





# A UNIVERSAL EXAMPLE FOR QUANTITATIVE SEMI-UNIFORM STABILITY

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**ABSTRACT.** We characterise quantitative semi-uniform stability for  $C_0$ -semigroups arising from port-Hamiltonian systems, complementing recent works on exponential and strong stability. With the result, we present a simple universal example class of port-Hamiltonian  $C_0$ -semigroups exhibiting arbitrary decay rates slower than  $t^{-1/2}$ . The latter is based on results from the theory of Diophantine approximation as the decay rates will be strongly related to approximation properties of irrational numbers by rationals given through cut-offs of continued fraction expansions.

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## 1. INTRODUCTION

Over the past fifty years operator semigroups have been established as an important framework in the context of evolution equations. While the theory may seem rather classical and largely developed, Borichev–Tomilov’s seminal work [6] on *polynomial stability* from 2010, inspired by an earlier result by Batty–Duyckaerts [5], initiated a tremendously active area of research. The interest in *semi-uniform stability*, which covers polynomial stability, arises from sharply quantifying energy-decay along classical solutions of linear partial differential equations (PDEs), where the stronger uniform exponential stability fails, but strong stability holds. Previously, this had been quantified on an ad-hoc basis, e.g. by exploiting the spectral decomposition of the involved operators [21]. Particular interest in such abstract operator-theoretic tools arises in wave equations, where different types of damping – for instance depending on the geometry – heavily influence asymptotic stability and accurate decay rates are desirable, see [9] and the references therein.

The success of Borichev–Tomilov’s result and its consequences also lies in the abstract framework, being characteristic of the power of operator theory, from which explicit rates arise naturally from estimates over the resolvent operator, encoding hard analysis from PDEs. The theory of *port-Hamiltonian systems* has a similar history. It developed from the paradigm that the energy flow should guide

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the modelling of dynamical systems, typically governed through Hamilton’s principle, thereby formalising the engineering formalism represented by *bond graphs*. In the last twenty years, port-Hamiltonian systems have emerged as a cornerstone framework for modelling and analysing physical phenomena, particularly those that conserve or dissipate energy. Since van der Schaft’s seminal work in [19], this framework has sparked a flurry of research and development across various domains, see [10] for a survey. By describing physical systems through partial differential equations and boundary conditions that define “ports”, port-Hamiltonian systems have become exemplary boundary control systems, especially for hyperbolic PDEs.

The starting point of this article is the question of “*How can we quantify decay rates of semi-uniformly stable port-Hamiltonian systems?*”. More precisely, we focus on the supposedly restrictive case of port-Hamiltonian systems in one-dimensional spatial variable in the spirit of [11]. Yet, the subtleties in assessing sharp decay rates are already present in such one-dimensional domains and are well documented in the literature [1,16,21]. Here, we fully characterise semi-uniform stability for this setting and even show that the class allows for rather arbitrary decay rates.

In particular, we revisit the stability problem for port-Hamiltonian systems, focusing on providing a quantified estimate of the resolvent growth function, which, as mentioned above, plays a significant role in characterising semi-uniform stability, see [9] for a survey. Our primary objective is to derive an estimate in terms of the matrix norm of a suitable inverse of a derived quantity, that has previously played a role in characterising exponential and strong stability of port-Hamiltonian systems in [18] and [20] respectively. This term encodes all (matrix) parameters which determine a port-Hamiltonian system in one spatial dimension. Such conditions also highlight the advantages of the (1-D-)port-Hamiltonian framework as this allows to reduce properties of a PDE to mere estimates on the algebraic building blocks. The structural insight developed in this first part then leads to the construction of a simple example showcasing algebraic decay with sharp decay rate estimates. To the best of our knowledge, this is indeed the simplest example admitting more complex decay behaviour than exponential available in the class of time-dependent partial differential equations and, thus, remarkable on its own.

The paper is organised as follows: Section 2 recalls known results about quantitative decay rates for  $C_0$ -semigroups in terms of the resolvent. In Section 3, we provide preliminaries on port-Hamiltonian systems and set our notation. The main abstract results of the paper are presented in Section 4. In particular, we obtain decay rates for our semigroup entirely in terms of the matrix norm of a suitable inverse associated with the corresponding port-Hamiltonian system. Next, in Section 5 we provide and apply our results to the elementary port-Hamiltonian example class mentioned above. We show that for (almost) any decay rate slower than  $t^{-1/2}$  there is an example in said class admitting this decay rate. To establish a complete analysis of our example, we need some preliminaries on Diophantine approximation which we state for completeness in the Appendix A. Note that the necessity of Diophantine approximations seems to be of no coincidence but rather a general feature of (algebraic) stability analysis of  $C_0$ -semigroups. Indeed, connections to Diophantine approximations and/or dynamical systems and number theory have been observed in the context of decay of orbits of solutions of time-dependent PDEs, for instance, in [16, Section 3], [8, Section 6B], and [20, Sections 4 and 5]. Finally, Sections 7, 8, and 9 contain the detailed proofs of the main results discussed in Section 5.

Throughout this manuscript, a function  $\psi$  is called *increasing*, if  $s \leq t$  implies  $\psi(s) \leq \psi(t)$ ;  $\psi$  is *decreasing*, if  $-\psi$  is increasing.

## 2. BACKGROUND ON QUANTIFIED SEMI-UNIFORM STABILITY

Let  $(U(t))_{t \geq 0}$  be a strongly continuous semigroup of bounded linear operators acting in some Banach space  $X$ , with  $A$  being its generator. Throughout this section, we assume that  $i\mathbb{R} \subseteq \rho(A)$ , where  $\rho(A)$  denotes the resolvent set of  $A$ ; we write  $\mathcal{R}(\lambda, A) := (\lambda - A)^{-1}$  for the resolvent of  $A$  at  $\lambda \in \mathbb{C}$ . Introduce for  $\eta \geq 0$ , the function

$$M(\eta) := \sup_{t \in [-\eta, \eta]} \|\mathcal{R}(it, A)\| \quad (\text{M})$$

associated to the resolvent and

$$M_{\log}(\eta) := M(\eta)(\log(1 + M(\eta)) + \log(1 + \eta)).$$

Further, we write  $M_{\log}^{-1}$  for the inverse of  $M_{\log}$  on  $[M_{\log}(0), \infty)$ .

