



# ADMISSIBLE OPERATORS FOR SUN-DUAL SEMIGROUPS

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ABSTRACT. We extend classical duality results by Weiss on admissible operators to settings where the dual semigroup lacks strong continuity. This is possible using the sun-dual framework, which is not immediate from the duality of the input and output maps. This extension enables the testing of admissibility for a broader range of examples, in particular for state space of continuous functions or  $L^1$ .

## 1. INTRODUCTION

Our starting point is linear time-invariant systems of the form

$$\Sigma(A, B, C) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geq 0 \\ y(t) = Cx(t), & t \geq 0 \\ x(0) = x_0; \end{cases}$$

where  $x(t)$  denotes the state of the system at time  $t$ ,  $u(t)$  denotes the input, and  $y(t)$  denotes the output. The state, input, and output spaces are denoted by  $X, U$ , and  $Y$  respectively and are assumed to be Banach spaces. Moreover,  $A$  is assumed to generate a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $X$ . For systems described by (time-dependent) PDEs and with controls and observations (measurements) acting on the spatial boundary, the operators  $B$  and  $C$  become “unbounded” with respect to the state space  $X$ , in the sense that only  $B \in \mathcal{L}(U, X_{-1})$  and  $C \in \mathcal{L}(X_1, Y)$ ; where  $X_{-1}$  denotes the extrapolation space associated to  $(T(t))_{t \geq 0}$  and  $X_1$  denotes the interpolation space  $\text{dom } A$ . This approach is explained in [20, 40, 41]; see also [42, 44, 46].

For each  $x_0 \in X$ , the system  $\Sigma(A, B, 0)$  has a mild solution in  $X_{-1}$  given by

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s) ds;$$

where  $(T_{-1}(t))_{t \geq 0}$  denotes the extrapolated semigroup on  $X_{-1}$ . In this case, it makes sense to ask whether the solution lies in  $X$ , which gives rise to the notion of the admissibility of control operators. Let  $Z$  be a placeholder for  $\mathbb{C}$  or  $L^p$  with  $p \in [1, \infty]$ . We say that  $B$  is a  $Z$ -admissible control operator if for some (equivalently, all)  $\tau > 0$ , the input map – defined as

$$\Phi_\tau : Z([0, \tau], U) \rightarrow X_{-1}, \quad u \mapsto \int_0^\tau T_{-1}(\tau-s)Bu(s) ds \quad (1.1)$$

satisfies  $\text{Rg } \Phi_\tau \subseteq X$ . Correspondingly, the solution of  $\Sigma(A, 0, C)$  is given by

$$y(t) = CT(t)x_0, \quad (x_0 \in X_1).$$

We say that  $C$  is a  $Z$ -admissible observation operator if the output map

$$\Psi_\tau : X_1 \rightarrow Z([0, \tau], Y), \quad x \mapsto CT(\cdot)x \quad (1.2)$$

has a bounded extension to  $X$  for some (equivalently, all)  $\tau > 0$ .

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The concept of admissible operators is fundamental to the investigation of infinite-dimensional systems, and their significance is especially pronounced in the realm of well-posed systems, where they facilitate the establishment of stability, controllability, and observability of such systems [30, 44, 47]. The theory of admissibility for the case  $Z = L^2$  and  $X$  being a Hilbert space is classical [29, 44, 46]. The case  $p \in (1, \infty)$  has also garnered significant attention [24, 25, 31, 50, 51]. In this context, the Weiss duality result [51, Theorem 6.9] plays an important role. In particular, under the assumption that the dual semigroup  $(T'(t))_{t \geq 0}$  is strongly continuous and  $p, q \in (1, \infty)$  are Hölder conjugates, it says that  $B$  is a  $L^p$ -admissible control operator if and only if  $B'$  is a  $L^q$ -admissible observation operator and analogously, for  $C$ 's. More recently, there is growing interest in “limit-case” admissibility [5, 28, 33, 36, 39], referring in particular to  $L^\infty$ -admissible control operators because of their importance in the study of input-to-state stability (ISS); see [28, 36].

The Weiss duality result enables the translation of various (negative) results between control and observation operators, especially when the state space is reflexive. In practice, however, there are multiple situations when the dual semigroup  $(T'(t))_{t \geq 0}$  is not strongly continuous on  $X'$  or when  $X$  has no pre-dual, for instance, when  $X$  is an  $L^1$ -space – which is often the case when studying  $L^1$ -admissibility for observation operators. As a result, various facts known for control operators cannot be translated to the observation operators and vice versa. An important example here is [50, Theorem 4.8] which says that if  $X$  is reflexive, then  $B$  is a  $L^1$ -admissible control operator if and only if  $B \in \mathcal{L}(U, X)$ . The reflexivity of  $X$  cannot be dropped as is shown in [50, Negative result 5.4] by taking a periodic left shift semigroup on  $L^1[0, 2\pi]$ . Since  $L^1[0, 2\pi]$  does not have a predual, the same example cannot be used to show the existence of an unbounded  $L^\infty$ -admissible observation operator. Similarly, the fact that all  $L^1$ -admissible control operators are those that map into the Favard space associated with the extrapolated semigroup [35, Corollary 17] cannot be dualized if the dual semigroup lacks strong continuity.

In operator semigroups, the classical approach to circumvent the above issue of strong continuity is restricting the dual semigroup to the closed subspace on which the dual semigroup is strongly continuous; the *sun-dual space*. Remarkably, this (still) allows for a rich theory, mostly developed in the 1980s, see [9–12, 14], as well as the monograph by van Neerven [48]. These works take motivation ranging from classical age population models over delay equations [15, 16] to models arising in neuroscience [43], where  $L^1$ - and sup-norms are naturally appearing. However, in the context of admissible operators and more generally infinite-dimensional systems theory, sun-duality has hardly been employed; see [26] for controllability results and [13] for some optimal control problems on non-reflexive spaces. This is the gap we would like to close in the present paper. Our original motivation for this lies in characterising  $Z$ -admissible operators, particularly for semigroups with a non-trivial sun-dual. Let us showcase why this is of interest: it is still an open question whether  $L^\infty$ -admissible control operators are always *zero-class*, i.e., whether  $\lim_{\tau \rightarrow 0^+} \|\Phi_\tau\|_{\mathcal{L}(L^\infty([0, \tau], U), X)} = 0$ , with  $\Phi_\tau$  defined in (1.1); see for instance, [28, Section 6]. On the other hand, the formally dual question can be answered in the negative [33, Example 26]: there exists  $L^1$ -admissible observation operators such that  $\lim_{\tau \rightarrow 0^+} \|\Psi_\tau\|_{\mathcal{L}(X, L^1([0, \tau], Y))} \neq 0$ . It is not possible to link these two settings by the usual duality as the involved function spaces are  $L^1$ -spaces and  $L^\infty$ -spaces. Moreover, by Lotz’s result [34], any strongly continuous semigroup on  $L^\infty[0, 1]$ , the dual of the state space of the mentioned counterexample, has a bounded generator, which readily implies zero-class admissibility. In [32], it was indeed shown that  $L^\infty$ -admissibility of  $B = A_{-1}$ , the extension of  $A$  to an operator from  $X$  to  $X_{-1}$ , implies that  $A$  is bounded, resting on deep results from the

