CHAPTER 2. Interval unavailability distribution for a 1-out-of-2 system with ample repair facilities[1]

Abstract

This paper deals with a two-unit reliability model having one operating unit and one cold standby unit and with ample repair facilities. An approximate method is given for the calculation of the probability distribution of the proportion of time that the system is available in a given time period. The approximate method first computes the mean time to failure and the mean time to repair and next approximates the up-and-down process of the two-unit reliability system by an alternating renewal process having exponentially distributed on and off periods. Numerical investigations show a satisfactory performance of this approximation. Also, a sensitivity analysis is given.

1. Model and approach

An important reliability model is the 1-out-of-2 model with repair. That is, we consider a two-unit system having one operating unit and one cold standby unit. If the operating unit fails, the other unit is switched into operation, while the failed unit enters repair. The system only goes down when a unit fails while the other unit is still under repair. There are ample repair facilities, so that both units can be simultaneously under repair. We assume perfect switch-over with no switch-over times and no start-up failures. The lifetimes of the units are independent and identically distributed variables with probability distribution function $F_L(x)$ and probability density $f_L(x)$. The repair times of the units are independent random variables with common probability distribution function $G_R(x)$ and are also independent of the lifetimes. Also, we make the technical assumption $\Pr\{L<R\}>0$ in order to avoid degenerate cases that are uninteresting from a practical point of view. In these, the generic variables L and R are used to denote a lifetime and a repair time of a unit. We denote by μ_L and μ_R the average lifetime and the average repair time of a unit, and in the sequel it will be assumed that $\mu_L \ll \mu_R$.

A lot of theoretical work has been done on the 1-out-of-2 system with repair, see e.g. the review in Birolini [2]. However, relatively little research has been done on the interval unavailability distribution, that is, the probability distribution function of the fraction of time that a system is down during a given time interval. In practical applications this distribution may be of great importance. For example, in a production environment sales

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contracts may require that a certain minimum level of production in a given time period is reached.

In this paper we focus on the derivation of computationally tractable results for the interval unavailability distribution for the case of general lifetimes and repair times. The analysis has to be approximate in order to obtain tractable results. The approximate approach uses the idea of approximating the two-unit system by a single component that is alternately on and off (cf. also Brouwers [3]), and heavily relies on renewal theoretical arguments. We shall give an outline of this approach.

Note first that the system is alternately up and down, where, however, both the up-periods and the down-periods are in general not independent and identically distributed random variables (except for the special case of exponential repair times). Let us define the key random variables τ_{SF} and τ_{SR}. The generic variables τ_{SF} and τ_{SR} are, respectively, distributed as the length of an arbitrary up-period and the length of an arbitrary down-period when the system has reached statistical equilibrium. In the sequel, the terminology "the system time to failure" and "the system time to repair" will be used for the random variables τ_{SF} and τ_{SR}. The approximate analysis will exploit the fact that in practical applications the average repair time μ_R of a failed unit is usually much smaller than the average lifetime μ_L of an operating unit. Thus, we can use a fundamental result from stochastic processes which says that under general conditions the time until the occurrence of a rare event in a regenerative stochastic process is approximately exponentially distributed, see Gertsbakh [4] and Keilson [7]. Therefore the analysis uses the approximation that the system time to failure is exponentially distributed. Also, the distribution of the system time to repair will be approximated by an exponential distribution, although the foregoing argument does not apply here. Nevertheless, since the system up-times strongly dominate the system down-times, it is reasonable to use only the first moment of the system time to repair.

Now the approximate method works as follows:

(i) Compute (approximately) the mean system time to failure and the mean system time to repair.

(ii) Approximate the process describing the up and down-times of the two-unit system by an alternating renewal process for a single component, where the on and off-times in the alternating renewal process are independent and exponentially distributed with respective means $E[\tau_{SF}]$ and $E[\tau_{SR}]$.