We recall the following two celebrated quantified estimates for semi-uniform stability.

**Theorem 2.1** (Batty–Duyckaerts, [5]; see the formulation in [6, Theorem 1.2]). *Assume that  $(U(t))_{t \geq 0}$  is bounded and  $i\mathbb{R} \subseteq \rho(A)$ . Then there exist  $c, t_0 > 0$  such that*

$$\|U(t)A^{-1}\| \leq \frac{c}{M_{\log}^{-1}(t/c)} \quad \text{for all } t \geq t_0.$$

The other main source on characterising stability rates we present here is the main result of [15]. For this, recall that a continuous function  $f : \mathbb{R}_+ \rightarrow (0, \infty)$  is said to have a *positive increase* if there are  $\alpha > 0$ ,  $c \in (0, 1]$ , and  $t_0 > 0$  such that

$$\frac{f(\lambda t)}{f(t)} \geq c\lambda^\alpha \quad (2.1)$$

whenever  $\lambda \geq 1$  and  $t \geq t_0$ . In particular, polynomials and *regularly varying* functions of positive index have a positive increase; see [4, 15].

**Theorem 2.2** (Rozendaal–Seifert–Stahn, [15, Theorem 1.1]). *If  $X$  is a Hilbert space,  $(U(t))_{t \geq 0}$  is bounded,  $i\mathbb{R} \subseteq \rho(A)$ , and the function associated to the resolvent in (M) is of positive increase, then there exist  $c, C, t_0 > 0$  such that*

$$\frac{c}{M^{-1}(t)} \leq \|U(t)A^{-1}\| \leq \frac{C}{M^{-1}(t)} \quad \text{for all } t \geq t_0.$$

In applications it might be challenging to directly identify  $M$  to be of positive increase, let alone the precise estimates for  $M$ . Thus, concerning applications, the following theorem is of somewhat more direct relevance.

**Theorem 2.3** (Rozendaal–Seifert–Stahn, [15, Theorem 3.2]). *Suppose that  $X$  is a Hilbert space,  $(U(t))_{t \geq 0}$  is bounded, and  $i\mathbb{R} \subseteq \rho(A)$ . Let  $m : [0, \infty) \rightarrow (0, \infty)$  be a continuous and increasing function of positive increase. If*

$$\|\mathcal{R}(is, A)\| \leq m(|s|) \quad \text{for all } s \in \mathbb{R},$$

*then there exist  $C, t_0 > 0$  such that*

$$\|U(t)A^{-1}\| \leq \frac{C}{m^{-1}(t)} \quad \text{for all } t \geq t_0.$$

Apart from Theorem 2.2, we also comment on the optimality of the stability estimates. The following theorem can be proved literally the same way as [2, Theorem 4.4.14(b)].

**Theorem 2.4.** *Assume that  $(U(t))_{t \geq 0}$  is bounded and  $i\mathbb{R} \subseteq \rho(A)$ . Let  $m : [0, \infty) \rightarrow (0, \infty)$  be lower-semi-continuous, right-continuous, and decreasing with  $\lim_{t \rightarrow \infty} m(t) = 0$ . Let  $m^* : (0, \infty) \rightarrow [0, \infty)$  be such that  $m(m^*(s)) \leq s$  with equality whenever  $s \in \text{range}(m)$ .*

If  $\|U(t)A^{-1}\| \leq m(t)$  for all  $t \geq 0$ , then for each  $c \in (0, 1)$ , there exist  $s_0, C \geq 0$  such that

$$\|\mathcal{R}(is, A)\| \leq Cm^* \left( \frac{c}{|s|} \right) \quad \text{whenever } |s| \geq s_0.$$

The above theorem implies the following lower bound estimate for the trajectories. (Note that this estimate is the one for the lower bound in Theorem 2.2 stated above.)

**Theorem 2.5** ([2, pg 281]). *Assume that  $(U(t))_{t \geq 0}$  is bounded,  $i\mathbb{R} \subseteq \rho(A)$ , and that the function associated to the resolvent defined in (M) is unbounded; let  $M^{-1}: (0, \infty) \rightarrow (0, \infty)$  be such that  $M(M^{-1}(s)) \leq s$  for all  $s \in (0, \infty)$  with equality for all  $s \in \text{range}(M)$ ,*

*Then there exist  $c, C, t_0 > 0$  such that*

$$\|U(t)A^{-1}\| \geq \frac{c}{M^{-1}(Ct)} \quad \text{for all } t \geq t_0.$$

*Moreover, if  $M$  is of positive increase, then there are  $c, T_0 > 0$  such that*

$$\|U(t)A^{-1}\| \geq \frac{c}{M^{-1}(t)} \quad \text{for all } t \geq t_0.$$

Whereas the last statement of this theorem follows from [15] given the first one, the first statement seems to be standard and was already observed in [2]. However, it was difficult for us to find an explicit proof for the latter other than the related [4, Corollary 6.11] or [8, Proposition 5.3]. For convenience, we shall provide the short proof here.

