v)$ to players i and j is equal to the difference in the components of $\pi^U(N,v)$, since

$$\tau_i^U(N,v) - \tau_j^U(N,v) = \pi_i^U(N,v) + \beta - \pi_j^U(N,v) - \beta = \pi_i^U(N,v) - \pi_j^U(N,v).$$

Let $(N,w) \in HS^{|N|}$ such that $(v+w)(S) = v(S)$ for all $|S| \leq U$ and $|S| = N$, then computation times for v and $v+w$ are identical, i.e., they fail at $U+1$, moreover $\pi^U(N,v+w) = \pi^U(N,v)$. So,

$$\begin{aligned} & \tau_i^U(N,v+w) - \tau_j^U(N,v+w) \\ =\ & \pi_i^U(N,v+w) + \widetilde{\beta} - \pi_j^U(N,v+w) - \widetilde{\beta} \\ =\ & \pi_i^U(N,v+w) - \pi_j^U(N,v+w) = \pi_i^U(N,v) - \pi_j^U(N,v). \end{aligned}$$

This in turn implies $\widetilde{\beta} = \beta$, which completes the proof. ∎

Proof of Lemma 18. Take $j \in N$, then for $k \in N\backslash\{j\}$ we have

$$\begin{aligned} \tau_k^1(N, v + w^{\{j\},\zeta}) &= \left(v + w^{\{j\},\zeta}\right)(\{k\}) + \frac{v(N) - \sum_{i=1}^{|N|} \left(v + w^{\{j\},\zeta}\right)(\{i\})}{|N|} \\ &= v(\{k\}) + \frac{v(N) - \sum_{i=1}^{|N|} v(\{i\}) - \zeta}{|N|} \\ &= v(\{k\}) - \frac{\zeta}{|N|} + \frac{v(N) - \sum_{i=1}^{|N|} v(\{i\})}{|N|} < \tau_k^1(N,v). \end{aligned}$$

Hence by efficiency $\tau_j^1(N, v + w^{\{j\},\zeta}) > \tau_j^1(N, v)$. This proves the statement of the lemma. ∎

Proof of Lemma 19 Let $j_1, j_2 \in N,\ \ j_1 \neq j_2$ and let $\{j_1, j_2\}$ satisfy

$$M_{j_1}^2(N, v) = v(\{j_1, j_2\}) - v(\{j_2\}).$$

Define

$$\varepsilon_{j_2} = v(\{j_1, j_2\}) - v(\{j_1\} - M_{j_2}^U(N, v).$$

Then, for $\zeta > 0$ sufficiently large

$$\begin{aligned}
& M_{j_1}^2(N, v + w^{\{j_1,j_2\},\zeta}) \\
= \ & \left(v + w^{\{j_1,j_2\},\zeta}\right)(\{j_1, j_2\}) - \left(v + w^{\{j_1,j_2\},\zeta}\right)(\{j_2\}) \\
= \ & v(\{j_1, j_2\}) + \zeta - v(\{j_2\}) = M_{j_1}^2(N, v) + \zeta, \\
& M_{j_2}^2(N, v + w^{\{j_1,j_2\},\zeta}) \\
= \ & \left(v + w^{\{j_1,j_2\},\zeta}\right)(\{j_1, j_2\}) - \left(v + w^{\{j_1,j_2\},\zeta}\right)(\{j_1\}) \\
= \ & v(\{j_1, j_2\}) + \zeta - v(\{j_1\}) = M_{j_2}^U(N, v) + \max\{\zeta + \varepsilon_{j_2}, 0\}, \\
& M_k^2(N, v + w^{\{j_1,j_2\},\zeta}) \\
= \ & \max_{S:|S|=2, k\in S} \left[\left(v + w^{\{j_1,j_2\},\zeta}\right)(S) - \left(v + w^{\{j_1,j_2\},\zeta}\right)(S\backslash\{k\})\right] \\
= \ & \max_{S:|S|=2, k\in S} [v(S) - v(S\backslash\{k\})] = M_k^2(N, v) \text{ for } k \in N\backslash\{j_1, j_2\}.
\end{aligned}$$