Then, the point-unavailability U_∞, defined as the long-run fraction of time the two-unit system is down, is given by

$$U_\infty = \frac{E[\tau_{SR}]}{E[\tau_{SF}]+E[\tau_{SR}]} \qquad (1)$$

To compute the stationary interval unavailability distribution, we use the following result: For an alternating renewal process in which the on and off-periods are exponentially distributed with means $E[\tau_{SF}]$ and $E[\tau_{SR}]$, define $\Omega(x;t_0)$ as the probability that the process will be on no more than an amount of time x during a given time interval of length t_0, when the process has reached statistical equilibrium. Then, by an obvious extension of Theorem 1 in Takács [8],

$$\Omega(x;t_0) = \frac{\nu_R}{\nu_L+\nu_R}\sum_{n=0}^{\infty} e^{-\nu_R(t_0-x)}\frac{[\nu_R(t_0-x)]^n}{n!}\left(1-\sum_{k=0}^{n} e^{-\nu_L x}\frac{[\nu_L x]^k}{k!}\right)$$
$$+\frac{\nu_L}{\nu_L+\nu_R}\sum_{n=0}^{\infty} e^{-\nu_R(t_0-x)}\frac{[\nu_R(t_0-x)]^n}{n!}\left(1-\sum_{k=0}^{n-1} e^{-\nu_L x}\frac{[\nu_L x]^k}{k!}\right) \qquad (2)$$

for $0\le x<t_0$, where $\nu_L=1/E[\tau_{SF}]$ and $\nu_R=1/E[\tau_{SR}]$. Note that $\Omega(t_0;t_0)=1$ and $\Omega(x;t_0)$ has a jump of $\nu_R e^{-\nu_L t_0}/(\nu_L+\nu_R)$ at $x=t_0$. The expression (2) is well- suited for numerical computations because of the easily calculated Poisson probabilities and the rapid convergence of the infinite summation. Define now for the two-unit reliability system the random variable

$U(t_0)$ = ***the proportion of the time that the system is down during a given time interval of length*** t_0,

assuming that the system has reached statistical equilibrium. Then the stationary interval unavailability distribution is approximately computed from

$$Pr\{U(t_0)<x\} \approx 1 - \Omega(t_0(1-x);t_0), \quad 0\le x\le 1 \qquad (3)$$

To conclude this section, we make the following remark. Numerical experiments indicate that the distributional form of the lifetime of a unit has a much larger effect on the various performance measures than the distributional form of the unit repair times. For example, consider the following numerical data. The lifetime L has mean $\mu_L=1$ and a squared coefficient of variation $c_L^2=0.5$, while the repair time R has mean $\mu_R=1/9$ and a squared coefficient of variation $c_R^2=0.5$ (hence the unit availability defined as the ratio $\mu_L/(\mu_L+\mu_R)$ equals 0.9). Note that the coefficient of variation of a random variable X is defined as

$c_X=\sigma(X)/E(X)$, the ratio of the standard deviation and the mean. Then, for a Weibull distributed repair time, we find for the respective cases of a gamma distributed lifetime and a Weibull distributed lifetime the values $E[\tau_{SF}]=35.8$ and $E[\tau_{SR}]=0.051$, $E[\tau_{SF}]=24.5$ and $E[\tau_{SR}]=0.053$, respectively. On the other hand, for a Weibull distributed lifetime, we find for the respective cases of a lognormal distributed repair time and a gamma distributed repair time the values $E[\tau_{SF}]=25.0$ and $E[\tau_{SR}]=0.053$, $E[\tau_{SF}]=25.1$ and $E[\tau_{SR}]=0.052$, respectively. It is noted that these values are simulated values. The empirical finding is that for practical purposes the repair time distribution is only relevant through its first two moments provided its coefficient of variation is not too large. In the 1-out-of-2 system it is in general not enough to use the lifetime distribution through its first two moments. Therefore, whenever necessary for the computation of $E[\tau_{SR}]$, we have fitted a computationally tractable distribution to the actual repair time distribution by matching the first few moments. However, we have used the actual lifetime distribution in the computations.

2. The approximation of $E[\tau_{SF}]$ and $E[\tau_{SR}]$

In this section, we derive approximate formulas for the averages $E[\tau_{SF}]$ and $E[\tau_{SR}]$. Unlike the 1-out-of-2 system with single repair facility dealt with in Barlow and Proschan [1] and Tijms [9], no tractable exact formula for $E[\tau_{SF}]$ and $E[\tau_{SR}]$ can in general be obtained for the 1-out-of-2 system with ample repair facilities.