*Proof of Theorem 2.5.* We define

$$\begin{aligned} N(t) &:= \sup\{\|U(\tau)A^{-1}\| : \tau \geq t\} \quad \text{for } t > 0 \quad \text{and} \\ N^*(s) &:= \min\{t \geq 0 : N(t) \leq s\} \quad \text{for } s > 0. \end{aligned}$$

As in [4, pg 903] it follows  $N(N^*(s)) \leq s$  with equality whenever  $s \in \text{range}(N)$ . By Theorem 2.4, we obtain that for all  $c \in (0, 1)$ , there exist  $s_0, C \geq 0$  such that

$$\|\mathcal{R}(is, A)\| \leq CN^* \left( \frac{c}{|s|} \right) \quad \text{whenever } |s| \geq s_0.$$

Since  $N$  is decreasing, so is  $N^*$ . Thus, the function defined in (M) satisfies

$$M(|s|) = M(s) \leq CN^* \left( \frac{c}{|s|} \right) \quad \text{whenever } |s| \geq s_0.$$

As  $M$  is unbounded, for  $t$  large enough, we may put  $t := M(s)/C$ . Moreover, by possibly further increasing  $s_0$ , we get  $c/|s| \in \text{range}(N)$  for all  $|s| \geq s_0$ . With  $s = M^{-1}(Ct)$  we thus get

$$N(t) = N(M(s)/C) \geq N \left( N^* \left( \frac{c}{|s|} \right) \right) = \frac{c}{|s|} = \frac{c}{M^{-1}(Ct)}. \quad \square$$

Finally, we recall an adapted optimality statement from [8].

**Proposition 2.6** ([8, Proposition 5.3]). *Suppose that  $(U(t))_{t \geq 0}$  is bounded with  $i\mathbb{R} \subseteq \rho(A)$ . Let  $m: [0, \infty) \rightarrow (0, \infty)$  be continuous, increasing, and unbounded. If*

$$\limsup_{|s| \rightarrow \infty} \frac{\|\mathcal{R}(is, A)\|}{m(|s|)} > 0,$$

*then there exists  $c > 0$  such that*

$$\limsup_{t \rightarrow \infty} m^{-1}(ct) \|U(t)A^{-1}\| > 0.$$

*Moreover, if  $m$  has positive increase, one may choose any  $c > 0$ .*

## 3. PORT-HAMILTONIAN SYSTEMS REVISITED

Fix  $d \in \mathbb{N}$  and let  $P_0, P_1$  be  $d$ -dimensional real square matrices such that  $P_0^* = -P_0$  and  $P_1$  is self-adjoint and invertible. Let  $\mathcal{H} : (a, b) \rightarrow M_{d, \text{sa}}(\mathbb{R})$  be a measurable function which is bounded above and strictly bounded away from 0 in the sense of positive definiteness, where  $M_{d, \text{sa}}(\mathbb{R}) = \{T \in \mathbb{R}^{d \times d} : T^\top = T\}$ . The Hilbert space  $H := L^2([a, b]; \mathbb{R}^d)$  is equipped with the norm

$$\|\cdot\|_H := \left\| \mathcal{H}^{1/2} \cdot \right\|_{L^2}.$$

Let  $\mathcal{A}$  be as in [18, Theorem 2.4]. In other words,  $\mathcal{A}$  is accretive and there exists a full rank matrix  $W \in M_{d, 2d}(\mathbb{R}) = \mathbb{R}^{d \times 2d}$  such that

$$\begin{aligned} \text{dom}(\mathcal{A}) &= \left\{ u \in H : \mathcal{H}u \in H^1([a, b]; \mathbb{R}^d), W \begin{bmatrix} (\mathcal{H}u)(b) \\ (\mathcal{H}u)(a) \end{bmatrix} = 0 \right\} \\ \mathcal{A}u &= P_1(\mathcal{H}u)' + P_0\mathcal{H}u. \end{aligned} \quad (3.1)$$

In particular,  $-\mathcal{A}$  generates a contraction semigroup  $\mathcal{T}$  on  $H$ . Moreover,  $\mathcal{A}$  has compact resolvent and hence, its spectrum consists of eigenvalues only [20, Theorem 2.2 and Lemma 3.3]. Here and in the sequel,  $\mathbb{1}_d$  denotes the identity matrix of dimension  $d$ . The fundamental matrix associated to the ODE-system

$$\begin{aligned} v'(x) &= -P_1^{-1}(it\mathcal{H}(x)^{-1} + P_0)v(x) && \text{for } x \in [a, b], t \in \mathbb{R}, \\ v(a) &= \mathbb{1}_d && \text{for } t \in \mathbb{R} \end{aligned}$$

is denoted by  $\Phi_t$ . It is proved in [20, Theorem 1.3] that semi-uniform stability and strong stability of  $\mathcal{T}$  are equivalent:

**Theorem 3.1** ([20, Theorem 1.3]). *Given the setting in the present section, the following conditions are equivalent.*

- (i) *The semigroup  $\mathcal{T}$  is strongly stable.*
- (ii) *There are no spectral values of  $\mathcal{A}$  on the imaginary axis, i.e.,  $i\mathbb{R} \subseteq \rho(\mathcal{A})$ .*
- (iii) *The matrix  $T_t := W \begin{bmatrix} \Phi_t(b) \\ \mathbb{1}_d \end{bmatrix}$  is invertible for each  $t \in \mathbb{R}$ .*

The following is implicitly proved in [18, Theorem 3.4]. For the benefit of the reader, we include the straightforward argument.

