References

1. Birolini, A. [1985]. *On the use of stochastic processes in modelling reliability problems.* Lecture notes in Economics and Mathematical Systems no. 252. Springer Verlag, Berlin.
2. Brouwers, J.J.H. [1986]. *Probabilistic descriptions of irregular system downtime.* Reliability Engineering 16.
3. Cox, D.R. [1955]. A use of complex probabilities in the theory of stochastic processes. *Proceedings of the Cambridge Philosophical Society* 51: 313-319.
4. Heijden, M.C. van der [1987]. Interval availability distribution for a 1-out-of-2 reliability distribution with repair. *Probability in the Engineering and Informational Sciences* 1: 211-224.
5. Takács, L. [1957]. On certain sojourn time problems in the theory of stochastic processes. *Acta Mathematica Academiae Scientiarum Hungaricae* 8: 169-191.
6. Tijms, H.C. [1986]. *Stochastic modelling and analysis: A computational approach.* Wiley, New York.
7. Whitt, W. [1982]. Approximating a point process by a renewal process I: Two basic methods. *Operations Research* vol. 30 no. 1: 125-147.
8. Yao, D.W. and J.A. Buzacott [1985]. Queueing models for a flexible machining station, part II: The method of Coxian phases. *European Journal of Operational Research* 19: 241-252.

CHAPTER 4. Unavailability probabilities for the 1-out-of-2 system with single repair facility

Abstract

A practically important problem is the evaluation of the interval unavailability distribution for repairable systems. In this paper we give a generally applicable approach to obtain the unavailability probabilities and apply this approach to the 1-out-of-2 system with cold standby redundancy and single repair facility. The first step of the approach is the computation of the moments of the up time and down time in the original system. Next we replace the original system by a single unit system that is alternately up and down, where one or more moments of the original up time and down time are matched. This approximation gives accurate results and requires much less computing time than the alternative approach of using the uniformization method for continuous-time Markov chains.

1. Introduction

Reliability analysis of repairable systems is an important subject on which much research has been done. An overview can for example be found in Birolini [1] and Yearout et al. [17]. In most of the literature the emphasis is on long-run performance measures such as the long run system availability. However, in practice one is often more interested in the transient behaviour of the system during a given time interval. For example, computer systems are sold with a guaranteed availability over a limited time period (cf. Goyal and Tantawi [4]) or a minimum level of gas delivery per period is agreed in sales contracts (cf. Van Rijn and Schornagel [11]). If such guarantees are violated, substantial penalties are incurred by the supplier. To establish the penalty risk, we need the *interval unavailability distribution.* This distribution gives the probability that the system unavailability in a given time interval of length T is less than some fraction x.

2. Model

This paper considers the 1-out-of-2 system with cold standby and a single repair facility. Only one unit is required for operation, while the other unit acts as a cold standby unit to prevent frequent system failures. If the operating unit breaks down, it is replaced immediately by the other unit if available. The failed unit is sent to a repair facility. The facility can repair only one unit at a time.

The unit lifetimes during operation L are independent and identically distributed (iid) random variables with general probability distribution function F(t) and mean μ_L. Also the unit repair times R are iid random variables with general probability distribution function G(t) and mean μ_R. We assume that the unit lifetimes and repair times are mutually independent. We focus on the interval unavailability distribution U(x;T), defined by

$$U(x;T) = \lim_{t\to\infty} Pr\{\textit{the fraction of time that the system is not available in } [t,t+T] \textit{ exceeds } x\}$$

where the interval length T is a given constant and $0 \le x \le T$.

Transient performance measures are usually difficult to calculate. If the system can be modelled by a continuous-time Markov chain, several numerical solution methods are available (see Reibman et al. [10] for a review). In particular, the uniformization technique as used by De Souza e Silva and Gail [12], [13] is worth mentioning. A drawback of these methods is that for practical problems the state space of the continuous-time Markov chain quickly becomes too large to keep the computations tractable. Then, simpler approximation techniques are required.

The approximate analysis of the 1-out-of-2 system is based on the idea of replacing the original system by a single unit system that is alternately up and down. This idea has been successfully used in Brouwers [2], Van der Heijden [6] and Van der Heijden and Schornagel [6]. In the approximate approach the up time and down time for the single unit system are chosen by matching a number of moments of the up time and down time in the original system. The simplest approximation is to match only the means and to use exponential distributions for the up time and the down time in the single unit system. This approximation is very well suited for quick engineering calculations and give rather accurate results. Better approximations are obtained by matching more moments using gamma distributions for the up time and down time. This approximation is somewhat less easy to apply than the simple exponential approximation, but its computational requirements are still relatively low.

The remaining part of this paper is structured as follows. Section 3 deals with the approximation of the system by a single unit system in which the up times and down times are assumed to be independent and exponentially distributed. Next the interval unavailability distribution is calculated using the results from Takács. This approach has been suggested by Van der Heijden [5]. A generalization of this method to gamma distributed up times and down times of the approximating single unit is discussed in section 4. It is based on the method by Van der Heijden and Schornagel [6] for the analysis of the k-out-of-n system. In section 5 we discuss how the uniformization

technique for continuous-time Markov chains can be used. The performance of the three methods is studied in section 6. As benchmark we use results from Monte Carlo simulation. We give our conclusions in section 7.

3. Approximation with exponential system up and down time

This method is proposed by Van der Heijden [5] who analysed the 1-out-of-2 system with *ample* repair facilities. The approximate method consists of three steps:

a. Calculate the mean system up time and the mean system down time.
b. Approximate the two-unit system by a single unit with exponentially distributed up and down times. We assume that the up and down times of the approximating single unit are independent and have respective means as calculated under a.
c. Calculate the system unavailability distribution for the single unit, based on the results of Takács [14] for an alternating renewal process.

The approximation is based on the following observations. The 1-out-of-2 cold standby system with single repair facility can be described by a regenerative process. Regeneration points are the points of time at which an operating unit fails and a standby unit is switched into operation. Now let us assume that Pr{L<R} is small as will be the case in most practical situations. In other words, the event that the operating unit fails while the other unit is still under repair is a *rare event*. A well-known result in the theory of stochastic processes says that the time until the occurrence of a rare event in a regenerative process is approximately exponentially distributed (see e.g. Gertsbakh [3] and Keilson [8]). Therefore, it is reasonable to approximate the system up time by an exponential distribution.