We first discuss the approximate computation of $E[\tau_{SR}]$. To do this, we observe that a down-period starts when the operating unit fails while the other unit is still under repair. Denote by $R_d(t)$ the probability distribution function of the residual repair time for the repair in progress at the moment that the other unit fails and enters repair. Then, the system time to repair exceeds t only if both the new repair of the second unit and the residual repair of the first unit exceed t, and so

$$Pr\{\tau_{SR}>t\} = [1-G_R(t)][1-R_d(t)], \quad t\geq 0 \tag{4}$$

An exact computation of $R_d(t)$ seems impossible except for exponential repair times and therefore we approximate this distribution function. Assuming that $\mu_R \ll \mu_L$, it is very likely that the repair of the first unit started at the moment that the other unit was put into operation. Thus, we use the approximation

$$1-R_d(t) \approx Pr\{R-L>t \mid R>L\}$$

and so, by familiar conditioning arguments,

$$1-R_d(t) \approx \frac{\int_0^\infty f_L(x)[1-G_R(x+t)]\,dx}{\int_0^\infty f_L(x)[1-G_R(x)]\,dx}, \quad t\geq 0 \tag{5}$$

where $f_L(x)$ denotes the density of the unit lifetime L. Note that this approximation is exact for the special case of exponentially distributed unit repair times. Using

$$E[X] = \int_0^\infty Pr\{X>x\}\,dx$$

for any nonnegative random variable X, we obtain from Eqs. (4) and (5) that

$$E[\tau_{SR}] \approx \frac{\int_0^\infty f_L(x)\int_0^\infty [1-G_R(t)][1-G_R(x+t)]\,dt\,dx}{\int_0^\infty f_L(x)[1-G_R(x)]\,dx} \tag{6}$$

The double integral in Eq. (6) can only be numerically handled for special repair time distributions. For deterministic, Erlangian and hyperexponential repair time distributions, the double integral can be reduced to a one-dimensional integral, see Appendix I for details. Therefore, in order to make Eq. (6) computationally tractable, we fit a tractable probability distribution function $\hat{G}_R(x)$ to the actual repair time distribution function $G_R(x)$ by matching two or three moments. To do this, we distinguish two cases:

(i) $0<c_R^2\leq 1$: Then $G_R(x)$ is approximated by a mixture of an Erlang(k-1) and an Erlang(k) distribution with the same scale parameter, i.e.,

$$\hat{G}_R(x) = q\left\{1-\sum_{j=0}^{k-2} e^{-\lambda x}\frac{(\lambda x)^j}{j!}\right\} + (1-q)\left\{1-\sum_{j=0}^{k-1} e^{-\lambda x}\frac{(\lambda x)^j}{j!}\right\}, \quad x\geq 0 \tag{7}$$

Here the integer $k\geq 2$ is chosen accordingly so that $1/k<c_R^2\leq 1/(k-1)$, while q and λ are chosen as

$$q = \frac{kc_R^2-\sqrt{k(1+c_R^2)-k^2c_R^2}}{1+c_R^2} \quad \text{and} \quad \lambda = \frac{k-q}{\mu_R} \tag{8}$$

so that the first two moments of $G_R(x)$ and $\hat{G}_R(x)$ coincide, see Tijms [9].

(ii) $c_R^2>1$: Then $G_R(x)$ is approximated by a hyperexponential distribution

$$\hat{G}_R(x) = 1 - qe^{-\lambda_1 x} - (1-q)e^{-\lambda_2 x}, \quad x\geq 0 \tag{9}$$

Here the parameters λ_1, λ_2 and q are chosen such that the first three moments of $G_R(x)$ and $\hat{G}_R(x)$ coincide. Thus, letting m_i denote the i^{th} moment of the repair time R,

$$\lambda_{1,2} = \frac{6m_1\beta}{\alpha+1.5\beta^2+3m_1^2\beta\pm\sqrt{(\alpha+1.5\beta^2-3m_1^2\beta)^2+18m_1^2\beta^3}}$$

$$q = \frac{\lambda_1(m_1\lambda_2-1)}{\lambda_2-\lambda_1} \tag{10}$$

where $\alpha=m_1m_3-1.5m_2^2$ and $\beta=m_2-2m_1^2$, see Whitt [11]. This match requires that $\alpha>0$ (as is always satisfied when $G_R(x)$ is the lognormal or gamma distribution). In case this requirement is not fulfilled, it is suggested to choose λ_1, λ_2 and q such that the first two moments of $G_R(x)$ and $\hat{G}_R(x)$ agree and the third moment of $\hat{G}_R(x)$ is as close as possible to m_3.