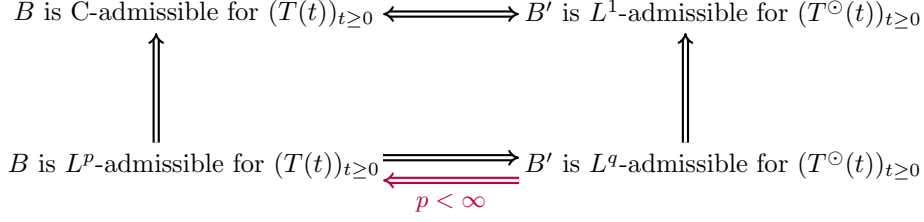


FIGURE 1. Duality between control operators  $B \in \mathcal{L}(U, X_{-1})$  and observation operators  $B' \in \mathcal{L}(X^\odot_1, U')$

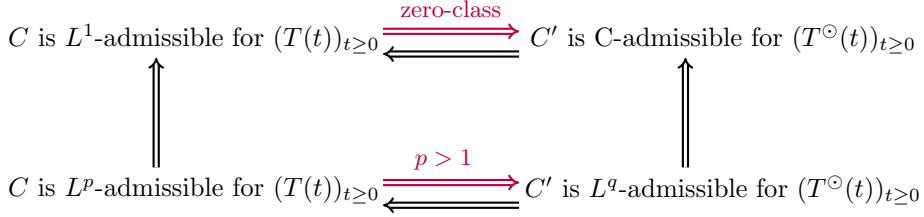


FIGURE 2. Duality for observation operators  $C \in \mathcal{L}(X_1, Y)$  with  $\text{Rg } C' \subseteq (X^\odot)_{-1}$  and control operators  $B' \in \mathcal{L}(Y', (X^\odot)_{-1})$ .

geometry of Banach spaces and a connection to maximal regularity for parabolic equations. Our results show that the sun-duality is the right framework to dualise these situations; in particular we show in Theorem 3.1 that C-admissibility of control operators  $B$  is the proper dual concept for  $B'$  being an  $L^1$ -admissible observation operator with respect to the sun-dual semigroup.

We note the connection of admissible operators to perturbation theory for operator semigroups, given by the classical Miyadera-Voigt and Desch-Schappacher theorems, see, for example, [21, Chapter 3]. In the system-theoretic context described above, these can in essence be phrased as follows: if perturbations  $C \in \mathcal{L}(X_1, X)$  or  $B \in \mathcal{L}(X, X_{-1})$  are zero-class  $L^1$ - or C-admissible, respectively, then the perturbed semigroup  $A + C$  or the part of  $A_{-1} + B$  in  $X$ , respectively, generate  $C_0$ -semigroups. It is worth mentioning that the sun-dual theory [9] originated from perturbation results around the same time. More precisely, in [9], see also [48, Theorems 3.2.6 and 4.3.5], it was shown that if  $B \in \mathcal{L}(X, X^{\odot \times})$ , then the part of  $A_{-1} + B$  in  $X$  generates a  $C_0$ -semigroup, where the space  $X^{\odot \times}$  can be isomorphically identified with the Favard space of the extrapolated semigroup on  $X_{-1}$ . We skip the definitions of those spaces but point out that  $\mathcal{L}(X, X^{\odot \times})$  is isomorphic to the set of  $L^1$ -admissible control operators from  $X$  to  $X_{-1}$ , in [35, Corollary 17].

Our duality results are given in Sections 3 and 4, generalising the duality result by Weiss from [51, Theorem 6.9], dropping any condition of the form  $X^\odot = X'$ . For convenience, we summarise the scenario in Figures 1 and 2. The article concludes with a prototypical example for which the limit-case admissibility is characterised.

**Preliminaries.** Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on a Banach space  $X$ . We use the notation  $X^\odot$ , to denote the subspace of  $X'$  where the dual semigroup  $(T(t)')_{t \geq 0}$  is strongly continuous. The restricted  $C_0$ -semigroup is as usual denoted by  $(T^\odot(t))_{t \geq 0}$ . For the theory of sun-dual semigroups, we refer the reader to [48].

Let  $U$  and  $Y$  be Banach spaces and let  $Z$  be a placeholder for  $C$  or  $L^p$ . For  $B \in \mathcal{L}(U, X_{-1})$ , we say that  $B$  is a *zero-class Z-admissible* control operator if the input map in (1.1) satisfies  $\lim_{\tau \downarrow 0} \|\Phi_\tau\|_{\mathcal{L}(Z([0, \tau], U), X)} = 0$ . Likewise,  $C \in \mathcal{L}(X_1, Y)$

is called a *zero-class Z-admissible* observation operator if the output map in (1.2) fulfils  $\lim_{\tau \downarrow 0} \|\Psi_\tau\|_{\mathcal{L}(X, \mathcal{Z}([0, \tau], Y))} = 0$ . For  $p \in [1, \infty]$ , we write  $\mathbb{C}_p(X, Y, (T(t))_{t \geq 0})$  for the subspace of  $\mathcal{L}(X_1, Y)$  of all  $L^p$ -admissible observation operators and set

$$\|C\|_{\mathbb{C}_p(X, Y, \tau)} := \|\Psi_\tau\|_{\mathcal{L}(X, L^p([0, \tau], Y))}.$$

Similarly,  $\mathbb{B}_p(U, X, (T(t))_{t \geq 0})$  denotes the subspace of  $\mathcal{L}(U, X_{-1})$  of all  $L^p$ -admissible control operators with

$$\|B\|_{\mathbb{B}_p(U, X, \tau)} := \|\Phi_\tau\|_{\mathcal{L}(L^p([0, \tau], U), X)}$$

For convenience, the notation  $\mathbb{B}_C(U, X, (T(t))_{t \geq 0})$  is sometimes used to denote the C-admissible control operators with

$$\|B\|_{\mathbb{B}_C(U, X, \tau)} := \|\Phi_\tau\|_{\mathcal{L}(C([0, \tau], U), X)}$$

denoting the corresponding norm.

## 2. CHARACTERISATION OF C-ADMISSIBILITY OF CONTROL OPERATORS

Let  $U$  be a Banach space and denote by  $T([0, \tau], U)$ , the space of all  $U$ -valued step functions on  $[0, \tau]$ , i.e., piecewise constant functions with finitely many pieces. Equipped with the supremum norm,  $T([0, \tau], U)$  becomes a normed space whose completion is the space of *regulated* functions  $\text{Reg}([0, \tau], U)$ . One can therefore define Reg-admissibility by replacing  $\mathcal{Z}$  by  $\text{Reg}$  in (1.1). Since every continuous function is regulated, it is immediate that Reg-admissibility implies C-admissibility. Actually, the two notions are even equivalent [5, Proposition 4.2]. The following result, which is an extension of [44, Theorem 10.2.2] adapting an argument of Travis [45, Lemma 3.1 and Proposition 3.1], see also [32, Proposition 2.2], characterizes the class of all C-admissible control operators.