**Lemma 3.2.** *Let  $t \in \mathbb{R}$  and  $u, f \in H$ . Then  $u = \mathcal{R}(it, -\mathcal{A})f$  if and only if*

$$(\mathcal{H}u)(x) = \Phi_t(x)(\mathcal{H}u)(a) + \Phi_t(x) \int_a^x \Phi_t(s)^{-1} P_1^{-1} f(s) \, ds \quad (3.2)$$

and

$$W \begin{bmatrix} \Phi_t(b) \\ \mathbb{1}_d \end{bmatrix} (\mathcal{H}u)(a) = -W \begin{bmatrix} \Phi_t(b) \int_a^b \Phi_t(s)^{-1} P_1^{-1} f(s) \, ds \\ 0 \end{bmatrix}. \quad (3.3)$$

*Proof.* Consider the operator

$$\begin{aligned} \text{dom}(A) &:= \left\{ v \in H^1([a, b], \mathbb{R}^d) : W \begin{bmatrix} v(b) \\ v(a) \end{bmatrix} = 0 \right\} \\ Av &:= P_1 v' + P_0 v \end{aligned}$$

on the space  $L^2([a, b]; \mathbb{R}^d)$ . Observe that  $(it + \mathcal{A})u = f$  if and only if  $\mathcal{H}u \in \text{dom}(A)$  and

$$(\mathcal{H}u)' = -P_1^{-1}(it\mathcal{H}^{-1} + P_0)\mathcal{H}u + P_1^{-1}f.$$

By definition of the fundamental matrix, the latter is equivalent to (3.2). Using this,

$$\begin{bmatrix} (\mathcal{H}u)(b) \\ (\mathcal{H}u)(a) \end{bmatrix} = \begin{bmatrix} \Phi_t(b) \\ \mathbb{1}_d \end{bmatrix} (\mathcal{H}u)(a) + \begin{bmatrix} \Phi_t(b) \int_a^b \Phi_t(s)^{-1} P_1^{-1} f(s) \, ds \\ 0 \end{bmatrix}.$$

In other words,  $\mathcal{H}u \in \text{dom}(A)$  is equivalent to (3.3).  $\square$

#### 4. THE FUNCTION IN (M) FOR PORT-HAMILTONIAN SYSTEMS

We adopt the notation and the general setting of port-Hamiltonian systems of Section 3. Recalling the definition of  $\Phi_t$ , we introduce for  $t \in \mathbb{R}$ ,

$$B_t := \|\Phi_t\|_\infty \quad \text{and} \quad B := \sup_{t \in \mathbb{R}} B_t$$

and recall condition

$$B < \infty \tag{B}$$

from [18]. The condition (B) is satisfied if, for instance,  $\mathcal{H}$  is of bounded variation [18, Theorem 6.7] or if  $\mathcal{H}(x)E_+ \subseteq E_+$  for a.e.  $x \in (a, b)$  with  $E_+$  being the spectral subspace of positive eigenvalues of  $P_1$ ; see [18, Theorem 6.3]. Our aim here is to give precise estimates on the function defined in (M). More precisely, we show the following result.

**Theorem 4.1.** *Assume that  $i\mathbb{R} \subseteq \rho(\mathcal{A})$  and that the condition (B) holds. Then for each  $t \in \mathbb{R}$ , the matrix  $T_t := W \begin{bmatrix} \Phi_t(b) \\ \mathbb{1}_d \end{bmatrix}$  is invertible with*

$$\|T_t^{-1}\| \leq C (\|\mathcal{R}(it, -\mathcal{A})\| + 1)$$

and

$$\|\mathcal{R}(it, -\mathcal{A})\| \leq \tilde{C} (\|T_t^{-1}\| + 1);$$

here

$$\tilde{C} := (b-a)B^2 \|S\| \|P_1\| \|P_1^{-1}\|^2 \max\{B \|W\|, (b-a)^{1/2}\},$$

while the constant  $C$  is given by

$$\left( \frac{1}{(b-a)^{3/2}} \|P_1\|^2 \|P_1^{-1}\|^2 B^3 (1+B) + 1 \right) \|W^+\| \max\left\{ (b-a)^{1/2} \|\mathcal{H}\|_\infty \|P_1\|, 1 \right\},$$

where  $W^+$  denotes the Moore–Penrose inverse of  $W$ .

**Remark 4.2** (Moore–Penrose inverse). Let  $d, r \in \mathbb{N}$ ,  $d \leq r$ , and  $D \in \mathbb{C}^{d \times r}$ . Assume that  $D$  has full rank (i.e.,  $\text{rank } D = d$ ). Then  $D$ , considered as a linear mapping from  $\mathbb{C}^r$  to  $\mathbb{C}^d$ , is onto and a right inverse is given by the Moore–Penrose inverse or pseudo-inverse  $D^+ = D^*(DD^*)^{-1}$ .

We split the proof of Theorem 4.1 across the following two lemmata. Observe the topological isomorphism of Hilbert spaces  $S: u \mapsto \mathcal{H}^{-1}u$  from  $L^2([a, b]; \mathbb{R}^d)$  to  $H$ ; see [18, Lemma 3.1].

**Lemma 4.3.** *Assume that  $i\mathbb{R} \subseteq \rho(\mathcal{A})$  and that the condition (B) holds. Then for each  $t \in \mathbb{R}$ , the matrix  $T_t := W \begin{bmatrix} \Phi_t(b) \\ \mathbb{1}_d \end{bmatrix}$  is invertible and*

$$\|\mathcal{R}(it, -\mathcal{A})\| \leq \tilde{C} (\|T_t^{-1}\| + 1)$$

with

$$\tilde{C} := (b-a)B^2 \|S\| \|P_1\| \|P_1^{-1}\|^2 \max\{B \|W\|, (b-a)^{1/2}\}.$$

*Proof.* First of all, recall that  $\|\Phi_t^{-1}\|_\infty \leq B_t \|P_1\| \|P_1^{-1}\| \leq B \|P_1\| \|P_1^{-1}\|$  from [18, Lemma 3.2] and from Theorem 3.1 that  $i\mathbb{R} \subseteq \rho(\mathcal{A})$  implies the invertibility of  $T_t$  for all  $t \in \mathbb{R}$ .