We turn next to the computation of $E[\tau_{SF}]$. As noted earlier, the successive system times to failure are not independent and identically distributed. Nevertheless, we can relate the distribution function of an arbitrary system time to failure to the distribution function of a specially chosen time to failure. To do so, note that the epochs at which one unit enters repair while the other unit is put into operation are regeneration epochs for the system. Let the random variable τ_{Ref} denote the time between such a regeneration epoch and the next epoch at which the system goes down. Suppose that the epoch T=0 is a regeneration epoch. The system time to failure from the regeneration epoch T=0 on is larger than t if one of the following two mutually exclusive events occurs:

(i) The lifetime of the newly installed unit exceeds t,

(ii) The lifetime of the newly installed unit equals x≤t, the repair time of the other unit is smaller than x and the system time to failure from the regeneration epoch x on exceeds t-x.

Hence we obtain the (defective) renewal equation (cf. also Gnedenko et al. [5]):

$$Pr\{\tau_{ReF}>t\} = 1-F_L(t) + \int_0^t f_L(x)G_R(x)Pr\{\tau_{ReF}>t-x\}\,dx, \quad t\geq 0 \tag{11}$$

Using similar arguments, it now follows that the probability distribution of an arbitrary system time to failure is given by

$$Pr\{\tau_{SF}>t\} = 1-F_L(t) + \int_0^t f_L(x)R_u(x)Pr\{\tau_{ReF}>t-x\}\,dx, \quad t\geq 0 \tag{12}$$

where $R_u(x)$ is the probability distribution function of R_U, the residual repair time of the unit still under repair at the start of an arbitrary system up-time. From Eqs. (11) and (12), it is a matter of algebra to derive

$$E[\tau_{SF}] = \mu_L\left\{1 + \frac{\int_0^\infty f_L(x)R_u(x)\,dx}{\int_0^\infty f_L(x)[1-G_R(x)]\,dx}\right\} \tag{13}$$

It remains to find the distribution function $R_u(x)$ of the random variable R_U. Except for exponential repair times, an exact tractable expression for $R_u(x)$ seems difficult to obtain. We give two possible approximations for $R_u(x)$. First, we can express $R_u(x)$ in terms of the distribution function $R_d(x)$ of the random variable R_D, which is defined as the residual repair time of the unit already under repair at the start of a system down-time, when the other unit enters a new repair time R. Some reflection shows that

$$Pr\{R_U>t\} = Pr\{R_D-R>t\} + Pr\{R-R_D>t\}$$

and so

$$1-R_u(t) = \int_0^\infty r_d(x)[1-G_R(x+t)]\,dx + \int_0^\infty g_R(x)[1-R_d(x+t)]\,dx, \quad t\geq 0 \tag{14}$$

where $r_d(x)$ denotes the density of the distribution function $R_d(x)$, which is approximately given by Eq. (5). Together, Eqs. (5), (13) and (14) lead to an approximation for $E[\tau_{SF}]$. However, this approximation is not tractable except for special cases such as deterministic repair times, for which case we find $E[\tau_{SF}]\approx\mu_L\{½+1/F_L(\mu_R)\}$.

Second, we can simply approximate $R_u(x)$ by the familiar equilibrium excess distribution function corresponding to the distribution function $G_R(x)$. That is, we approximate $R_u(x)$ by

$$R_u(x) \approx \frac{1}{\mu_R} \int_0^x [1-G_R(y)]dy, \quad x \geq 0 \tag{15}$$

Together, Eqs. (13) and (15) result in the approximation

$$E[\tau_{SF}] \approx \mu_L \left\{ 1 + \frac{\int_0^\infty [1-G_R(y)][1-F_L(y)]dy}{\mu_R \int_0^\infty f_L(x)[1-G_R(x)]dx} \right\} \tag{16}$$

This formula will be used in the approximation method. In view of the crucial importance of having a good approximation to $E[\tau_{SF}]$, we use in Eq. (16) the actual repair time distribution G(x). It should be noted that the evaluation of the formulas (6) and (16) require in general numerical integration for a one-dimensional integral. To do this, effective techniques for numerical integration are available (see also the numerical integration methods discussed in Jagerman [6] and Appendix B of Van Hoorn [10]).