Next, we find minimal rights vector $m^2(N, v + w^{\{j_1,j_2\},\zeta})$ for $\zeta > 0$ sufficiently large. Since

$$\begin{aligned}
& \left(v + w^{\{j_1,j_2\},\zeta}\right)(\{j_1, j_2\}) - M_{j_2}^2(N, v + w^{\{j_1,j_2\},\zeta}) \\
= \ & v(\{j_1, j_2\}) + \zeta - [v(\{j_1, j_2\}) + \zeta - v(\{j_1\})] = v(\{j_1\}), \\
& \left(v + w^{\{j_1,j_2\},\zeta}\right)(\{j_1, j_2\}) - M_{j_1}^2(N, v + w^{\{j_1,j_2\},\zeta}) \\
= \ & v(\{j_1, j_2\}) + \zeta - [v(\{j_1, j_2\}) + \zeta - v(\{j_2\})] = v(\{j_2\}),
\end{aligned}$$

we have

$$\begin{aligned}
& m_{j_1}^2(N, v + w^{\{j_1,j_2\},\zeta}) = \max\{v(\{j_1\}) - v(\emptyset), v(\{j_1\})\} = v(\{j_1\}) \\
& m_{j_2}^2(N, v + w^{\{j_1,j_2\},\zeta}) = \max\{v(\{j_2\}) - v(\emptyset), v(\{j_2\})\} = v(\{j_2\}) \\
& m_k^2(N, v + w^{\{j_1,j_2\},\zeta}) \\
= \ & \max_{S:S=\{k,k^*\}\subset N} \left[\left(v + w^{\{j_1,j_2\},\zeta}\right)(\{k, k^*\}) - M_{k^*}^2\left(N, v + w^{\{j_1,j_2\},\zeta}\right)\right] \\
\leq \ & \max_{S:S=\{k,k^*\}\subset N} \left[v(\{k, k^*\}) - M_{k^*}^2(N, v)\right] = m_k^2(N, v) \text{ for } k \in N\backslash\{j_1, j_2\}.
\end{aligned}$$ ∎

Proof of Lemma 20. Take $j \in N$ arbitrarily and $S^* \ni j$ with $|S^*| = U \geq 2$ such that

$$v(S^*) - v(S^*\backslash\{j\}) = \max_{S:|S|=U, j\in S} v(S) - v(S\backslash\{j\}) = M_j^U(N, v).$$

Then, since $v+w^{S^*,\zeta}$ differs only from v for the set S^*, we have

$$\begin{aligned} M_j^U(N, v+w^{S^*,\zeta}) &= (v+w^{S^*,\zeta})(S^*) - (v+w^{S^*,\zeta})(S^*\backslash\{j\}) \\ &= v(S^*) + \zeta - v(S^*\backslash\{j\}) = M_j^U(N,v) + \zeta. \end{aligned}$$

For all $k \in N\backslash S^*$ we have

$$\begin{aligned} M_k^U(N, v+w^{S^*,\zeta}) &= \max_{S:|S|=U, k\in S} \left[(v+w^{S^*,\zeta})(S) - (v+w^{S^*,\zeta})(S\backslash\{k\})\right] \\ &= \max_{S:|S|=U, k\in S} [v(S) - v(S\backslash\{k\})] = M_k^U(N,v). \end{aligned}$$

Finally, for all $k \in S^*\backslash\{j\}$ let $\varepsilon_k = v(S^*) - v(S^*\backslash\{k\}) - M_k^U(N,v)$, and observe that $\varepsilon_k \leq 0$, then we have

$$\begin{aligned} & \left(v+w^{S^*,\zeta}\right)(S^*) - \left(v+w^{S^*,\zeta}\right)(S^*\backslash\{k\}) \\ = \;& v(S^*) + \zeta - v(S^*\backslash\{k\}) \\ = \;& M_k^U(N,v) + \zeta + \varepsilon_k \text{ hence} \\ & M_k^U(N, v+w^{S^*,\zeta}) \\ = \;& \max[M_k^U(N,v) + \zeta + \varepsilon_k, M_k^U(N,v)] \\ = \;& M_k^U(N,v) + \max[\zeta + \varepsilon_k, 0] \geq M_k^U(N,v). \end{aligned}$$