Let  $X$  and  $U$  be Banach spaces. Recall that the *semivariation* of a function  $f : [0, \tau] \rightarrow \mathcal{L}(U, X)$  is defined as

$$\text{SV}_0^\tau(f) := \sup_{\substack{\|u_i\|_U \leq 1 \\ 0=t_1 < t_2 < \dots < t_n = \tau \\ n \in \mathbb{N}}} \left\| \sum_{i=1}^n (f(t_i) - f(t_{i-1}))u_i \right\|_X$$

and  $f$  is said to be of *bounded semivariation* on  $[0, \tau]$  if  $\text{SV}(f) < \infty$ . Moreover, the *variation* of  $f$  is given by

$$\text{var}_0^\tau(f) := \sup_{\substack{0=t_1 < t_2 < \dots < t_n = \tau \\ n \in \mathbb{N}}} \sum_{i=1}^n \|f(t_i) - f(t_{i-1})\|_{\mathcal{L}(U, X)}$$

and  $f$  is said to have *bounded variation* on  $[0, \tau]$  if  $\text{var}(f) < \infty$ . Clearly,  $\text{SV}_0^\tau(f) \leq \text{var}_0^\tau(f)$ . A thorough treatment of functions of bounded variation can be found in [4] and for functions of bounded semivariation, we refer to the survey [37].

**Proposition 2.1.** Let  $X$  and  $U$  be Banach spaces,  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ , let  $\lambda \in \rho(A)$ , and let  $\tau > 0$ . For the control operator  $B \in \mathcal{L}(U, X_{-1})$ , the following are equivalent.

- (i) The control operator  $B$  is C-admissible.
- (ii) The function  $T(\cdot)R(\lambda, A_{-1})B$  is of bounded semivariation on  $[0, \tau]$ .
- (iii) For each  $x' \in X'$  with  $\|x'\| \leq 1$ , the function  $B'R(\bar{\lambda}, A')T(\cdot)'x'$  is of bounded variation on  $[0, \tau]$ .
- (iv) The control operator  $B$  is Reg-admissible.

Moreover, setting  $F(\cdot) := T(\cdot)R(\lambda, A_{-1})B$ , we have that

$$(1 - \|\lambda R(\lambda, A)\|) \|B\|_{\mathbb{B}_C(U, X, \tau)} \leq \text{var}_0^\tau(F(\cdot)'x')$$

and

$$\text{var}_0^\tau(F(\cdot)'x') \leq 2 \text{SV}_0^\tau(F) \leq 2 \|1 - \lambda R(\lambda, A)\| \|B\|_{\mathbb{B}_C(U, X, \tau)}$$

for all  $x' \in X'$  with  $\|x'\| \leq 1$ .

*Proof.* The equivalence of (iii) and (iv) and the first inequality is proved in [44, Theorem 10.2.2], whereas implication (iv)  $\Rightarrow$  (i) is obvious.

“(i)  $\Rightarrow$  (ii)”: Let  $B$  be  $C$ -admissible and for simplicity, suppose  $\tau = 1$ . Consider a partition  $0 = t_0 < t_1 < \dots < t_n = \tau$  of  $[0, 1]$  and let  $\epsilon < \min |t_i - t_{i-1}|$ . Fix arbitrary elements  $u_1, \dots, u_{n+1} \in U$  with  $\|u_i\| \leq 1$  and define  $u_\epsilon : [0, 1] \rightarrow X$  as

$$u_\epsilon(s) := \begin{cases} u_i, & t_{i-1} \leq s \leq t_i - \epsilon \\ u_{i+1} + (u_{i+1} - u_i) \frac{s - t_i}{\epsilon}, & t_i - \epsilon \leq s \leq t_i. \end{cases}$$

Since  $u_\epsilon$  is continuous, so admissibility implies that  $\Phi_1 u_\epsilon \in X$ . For simplicity, we set  $w_i = R(\lambda, A_{-1})B u_i \in \text{dom}(A_{-1}) = X$  for  $1 \leq i \leq n_1$ . Then we can write

$$\begin{aligned} (\lambda R(\lambda, A) - 1)\Phi_1 u_\epsilon &= AR(\lambda, A)\Phi_1 u_\epsilon \\ &= A \int_0^1 T(1-s)R(\lambda, A_{-1})B u_\epsilon(s) \, ds \\ &= A \left[ \sum_{i=1}^n \int_{t_{i-1}}^{t_i} T(1-s)w_i \, ds \right. \\ &\quad \left. + \int_{t_i-\epsilon}^{t_i} T(1-s)(w_{i+1} + (w_{i+1} - w_i) \frac{s - t_i}{\epsilon}) \, ds \right] \end{aligned}$$

Now, we can repeat the computations in the proof of [45, Proposition 3.1] to deduce that  $\|\sum_{i=1}^n (T(t_i) - T(t_{i-1}))R(\lambda, A_{-1})B u_i\|$  can be estimated from above by

$$\|(\lambda R(\lambda, A) - 1)\Phi_1\| + \sum_{i=1}^n \left\| \frac{1}{\epsilon} \int_{t_i-\epsilon}^{t_i} T(1-s)(w_{i+1} - w_i) \, ds - T(1-t_i)(w_{i+1} - w_i) \right\|$$

Taking  $\epsilon \rightarrow 0$  yields that  $F$  is of bounded semivariation and

$$\text{SV}_0^\tau(F) \leq \|1 - \lambda R(\lambda, A)\| \|B\|_{\mathbb{B}_C(U, X, \tau)}.$$

“(ii)  $\Rightarrow$  (iii)”: Let  $x' \in X'$  with  $\|x'\| \leq 1$ . We need to show that  $F(\cdot)'x'$  is of bounded variation. Consider a partition  $0 = t_0 < t_1 < \dots < t_n = \tau$  of  $[0, \tau]$ . For each  $1 \leq i \leq n$ , choose  $u_i \in U$  with  $\|u_i\| \leq 1$  such that

$$\begin{aligned} \frac{1}{2} \|F(t_i)'x' - F(t_{i-1})'x'\|_{U'} &\leq \langle F(t_i)'x' - F(t_{i-1})'x', u_i \rangle \\ &= \langle x', (F(t_i) - F(t_{i-1}))u_i \rangle. \end{aligned}$$

Employing bounded semivariation of  $F(\cdot)$  together with

$$\begin{aligned} \sum_{i=1}^n \|F(t_i)'x' - F(t_{i-1})'x'\|_{U'} &\leq 2 \left\langle x', \sum_{i=1}^n (F(t_i) - F(t_{i-1}))u_i \right\rangle \\ &\leq 2 \left\| \sum_{i=1}^n (F(t_i) - F(t_{i-1}))u_i \right\|. \end{aligned}$$

yields bounded variation of  $F(\cdot)'x'$  and  $\text{var}_0^\tau(F(\cdot)'x') \leq 2 \text{SV}_0^\tau(F)$ .  $\square$

## 3. DUALITY RESULTS FOR CONTROL OPERATORS

In [51, Theorem 6.9], Weiss explored the dual relationship between  $L^p$ -admissible observation operators and  $L^q$ -admissible control operators for Hölder conjugates  $p$  and  $q$ . The result, however, assumes strong continuity of the dual semigroup. Restricting to the sun-dual space – the space of strong continuity of the dual semigroup – it is natural to ask whether [51, Theorem 6.9] can be appropriately generalised. We explore this for control operators in the present section.

In our first result, we show that C-admissibility of the control is equivalent to  $L^1$ -admissibility of the dual observation operator. Keeping Proposition 2.1 in mind, the proof of the necessity in the reflexive case was given in [44, Theorem 10.2.2] and the converse for the case  $X^\circ = X'$  was indicated in [51, Remark 6.10].