3. Numerical results and conclusions

In this section, we present some numerical results. To test the quality of the approximations, we have used computer simulation.

In Table 1 we give the approximate and the simulated values of $E[\tau_{SR}]$ and $E[\tau_{SF}]$. The approximate values are computed using the formulas (6) and (16), where the evaluation of formula (6) uses a two or three moment fit to the repair time distribution according to Eqs. (7)-(10). In all cases, the half length of the 95% confidence interval of the simulated value is smaller than a fraction 0.05 of the simulated value. In the examples of Table 1, the unit lifetime L has a Weibull distribution, whose squared coefficient of variation is varied as c_L^2=0.25, 0.50, 1.0 and 2.0, while for the unit repair time we consider the deterministic distribution (c_R^2=0), the Weibull distribution (c_R^2=0.25, 0.50, 0.75 and 1.0) and the lognormal distribution (c_R^2=2.0 and 4.0). Note that the Weibull distribution with a squared coefficient of variation of 1 is the exponential distribution. In all examples, the normalization μ_L=1 is used. The unit availability which is defined as $\mu_L/(\mu_L+\mu_R)$, is varied as 0.80 (μ_R= 1/4) and 0.90 (μ_R=1/9).

Unit availability	c_R^2		c_L^2=0.25 $E[\tau_{SF}]$	c_L^2=0.25 $E[\tau_{SR}]$	c_L^2=0.50 $E[\tau_{SF}]$	c_L^2=0.50 $E[\tau_{SR}]$	c_L^2=1.00 $E[\tau_{SF}]$	c_L^2=1.00 $E[\tau_{SR}]$	c_L^2=2.00 $E[\tau_{SF}]$	c_L^2=2.00 $E[\tau_{SR}]$
0.8	0.00	sim	25.2	0.080	9.3	0.103	5.0	0.126	3.3	0.142
		app	24.9	0.081	9.5	0.105	5.0	0.130	3.3	0.154
	0.25	sim	20.1	0.103	9.1	0.115	5.0	0.124	3.2	0.140
		app	19.9	0.104	9.0	0.116	5.0	0.128	3.4	0.139
	0.50	sim	17.1	0.111	8.6	0.122	5.0	0.122	3.3	0.132
		app	16.9	0.115	8.6	0.121	5.0	0.127	3.4	0.132
	0.75	sim	15.1	0.121	8.3	0.122	4.9	0.125	3.3	0.123
		app	15.0	0.120	8.3	0.123	5.0	0.126	3.4	0.129
	1.00	sim	13.7	0.125	8.1	0.125	5.0	0.125	3.4	0.125
		app	13.7	0.125	8.1	0.125	5.0	0.125	3.5	0.125
	2.00	sim	12.8	0.141	8.0	0.134	5.1	0.123	3.5	0.118
		app	13.0	0.136	8.1	0.129	5.0	0.121	3.3	0.114
	4.00	sim	11.6	0.142	7.8	0.137	5.1	0.120	3.4	0.111
		app	11.7	0.144	7.8	0.127	5.0	0.111	3.3	0.100
0.9	0.00	sim	137	0.036	28.0	0.045	10.0	0.056	5.2	0.064
		app	131.8	0.036	28.0	0.046	10.0	0.057	5.1	0.067
	0.25	sim	103	0.044	26.5	0.051	10.1	0.055	5.1	0.060
		app	99.9	0.046	25.7	0.051	10.0	0.056	5.5	0.061
	0.50	sim	83	0.051	24.5	0.053	10.0	0.056	5.3	0.058
		app	84.3	0.051	24.5	0.053	10.0	0.056	5.6	0.058
	0.75	sim	72	0.053	24.0	0.055	10.1	0.057	5.3	0.055
		app	71.7	0.053	23.4	0.054	10.0	0.056	5.6	0.057
	1.00	sim	63	0.056	22.7	0.056	10.0	0.056	5.4	0.056
		app	62.6	0.056	22.5	0.056	10.0	0.056	5.7	0.056
	2.00	sim	49	0.067	22.1	0.059	10.0	0.056	5.4	0.053
		app	49.7	0.061	22.0	0.058	10.1	0.055	5.4	0.051
	4.00	sim	38	0.072	20.0	0.063	10.1	0.056	5.7	0.049
		app	39.2	0.069	20.3	0.061	9.9	0.052	5.3	0.046

Table 1. The simulated and approximate values of $E[\tau_{SR}]$ and $E[\tau_{SF}]$.