Let  $f \in H$  and  $t \in \mathbb{R}$ . Setting  $v := S^{-1}\mathcal{R}(it, -\mathcal{A})f$ , we obtain from (3.2) that

$$\begin{aligned} \|\mathcal{R}(it, -\mathcal{A})f\| &\leq \|S\| \|v\|_{L^2} \\ &\leq (b-a)^{1/2} B_t \|S\| \left( \|v(a)\| + (b-a) B_t \|P_1\| \|P_1^{-1}\|^2 \|f\|_{L^2} \right). \end{aligned}$$

On the other hand, (3.3) and invertibility of  $T_t$  together imply that

$$\begin{aligned} \|v(a)\| &= \left\| T_t^{-1} W \begin{bmatrix} \Phi_t(b) \int_a^b \Phi_t(s)^{-1} P_1^{-1} f(s) \, ds \\ 0 \end{bmatrix} \right\| \\ &\leq (b-a)^{1/2} B_t^2 \|W\| \|P_1\| \|P_1^{-1}\|^2 \|T_t^{-1}\| \|f\|_{L^2}. \end{aligned} \quad (4.1)$$

It follows that

$$\|\mathcal{R}(it, -\mathcal{A})\| \leq (b-a) B^2 \|S\| \|P_1\| \|P_1^{-1}\|^2 \left( B \|W\| \|T_t^{-1}\| + (b-a)^{1/2} \right)$$

and hence the assertion.  $\square$

**Lemma 4.4.** *Let  $t \in \mathbb{R}$ . If  $T_t := W \begin{bmatrix} \Phi_t(b) \\ \mathbb{1}_d \end{bmatrix}$  is invertible, then*

$$\|T_t^{-1}\| \leq C_t \left( (b-a)^{1/2} \|\mathcal{H}\|_\infty \|P_1\| \|\mathcal{R}(it, -\mathcal{A})\| + 1 \right);$$

where

$$C_t := \left( \frac{1}{(b-a)^{3/2}} \|P_1^{-1}\|^2 \|P_1\|^2 B_t^3 (1 + B_t) + 1 \right) \|W^+\|.$$

*Proof.* By compactness of the unit ball in  $\mathbb{R}^d$ , we find  $z \in \mathbb{R}^d$  with  $\|z\| = 1$  such that

$$\|T_t^{-1}z\| = \|T_t^{-1}\|.$$

Since  $W$  has full rank, by Remark 4.2, the element  $[z_1 \ z_2]^\top := W^+z$  satisfies

$$z = W \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Next, we set

$$y := -z_1 + \Phi_t(b)z_2 \quad \text{and} \quad f := (b-a)^{-1} P_1 \Phi_t(\cdot) \Phi_t(b)^{-1} y.$$

Of course,  $\|y\| \leq (1 + B_t) \|W^+z\|$  and furthermore, by [18, Lemma 3.2] we have  $f \in L^\infty([a, b], \mathbb{R}^d)$  with

$$\begin{aligned} \|f\|_\infty &\leq (b-a)^{-1} \|P_1\| B_t \left( \|P_1^{-1}\| \|P_1\| \|\Phi_t(b)\| \right) \|y\| \\ &\leq (b-a)^{-1} B_t^2 (1 + B_t) \|P_1\|^2 \|P_1^{-1}\| \|W^+z\|. \end{aligned} \quad (4.2)$$

Next, (3.3) along with our choice of  $f$  yields that  $u := \mathcal{R}(it, -\mathcal{A})f$  satisfies

$$\begin{aligned} T_t(\mathcal{H}u)(a) &= -W \begin{bmatrix} y \\ 0 \end{bmatrix} = W \begin{bmatrix} z_1 - \Phi_t(b)z_2 \\ 0 \end{bmatrix} \\ &= W \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - W \begin{bmatrix} \Phi_t(b) \\ \mathbb{1}_d \end{bmatrix} z_2 \\ &= z - T_t z_2. \end{aligned}$$

Invertibility of  $T_t$  thus gives  $T_t^{-1}z = (\mathcal{H}u)(a) + z_2$  and whence

$$\|T_t^{-1}\| = \|T_t^{-1}z\| \leq \|(\mathcal{H}u)(a)\| + \|z_2\| \leq \|(\mathcal{H}u)(a)\| + \|W^+z\|. \quad (4.3)$$

Next, we want to estimate  $\|(\mathcal{H}u)(a)\|$  for which we recall from Lemma 3.2 that

$$(\mathcal{H}u)(a) = \Phi_t(x)^{-1}(\mathcal{H}u)(x) - \int_a^x \Phi_t(s)P_1^{-1}f(s) \, ds.$$

Therefore, we can once again use [18, Lemma 3.2] to deduce that

$$\begin{aligned} \|(\mathcal{H}u)(a)\| &\leq (b-a)^{-1/2} \|(\mathcal{H}u)(a)\|_2 \\ &= (b-a)^{-1/2} B_t \|P_1^{-1}\| (\|P_1\| \|\mathcal{H}u\|_{L^2} + \|f\|_\infty) \\ &= (b-a)^{-1/2} B_t \|P_1^{-1}\| (\|P_1\| \|\mathcal{H}\mathcal{R}(it, -\mathcal{A})f\|_{L^2} + \|f\|_\infty) \\ &\leq (b-a)^{-1/2} B_t \|P_1^{-1}\| (\|\mathcal{H}\|_\infty \|P_1\| \|\mathcal{R}(it, -\mathcal{A})\| \|f\|_{L^2} + \|f\|_\infty) \\ &\leq (b-a)^{-1/2} B_t \|P_1^{-1}\| \left( (b-a)^{1/2} \|\mathcal{H}\|_\infty \|P_1\| \|\mathcal{R}(it, -\mathcal{A})\| + 1 \right) \|f\|_\infty. \end{aligned}$$

In particular, we infer from (4.2) that  $\|(\mathcal{H}u)(a)\|$  can be estimated from above by

$$\left( C_t \|W^+\|^{-1} - 1 \right) \left( (b-a)^{1/2} \|\mathcal{H}\|_\infty \|P_1\| \|\mathcal{R}(it, -\mathcal{A})\| + 1 \right) \|W^+z\|.$$

Substituting into (4.3) and using  $\|z\| = 1$ , we deduce the assertion.  $\square$

*Proof of Theorem 4.1.* The invertibility of each  $T_t$  and the latter estimate are contents of Lemma 4.3. Conversely, Lemma 4.4 provides the desired estimate for fixed  $t$ .  $\square$

An immediate application of Theorem 4.1 yields the following desired computation of (M) in the present context.