So, we have proven $\gamma_{M^U(N,v+w^{S^*,\zeta})} > \gamma_{M^U(N,v)}$. For the other part, we focus on $m^U(N,v)$. For all $k \in N\backslash S^*$ we have for any $S \cap S^* \ni k$

$$\begin{aligned} & (v+w^{S^*,\zeta})(S) - \sum_{i\in S\backslash\{k\}} M_i^U(N, v+w^{S^*,\zeta}) \\ = \;& v(S) - \sum_{i\in(S\cap S^*)\backslash\{k\}} M_i^U(N, v+w^{S^*,\zeta}) - \sum_{i\in(S\backslash S^*)\backslash\{k\}} M_i^U(N, v+w^{S^*,\zeta}) \\ = \;& v(S) - \sum_{i\in(S\backslash S^*)\backslash\{k\}} M_i^U(N,v) - \sum_{i\in(S\cap S^*)\backslash\{k\}} M_i^U(N, v+w^{S^*,\zeta}) \\ \leq \;& v(S) - \sum_{i\in(S\backslash S^*)\backslash\{k\}} M_i^U(N,v) - \sum_{i\in(S\cap S^*)\backslash\{k\}} M_i^U(N,v) \\ = \;& v(S) - \sum_{i\in S\backslash\{k\}} M_i^U(N,v). \end{aligned}$$

For $k \in N\backslash S^*$, $S \cap S^* = \emptyset$, and $S \ni k$ we have

$$(v+w^{S^*,\zeta})(S) - \sum_{i\in S\backslash\{k\}} M_i^U(N, v+w^{S^*,\zeta}) = v(S) - \sum_{i\in S\backslash\{k\}} M_i^U(N,v).$$

Combining this, $k \in N\backslash S^*$ implies that

$$\begin{aligned} & m_k^U(N, v+w^{S^*,\zeta}) \\ = \;& \max_{S:|S|=U, k\in S} \left[(v+w^{S^*,\zeta})(S) - \sum_{i\in S\backslash\{k\}} M_i^U(N, v+w^{S^*,\zeta})\right] \\ \leq \;& \max_{S:|S|=U, k\in S} \left[v(S) - \sum_{i\in S\backslash\{k\}} M_i^U(N,v))\right] = m_k^U(N,v). \end{aligned}$$

Consider $k \in S^*$, then we have

$$\begin{aligned}
& m_k^U(N, v + w^{S^*,\zeta}) \\
= & \max_{S:|S|=U, k\in S} \left[(v + w^{S^*,\zeta})(S) - \sum_{i\in S\backslash\{k\}} M_i^U(N, v + w^{S^*,\zeta}) \right] \\
= & v(S^*) + \zeta - \sum_{i\in S\backslash\{k\}} M_i^U(N, v + w^{S^*,\zeta}) \\
= & v(S^*) + \zeta - \sum_{i\in S\backslash\{k\}} \left[M_i^U(N, v) + \max[\zeta + \varepsilon_i, 0] \right] \\
= & v(S^*) - \sum_{i\in S\backslash\{k\}} M_i^U(N, v) + \zeta - \sum_{i\in S\backslash\{k\}} \max[\zeta + \varepsilon_i, 0] \\
\leq & \max_{S:|S|=U, k\in S} \left[v(S) - \sum_{i\in S\backslash\{k\}} M_i^U(N, v)) \right] + \zeta - \sum_{i\in S\backslash\{k\}} \max[\zeta + \varepsilon_i, 0] \\
= & m_k^U(N, v) + \zeta - \sum_{i\in S\backslash\{k\}} \max[\zeta + \varepsilon_i, 0].
\end{aligned}$$

For $\zeta > -\min_{i\in S^*} \varepsilon_i$ we have

$$\begin{aligned}
m_k^U(N, v + w^{S^*,\zeta}) & \leq m_k^U(N, v) + \zeta - \sum_{i\in S\backslash\{k\}} \max[\zeta + \varepsilon_i, 0] \\
& \leq m_k^U(N, v) + (2 - |S^*|) \cdot \zeta - \sum_{i\in S\backslash\{k\}} \varepsilon_i \leq m_k^U(N, v).
\end{aligned}$$

Summarizing, for any $\zeta > 0$, we have $\gamma_{M^U(N,v+w^{S^*,\zeta})} > \gamma_{M^U(N,v)}$ and for $\zeta > -\min_{i\in S^*} \varepsilon_i$ we have $\gamma_{m^U(N,v+w^{S^*,\zeta})} \leq \gamma_{m^U(N,v)}$. ■