**Theorem 3.1.** Let  $X$  and  $U$  be Banach spaces and let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ .

A control operator  $B \in \mathcal{L}(U, X_{-1})$  is (zero-class) C-admissible if and only if the observation operator  $B' \in \mathcal{L}((X^\circ)_1, U')$  is (zero-class)  $L^1$ -admissible.

We point out that the observation operator considered above is actually the restriction of  $B' : (X_{-1})' \rightarrow U'$  to the interpolation space  $(X_{-1})^\circ = (X^\circ)_1$ . For this reason, while our result may seem like a straightforward generalisation, we emphasise that the invariance of the zero-class property is not necessarily expected.

*Proof of Theorem 3.1.* First, assume that  $B \in \mathcal{L}(U, X_{-1})$  is C-admissible with input operator  $\Phi_\tau$ . Fix  $\lambda \in \rho(A)$  and  $x^\circ \in \text{dom}(A^\circ)$ . Setting  $F(\cdot) := B'R(\bar{\lambda}, A')T^\circ(\cdot)$ , the continuity of  $s \mapsto B'T^\circ(s)x^\circ$  allows us to compute

$$\begin{aligned} \int_0^\tau \|B'T^\circ(s)x^\circ\| \, ds &= \int_0^\tau \|F(s)(\bar{\lambda} - A')x^\circ\| \, ds \\ &\leq |\lambda| \int_0^\tau \|F(s)x^\circ\| \, ds + \int_0^\tau \left\| \frac{d}{ds} F(s)x^\circ \, ds \right\| \\ &\leq \tau |\lambda| \|F(0)\| \|x^\circ\| + (\tau |\lambda| + 1) \text{var}_0^\tau(F(\cdot)x^\circ); \end{aligned}$$

where we've used that  $\int_0^\tau \left\| \frac{d}{ds} F(s)x^\circ \, ds \right\|$  is the total variation of  $F(\cdot)x^\circ$  on  $[0, \tau]$ . Due to C-admissibility of  $B$ , we can now apply Proposition 2.1(iii) to obtain that

$$\int_0^\tau \|B'T^\circ(s)x^\circ\| \, ds \leq \tau |\lambda| \|F(0)\| \|x^\circ\| + C_\tau \|B\|_{\mathbb{B}_C(U, X, \tau)} \|x^\circ\| \quad (3.1)$$

with  $C_\tau := 2(\tau |\lambda| + 1) \|1 - \lambda R(\lambda, A)\|$ . It follows that  $B' \in \mathbb{C}_1(X^\circ, U', (T^\circ(t))_{t \geq 0})$ .

Conversely, let  $B' \in \mathbb{C}_1(X^\circ, U', (T^\circ(t))_{t \geq 0})$ . We show that  $B$  is Reg-admissible. First of all, for  $u \in T([0, \tau], U)$  – the space of  $U$ -valued step functions, of course

$$\Phi_\tau u := \int_0^\tau T_{-1}(\tau - t)Bu(t) \, dt \in X.$$

By density of step functions in regulated functions, we therefore only need to show that there exists  $K > 0$  such that  $\|\Phi_\tau u\|_X \leq K \|u\|_\infty$  for all  $u \in T([0, \tau], U)$ .

To this end, fix  $x^\circ \in \text{dom}(A^\circ)$ , and for each  $u \in T([0, \tau], U)$  estimate

$$\begin{aligned} |\langle \Phi_\tau u, x^\circ \rangle| &= \left| \left\langle \int_0^\tau T_{-1}(\tau - t)Bu(t) \, dt, x^\circ \right\rangle \right| \\ &= \left| \int_0^\tau \langle u(t), B'(T_{-1})^\circ(\tau - t)x^\circ \rangle \, dt \right| \\ &= \left| \int_0^\tau \langle u(t), B'T^\circ(\tau - t)x^\circ \rangle \, dt \right| \\ &\leq \|B'\|_{\mathbb{C}_1(X^\circ, U', \tau)} \|x^\circ\| \|u\|_\infty \end{aligned}$$

by Hölder's inequality. From the density of  $\text{dom}(A^\odot)$  in  $X^\odot$ , we infer that the above inequality also holds for each  $x^\odot \in X^\odot$ . Consequently, the norming property of the sun-dual [48, Theorem 1.3.5] yields the desired estimate:

$$\|\Phi_\tau u\|_X \leq \|B'\|_{\mathbb{C}_1(X^\odot, U', \tau)} \limsup_{t \downarrow 0} \|T(t)\| \|u\|_\infty. \quad (3.2)$$

Lastly, the zero-class equivalence can be seen immediately from the two estimates (3.1) and (3.2).  $\square$

Next, we generalise [51, Theorem 6.9(ii)] to go from  $L^p$ -admissibility of the control to  $L^q$ -admissibility of its dual, where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Theorem 3.2.** Let  $X$  and  $U$  be Banach spaces, let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ , and let  $B \in \mathcal{L}(U, X_{-1})$ .

For Hölder conjugates  $p, q \in [1, \infty]$ , if  $B \in \mathbb{B}_p(U, X, (T(t))_{t \geq 0})$ , then  $B' \in \mathbb{C}_q(X^\odot, U', (T^\odot(t))_{t \geq 0})$  with

$$\|B'\|_{\mathbb{C}_q(X^\odot, U', \tau)} \leq \|B\|_{\mathbb{B}_p(U, X, \tau)}.$$

The converse is true if  $p < \infty$ .

**Remarks 3.3.** (a) The condition  $p < \infty$  cannot be dropped in the converse part of Theorem 3.2; see [32, Remark 2.4]. The example in the reference – which has also appeared in the context of maximal regularity (see, [17, Page 48] and [32, Example 2.3]) and admissibility [5, Remark 4.9 and Page 22] – satisfies  $X^\odot = X'$ .

(b) Actually, if  $X$  is reflexive, then  $L^1$ -admissibility of the dual of a control operator  $B$  does imply  $L^\infty$ -admissibility of  $B$  [51, Theorem 6.9(ii)]. This begs the questions whether sun-reflexivity of the semigroup is a sufficient condition to obtain the converse in Theorem 3.2 for the case  $p = \infty$ . However, an evidence to the contrary is again provided by the example in [32, Remark 2.4].

*Proof of Theorem 3.2.* Let  $B \in \mathbb{B}_p(U, X, (T(t))_{t \geq 0})$ . By the norming property of Banach space valued  $L^p$ -spaces [27, Proposition 1.3.1], we can compute the norm

$$\begin{aligned} \|B'T^\odot(\cdot)x^\odot\|_q &= \sup_{\substack{u \in L^p([0, \tau], U) \\ \|u\|_p \leq 1}} \left| \int_0^\tau \langle u(t), B'T^\odot(\tau - t)x^\odot \rangle dt \right| \\ &= \sup_{\substack{u \in L^p([0, \tau], U) \\ \|u\|_p \leq 1}} \left| \int_0^\tau \langle u(t), B'(T_{-1})^\odot(\tau - t)x^\odot \rangle dt \right| \\ &= \sup_{\substack{u \in L^p([0, \tau], U) \\ \|u\|_p \leq 1}} \left| \left\langle \int_0^\tau T_{-1}(\tau - t)Bu(t) dt, x^\odot \right\rangle \right| \\ &\leq \|B\|_{\mathbb{B}_p(U, X, \tau)} \|x^\odot\| \end{aligned}$$

for all  $x^\odot \in \text{dom}(A^\odot)$  and so  $B' \in \mathbb{C}_q(X^\odot, U', (T^\odot(t))_{t \geq 0})$ .