For the approximation of the system down time by an exponential distribution, a similar reasoning cannot be given. In fact, simulation results show that in many situations the system down time is clearly non-exponential distributed (except for exponentially distributed unit repair times). However, it is reasonable to expect that the distributional form of the system down time has only a minor effect on the accuracy of the approximation, since the system up time will typically dominate system down time. Therefore, we also use an exponential distribution for the system down times.

To apply the method, we need the mean system up time $E[\tau_{up}]$, the mean system down time $E[\tau_{down}]$ and an expression for the interval unavailability distribution U(x;T) for a single unit with exponential distributed up and down times. These three measures are discussed next.

3.1. *The mean system up time $E[\tau_{up}]$*

We observe that at the start of each system up time a unit lifetime and a unit repair time start. Hence the system up time exceeds t if and only if one of the following two mutually exclusive events occurs:

(i) The unit lifetime exceeds t,

(ii) The unit lifetime equals x<t, the repair of the other unit is finished before x and the system up time from the regeneration point x on exceeds t-x.

Therefore we can write down a defective renewal equation for τ_{up}:

$$Pr\{\tau_{up}>t\} = 1-F(t) + \int_0^t f(x)\, G(x)\, Pr\{\tau_{up}>t-x\}\, dx \qquad (1)$$

Now we integrate both sides of (1) over t. After some rearrangement of terms we obtain

$$E[\tau_{up}] = \frac{E[L]}{\int_0^\infty f(x)*[1-G(x)]dx} \qquad (2)$$

3.2. *The mean system down time $E[\tau_{down}]$*

We note that the system down period is distributed as the residual repair time of the repair in progress when the operating unit fails. Because this repair started simultaneously with the lifetime of the operating unit, we have that $Pr\{\tau_{down}>t\} = Pr\{R-L>t \mid R>L\}$. By familiar conditioning arguments, it can be derived that

$$Pr\{\tau_{down}>t\} = \frac{\int_0^\infty f(x)[1-G(x+t)]\, dx}{\int_0^\infty f(x)[1-G(x)]\, dx} \qquad (3)$$

Integration of both sides over t yields after some algebra

$$E[\tau_{down}] = \frac{\int_0^\infty F(x)*[1-G(x)]dx}{\int_0^\infty f(x)*[1-G(x)]dx} \qquad (4)$$

Now we can calculate $E[\tau_{up}]$ and $E[\tau_{down}]$ exactly using standard numerical integration. It is remarked that the relations (2) and (4) imply the following well-known expression for the long term unavailability U_∞:

$$U_\infty = \frac{\int_0^\infty F(x)*[1-G(x)]dx}{\int_0^\infty [1-F(x)*G(x)]dx} \qquad (5)$$

3.3. *The interval unavailability distribution based on exponential distributions*

To derive an expression for the interval unavailability distribution, we first give two important results of Takács [14]. Consider a renewal process that alternates between two states, representing the on and off state of a system. Assume that the sojourn times in both states are independent and have probability distribution functions $F_{on}(x)$ and $G_{off}(x)$ with respective means μ_{on} and μ_{off}. Define the probability distribution functions $H_{on}(x;t)$ and $H_{off}(x;t)$ as

$$H_{on}(x;t) = Pr\{\text{the system is on at time } t \text{ and the remaining on-time is } \le x\}$$

$$H_{off}(x;t) = Pr\{\text{the system is off at time } t \text{ and the remaining off-time is } \le x\}$$

According to Takács [14] we have that

$$\lim_{t\to\infty} H_{on}(x;t) = \frac{\mu_{on}}{\mu_{on}+\mu_{off}} * F_{on}^{res}(x) = \frac{\mu_{on}}{\mu_{on}+\mu_{off}} * \frac{1}{\mu_{on}} \int_0^\infty [1-F_{on}(y)]\, dy \qquad (6)$$

where $F_{on}^{res}(x)$ denotes the well-known equilibrium excess distribution corresponding to $F_{on}(x)$ (see e.g. Tijms[15]). For $H_{off}(x;t)$ we have an analogous expression. Now denote by $\Theta(t)$ the cumulative time that the system is on in the time interval [0,t] given that an on-period starts at time 0. In Takács [14] it is shown that

$$Pr\{\Theta(t)\le x\} = \sum_{n=0}^\infty [G_{off}^{(n)}(t-x) - G_{off}^{(n+1)}(t-x)]\, F_{on}^{(n)}(x), \quad 0\le x<t \qquad (7)$$

where $F_{on}^{(n)}(x)$ and $G_{off}^{(n)}(x)$ denote the n-fold convolution of $F_{on}(x)$ respectively $G_{off}(x)$. The expressions (6) and (7) form the basis to derive the interval unavailability distribution. For convenience we define $\Psi(x,T)$ by

$$\Psi(x;T) = \lim_{t\to\infty} Pr\{\textit{the TOTAL time that the system is not available in } [t,t+T] \textit{ exceeds } x\}$$

From (6) and (7) it can easily be derived that

$$\Psi(x;T) = \frac{\mu_{on}}{\mu_{on}+\mu_{off}} \sum_{n=0}^{\infty} [G_{off}^{(n)}(T-x) - G_{off}^{(n+1)}(T-x)]\ F_{on}^{res} * F_{on}^{(n)}(x)$$
$$+ \frac{\mu_{off}}{\mu_{on}+\mu_{off}} \sum_{n=0}^{\infty} [G_{off}^{res} * G_{off}^{(n)}(T-x) - G_{off}^{res} * G_{off}^{(n+1)}(T-x)]\ F_{on}^{(n+1)}(x)$$
$$+ \frac{\mu_{off}}{\mu_{on}+\mu_{off}} [1 - G_{off}^{res}(T-x)] \qquad (8)$$

where the symbol $*$ denotes the convolution of two probability distribution functions. Now it is immediately clear from the definition of Ψ(x,T) that the interval unavailability distribution is obtained by