The results in Table 1 show a satisfactory performance of the approximations to $E[\tau_{SR}]$ and $E[\tau_{SF}]$ for most practical situations. It is pointed out that the approximation to $E[\tau_{SR}]$ becomes less accurate when c_R^2 gets large, particularly when the actual repair time distribution is not used in Eq. (6). We remark that in practice c_R^2 is usually not larger than 2. Also, the results confirm that the distributional form of the unit lifetime has a much larger effect on the performance measures than the distributional form of the unit repair time. Further, it is interesting to observe that, for the important case of an exponentially distributed unit lifetime, the first moment of the unit repair time is sufficient for practical purposes (cf. Tijms [9] for a similar empirical finding for the measure of system reliability). Actually, from insensitivity results for stochastic networks it can be shown that the particular measure U_∞ is insensitive to the distributional form of the unit repair time when the unit lifetime is exponentially distributed (see exercise 2.27 in Tijms [9]). These findings are also clearly illustrated in the figures 1 and 2, in which we display the (simulated) values of the long term unavailability U_∞ when the unit availability equals respectively 0.9. We use the deterministic and Weibull distribution both for the lifetime and the repair time. In the figures, we have presented the long-term unavailabilities as a percentage, so $100{*}U_\infty\%$ rather than U_∞.

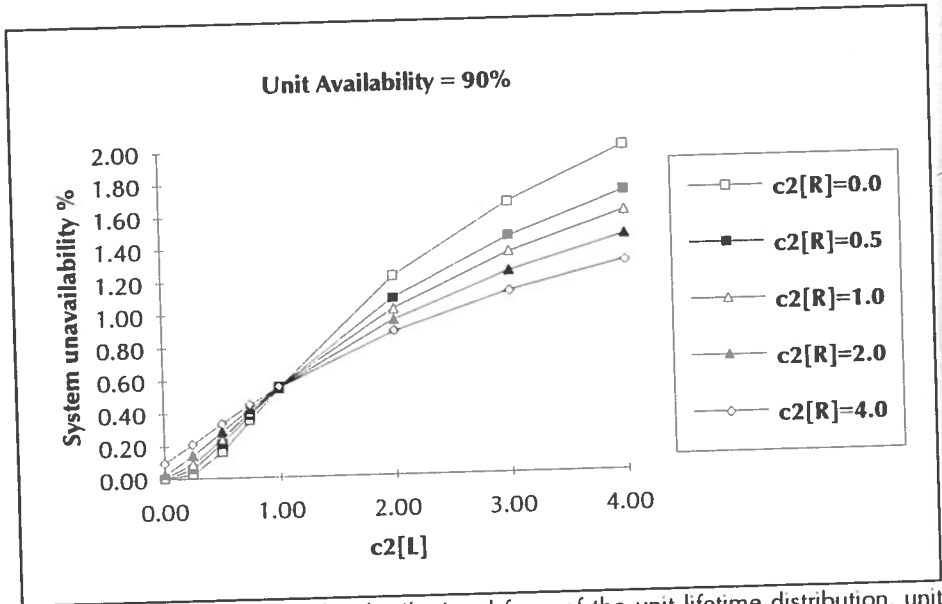

Figure 1. The effect of the distributional form of the unit lifetime distribution, unit availability = 0.90%