Conversely, let  $B' \in \mathbb{C}_q(X^\odot, U', (T^\odot(t))_{t \geq 0})$ . Employing Hölder's inequality, we can argue exactly as in Theorem 3.1, to obtain a constant  $K > 0$  such that  $\|\Phi_\tau u\| \leq K \|u\|_p$  for all step functions  $u \in T([0, \tau], U)$ . If  $p < \infty$ , this implies that  $B$  is  $L^p$ -admissible by density of the step functions in  $L^p([0, \tau], U)$ .  $\square$

#### 4. DUALITY RESULTS FOR OBSERVATION OPERATORS

In this section, we look at dual of observation operators, i.e., analogous to [51, Theorem 6.9(i)], we ask whether the equivalence

$$C \in \mathbb{C}_p(X, Y, (T(t))_{t \geq 0}) \stackrel{?}{\iff} C' \in \mathbb{B}_q(Y', X^\odot, (T^\odot(t))_{t \geq 0}) \quad (4.1)$$



holds for Hölder conjugates  $p, q \in [1, \infty]$ . Note that given  $C \in \mathcal{L}(X_1, Y)$ , we only know that  $\text{Rg } C' \subseteq (X_1)' = (X')_{-1}$ . Therefore, in order for the second inclusion in (4.1) to be meaningful, we must a priori assume that  $\text{Rg } C' \subseteq (X^\odot)_{-1}$ . Adapting the arguments of [51, Theorem 6.9], we first settle the reverse implication in (4.1):

**Theorem 4.1.** Let  $X$  and  $Y$  be Banach spaces, let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ , and let  $C \in \mathcal{L}(X_1, Y)$  be such that  $C'(Y') \subseteq (X_1)^\odot = (X^\odot)_{-1}$ .

Let  $p, q \in [1, \infty]$  be Hölder conjugates. If  $C' \in \mathbb{B}_q(Y', X^\odot, (T^\odot(t))_{t \geq 0})$ , then  $C \in \mathbb{C}_p(X, Y, (T(t))_{t \geq 0})$  with  $\|C\|_{\mathbb{C}_p(X, Y, \tau)} \leq \|C'\|_{\mathbb{B}_q(Y', X^\odot, \tau)}$ .

*Proof.* First, suppose that  $C' \in \mathbb{B}_q(Y', X^\odot, (T^\odot(t))_{t \geq 0})$  and fix  $\tau > 0$ . The norm of the output map corresponding to  $C$  can then be estimated as

$$\begin{aligned} \|CT(\cdot)x\|_p &= \sup_{\substack{y \in L^q([0, \tau], Y') \\ \|y\|_q \leq 1}} \left| \int_0^\tau \langle CT(\tau-t)x, y(t) \rangle dt \right| \\ &= \sup_{\substack{y \in L^q([0, \tau], Y') \\ \|y\|_q \leq 1}} \left| \left\langle x, \int_0^\tau T'(\tau-t)C'y(t) dt \right\rangle \right| \\ &= \sup_{\substack{y \in L^q([0, \tau], Y') \\ \|y\|_q \leq 1}} \left| \left\langle x, \int_0^\tau (T^\odot)_{-1}(\tau-t)C'y(t) dt \right\rangle \right| \\ &\leq \|C'\|_{\mathbb{B}_q(Y', X^\odot, \tau)} \|x\| \end{aligned}$$

for all  $x \in \text{dom}(A)$ ; the first equality is obtained employing norming property of Banach space valued  $L^p$ -spaces [27, Proposition 1.3.1], treating  $CT(\cdot)x$  as an element of  $L^p([0, \tau], Y'')$ . So,  $C \in \mathbb{C}_p(X, Y, (T(t))_{t \geq 0})$  with the asserted inequality.  $\square$

**Remark 4.2.** For  $p = 1$ , Theorem 4.1 can be strengthened as follows: if the control operator  $C' \in \mathcal{L}(Y', (X^\odot)_{-1})$  is (zero-class) C-admissible, then the observation operator  $C \in \mathcal{L}(X_1, Y)$  is (zero-class)  $L^1$ -admissible. Indeed, (zero-class) C-admissibility of  $C'$  implies (zero-class)  $L^1$ -admissibility of  $C'' \in \mathcal{L}((X^\odot)^\odot_1, Y'')$  by Theorem 3.1, and in turn, the claim.

The forward implication in (4.1) is slightly subtle. While the case  $p > 1$  yields the desired implication, the case  $p = 1$  requires an additional assumption of zero-class admissibility, which emerges organically from our proof technique. Moreover, we are only able to show C-admissibility of the control  $C'$  in this case.

**Theorem 4.3.** Let  $X$  and  $Y$  be Banach spaces, let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$ , let  $C \in \mathcal{L}(X_1, Y)$  be such that  $C'(Y') \subseteq (X_1)^\odot = (X^\odot)_{-1}$ , and let  $p, q \in [1, \infty]$  be Hölder conjugates.

Assume that the observation operator  $C$  is  $L^p$ -admissible. If  $p > 1$ , then the control operator  $C' \in \mathcal{L}(Y', (X^\odot)_{-1})$  is  $L^q$ -admissible. If  $p = 1$  and the admissibility of  $C$  is zero-class, then the control operator  $C'$  is C-admissible. In these cases,

$$\|C'\|_{Z(Y', (X^\odot)_{-1}, \tau)} \leq \|C\|_{\mathbb{C}_p(X, Y, \tau)}$$

with  $Z = \mathbb{B}_q$  and  $Z = \mathbb{B}_C$  respectively.

*Proof.* Let  $C \in \mathbb{C}_p(X, Y, (T(t))_{t \geq 0})$ . For each  $\tau > 0$ , the input operator  $\Phi_\tau : L^q([0, \tau], Y') \rightarrow (X^\odot)_{-1}$  given by

$$\Phi_\tau : y \mapsto \int_0^\tau (T^\odot)_{-1}(\tau-t)C'y(t) dt,$$



is well-defined because  $C'(Y') \subseteq (X^\odot)_{-1}$  and the (extrapolated) sun-dual semigroup is strongly continuous. Note from the proof of the first part [51, Theorem 6.9(i)] – observing that the strong continuity of the dual semigroup was needed in this part of the argument merely for the integral in the definition of the input operator  $\Phi_\tau$  to be well-defined – that  $\text{Rg } \Phi_\tau \subseteq X'$  and

$$\|\Phi_\tau\|_{\mathcal{L}(L^q([0,\tau],Y'),X')} \leq \|C\|_{\mathbb{C}_p(X,Y,\tau)} \quad (4.2)$$