$$U(x;T) = \Psi[T(1-x);T] \qquad (9)$$

However, expression (8) is computationally only useful in some special cases for which all convolutions can easily be evaluated. In particular, if both on and off times are exponentially distributed with parameters $\nu_{on}=1/\mu_{on}$ respectively $\nu_{off}=1/\mu_{off}$, it can easily be shown that

$$\Psi(x;T) = \frac{\nu_{off}}{\nu_{on}+\nu_{off}} \sum_{n=0}^{\infty} e^{-\nu_{off}(T-x)} \frac{[\nu_{off}\,(T-x)]^n}{n!} \left(1 - \sum_{k=0}^{n} e^{-\nu_{on}x} \frac{[\nu_{on}x]^k}{k!}\right)$$
$$+ \frac{\nu_{on}}{\nu_{on}+\nu_{off}} \sum_{n=0}^{\infty} e^{-\nu_{off}(T-x)} \frac{[\nu_{off}\,(T-x)]^n}{n!} \left(1 - \sum_{k=0}^{n-1} e^{-\nu_{on}x} \frac{[\nu_{on}x]^k}{k!}\right) \qquad (10)$$

Values for the interval unavailability distribution are obtained by truncating the infinite sums in (8) or (10) after some integer N. Define the truncation error $\varepsilon_N(x;T)$ for the general expression (8) as

$$\varepsilon_N(x;T) = \frac{\mu_{on}}{\mu_{on}+\mu_{off}} \sum_{n=N+1}^{\infty} [G_{off}^{(n)}(T-x) - G_{off}^{(n+1)}(T-x)]\ F_{on}^{res} * F_{on}^{(n)}(x)$$
$$+ \frac{\mu_{off}}{\mu_{on}+\mu_{off}} \sum_{n=N+1}^{\infty} [G_{off}^{res} * G_{off}^{(n)}(T-x) - G_{off}^{res} * G_{off}^{(n+1)}(T-x)]\ F_{on}^{(n+1)}(x)$$

The following criterion can be given to ensure that $\varepsilon_N(x;T)$ does not exceed some pre-specified level:

$$\varepsilon_N(x;T) \le \frac{\mu_{on}}{\mu_{on}+\mu_{off}} G_{off}^{(N+1)}(T-x)\ F_{on}^{res} * F_{on}^{(N+1)}(x) + \frac{\mu_{off}}{\mu_{on}+\mu_{off}} G_{off}^{res} * G_{off}^{(N+1)}(T-x)\ F_{on}^{(N+2)}(x) \qquad (11)$$

This upper bound for $\varepsilon_N(x;T)$ is a straightforward extension of the result by Wartenhorst [16] for the truncation of the infinite sum in (7). Summarized, the interval unavailability distribution U(x;T) is obtained by calculating $E[\tau_{up}]$ and $E[\tau_{down}]$ according to (2) respectively (4). Subsequently we evaluate U(x;T) for all values x and intervals T as desired using (10) where we take $\nu_{on}=1/E[\tau_{up}]$ and $\nu_{off}=1/E[\tau_{down}]$.

4. Approximation with gamma distributed system up and down time

As pointed out in the previous section, the system up time is not exponentially distributed. Further it is not clear under which conditions the event R>L is rare enough to ensure that the exponential distribution is a good approximation for the system up time distribution. Therefore it seems useful to study a refined method at the cost of increasing complexity and, probably, increasing computing time.

An obvious extension is to use higher moments of τ_{up} and τ_{down} in the approximations. This approach is followed by Van der Heijden and Schornagel [6] for the k-out-of-n multistate system with phase-type unit lifetimes and repair times. They use expression (8) with gamma distributed system up time and down time. To apply this method to the 1-out-of-2 cold standby system with single repair facility, we extend (2) and (4) to the higher moments of τ_{up} and τ_{down}. Further we discuss the use of (8) with gamma distributions.

4.1. The higher moments of the system up time τ_{up}

It is straightforward to derive all moments of τ_{up} from (1). We multiply both sides of (1) by $n*t^{n-1}$ and integrate over t. After some rearrangement of terms we find that

$$E[\tau_{up}^n] = \frac{E[L^n] + \sum_{i=1}^{n-1} \binom{n}{i} E[\tau_{up}^n] * \left(E[L^{n-i}] - \int_0^\infty x^{n-i} f(x)[1-G(x)]dx\right)}{\int_0^\infty f(x)[1-G(x)]\,dx}, \quad n \ge 1 \qquad (12)$$

Hence the moments of τ_{up} can be obtained recursively using numerical integration of one-dimensional integrals only.

4.2. *The higher moments of the system down time τ_{down}*

Similar to τ_{up}, we can derive the higher moments of τ_{down} from (3). After some algebra we find:

$$E[\tau_{down}^n] = \frac{E[R^n] - n(n-1)\int_0^\infty [1-F(x)] \int_x^\infty (y-x)^{n-2}[1-G(x)]\,dy\,dx}{\int_0^\infty f(x)[1-G(x)]\,dx}, \quad n \geq 2 \tag{13}$$

We see that for $n \geq 2$ a double integral has to be evaluated, which is difficult from a numerical point of view. However, the double integral reduces to a one-dimensional integral for some special cases. Here we consider deterministic, Hyperexponentially and Erlangian distributed unit repair times. We denote the double integral by

$$I_n = \int_0^\infty [1-F(x)] \int_x^\infty (y-x)^{n-2}[1-G(y)]\,dy\,dx, \quad n \geq 2 \tag{14}$$

(i) Deterministic unit repair times

Then the double integral reduces to

$$I_n = \int_0^{\mu_R} [1-F(x)] \frac{(\mu_R - x)^{n-1}}{n-1}\,dx \tag{15}$$

(ii) Hyperexponentially distributed unit repair times

The Hyperexponential distribution is described by the following function:

$$\hat{G}(t) = 1 - qe^{\lambda_1 t} - (1-q)e^{-\lambda_2 t}, \quad t \geq 0 \tag{16}$$