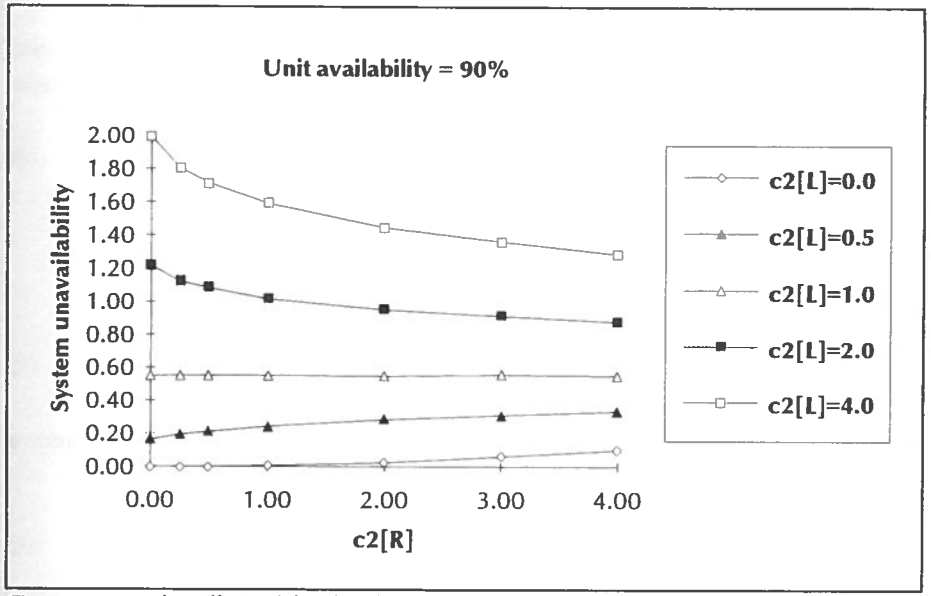

Figure 2. The effect of the distributional form of the unit repair time distribution, unit availability = 0.90%

In the tables 2 and 3, we give the approximate and simulated values for the long-term unavailability U_∞ and for a number of stationary interval unavailability probabilities. For the simulated probabilities, the half length of the 95% confidence interval never exceeds 0.01 and is about 0.002 for the probabilities very close to 1. In all examples, we take a Weibull distributed unit lifetime with $\mu_L=1$ and $c_L^2=0.5$. For the unit repair times, we consider again the deterministic distribution ($c_R^2=0$), the Weibull distribution ($c_R^2=0.75$), and the lognormal distribution ($c_R^2=2.0$ and 4.0). The unit availability has the value 0.9 (i.e., $\mu_R=1/9$). For the stationary interval unavailability distribution, we consider an interval whose length t_0 equals μ_L (Table 2) and an interval whose length t_0 equals $10\mu_L$ (Table 3). In the tables, we give $P_x=\Pr\{U(t_0)\leq x/100\}$ for several values of x. Again, the results in the Tables 2 and 3 indicate that the approximation of Eq. (2) for the stationary interval unavailability distribution is sufficiently accurate for practical purposes.

c_R^2		$100U_\infty\%$	P_0	P_5	P_{10}	P_{15}	P_{20}	P_{25}	P_{30}
0.00	sim	1.08	0.888	0.921	0.949	0.972	0.989	1.000	1.000
	app	1.09	0.890	0.933	0.959	0.975	0.985	0.991	0.995
0.75	sim	1.44	0.875	0.912	0.942	0.962	0.977	0.985	0.991
	app	1.46	0.874	0.917	0.946	0.965	0.977	0.986	0.991
2.00	sim	1.67	0.877	0.914	0.940	0.960	0.972	0.982	0.989
	app	1.57	0.870	0.913	0.942	0.962	0.975	0.984	0.989
4.00	sim	1.82	0.865	0.912	0.942	0.961	0.971	0.980	0.984
	app	1.59	0.866	0.911	0.941	0.962	0.975	0.984	0.989

Table 2. The simulated and approximate values for the stationary interval unavailability distribution for $t_0=\mu_L$ ($\mu_L=1$, $c_L^2=0.5$, $\mu_R=1/4$).

c_R^2		$100U_\infty\%$	P_0	P_1	P_2	P_3	P_4	P_5	P_{10}
0.00	sim	1.08	0.349	0.565	0.787	0.919	0.973	0.992	1.000
	app	1.09	0.345	0.627	0.796	0.892	0.944	0.972	0.999
0.75	sim	1.44	0.311	0.540	0.719	0.833	0.904	0.947	0.998
	app	1.46	0.296	0.547	0.719	0.831	0.900	0.943	0.997
2.00	sim	1.67	0.317	0.549	0.712	0.812	0.875	0.919	0.986
	app	1.57	0.286	0.528	0.699	0.813	0.886	0.932	0.996
4.00	sim	1.82	0.326	0.573	0.720	0.807	0.867	0.901	0.972
	app	1.59	0.275	0.520	0.693	0.810	0.885	0.932	0.996

Table 3. The simulated and approximate values for the stationary interval unavailability distribution for $t_0=10\mu_L$ ($\mu_L=1$, $c_L^2=0.5$, $\mu_R=1/4$)

Appendix 1. $E[\tau_{SR}]$ for some special cases

The formula of Eq. (6) for $E[\tau_{SR}]$ can easily be evaluated for the following repair time distributions.