**Corollary 4.5.** *Let  $-\mathcal{A}$  be the generator of the contraction (port-Hamiltonian) semigroup  $\mathcal{T}$  defined in Section 3 such that  $i\mathbb{R} \subseteq \rho(\mathcal{A})$  and condition (B) on the fundamental matrices holds. With  $T_t := W \begin{bmatrix} \Phi_t(b) \\ \mathbb{1}_d \end{bmatrix}$  for  $t \in \mathbb{R}$ , there exist  $c, C > 0$  such that for all  $\eta \geq 0$*

$$cm(\eta) \leq M(\eta) \leq Cm(\eta),$$

where

$$m(\eta) := \sup_{t \in [-\eta, \eta]} \|T_t^{-1}\|$$

and  $M$  is the function associated to the resolvent defined in (M).

**Remark 4.6.** If the function  $m$  defined in Corollary 4.5 is of positive increase, then so is the function  $M$  associated to the resolvent defined in (M). Indeed, let  $f$  be of positive increase as in (2.1) and let  $g: (0, \infty) \rightarrow (0, \infty)$  satisfy

$$df(t) \leq g(t) \leq Df(t) \quad (t \geq t_0),$$

for some  $d, D, t_0 > 0$ . Then letting  $\alpha > 0$ ,  $c \in (0, 1]$  according to (2.1) yields for  $t \geq t_0$  and  $\lambda \geq 1$  that

$$\frac{g(\lambda t)}{g(t)} \geq \frac{df(\lambda t)}{Df(t)} = \frac{d}{D} c \lambda^\alpha.$$

In other words,  $g$  is of positive increase as well.

## 5. A UNIVERSAL EXAMPLE FOR STABILITY TYPES

The main example we focus on in the following is a slight generalisation of [18, Example 4.3]. In fact, using the notation to follow,  $\alpha = \sqrt{2}$  was taken in [18, Example 4.3]. For the general case, let  $\alpha \in (0, \infty)$ . On  $L^2([0, 1]; \mathbb{R}^2)$  consider the port-Hamiltonian system obtained by setting

$$\mathcal{H} := \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}^{-1}, \quad P_1 := \mathbb{1}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}, \quad P_0 := 0, \quad \text{and } W := [M \quad \mathbb{1}_2]; \quad (5.1)$$



where  $M := \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . By [18, Theorem 2.4(iv) and Example 4.3], the operator  $-\mathcal{A}$  given by (3.1) generates a contraction semigroup  $\mathcal{T}^\alpha$  with

$$T_{t,\alpha} = Me^{it\mathcal{H}^{-1}} + \mathbb{1}_2 \quad \text{and} \quad \det(T_{t,\alpha}) = 1 + \frac{1}{2}(e^{it} + e^{i\alpha t}); \quad (5.2)$$

here  $T_{t,\alpha}$  denotes the matrix  $T_t$  in Corollary 4.5 corresponding to  $\mathcal{T}^\alpha$ .

Note that in the present case,  $\mathcal{H}$  is constant, and, thus, of bounded variation, so that the condition (B) on the fundamental matrices is satisfied. An application of Theorem 3.1 yields the following statement.

**Proposition 5.1.** *The following conditions are equivalent:*

- (i) *The semigroup  $\mathcal{T}^\alpha$  is strongly stable.*
- (ii) *The number  $\alpha$  is irrational.*

In fact, it was also shown in [18, Example 4.3] that  $\mathcal{T}^\alpha$  is not exponentially stable for  $\alpha = \sqrt{2}$ . Actually, even more is true:

**Proposition 5.2.** *If  $\alpha \in (0, \infty)$  is irrational, then  $\mathcal{T}^\alpha$  is not exponentially stable.*

*Proof.* We know from [18, Theorem 3.5] that  $\mathcal{T}^\alpha$  is exponentially stable if and only if the function  $m_\alpha$  associated to  $T_{t,\alpha}^{-1}$  defined in Corollary 4.5 is uniformly bounded. However, for irrational  $\alpha$ , we have from Lemma 7.1 below that  $m_\alpha$  is unbounded, and whence the claim.  $\square$

We examine the system for non-exponential stability. We denote the function  $m$  in Corollary 4.5 corresponding to  $\mathcal{T}^\alpha$  by  $m_\alpha$ .

**Theorem 5.3.** *Let  $\varepsilon > 0$ . Then for Lebesgue a.e.  $\alpha \in (0, \infty)$  there exist  $C, \eta_0 > 0$  such that for all  $\eta \in [\eta_0, \infty)$ , we have*

$$m_\alpha(\eta) \leq C\eta^2(\log \eta)^{2+\varepsilon}.$$

Moreover, for all irrational  $\alpha \in (0, \infty)$  there is  $c > 0$  such that for all  $t_0 > 0$  there exists  $t \geq t_0$  with

$$ct^2 \leq \|T_{t,\alpha}^{-1}\|.$$

More information can be given if  $\alpha \in (0, \infty)$  is *badly approximable*, that is, there is  $c > 0$  such that for all  $p, q \in \mathbb{Z}$  with  $q \neq 0$ , we have

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^2}.$$

**Theorem 5.4.** *Let  $\alpha \in (0, \infty)$  be badly approximable. Then there exists  $c, C, \eta_0 > 0$  such that for all  $\eta \geq \eta_0$ , we have*

$$c\eta^2 \leq m_\alpha(\eta) \leq C\eta^2.$$

In particular,  $m_\alpha$  is of positive increase (see Remark 4.6).