Substituting this in (14) gives after some manipulations

$$I_n = (n-2)! * \left(\frac{q * Z_0(\lambda_1)}{\lambda_1^{n-1}} + \frac{(1-q) * Z_0(\lambda_2)}{\lambda_2^{n-1}} \right) \tag{17}$$

where we define the function $Z_i(\lambda)$ as

$$Z_i(\lambda) = \int_0^\infty [1-F(x)] \frac{(\lambda x)^i}{i!} e^{-\lambda x}\,dx \tag{18}$$

(iii) Erlang unit repair times

A mixture of Erlang distributions is defined by

$$\hat{G}(t) = 1 - q \sum_{j=0}^{k-2} e^{-\lambda x} \frac{(\lambda x)^j}{j!} - (1-q) \sum_{j=0}^{k-1} e^{\lambda x} \frac{(\lambda x)^j}{j!} \tag{19}$$

For this distribution we find

$$I_n = \frac{(n-2)!}{\lambda^{n-1}} \sum_{l=0}^{n+k-4} Z_l(\lambda) \sum_{i=0}^{\min(l,n-2)} A_{il}(n) * [B_{l-i,n-i-2}(k-1) + (1-q)\frac{(n+k-i-3)!}{(k-1)!}]$$
$$+ \frac{(n-2)!}{\lambda^{n-1}} (1-q) * Z_{n+k-3}(\lambda) \sum_{i=0}^{n-2} A_{i,n+k-3}(n) * B_{n+k-i-3,n-i-2}(k) \tag{20}$$

where $Z_l(\lambda)$ is given by (18) and the functions $A_{il}(n)$ and $B_{jm}(k)$ are defined by

$$A_{il}(n) = \binom{l}{i} \frac{(-1)^i}{(n-2-i)!}, \qquad B_{jm}(k) = \sum_{h=\max[j-m,0]}^{k-1} \frac{(h+m)!}{h!}$$

4.3. *The interval unavailability distribution based on gamma distributions*

Once we have the higher moments of τ_{up} and τ_{down} we can fit probability distributions $F_{on}(t)$ and $G_{off}(t)$ to these moments and use (8) to approximate the interval unavailability distribution. For this we need probability distribution functions $F_{on}(t)$ and $G_{off}(t)$ for which convolutions can be evaluated easily, which is true for gamma distributions. However, a disadvantage is that $F_{on}^{res}(t)$ and $G_{off}^{res}(t)$ are *not* gamma distributions, creating problems for the evaluation of $F_{on}^{res} * F_{on}(t)$ and $G_{off}^{res} * G_{on}(t)$. However, this numerical problem can be resolved rather easily, see Appendix IV of Van der Heijden and Schornagel [6].

5. The uniformization technique

De Souza e Silva and Gail [12], [13] use the powerful uniformization technique to calculate the interval unavailability distribution. It is based on the modelling of a system as a continuous-time Markov chain. Here we give a rough outline of their method. For details we refer to the two papers by De Souza e Silva and Gail. Next we discuss how to apply the uniformization technique to the 1-out-of-2 cold standby system with single repair facility.

5.1. Outline of the method

Suppose that we can model a repairable system as a continuous-time Markov chain. Then we can calculate transient performance measure as follows. We can modify the Markov chain model in such a way, that the total transition rate out of each state is equal. This is called *uniformization* of the Markov chain. We randomize the original Markov chain in two steps:

a. Choose the state with the highest sum of transition rates out of this state. Denoting by λ_{ij} the transition rate from state i to state j, we have defined this maximal transition rate Λ as

$$\Lambda = \underset{i}{Max}\left\{\sum_j \lambda_{ij}\right\}$$

b. Add to each state i in the original model a fictitious transition to itself with rate:

$$\Lambda - \sum_{j;\, j\neq i} \lambda_{ij}$$

The total transition rate out of each state equals Λ. At each transition out of state i, another state j is entered with probability $P_{ij} = \lambda_{ij}/\Lambda$. With probability $P_{ii} = 1 - (\Sigma_{i\neq j}\, \lambda_{ij})/\Lambda$, a fictitious transition back to state i is made.

Now it is easy to obtain an expression for the cumulative up time distribution $Pr\{\Theta(t)\leq x\}$ by conditioning on the number of state transitions in the interval [0,t]. De Souza e Silva and Gail [12] show that

$$Pr\{\Theta(t)\leq x\} = \sum_{n=0}^{\infty} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}\sum_{k=0}^{n}\Omega[n,k]\sum_{i=k}^{n}\binom{n}{i}\left(\frac{x}{t}\right)^i\left(1-\frac{x}{t}\right)^{n-i} \qquad (21)$$

where $\Omega[n,k]$ denotes the probability that the randomized Markov chain visits k states in which the system is up, given that n transitions have occurred ($0\leq k\leq n+1$). The infinite sum in (21) can be truncated after N yielding a truncation error $\varepsilon(N)$ that satisfies

$$\epsilon(N) \leq 1 - \sum_{n=0}^{N} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} \qquad (22)$$

The probabilities $\Omega[n,k]$ can be calculated from the joint distribution $\Omega[n,k;J]$, denoting the probability that the randomized Markov chain visits is k times in operational states out of n steps and that I is the state visited in the last transition:

$$\Omega[n,k] = \sum_J \Omega[n,k;J] \qquad (23)$$

For the joint probabilities $\Omega[n,k;J]$ a recursion scheme can be made. To do this, we divide the set of all possible system states in two subsets *U* and *D*, denoting the states in which the system is up respectively down. Then we can calculate all probabilities $\Omega[n,k;J]$ from

$$\Omega[n,k;J] = \begin{cases}\sum_I \Omega[n-1,k-1;I]P_{ji} & \text{if } J\in U\\ \sum_I \Omega[n-1,k;I]P_{ji} & \text{if } J\in D\end{cases} \qquad (24)$$

with as initial conditions for $\Omega[0,1;J]$ and $\Omega[0,0;J]$

$$\Omega[0,1;J] = \begin{cases}\pi_J(0) & \text{if } J\in U\\ 0 & \text{otherwise}\end{cases} \qquad (25^a)$$

$$\Omega[0,0;J] = \begin{cases}\pi_J(0) & \text{if } J\in F\\ 0 & \text{otherwise}\end{cases} \qquad (25^b)$$

where $\pi_J(0)$ denote the initial state probabilities representing the state of the system at time 0. If we are interested in the interval unavailability distribution, we should take the steady state probabilities of the randomized Markov chain for $\pi_J(0)$.