(i) The repair time is deterministic. Then,

$$E[\tau_{SR}] \approx \frac{1}{F_L(\mu_R)} \int_0^{\mu_R} F_L(x)\,dx \tag{A1}$$

(ii) The repair time has an exponential distribution. Then,

$$E[\tau_{SR}] = \frac{\mu_R}{2} \tag{A2}$$

(iii) The repair time has a mixture of an Erlang(k-1) and an Erlang(k) distribution with the same scale parameter λ (see Eq. (7)). Then,

$$E[\tau_{SR}] \approx \sum_{j=0}^{k-1} \alpha_j \left\{ \sum_{i=0}^{k-2} \frac{\binom{i+j}{i}}{2^{i+j+1}\lambda} + (1-q)\frac{\binom{k-1+j}{k-1}}{2^{k+j}\lambda} \right\} \tag{A3}$$

where

$$\alpha_j = \frac{\sum_{i=j}^{k-2} \beta_{i-j} + (1-q)\beta_{k-1-j}}{\sum_{i=0}^{k-2} \beta_i + (1-q)\beta_{k-1}} \qquad with \quad \beta_i = \int_0^{\infty} \frac{(\lambda x)^i e^{-\lambda x}}{i!} f_L(x)\,dx$$

(iv) The repair time has a hyperexponential distribution with density

$$q\lambda_1 e^{-\lambda_1 x} + (1-q)\lambda_2 e^{-\lambda_2 x}$$

Then,

$$E[\tau_{SR}] \approx \frac{qr}{2\lambda_1} + \frac{q+r-2qr}{\lambda_1+\lambda_2} + \frac{(1-q)(1-r)}{2\lambda_2}$$

where

$$r = \frac{qL_f(\lambda_1)}{qL_f(\lambda_1) + (1-q)L_f(\lambda_2)} \qquad \text{with} \quad L_f(s) = \int_0^\infty e^{-st} f_L(t)\,dt \tag{A4}$$

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CHAPTER 3. Interval uneffectiveness distribution for a k-out-of-n multistate system with repair[1]

Abstract

This paper deals with a parallel load-sharing reliability system with cold standby redundancy and ample repair facilities. That is, we have n identical parallel units, of which at most k units are operating simultaneously. If less than k units are available, the system operates at a proportionally reduced level. For this system, an approximate method is given for the calculation of the probability distribution of that proportion of the system capacity that cannot be used in a given time period. The method is based on an approximation of the k-out-of-n multistate system by a two-state single component. Validation of the approximation using Monte Carlo simulation shows satisfactory performance. Also, sensitivity results are given, showing in particular a decreasing sensitivity of the measures of performance to the distributional form of the unit lifetimes and repair times as the size of the system increases. Furthermore, it is found that the effect of the distributional form of the unit lifetimes dominates that of the unit repair times.

1. Introduction

In this paper, we consider the k-out-of-n multistate reliability model with repair. That is, we have a system consisting of n identical units, each having a capacity of 100/k%. Hence k units are required to be available for running the system at full capacity. If more than k units are available, the superfluous units are put on cold standby and cannot fail. If less than k units (say i) are available, the system operates at a proportionally reduced level, i.e. at 100i/k%. The model under consideration has ample repair facilities, so all failed units can be under repair simultaneously. We assume perfect switch-over with no start-up failures and no switch-over times. The lifetimes as well as the repair times of the units are independent, identically distributed random variables, both with probability distributions belonging to the important class of phase-type distributions with two phases. Further, the assumtion is made that $\mu_R << \mu_L$, where μ_L and μ_R denote the mean unit lifetime and the mean unit repair time respectively. It is noted that this assumption is no restriction for most practical situations.

[1]: This chapter is written together with A. Schornagel and has appeared in the *European Journal of Operational Research* 36: 66-77, 1988.