Even though badly approximable numbers form a Lebesgue null set [13, Theorem 29], their Hausdorff dimension is 1, see [17]. In the sense of the Lebesgue measure,  $m_\alpha$  does not have positive increase a.e., as the next result confirms.

**Theorem 5.5.** *The set*

$$\{\alpha \in (0, \infty) : m_\alpha \text{ is of positive increase}\}$$

*is a Lebesgue null set.*

Next, we turn to more particular constructions allowing for more specific asymptotic behaviour of the resolvent/semigroup.

**Theorem 5.6.** *If  $\gamma: (0, \infty) \rightarrow (0, \infty)$  is increasing, then there exist  $\alpha \in (0, \infty)$  and  $c > 0$  such that for all  $t_0 > \pi$ , there exists  $t \geq t_0$  with*

$$c\gamma(t - \pi) \leq \|T_{t,\alpha}^{-1}\|.$$

*In addition, if*

$$\lim_{t \rightarrow \infty} t^{-2}\gamma(t) = \infty,$$

*then there exist  $C, \eta_0 > 0$  such that*

$$m_\alpha(\eta) \leq C\gamma(\eta + \pi) \quad (\eta \geq \eta_0).$$

**Remark 5.7.** In Theorem 5.6, even if  $\gamma$  is of positive increase,  $m_\alpha$  might not be. Indeed, it will follow from the proofs of Theorems 5.5 and 5.6 that the constructed  $\alpha$  in Example 8.3 satisfies a property resulting in  $m_\alpha$  failing to be of positive increase.

The proofs of the above results are based on approximations of irrational numbers by rational ones. We prove the corresponding number theoretical results in the subsequent sections estimating  $\det(T_{t,\alpha})$  accordingly. For now, a combination of Corollary 4.5 and the stability theorems from Section 2 yields the following statements on the asymptotic behaviour of  $\mathcal{T}^\alpha$ .

**Corollary 5.8.** *For the port-Hamiltonian system defined by (5.1), the following hold.*

- (a) *For Lebesgue a.e.  $\alpha \in (0, \infty)$  and all  $\varepsilon > 0$ , there exist  $c, C, t_0 > 0$  with*

$$\|\mathcal{T}^\alpha(t)\mathcal{A}^{-1}\| \leq \frac{C}{\delta^{-1}(t)} \quad \text{for all } t \geq t_0;$$

*where  $\delta(t) := t^2(\log t)^{2+\varepsilon}$ .*

- (b) *If  $\alpha \in (0, \infty)$  is badly approximable, then there exist  $c, C, t_0 > 0$  such that*

$$\frac{c}{\sqrt{t}} \leq \|\mathcal{T}^\alpha(t)\mathcal{A}^{-1}\| \leq \frac{C}{\sqrt{t}} \quad \text{for all } t \geq t_0.$$

- (c) *If  $\gamma: (0, \infty) \rightarrow (0, \infty)$  is increasing with  $\lim_{t \rightarrow \infty} t^{-2}\gamma(t) = \infty$ , then there exist  $\alpha \in (0, \infty)$  and  $c, t_0 > 0$  such that*

$$\|\mathcal{T}^\alpha(t)\mathcal{A}^{-1}\| \leq \frac{c}{\gamma_{\log}^{-1}(t/c)} \quad \text{for all } t \geq t_0;$$

*where*

$$\gamma_{\log}(\eta) := \gamma(\eta)(\log(1 + \gamma(\eta)) + \log(1 + \eta)) \quad (\eta > 0).$$

*In addition, if  $\gamma$  is of positive increase, then the latter estimate can be replaced by*

$$\|\mathcal{T}^\alpha(t)\mathcal{A}^{-1}\| \leq \frac{c}{\gamma^{-1}(t)}$$

*for all  $t \geq t_0$  and for some  $c > 0$ .*

*Proof.* Note that  $t \mapsto \delta(t) := t^2(\log t)^{2+\varepsilon}$  is of positive increase. Therefore, part (a) follows by combining Theorem 5.3, Corollary 4.5, and Theorem 2.3.

(b) If  $\alpha$  is badly approximable, then  $m_\alpha$  is of positive increase by Theorem 5.4. Due to Remark 4.6 and Corollary 4.5, we can apply Theorem 2.2 to obtain the desired estimate with  $m_\alpha^{-1}(t)$  instead of  $\sqrt{t}$ . Employing Theorem 5.4 once again yields the result.

(c) The first estimate follows from Theorem 2.1, Corollary 4.5, and Theorem 5.6. Furthermore, if  $\gamma$  is of positive increase, then the second inequality is a consequence of Theorem 2.3, Corollary 4.5, and Theorem 5.6.  $\square$

The stability rates in Corollary 5.8(b) are optimal, while in (a) they are optimal up to a logarithmic factor. The optimality of the rates in (c) holds in the following sense.

**Corollary 5.9.** *For the port-Hamiltonian system defined by (5.1), the following hold.*

(a) *For all irrational  $\alpha \in (0, \infty)$ , we have*

$$\limsup_{t \rightarrow \infty} \sqrt{t} \|\mathcal{T}^\alpha(t)\mathcal{A}^{-1}\| > 0.$$

(b) *If  $\gamma: (0, \infty) \rightarrow (0, \infty)$  is increasing with  $\lim_{t \rightarrow \infty} t^{-2}\gamma(t) = \infty$ , then there exist  $\alpha \in (0, \infty)$  and  $c > 0$  such that*

$$\limsup_{t \rightarrow \infty} \gamma^{-1}(ct) \|\mathcal{T}^\alpha(t)\mathcal{A}^{-1}\| > 0.$$

*Furthermore, if  $\gamma$  is of positive increase, any  $c > 0$  may be chosen.*

*Proof.* Let  $\alpha \in (0, \infty)$  be irrational. Then Lemma 7.1 below shows that  $m_\alpha$  is unbounded. Thus, (a) is a consequence of Corollary 4.5 and Theorems 2.5 and 5.3. On the other hand, (b) follows from Theorem 5.6, Corollary 4.5, and Proposition 2.6.  $\square$

The proofs of Theorems 5.3, 5.6, 5.4, and 5.5 are postponed to Sections 7, 8, and 9 respectively.