Proof of Lemma 21 Take $j \in N$ arbitrarily, and take $S^* \ni j$ with $|S^*| = U \geq 2$ such that

$$v(S^*) - v(S^*\backslash\{j\}) = M_j^U(N, v).$$

Let for $k \in N\backslash\{j\}$ the set S^{**} satisfy $|S^{**}| = |S^*| = U$ and

$$\begin{aligned}
m_k^U(N, v + w^{S^*,\zeta}) & = \max_{S:|S|=U, k\in S} \left[v(S) - \sum_{i\in S\backslash\{k\}} M_i^U(N, v + w^{S^*,\zeta}) \right] \\
& = \left(v + w^{S^*,\zeta} \right)(S^{**}) - \sum_{i\in S^{**}\backslash\{k\}} M_i^U(N, v + w^{S^*,\zeta}).
\end{aligned}$$

If $S^* \cap S^{**} = \emptyset$ or $S^* \cap S^{**} = \{k\}$, then

$$\begin{aligned}
m_k^U(N, v + w^{S^*,\zeta}) & = \left(v + w^{S^*,\zeta} \right)(S^{**}) - \sum_{i\in S^{**}\backslash\{k\}} M_i^U(N, v) \\
& = m_k^U(N, v) = \underline{m}_k^U(N, v + w^{S^*}).
\end{aligned}$$

Moreover,

$$\lim_{\zeta\to\infty} m_k^U(N, v + w^{S^*,\zeta}) = m_k^U(N, v) = \underline{m}_k^U(N, v + w^{S^*}).$$

If $S^* \cap S^{**}\backslash\{k\} \neq \emptyset$, then

$$\begin{aligned}
& \left(v + w^{S^*,\zeta}\right)(S^{**}) - \sum_{i \in S^{**}\backslash\{k\}} M_i^U(N, v + w^{S^*,\zeta}) \\
= \quad & v(S^{**}) - \sum_{i \in S^{**}\backslash\{k\}} M_i^U(N, v + w^{S^*,\zeta}) \\
= \quad & v(S^{**}) - \sum_{i \in S^{**}\backslash\{k\}} M_i^U(N, v) - (|S^{**} \cap S^*| - 1) \cdot \zeta - \sum_{i \in S^{**} \cap S^* \backslash\{k,j\}} \varepsilon_i.
\end{aligned}$$

Then, for sufficiently large ζ we have

$$\left(v + w^{S^*,\zeta}\right)(S^{**}) - \sum_{i \in S^{**}\backslash\{k\}} M_i^U(N, v + w^{S^*,\zeta}) < \underline{m}_k^U(N, v + w^{S^*}).$$

This contradiction implies that $S^* \cap S^{**}\backslash\{k\} = \emptyset$. Hence,

$$\lim_{\zeta \to \infty} m^U(N, v + w^{S^*,\zeta}) = \underline{m}^U(N, v + w^{S^*}). \qquad \blacksquare$$

Proof of Lemma 22. Take $j \in N$ arbitrarily, and take $S^* \ni j$ with $|S^*| = U > 2$ such that

$$v(S^*) - v(S^*\backslash\{j\}) = M_j^U(N, v).$$

We want to find out for which cases $\overline{\tau}^U(N, v+w^{S^*,\zeta_1}) = \overline{\tau}^U(N, v+w^{S^*,\zeta_2})$ while $\zeta_1 \neq \zeta_2$ if there are any. First, we intend to take $\zeta_1, \zeta_2 > \underline{\zeta}$ and we rewrite

$$\begin{aligned}
& \overline{\tau}^U(N, v + w^{S^*,\zeta_1}) \\
= \quad & \underline{m}^U(N, v + w^{S^*}) + \\
& \frac{\gamma_{\underline{m}^U(N,v+w^{S^*})} - v(N)}{\gamma_{\underline{m}^U(N,v+w^{S^*})} - \gamma_{M^U(N,v+w^{S^*,\zeta_1})}} \left[M^U(N, v + w^{S^*,\zeta_1}) - \underline{m}^U(N, v + w^{S^*})\right], \\
& \overline{\tau}^U(N, v + w^{S^*,\zeta_2}) \\
= \quad & \underline{m}^U(N, v + w^{S^*}) + \\
& \frac{\gamma_{\underline{m}^U(N,v+w^{S^*})} - v(N)}{\gamma_{\underline{m}^U(N,v+w^{S^*})} - \gamma_{M^U(N,v+w^{S^*,\zeta_2})}} \left[M^U(N, v + w^{S^*,\zeta_2}) - \underline{m}^U(N, v + w^{S^*})\right].
\end{aligned}$$