Summarized, the uniformization method works as follows.

a. Model the system as a continuous-time Markov chain.
b. Uniformize the leaving rates at this Markov chain.
c. Calculate the steady-state probabilities ϕ_J for the randomized Markov chain.
d. Calculate the joint probabilities $\Omega[n,k;J]$ with $\pi_J(0)=\phi_J$ as initial state probabilities (see (24), (25^a) and (25^b)).

e. Aggregate the joint probabilities to obtain the marginal probabilities $\Omega[n,k]$ (see (23)).

f. Calculate the cumulative up time distribution from (22) with (21) as stop criterion.

We remark that a slightly modified, timesaving version of the recursive procedure exists. We used this modified version for our numerical experiments. We refer to De Souza e Silva and Gail [12] for details.

It is clear that the procedure based on the uniformization technique is easy to implement for general models. A drawback is that the state space of the Markov chain may grow beyond of control, particularly when the unit lifetimes and repair times have small coefficients of variation. The computation time increases strongly if the total transition rate Λ is large compared to the interval length t. Then a large number of terms is required for the truncation of the infinite sum.

5.2. Application of the uniformization technique to the 1-out-of-2 system

To model the 1-out-of-2 system, we have to approximate the unit lifetime and repair time distribution by a phase-type distribution. An obvious way to do this is to match the first two moments to a Hyperexponential distribution for $c^2>1$ (defined by (16)) or to a mixture of Erlang distributions for $c^2<1$ (defined by (19)). However, the state space increases strongly for small values of c^2. Therefore we choose the approach as suggested by Nojo and Watanabe [9]. They present a method to approximate a positive random variable by a two-stage continuous-time Markov chain regardless how small the coefficient of variation of the random variable is. This Markov chain is displayed in Figure 1.

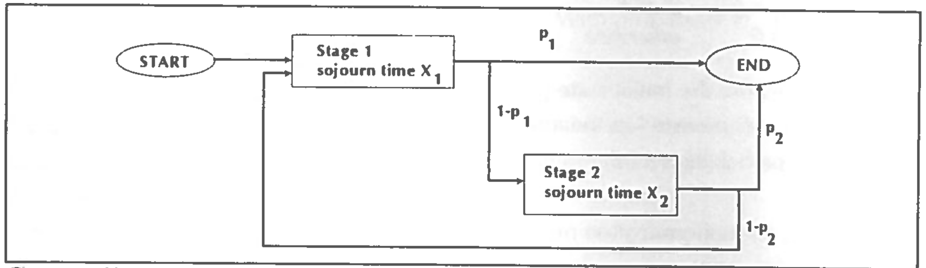

Figure 1. Two stage process of Nojo and Watanabe

The sojourn time in stage k is exponentially distributed with mean $1/\gamma_k$ (k=1,2). For their moment matching method, they assume that $\gamma_1=\gamma_2=\gamma$. First the process enters stage 1. Then the process is finished with probability p_1, while it enters stage 2 with probability $1-p_1$. When leaving stage 2, the process is finished with probability p_2 and it returns to stage 1 with probability $1-p_2$. An approximating stochastic process is given by the passage time from START to END. Negative values for p_1 and p_2 are allowed, therefore Nojo and Watanabe call this the NP-distribution (Negative branching Probability process).

A drawback of this method is that the two-stage process as shown in Figure 1 does *not* always represent a probability distribution function. This is the case for $c^2<0.5$, see the addendum in Chapter 1. We note that the NP-distribution is identical to the Hyperexponential distribution or the mixture of an Exponential and an Erlang(2) distribution with the same scale parameter if these conditions are satisfied. Despite the questionable modelling, we choose to use the NP-distribution, because Nojo and Watanabe report excellent approximate results for the long-run unavailability in the 1-out-of-2 system. This is even true when the two-stage process does not represent a probability distribution function.

Thus we approximate the unit lifetime and repair time distribution by NP-distribution with parameters (γ, p_1, p_2) respectively (δ, q_1, q_2). Then we can describe the 1-out-of-2 system by a continuous-time Markov chain with only 8 states. We define the system state as (i,j) where

$$i = \begin{cases} 1 & \textit{if the operating unit is in the first phase of its lifetime,} \\ 2 & \textit{if the operating unit is in the second phase of its lifetime,} \\ 3 & \textit{if the operating unit has failed and the repair facility is occupied} \end{cases}$$

$$j = \begin{cases} 1 & \textit{if the second unit is in the first phase of its repair time,} \\ 2 & \textit{if the second unit is in the second phase of its repair time,} \\ 3 & \textit{if the second unit is available and acts as a cold stanby} \end{cases}$$

Note that all combinations (i,j) are possible except (3,3), hence we have 8 system states that can be visited. The transition rates in the original continuous-time Markov chain are shown in Table 1. We proceed as follows. First we randomize the Markov chain. Then we calculate the stationary state probabilities ϕ_l (l=(i,j)) using the familiar solving technique. Now we can calculate the interval unavailability distribution using (21)-(25).

From state	To state (1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)
(1,1)	0	$q_1\delta$	$(1-q_1)\delta$	$p_1\gamma$	0	0	$(1-p_1)\gamma$	0
(1,2)	$q_2\delta$	0	$(1-q_2)\delta$	0	$p_1\gamma$	0	0	$(1-p_1)\gamma$
(1,3)	$(1-p_1)\gamma$	0	0	0	0	$p_1\gamma$	0	0
(2,1)	$p_2\gamma$	0	0	0	$q_1\delta$	$(1-q_1)\delta$	$(1-p_2)\gamma$	0
(2,2)	0	$p_2\gamma$	0	$q_2\delta$	0	$(1-q_2)\delta$	0	$(1-p_2)\gamma$
(2,3)	$(1-p_2)\gamma$	0	$p_2\gamma$	0	0	0	0	0
(3,1)	$(1-q_1)\delta$	0	0	0	0	0	0	$q_1\delta$
(3,2)	$(1-q_2)\delta$	0	0	0	0	0	$q_2\delta$	0

Table 1. Transition rates in the original (non-randomized) continuous-time Markov chain

6. Numerical comparison of methods

In this section we test the three methods on accuracy and computing speed. We choose the following parameters for our numerical experiments. The unit lifetimes are Weibull distributed with E[L]=1 for all cases. The squared coefficient of variation equals c_L^2=0.5 or 1.0. Hence we have exponential unit lifetimes in the latter case. The unit repair times have mean E[R]=0.25, so the unit availability equals 0.80. The unit repair times are deterministic (c_R^2=0) or have a gamma distribution (c_R^2=0.5, 1, 2 or 4). For the interval length we choose T=0.25, 1, 3 and 10. However, we show the approximation accuracy only for the most interesting cases, T=1 and T=3, to keep the tables clear. The other intervals did not show significantly different results for the approximation accuracy.