Next, assume that  $p > 1$ , fix an element  $y \in L^q([0,\tau],Y')$ , and extend  $y$  by 0 outside  $[0,\tau]$ . For  $s \geq 0$ , we write

$$y_s := y(\cdot + s) \in L^q([0,\tau],Y').$$

For each  $0 \leq s < \tau$ , we obtain

$$\begin{aligned} \|T'(s)\Phi_\tau y - \Phi_\tau y\|_{X'} &= \left\| \int_0^\tau (T')_{-1}(\tau + s - t)C'y(t) dt - \Phi_\tau y \right\|_{X'} \\ &= \left\| \int_0^{\tau+s} (T')_{-1}(\tau + s - t)C'y(t) dt - \Phi_\tau y \right\|_{X'} \\ &= \left\| \int_0^s (T')_{-1}(\tau + s - t)C'y(t) dt + \Phi_\tau y_s - \Phi_\tau y \right\|_{X'} \\ &\leq \|T'(\tau)\Phi_s y\|_{X'} + \|\Phi_\tau\|_{\mathcal{L}(L^q([0,\tau],Y'),X')} \|y_s - y\|_{L^q([0,\tau],Y')} \\ &\leq \|C\|_{\mathbb{C}_p(X,Y,\tau)} \left( \|T'(\tau)\| \|y\|_{L^q([0,s],Y')} + \|y_s - y\|_{L^q([0,\tau],Y')} \right); \end{aligned}$$

where we've used the fact that  $\Phi_s y \in X'$  for the first inequality and (4.2) along with  $s < \tau$  for the second. As  $q < \infty$ , both norms involving  $y$  converge to 0 as  $s \downarrow 0$ . By definition of the sun-dual space, we infer that  $\Phi_\tau y \in X^\odot$  and conclude the  $L^q$ -admissibility of  $C'$ .

On the other hand, if  $C$  is zero-class  $L^1$ -admissible, instead extend  $y \in C([0,\tau],Y')$  constantly outside  $[0,\tau]$ , so that again

$$y_s := y(\cdot + s) \in C([0,\tau],Y').$$

This time, for  $0 \leq s < \tau$ , one can show

$$T'(s)\Phi_\tau y - \Phi_\tau y = T'(\tau)\Phi_s y - \Phi_s y_\tau + \Phi_\tau(y_s - y);$$

considering the restriction of  $\Phi_\tau$  to  $C([0,\tau],Y')$ . Using zero-class admissibility and continuity of  $y$ , we may again deduce that  $\Phi_\tau y \in X^\odot$ .

Lastly, the claimed inequality holds by (4.2) and the closed graph theorem.  $\square$

We don't know whether the zero-class assumption in Theorem 4.3 can be dropped in the case of  $p = 1$ , nor do we know if the assertion can be strengthened to  $L^\infty$ -admissibility of  $C$ . While,  $X^\odot = X'$  is sufficient for both [51, Theorem 6.9(i)], another situation can be constructed for the case of positive systems. In light of the recent interest on infinite-dimensional positive systems [5, 18, 19, 52], we find it worthwhile to mention it in the following.

A non-empty subset  $X_+$  of a Banach space  $X$  is called a *cone* if  $\alpha X_+ + \beta X_+ \subseteq X_+$  for all  $\alpha, \beta \geq 0$  and  $X_+ \cap (-X_+) = \{0\}$ . The cone  $X_+$  induces a natural order on  $X$  given by  $x \leq y$  if and only if  $y - x \in X_+$ . The Banach space  $X$  together with a closed cone  $X_+$  is called an *ordered Banach space*. We refer to [1–3] for the theory of ordered Banach spaces.

Closed subspaces of ordered Banach spaces are endowed with the induced order rendering them ordered Banach spaces as well. The cone  $X_+$  is called *generating* if  $X = X_+ - X_+$  and it is called *normal* if there exists  $M \geq 1$  such that  $0 \leq x \leq y$  implies  $\|x\| \leq M \|y\|$ . For example,  $L^p(\Omega, \mu)$  with  $p \in [1, \infty]$ ,  $C(K)$  for a compact set  $K$ , and  $C_0(L)$  for a locally compact set  $L$  are ordered Banach spaces with the

canonical cone being generating and normal. The norm on  $X$  is said to be *additive on the positive cone* if

$$\|x + y\| = \|x\| + \|y\| \quad \text{for all } x, y \in E_+.$$

Both finite-dimensional spaces and  $L^1(\Omega, \mu)$  fall into this category. A non-empty set  $C \subseteq X_+$  is called a *face* of  $X_+$  if  $0 \leq y \leq x$  implies  $y \in C$  for all  $x, y \in X$ . Lastly, an operator  $T$  between ordered Banach spaces  $X$  and  $Y$  is called *positive* if  $TX_+ \subseteq Y_+$ . The set of positive linear functionals on  $X$  form a cone and turn  $X'$  into an ordered Banach space. A  $C_0$ -semigroup on an ordered Banach space is called *positive* if each semigroup operator is positive. In fact, the associated extrapolation space is also an ordered Banach space. For the definition and a detailed analysis of the order on the extrapolation space, we refer to [5, Section 2.2] and [6, Section 4].

**Theorem 4.4.** Suppose that  $X$  and  $Y$  are ordered Banach spaces such that  $X$  has a generating and normal cone. Let  $(T(t))_{t \geq 0}$  be a positive  $C_0$ -semigroup on  $X$  and  $C \in \mathcal{L}(X_1, Y)$  be such that  $C'(Y') \subseteq (X_1)^\odot = (X^\odot)_{-1}$ .

If  $C$  is positive,  $(X^\odot)_+$  is a face of  $X'_+$ , and the norm on  $Y$  is additive on the positive cone, then

$$C \in \mathbb{C}_p(X, Y, (T(t))_{t \geq 0}) \Rightarrow C' \in \mathbb{B}_q(Y', X^\odot, (T^\odot(t))_{t \geq 0})$$

for Hölder conjugates  $p, q \in [1, \infty]$ .

An ordered Banach space is called a *Banach lattice* if any two elements have a supremum and  $\sup\{-x, x\} \leq \sup\{-y, y\}$  implies  $\|x\| \leq \|y\|$ . If  $X$  is a Banach lattice, then various sufficient conditions under which  $(X^\odot)_+$  is a face of  $X'_+$  are given in [48, Chapter 8]. In particular, this is the case when  $X = C(K)$ .

*Proof of Theorem 4.4.* For  $\tau > 0$ , let  $\Phi_\tau : L^q([0, \tau], Y') \rightarrow (X_1)^\odot$  be given by

$$y \mapsto \int_0^\tau (T^\odot)_{-1}(\tau - s)C'y(s) ds.$$

As  $C$  is positive, so is its dual  $C'$ . Together with positivity of the semigroup, this ensures positivity of  $\Phi_\tau$ . As explained in the proof of Theorem 4.3, we know from [51, Theorem 6.9(i)] that  $\text{Rg } \Phi_\tau \subseteq X'$  and we're left to show that  $\text{Rg } \Phi_\tau \subseteq X^\odot$ .