Note that the proof of Lemma 20 implies that for $k \in N\backslash S^*$, and any $\zeta > 0$, we have $M_k^U(N, v + w^{S^*,\zeta}) = M_k^U(N, v)$ which implies in turn that for $k \in N\backslash S^*$ and $\zeta_1, \zeta_2 > \underline{\zeta}$, we have

$$\begin{aligned}
0 \quad = \quad & \overline{\tau}_k^U(N, v + w^{S^*,\zeta_1}) - \tau_k^U(N, v + w^{S^*,\zeta_2}) \\
= \quad & \frac{\gamma_{\underline{m}^U(N,v+w^{S^*})} - v(N)}{\gamma_{M^U(N,v+w^{S^*,\zeta_1})} - \gamma_{\underline{m}^U(N,v+w^{S^*})}} \left[M_k^U(N, v) - \underline{m}_k^U(N, v + w^{S^*})\right] - \\
& \frac{\gamma_{\underline{m}^U(N,v+w^{S^*})} - v(N)}{\gamma_{M^U(N,v+w^{S^*,\zeta_2})} - \gamma_{\underline{m}^U(N,v+w^{S^*})}} \left[M_k^U(N, v) - \underline{m}_k^U(N, v + w^{S^*})\right].
\end{aligned}$$

Since for $\zeta_1 \neq \zeta_2$

$$\frac{\gamma_{\underline{m}^U(N,v+w^{S^*})} - v(N)}{\gamma_{M^U(N,v+w^{S^*,\zeta_1})} - \gamma_{\underline{m}^U(N,v+w^{S^*})}} \neq \frac{\gamma_{\underline{m}^U(N,v+w^{S^*})} - v(N)}{\gamma_{M^U(N,v+w^{S^*,\zeta_2})} - \gamma_{\underline{m}^U(N,v+w^{S^*})}},$$

the above can only hold if (recall that case (a) excludes $v(N) = \gamma_{\underline{m}^U(N,v+w^{S^*})}$)

$$0 = M_k^U(N,v) - \underline{m}_k^U(N, v + w^{S^*}) \text{ for all } k \in N \backslash S^*.$$

Note furthermore that the latter expression implies that

$$\gamma_{M^U(N,v+w^{S^*,\zeta})} - \gamma_{\underline{m}^U(N,v+w^{S^*})} = \sum_{k \in S^*} \left[M_k^U(N, v + w^{S^*,\zeta}) - \underline{m}_k^U(N, v + w^{S^*}) \right].$$

Now, we turn to $k \in S^*$ for which we recall

$$M_k^U(N, v + w^{S^*,\zeta}) = M_k^U(N,v) + \max\{\zeta + \varepsilon_k, 0\}.$$

Define

$$\begin{aligned} A_k &= M_k^U(N,v) + \varepsilon_k - \underline{m}_k^U(N, v + w^{S^*}), \\ B &= \sum_{k \in S^*} A_k, \\ C &= \gamma_{\underline{m}^U(N,v+w^{S^*})} - v(N). \end{aligned}$$

Hence, for ζ sufficiently large we have

$$\gamma_{M^U(N,v+w^{S^*,\zeta})} - \gamma_{\underline{m}^U(N,v+w^{S^*})} = \sum_{k \in S^*} [A_k + \zeta] = \sum_{k \in S^*} A_k + |S^*|\zeta.$$

Then for all $k \in S^*$

$$\begin{aligned} 0 &= \overline{\tau}_k^U(N, v + w^{S^*,\zeta_1}) - \overline{\tau}_k^U(N, v + w^{S^*,\zeta_2}) \\ &= C \left[\frac{A_k + \zeta_1}{B + |S^*|\zeta_1} - \frac{A_k + \zeta_2}{B + |S^*|\zeta_2} \right] \\ &= \frac{C}{(B + |S^*|\zeta_1)(B + |S^*|\zeta_2)} (\zeta_1 - \zeta_2)(B - A_k|S^*|), \end{aligned}$$