6.1. Approximation accuracy

To test the approximation accuracy, we used results from Monte Carlo simulation as benchmark. For the numerical integration in the expressions (2), (4), (12) and (13), we used Adaptive Gauss Quadrature (finite integrals) and the Gauss-Laguerre method (infinite integrals). The specific application of the latter method to integrals of the type $Z_i(\lambda)$ as defined by (18) is discussed in Van Hoorn [7]. However, we experienced serious problems with numerical stability if i is not very small. This applies to the case of Erlang distributed unit repair times, in particular to the third and fourth moment of τ_{down}.

In the tables 2-5 we show the long term unavailability U_∞ and some points Q_x (=U(100x%;T)) from the interval unavailability distribution. We have that c_L^2=0.5 in Table 2 and 3, while c_L^2=1.0 in Table 4 and 5. The interval length equals T=1 in Table 2 and 4 and T=3 in Table 3 and 5. The half lengths of the 95% confidence intervals for Q_x never exceed 0.002 and are often less than 0.001. For the long term unavailability U_∞ we have that the half lengths of the 95% confidence intervals is less than 0.02. The results in the tables are indicated with respectively

- "Sim" (results from Monte Carlo simulation),
- "Tak M/M" (formula of Takács with exponential on and off times),
- "Tak G/G" (formula of Takács with gamma distributed on and off times),
- "SSG" (uniformization method by De Souza e Silva and Gail)

Note that $E[\tau_{up}]$ and $E[\tau_{down}]$ are calculated exactly, hence the values for the long term unavailability U_∞ in Table 2-5 are exact for the methods "Tak M/M" and "Tak G/G". The uniformization methods gives only approximate results, because the unit lifetime and repair time distributions are approximated by NP-distributions.

c_R^2	Method	U_∞	Q_0	Q_2	Q_5	Q_{10}	Q_{25}
0.00	Sim	1.16	0.110	0.097	0.079	0.052	0.003
	Tak M/M	1.16	0.117	0.096	0.071	0.043	0.009
	Tak G/G	1.16	0.110	0.104	0.084	0.047	0.004
	SSG	1.15	0.119	0.106	0.088	0.062	0.019
0.50	Sim	2.09	0.126	0.113	0.097	0.073	0.029
	Tak M/M	2.09	0.131	0.116	0.097	0.072	0.028
	Tak G/G	2.09	0.125	0.115	0.098	0.073	0.027
	SSG	2.06	0.124	0.112	0.095	0.072	0.028
1.00	Sim	2.99	0.138	0.126	0.111	0.089	0.046
	Tak M/M	2.99	0.142	0.130	0.114	0.091	0.045
	Tak G/G	2.99	0.137	0.126	0.110	0.089	0.045
	SSG	2.96	0.136	0.125	0.110	0.088	0.045
2.00	Sim	4.57	0.154	0.144	0.131	0.112	0.070
	Tak M/M	4.57	0.157	0.148	0.135	0.115	0.071
	Tak G/G	4.57	0.153	0.142	0.128	0.109	0.069
	SSG	4.57	0.153	0.142	0.129	0.110	0.070
4.00	Sim	6.98	0.172	0.164	0.153	0.137	0.100
	Tak M/M	6.97	0.174	0.167	0.157	0.142	0.103
	Tak G/G	6.97	0.171	0.160	0.149	0.133	0.098
	SSG	6.98	0.168	0.159	0.148	0.132	0.099

Table 2. Comparison between simulated and approximate unavailability distribution of an interval with length 1 for c_L^2=0.5

c_R^2	Method	U_∞	Q_0	Q_2	Q_5	Q_{10}	Q_{25}
0.00	Sim	1.16	0.279	0.202	0.098	0.012	0.000
	Tak M/M	1.16	0.294	0.179	0.084	0.023	0.000
	Tak G/G	1.16	0.279	0.210	0.082	0.012	0.000
	SSG	1.15	0.313	0.234	0.138	0.049	0.001
0.50	Sim	2.09	0.302	0.235	0.155	0.072	0.005
	Tak M/M	2.09	0.316	0.236	0.151	0.071	0.006
	Tak G/G	2.09	0.302	0.238	0.154	0.070	0.005
	SSG	2.06	0.298	0.232	0.153	0.071	0.005
1.00	Sim	2.99	0.317	0.259	0.190	0.112	0.021
	Tak M/M	2.99	0.330	0.267	0.193	0.112	0.020
	Tak G/G	2.99	0.317	0.258	0.189	0.111	0.021
	SSG	2.96	0.314	0.256	0.188	0.111	0.021
2.00	Sim	4.57	0.333	0.285	0.229	0.161	0.056
	Tak M/M	4.57	0.343	0.298	0.241	0.167	0.053
	Tak G/G	4.57	0.333	0.281	0.226	0.160	0.056
	SSG	4.57	0.330	0.281	0.226	0.161	0.057
4.00	Sim	6.98	0.343	0.307	0.263	0.207	0.103
	Tak M/M	6.97	0.348	0.318	0.277	0.220	0.105
	Tak G/G	6.97	0.342	0.296	0.252	0.198	0.099
	SSG	6.98	0.333	0.292	0.251	0.203	0.107