Due to [8, Proposition 1.4.2(3) and (2)], the assumption on  $Y$  implies that  $Y'$  has a unit, say  $e$ , i.e.,  $Y' = \bigcup_{\lambda > 0} [-\lambda e, \lambda e]$ . Denoting by  $\mathbf{e} \in L^q([0, \tau], Y')$ , the constant function taking value  $e$ , we have  $\Phi_\tau \mathbf{e} \in \text{dom}((A^\odot)_{-1}) = X^\odot$ . As  $(X^\odot)_+$  is a face of  $X'_+$  and  $\Phi_\tau$  is positive, it follows that  $\Phi_\tau y \in X^\odot$  for all  $y \in L^q([0, \tau], Y')_+$ . Finally, as  $Y'$  has a unit, its cone – and in turn the cone of  $L^q([0, \tau], Y')$  – is generating [38, Lemma 2]. It follows that  $\Phi_\tau y \in X^\odot$  for all  $y \in L^q([0, \tau], Y')$ .  $\square$

**Remark 4.5.** In Theorem 4.4, one can argue as in [5, Theorem 4.11] to weaken the assumption of positivity of  $C$  to the condition:  $C'$  maps the unit ball of  $Y'$  into an order bounded subset of  $(X_1)'$ . Sufficient conditions for this property are available in [5, Proposition A.1].

## 5. AN EXAMPLE

Throughout this section, let  $(R(t))_{t \geq 0}$  be the nilpotent right-shift semigroup on  $X := L^1[0, 1]$  with generator  $A$ . Consider the space of test functions

$$\tilde{\mathcal{D}} := \{\phi \in C^\infty[0, 1] : \phi(0) = 0\}$$

and for  $g, f$  in the dual space  $(\tilde{\mathcal{D}})'$ , write

$$g = \tilde{\partial}f : \Leftrightarrow \langle f, \phi' \rangle = -\langle g, \phi \rangle \quad \text{for all } \phi \in \tilde{\mathcal{D}}.$$

This allows for a convenient description of the extrapolation space corresponding to the dual:

**Proposition 5.1.** The extrapolation space associated to the left shift semigroup on  $X'$  is given by

$$(X')_{-1} = \{g \in (\tilde{\mathcal{D}})' : g = \tilde{\partial}f \text{ for some } f \in L^\infty[0, 1]\}. \quad (5.1)$$

*Proof.* We proceed as in [52, Example 3.2.7]: Let  $j : X' \rightarrow (\tilde{\mathcal{D}})'$  be the canonical embedding, i.e.,  $\langle j(g), \phi \rangle = \int_0^1 g(x)\phi(x) dx$  for all  $\phi \in \tilde{\mathcal{D}}$ . For each  $g \in X'$  and  $\phi \in \tilde{\mathcal{D}}$ , we have

$$\langle j(g), \phi \rangle = \int_0^1 ((A')^{-1}g)'(x)\phi(x) dx = - \int_0^1 ((A')^{-1}g)(x)\phi'(x) dx;$$

here we've used that each  $f \in \text{dom } A'$  satisfies  $f(1) = 0$  and  $A'f = f'$ . Thus,  $j$  can be extended to  $j_{-1} : (X')_{-1} \rightarrow (\tilde{\mathcal{D}})'$  as

$$\langle j_{-1}(g), \phi \rangle := - \langle (A')_{-1}^{-1}g, \phi' \rangle \quad \left( g \in (X')_{-1}, \phi \in \tilde{\mathcal{D}} \right).$$

Next, let  $g \in (X')_{-1}$  such that  $j_{-1}(g) = 0$  and set  $h := -(A')_{-1}^{-1}g$ . Then  $\alpha := \int_0^1 h(x) dx$  and  $\phi(t) := \int_0^t (h(x) - \alpha) dx$  satisfy  $\phi \in \tilde{\mathcal{D}}$  and  $\int_0^1 \alpha h(x) dx = \alpha^2$ . Thus,

$$0 = \langle j_{-1}(g), \phi \rangle = \langle h, \phi' \rangle = \int_0^1 h(x)(h(x) - \alpha) dx = \int_0^1 (h(x) - \alpha)^2 dx,$$

which shows that  $h = \alpha$ . Next, choose  $\phi \in \tilde{\mathcal{D}}$  with  $\phi(1) \neq 0$ . Again using  $\langle j_{-1}(g), \phi \rangle = 0$ , it follows that  $h = \alpha = 0$ . Injectivity of  $(A')_{-1}^{-1}$  now implies that  $g = 0$ . Summarising,  $j_{-1}$  is actually an embedding and the following diagram

$$\begin{array}{ccc} X' & \xleftarrow{(A')_{-1}^{-1}} & (X')_{-1} \\ j \downarrow & & \downarrow j_{-1} \\ (\tilde{\mathcal{D}})' & \xrightarrow{\tilde{\partial}} & (\tilde{\mathcal{D}})' \end{array}$$

commutes. This proves that  $(X')_{-1}$  is contained in the set on the right in (5.1). Conversely, let  $g \in (\tilde{\mathcal{D}})'$  such that  $g = \tilde{\partial}f$  for some  $f \in X'$ . Bijectivity of  $(A')_{-1}$  implies that there exists  $h \in (X')_{-1}$  such that  $(A')_{-1}^{-1}h = f$ . Commutativity of the above diagram implies that  $j_{-1}(h) = \tilde{\partial}(j(f))$  and so  $h$  can be identified with  $\tilde{\partial}f$  which means  $g = h \in (X')_{-1}$ . This verifies (5.1).  $\square$

Having the description of the dual extrapolation space at hand enables us to describe all (zero-class)  $L^1$ -admissible observation functionals associated to the right shift semigroup on  $X$ . In what follows,  $\text{BV}[0, 1]$  denotes the space of functions of bounded variation on  $[0, 1]$ .

**Proposition 5.2.** The complex-valued  $L^1$ -admissible observation operators are

$$\mathbb{C}_1(X, \mathbb{C}, (R(t))_{t \geq 0}) = \{C \in \mathcal{L}(X_1, \mathbb{C}) : C'(1) = \tilde{\partial}c \text{ for some } c \in \text{BV}[0, 1]\}. \quad (5.2)$$

Moreover, the admissibility of  $C \in \mathbb{C}_1(X, \mathbb{C}, (R(t))_{t \geq 0})$  is zero-class if and only if  $\tilde{\partial}c$  has no atomic part.

*Proof.* Let  $C \in \mathcal{L}(X_1, \mathbb{C})$ . Since  $C'(\mathbb{C}) \subseteq (X_1)' = (X')_{-1}$ , we obtain from (5.1), an element  $c \in L^\infty[0, 1]$  such that  $C'(1) = \tilde{\partial}c$ . Equivalently,

$$Cf = \langle C'(1), f \rangle = \left\langle \tilde{\partial}c, f \right\rangle_{(X_1)', X_1} = \left\langle \tilde{\partial}c, f \right\rangle_{(\tilde{\mathcal{D}})', \tilde{\mathcal{D}}}$$

for all  $f \in \tilde{\mathcal{D}}$ . In particular, for each  $f \in \tilde{\mathcal{D}}$ , we have

$$CR(s)f = \left\langle \tilde{\partial}c, R(s)f \right\rangle_{(\tilde{\mathcal{D}})', \tilde{\mathcal{D}}} = \left\langle \tilde{\partial}c, \bar{f}(s - \cdot) \right\rangle_{(\tilde{\mathcal{D}})', \tilde{\mathcal{D}}} = (\bar{f} \star \tilde{\partial}c)(s);$$

where  $\bar{f}(x) = f(-x)$  and  $f$  is extended by 0 outside  $[0, 1]$ . This means that  $C \in \mathcal{C}_1(X, \mathbb{C}, (R(t))_{t \geq 0})$  if and only if  $\tilde{\partial}c$  lies in  $\mathcal{M}[0, 1]$ , the space of measures of bounded variation, see [23, Theorem 2.5.8]. Since  $g = \tilde{\partial}f$  implies that  $g = \partial f$ , so  $\tilde{\partial}c \in \mathcal{M}[0, 1]$  is equivalent to  $c \in \text{BV}[0, 1]$  due to [4, Proposition 3.6]. The equality in (5.2) is now immediate.