which, as $\zeta_1 \neq \zeta_2$, in turn implies for all $k \in S^*$

$$B - A_k|S^*| = 0 \Longrightarrow A_k = \frac{1}{|S^*|} B.$$

So, let $k, l \in S^*$, then

$$\begin{aligned} & v(S^*) - v(S^* \backslash \{k\}) - \underline{m}_k^U(N, v + w^{S^*}) \\ =\ & v(S^*) - v(S^* \backslash \{l\}) - \underline{m}_l^U(N, v + w^{S^*}) \Longrightarrow \\ & v(S^* \backslash \{l\}) - v(S^* \backslash \{k\}) \\ =\ & \underline{m}_k^U(N, v + w^{S^*}) - \underline{m}_l^U(N, v + w^{S^*}) \end{aligned}$$

Summarizing matters, we find $\overline{\tau}^U(N, v + w^{S^*,\zeta_1}) = \overline{\tau}^U(N, v + w^{S^*,\zeta_2})$ while $\zeta_1 \neq \zeta_2$ if the following two conditions are met simultaneously

$$\begin{aligned} v(S^* \backslash \{l\}) - v(S^* \backslash \{k\}) &= \underline{m}_k^U(N, v + w^{S^*}) - \underline{m}_l^U(N, v + w^{S^*}) \text{ if } k, l \in S^* \\ M_k^U(N,v) &= \underline{m}_k^U(N, v + w^{S^*}) \text{ if } k \in N \backslash S^*. \qquad \blacksquare \end{aligned}$$

Proof of Lemma 23. Recall that we are in case ($c\prime$) which means

$$\tau^U(N, v + w^{S^*,\zeta}) = \underline{m}^U(N, v + w^{S^*}) - \frac{\gamma_{\underline{m}^U(N,v+w^{S^*})} - v(N)}{|N|} \cdot (1, ..., 1) \,.$$

For the **originating game** (N, v) we have three possibilities

$$\tau^U(N,v) = \begin{cases} \overline{\tau}^U(N,v) \\ m^U(N,v) - \frac{\gamma_{m^U(N,v)} - v(N)}{|N|} \cdot (1, ..., 1) \\ M^U(N,v) - \frac{\gamma_{M^U(N,v)} - v(N)}{|N|} \cdot (1, ..., 1) \end{cases} .$$

Next, we consider the middle possibility. The equality $\tau^U(N, v + w^{S^*,\zeta}) = \tau^U(N, v)$ is equivalent to

$$\begin{aligned} & \underline{m}^U(N, v + w^{S^*}) - \frac{\gamma_{\underline{m}^U(N,v+w^{S^*})} - v(N)}{|N|} \cdot (1, ..., 1) \\ = \; & m^U(N,v) - \frac{\gamma_{m^U(N,v)} - v(N)}{|N|} \cdot (1, ..., 1) \Longrightarrow \\ & \underline{m}^U(N, v + w^{S^*}) - m^U(N,v) = \frac{\gamma_{\underline{m}^U(N,v+w^{S^*})} - \gamma_{m^U(N,v)}}{|N|} \cdot (1, ..., 1) \,. \end{aligned}$$

The validity of the statement of the lemma for the other two possibilities can be demonstrated analogously. ■

Proof of Prop 24 For $U = 1$, Lemma 18 validates the statement of the proposition. For $U \geq 2$ and let (N, v) fulfill the requirements of the statement of the proposition. For any $j \in N$ we can find a set $S^* \ni j$ with $|S^*| = U \geq 2$ such that

$$v(S^*) - v(S^* \backslash \{j\}) = M_j^U(N, v).$$

Then, for $\zeta \to \infty$, as we can make $\gamma_{M^U(N,v+w^{S^*,\zeta})}$ arbitrarily large by Lemma 20 we know that for $\tau^U(N, v + w^{S^*,\zeta})$ only cases (a') and (c') can occur. If we end up in case ($a\prime$) we know by a'-admissibility that the conditions of Lemma 22 are violated, hence a ζ can be found for some combination j, S^* such that $\tau^U(N, v + w^{S^*,\zeta}) \neq \tau^U(N, v)$. If we end up in case ($c\prime$) we know by alignment-admissibility that the conditions of Lemma 23 are violated, hence for each $U \geq 2$ a ζ can be found for some combination j, S^* such that $\tau^U(N, v + w^{S^*,\zeta}) \neq \tau^U(N, v)$. This proves the statement of the lemma.

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