Table 3. Comparison between simulated and approximate unavailability distribution of an interval with length 3 for c_L^2=0.5

c_R^2	Method	U_∞	Q_0	Q_2	Q_5	Q_{10}	Q_{25}
0.00	Sim	2.80	0.195	0.179	0.157	0.120	0.020
	Tak M/M	2.80	0.221	0.190	0.151	0.102	0.031
	Tak G/G	2.80	0.194	0.189	0.170	0.117	0.021
	SSG	2.80	0.217	0.201	0.176	0.135	0.050
0.50	Sim	3.87	0.199	0.184	0.162	0.128	0.058
	Tak M/M	3.86	0.221	0.198	0.168	0.128	0.054
	Tak G/G	3.86	0.198	0.185	0.163	0.128	0.055
	SSG	3.86	0.199	0.184	0.161	0.128	0.058
1.00	Sim	4.96	0.202	0.187	0.167	0.137	0.074
	Tak M/M	4.76	0.220	0.202	0.178	0.143	0.073
	Tak G/G	4.76	0.200	0.186	0.165	0.136	0.073
	SSG	4.76	0.202	0.187	0.167	0.137	0.074
2.00	Sim	6.24	0.206	0.192	0.175	0.150	0.095
	Tak M/M	6.24	0.220	0.206	0.188	0.160	0.098
	Tak G/G	6.24	0.204	0.188	0.170	0.146	0.093
	SSG	6.22	0.206	0.191	0.172	0.147	0.095
4.00	Sim	8.33	0.209	0.198	0.184	0.164	0.119
	Tak M/M	8.36	0.218	0.209	0.196	0.176	0.125
	Tak G/G	8.36	0.208	0.192	0.178	0.159	0.117
	SSG	8.24	0.209	0.195	0.178	0.156	0.115

Table 4. Comparison between simulated and approximate unavailability distribution of an interval with length 1 for c_L^2=1.0

c_R^2	Method	U_∞	Q_0	Q_2	Q_5	Q_{10}	Q_{25}
0.00	Sim	2.80	0.443	0.371	0.251	0.075	0.001
	Tak M/M	2.80	0.499	0.356	0.211	0.084	0.004
	Tak G/G	2.80	0.444	0.394	0.229	0.072	0.001
	SSG	2.80	0.450	0.369	0.241	0.095	0.001
0.50	Sim	3.87	0.439	0.367	0.268	0.148	0.019
	Tak M/M	3.86	0.488	0.385	0.267	0.142	0.018
	Tak G/G	3.86	0.439	0.372	0.267	0.144	0.017
	SSG	3.86	0.439	0.367	0.268	0.148	0.019
1.00	Sim	4.96	0.434	0.366	0.283	0.181	0.043
	Tak M/M	4.76	0.477	0.396	0.298	0.182	0.036
	Tak G/G	4.76	0.434	0.366	0.282	0.179	0.041
	SSG	4.76	0.434	0.366	0.283	0.181	0.043
2.00	Sim	6.24	0.425	0.366	0.298	0.216	0.081
	Tak M/M	6.24	0.459	0.402	0.327	0.229	0.073
	Tak G/G	6.24	0.424	0.360	0.295	0.214	0.080
	SSG	6.22	0.424	0.360	0.292	0.213	0.082
4.00	Sim	8.33	0.407	0.359	0.308	0.243	0.124
	Tak M/M	8.36	0.431	0.393	0.342	0.269	0.125
	Tak G/G	8.36	0.406	0.348	0.297	0.235	0.120
	SSG	8.24	0.407	0.344	0.289	0.232	0.127

Table 5. Comparison between simulated and approximate unavailability distribution of an interval with length 3 for c_L^2=1.0

A summary of the approximation accuracy for all interval (including T=0.25 and T=10) is given in Table 6. This summary is based on all cases analyzed, including the interval length T=0.25 and T=10. For each case, the following 32 points from the interval unavailability distribution were calculated:

- x=0%-5% with steps of 0.5%,
- x=5%-10% with steps of 1%
- x=10%-25% with steps of 2.5%,
- x=25%-50% with steps of 5%,
- x=50%-100% with steps of 10%.

We give two statistics for the differences between M simulated values Q_m(sim) and M approximated values Q_m(appr), m=1..M:

a. the Mean Absolute Deviation (MAD), defined as

$$MAD = \frac{1}{M}\sum_{m=1}^{M}|Q_m(sim) - Q_m(appr)| \qquad (26)$$

b. the Maximal Absolute Error (MAE), defined as

$$MAE = \max_{m=1,\ldots,M} |Q_m(sim) - Q_m(appr)| \qquad (27)$$

Summary statistic	Method	Length of interval (T) 0.25	1	3	10	All intervals
Mean Absolute Deviation	Tak M/M	0.0011	0.0051	0.0099	0.0105	0.0067
	Tak G/G	0.0011	0.0024	0.0033	0.0021	0.0022
	SSG	0.0010	0.0035	0.0049	0.0065	0.0040
Maximal Absolute Error	Tak M/M	0.005	0.026	0.056	0.057	0.057
	Tak G/G	0.005	0.014	0.027	0.026	0.027
	SSG	0.004	0.030	0.051	0.076	0.076

Table 6. Summary of approximation accuracy

Table 6 shows that the most accurate results are obtained using the formula of Takács with gamma distributed on- and off periods. The use of exponential on-and off times gives less accurate results. This is caused by the fact that the system on period is not almost exponentially distributed in many cases. Extensive numerical analysis showed that $c^2[\tau_{up}]$ may differ significantly from 1 (the value for the exponential distribution), in particular if c_L^2 is rather high (say ≥1) or the unit availability is quite low (say less thatn 90%). Apparently the event R>L is not "rare" enough in these cases. The method based on the uniformization technique gives no exact results since the unit lifetime and repair time distributions are replaced by NP-distributions. The effect of this approximation is particularly significant when the two-stage process does not represent a probability distribution function (e.g. for c_R^2=0). For other cases the results of the uniformization method are about as accurate as the "Tak G/G" method. This shows that the uniformization method is useful, but that one should take care to use the NP-distribution if the conditions to represent a probability distribution are not satisfied.