Next, by the semigroup law, zero-class  $L^1$ -admissibility of  $C$  is equivalent to  $\lim_{\tau \downarrow 0} \int_{\xi}^{\xi+\tau} |CR(s)f| ds = 0$  for each  $\xi \in [0, 1]$ . Therefore, the above computations show that  $C$  is zero-class admissible  $L^1$ -admissible if and only if there exists  $c \in \text{BV}[0, 1]$  such that  $C'(1) = \tilde{\partial}c$  and

$$\lim_{\tau \downarrow 0} \left\| f \star \tilde{\partial}c \right\|_{L^1([\xi, \xi+\tau], \mathbb{C})} = 0 \quad \text{for all } \xi \in [0, 1] \text{ and } f \in \tilde{\mathcal{D}}. \quad (5.3)$$

Now, let  $c \in \text{BV}[0, 1]$  and set  $\mu := \tilde{\partial}c$ . Using the Radon-Nikodym decomposition, we write  $\mu = \mu_a + \mu_j + \mu_c$  where  $\mu_a$  is absolutely continuous with respect to the Lebesgue measure,  $\mu_c$  is non-atomic part, and  $\mu_j$  is the purely atomic part; see [4, Section 3.2]. If  $\mu_j = 0$ , then a variation of Young's convolution inequality [22, Page 54] and mutual singularity of the measures, gives for each  $f \in \tilde{\mathcal{D}}$  that

$$\begin{aligned} \left\| f \star \tilde{\partial}c \right\|_{L^1([0, \tau], \mathbb{C})} &\leq \|\mu\|_{\mathcal{M}[0, \tau]} \|f\|_{L^1[0, 1]} \\ &= (|\mu_a|([0, \tau]) + |\mu_c|([0, \tau])) \|f\|_{L^1[0, 1]}, \end{aligned}$$

which converges to 0 as  $\tau \downarrow 0$ . On the other hand, if  $\mu_j \neq 0$ , then there exists  $\xi \in [0, 1]$  such that  $|\mu_j|(\{\xi\}) \neq 0$ . Therefore,  $|\mu_j|([\xi, \xi + \tau]) \not\rightarrow 0$  as  $\tau \downarrow 0$ . Once again, the mutual singularity of the measures yields

$$\|\mu\|_{\mathcal{M}[\xi, \xi+\tau]} = |\mu_j|([\xi, \xi + \tau]) + |\mu_a|([\xi, \xi + \tau]) + |\mu_c|([\xi, \xi + \tau]).$$

Noting as above that  $|\mu_a|([\xi, \xi + \tau]) + |\mu_c|([\xi, \xi + \tau]) \rightarrow 0$  as  $\tau \downarrow 0$ , it follows that  $\lim_{\tau \downarrow 0} \|\mu\|_{\mathcal{M}[\xi, \xi+\tau]} \neq 0$ . A suitable choice of  $f \in \tilde{\mathcal{D}}$  thus implies that  $\lim_{\tau \downarrow 0} \left\| f \star \tilde{\partial}c \right\|_{L^1([\xi, \xi+\tau], \mathbb{C})} \neq 0$  and so the  $L^1$ -admissibility is not zero-class. An appeal to (5.3) now shows that the zero-class  $L^1$ -admissibility of  $C$  is equivalent to the existence of  $c \in \text{BV}[0, 1]$  such that  $C'(1) = \tilde{\partial}c$  has no atomic part.  $\square$

Recall from [49, Example 1.1(ii)] that the sun-dual semigroup associated to  $(R(t))_{t \geq 0}$  is the nilpotent left translation semigroup  $(L(t))_{t \geq 0}$  on

$$X^\circ = \{f \in C[0, 1] : f(1) = 0\}.$$

Moreover, we know from [7, Example 5.1] that

$$(X^\circ)_{-1} = \{g \in \mathcal{D}' : g = \partial f \text{ for some } f \in X^\circ\};$$

where  $\mathcal{D}$  is the usual space of test functions on  $[0, 1]$ , i.e.,

$$\mathcal{D} := \{\phi \in C^\infty[0, 1] : \phi(0) = \phi(1) = 0\}.$$

While the direct computation of admissible control operators associated to  $(L(t))_{t \geq 0}$  on  $X^\circ$  is tedious, our results in the prequel along with the analysis in the present section allow us to characterise all C-admissible control operators:

**Proposition 5.3.** The set  $\mathbb{B}_C(\mathbb{C}, X^\circ, (L(t))_{t \geq 0})$  can be described as

$$\{C' : C \in \mathcal{L}(X_1, \mathbb{C}) \text{ and } C'(1) = \partial b = \tilde{\partial}c \text{ for some } b \in X^\circ, c \in \text{BV}[0, 1]\}.$$

*Proof.* Firstly, let  $C \in \mathcal{L}(X_1, Y)$  be such that  $C'(1) = \partial b = \tilde{\partial}c$  for some  $b \in X^\circ$  and  $c \in \text{BV}[0, 1]$ . By description of  $(X^\circ)_{-1}$ , we get  $C'(C) \subseteq (X^\circ)_{-1}$ . In fact, the continuity of  $b$ , in particular, also means that  $\tilde{\partial}c = \partial b$  has no atomic part.

Whence,  $C$  is zero-class  $L^1$ -admissible by Proposition 5.2 and so we deduce from Theorem 4.3 that  $C' \in \mathbb{B}_C(\mathbb{C}, X^\odot, (L(t))_{t \geq 0})$ .

Conversely, let  $B \in \mathbb{B}_C(\mathbb{C}, X^\odot, (L(t))_{t \geq 0})$ , then sun-reflexivity of  $X$  – which is known from [49, Example 1.3(ii)] – along with Theorem 3.1 yields that  $B' \in \mathbb{C}_1(X, \mathbb{C}, (R(t))_{t \geq 0})$ . Thus there exists  $c \in \text{BV}[0, 1]$  such that  $B'(1) = \tilde{\partial}c$ . Also, as  $B''(1) = B(1) \in (X^\odot)_{-1}$ , so there exists  $b \in X^\odot$  such that  $B''(1) = \partial b$ . Thus,  $B = B''$  has the desired form.  $\square$

**Remarks 5.4.** (a) The proof of Proposition 5.3 even shows that the admissibility of each element of  $\mathbb{B}_C(\mathbb{C}, X^\odot, (L(t))_{t \geq 0})$  is zero-class; cf. Theorem 4.3.

(b) It can be inferred from [5, Corollary 4.8] that every positive  $B \in \mathcal{L}(U, (X^\odot)_{-1})$  is zero-class  $C$ -admissible.

(c) While  $g = f$  implies  $g = \partial f$ , the converse is not true in general. Therefore, we don't know whether in the description of  $\mathbb{B}_C(\mathbb{C}, X^\odot, (L(t))_{t \geq 0})$  from Proposition 5.3 we can simply write  $C'(1) = \partial b$  for some  $b \in X^\odot \cap \text{BV}[0, 1]$ .

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