6.2. Computing speed

Next to accuracy of the approximations, the computing speed is relevant. We measured the time required to calculate the 32 points of the interval unavailability distribution for all cases and methods. The calculations are carried out on an IBM-compatible PC with an Intel 386DX / 25 MHz processor with numerical co-processor. Table 7 shows indicative values for the average per set of 32 points in seconds.

It is clear that the least accurate method, based on exponentially distributed on and off times, is by far the fastest method. The value of T has hardly any effect on the computing time. If we use gamma distributions instead of exponential distributions, the computing time increases because many times the incomplete gamma function has to be evaluated. As T increases, more terms of the infinite sum in (8) and thus more evaluations of the incomplete gamma function are required. This explains why the computing time increases with T though the increase is within reasonable bounds.

	Method	Length of interval (T) 0.25	1	3	10	Average over all intervals
Time (sec.)	Tak M/M	0.12	0.16	0.12	0.18	0.14
	Tak G/G	2.47	3.62	4.66	6.95	4.43
	SSG	2.52	8.08	27.72	108.7	36.76

Table 7. CPU time to calculate 32 points of the interval unavailability distribution (average values over all c_L^2 and c_R^2)

The uniformization method however shows a much stronger dependency of the computational effort on the interval length T. As T grows, more terms of the infinite sum in (21) are required ad considerably more values Ω[n,k] have to be calculated. This is the main reason to prefer the "Tak G/G" method over the uniformization method, in particular when the interval length T is large compared to the mean unit lifetime.

6.3. Sensitivity analysis

Finally we mention that the sensitivity of the system performance to the shape of the unit lifetime and repair time distribution was extensively analyzed. Main conclusions from this sensitivity analysis are:

1. the shape of the unit lifetime and repair time distributions have a large effect on the system up time and down time distribution. Just assuming exponential distributions for L and R, as is done by Brouwers [2], can not be justified.
2. The shape of the unit lifetime distribution has a larger effect on U_∞ than the shape of the unit repair time distribution. However, c_R^2 has a larger effect than in the ample repair facility case (compare to Van der Heijden [5]). Therefore one should be more careful to assume exponentially distributed repair times when analyzing the performance of this reliability system, in particular when considering the performance measure U_∞.
3. Even the third moment of the unit lifetimes may have a significant effect on the long term unavailability if c_L^2 differs strongly from 1 (e.g. c_L^2=0.5). The sensitivity of U_∞ to the shape of the unit repair time distribution is mainly restricted to the first two moments, unless the variation in the unit repair times is very high (say c_R^2≥2).

7. Conclusions

Three methods to calculate the interval unavailability distribution were studied in this paper. These methods were applied to the 1-out-of-2 cold standby system with single repair facility. Based on our numerical experiments we conclude:

a. The formula of Takács with exponential on and off times (cf. Van der Heijden [5]) is easy to implement, very fast and reasonably accurate. It is the preferred method for quick engineering calculations.

b. The accuracy of the former method is significantly improved by using gamma distributed on- and off times. This method is based on the approach by Van der Heijden and Schornagel [6].

c. The uniformization technique as used by De Souza e Silva and Gail [12], [13] is generally applicable and is not very difficult to implement. Accurate results are obtained, but the computational effort of the method makes it less suitable for use in practice. Moreover, the uniformization technique works only satisfactorily if the unit lifetime and repair time distributions can be modelled by two-phase processes that represent actual probability distributions.

References

1. Birolini, A. [1985]. *On the use of stochastic processes in modelling reliability problems.* Lecture notes in Economics and Mathematical Systems no. 252. Springer Verlag, Berlin.
2. Brouwers, J.J.H. [1986]. *Probabilistic descriptions of irregular system down time.* Reliability Engineering 16.
3. Gertsbakh, *I. [1984].* Asymptotic methods in reliability: A review. *Advances in Applied Probability* 16: 147-175.
4. Goyal, A. and A.N. Tantawi [1988]. A measure of guaranteed availability and its numerical evaluation. *IEEE transactions on computers* 37, 25-32.
5. van der Heijden, M.C. [1987]. Interval availability distribution for a 1-out-of-2 reliability system with repair. *Probability in the Engineering and Informational Sciences* 1: 211-224.
6. van der Heijden, M.C. and A. Schornagel [1988]. Interval uneffectiveness distribution for a k-out-of-n multistate reliability system with repair. *European Journal of Operational Research* 36: 66-77.
7. van Hoorn, M.H. [1984]. *Algorithms and approximations for queueing systems.* CWI tract. no. 8, Centre for Mathematics and Computer Science, Amsterdam.
8. Keilson, J. [1979]. *Markov chain models: Rarity and exponentiality.* Springer Verlag, Berlin.
9. Nojo, S. and H. Watanabe (1987). A new stage method getting arbitrary coefficient of variation by two stages. *The transactions of the IEICE* vol. E 70 no. 1: 33-36.
10. Reibman, A., R. Smith and K. Trivedi [1989]. Markov and Markov reward model transient analysis: An overview of numerical approaches. *European Journal of Operational Research* 40: 257-267.
11. van Rijn, C.F.H and A. Schornagel [1988]. STAMP - A new technique to assess the probability distribution of system effectiveness. *The SRS Quarterly Digest, supplement: Proceedings of the seventeenth annual meeting of the systems reliability service associate members 29-30 october 1987*: 50-62.
12. de Souza e Silva, E. and H.R. Gail [1986]. Calculating cumulative operational time distributions of repairable computer systems. *IEEE transactions on computers* vol. C-35 no. 4: 322-332.
13. de Souza e Silva, E. and H.R. Gail [1989]. Calculating availability and performability measures of repairable computer systems using randomization. *Journal of Association for Computing Machinery* 36: 171-193.
14. Takács, L. [1957]. On certain sojourn time problems in the theory of stochastic processes. *Acta Mathematica Academiae Scientiarum Hungaricae* 8: 169-191.
15. Tijms, H.C. [1986]. *Stochastic modelling and analysis: A computational approach.* Wiley, New York.
16. Wartenhorst, P. [1993]. *Performance analysis of repairable systems,* Ph.D. dissertation, Tilburg University.
17. Yearout, R.D., P. Reddy and D.L. Grosh [1986]. Standby redundancy in reliability - A review. *IEEE Transactions on Reliability* R-35: 286-292.