## 6. A MEANS TO QUANTIFY ODD OVER ODD RATIONAL APPROXIMATIONS

For the proofs of our main results related to the example in Section 5, rational approximations of irrational numbers, so that both numerator and denominator are odd integers, are the decisive objects to understand. Put differently, for  $\alpha \in \mathbb{R}$ , the function

$$h(t) := |2 + e^{i\pi t} + e^{i\pi\alpha t}| \quad (t \in \mathbb{R}) \quad (6.1)$$

can serve as a means to quantify the error for odd/odd rational approximations of  $\alpha$  in the following sense.

**Theorem 6.1.** *If  $\alpha \in \mathbb{R}$  is irrational, then there exist  $c, C > 0$  such that*

$$c \left( \min_{u \text{ odd}} |v\alpha - u| \right)^2 \leq \inf_{t \in [v-1, v+1]} h(t) \leq C \left( \min_{u \text{ odd}} |v\alpha - u| \right)^2$$

*for all odd integers  $v$ .*

The proof is subdivided into two partial results – one lemma for the upper and one lemma for the lower bound.

**Lemma 6.2.** *If  $\alpha \in \mathbb{R}$  is irrational, then there exists  $C > 0$  such that*

$$\inf_{t \in [v-1, v+1]} h(t) \leq C \left( \min_{u \text{ odd}} |v\alpha - u| \right)^2$$

*for all odd integers  $v$ .*

*Proof.* Fix an odd  $v \in \mathbb{Z}$  and choose an odd  $u \in \mathbb{Z}$  so that  $|v\alpha - u|$  is minimal. If

$$|v\alpha - u| \geq \frac{1}{3\pi} \min\{|1 + \alpha|, 1\} =: c_0,$$

then the upper bound is trivially satisfied for any  $C \geq 4/c_0^2$  because  $|h(t)| \leq 4$ . On the other hand, if

$$|v\alpha - u| < c_0,$$

we set

$$\delta := -\frac{v\alpha - u}{1 + \alpha} \quad \text{and} \quad t_0 := v + \delta.$$

In particular,  $\alpha t_0 = u - \delta$  and

$$|\delta| = \frac{|v\alpha - u|}{|1 + \alpha|} < \frac{c_0}{|1 + \alpha|} \leq \frac{1}{3\pi}.$$

Therefore,  $t_0 \in (v - 1, v + 1)$  and, furthermore,  $\pi |\delta| e^{\pi|\delta|} \leq 1/2$ . Thus,

$$|e^{i\pi\delta} - (1 + i\pi\delta - (\pi\delta)^2/2)| \leq (\pi\delta)^2 \pi |\delta| e^{\pi|\delta|} \leq \frac{1}{2}(\pi\delta)^2.$$

Since  $u$  and  $v$  are odd, we obtain

$$\begin{aligned} h(t_0) &= |e^{i\pi t_0} + e^{i\pi t_0 \alpha} + 2| = |e^{iv\pi + i\pi\delta} + e^{iu\pi - i\pi\delta} + 2| = |-e^{i\pi\delta} - e^{-i\pi\delta} + 2| \\ &\leq |-(1 + i\pi\delta - (\pi\delta)^2/2) - (1 - i\pi\delta - (\pi\delta)^2/2) + 2| + (\pi\delta)^2 \\ &\leq 2(\pi\delta)^2 = 2 \left( \frac{\pi |v\alpha - u|}{|1 + \alpha|} \right)^2. \end{aligned}$$

Setting

$$C := \max \left\{ \frac{4}{c_0^2}, \frac{2\pi^2}{|1 + \alpha|^2} \right\} = \frac{36\pi^2}{\min\{|1 + \alpha|^2, 1\}},$$

we obtain

$$\inf_{t \in [v-1, v+1]} h(t) \leq h(t_0) \leq C |v\alpha - u|^2. \quad \square$$

The lower bound estimate requires the following elementary observation.

**Lemma 6.3.** *Let  $\alpha \in \mathbb{R}$ . Then there exists  $C > 0$  such that for any  $\delta_1, \delta_2 \in \mathbb{R}$  with  $|\delta_1|, |\delta_2| \leq 1/2$ , we have*

$$(\delta_2 - \alpha\delta_1)^2 \leq C \max\{|\delta_1 + \delta_2|, \delta_1^2 + \delta_2^2\}.$$

*Proof.* Setting  $\alpha =: -1 - \beta$ , we get

$$\begin{aligned} (\delta_2 - \alpha\delta_1)^2 &= |\delta_2 + \delta_1 + \beta\delta_1|^2 \leq (|\delta_1 + \delta_2| + |\beta| |\delta_1|)^2 \\ &= |\delta_1 + \delta_2|^2 + 2|\delta_1 + \delta_2| |\beta| |\delta_1| + |\beta|^2 |\delta_1|^2 \\ &\leq 4 \max\{|\beta|^2, 1\} \max\{|\delta_1 + \delta_2|, \delta_1^2 + \delta_2^2\}. \quad \square \end{aligned}$$

Next, we turn to state and prove the lower bound estimates.

**Lemma 6.4.** *If  $\alpha \in \mathbb{R}$  is irrational, then there exists  $c > 0$  such that*

$$c \left( \min_{u \text{ odd}} |v\alpha - u| \right)^2 \leq \inf_{t \in [v-1, v+1]} h(t)$$

for all odd integers  $v$ .

*Proof.* Let  $v \in \mathbb{Z}$  be odd and fix  $t \in [v - 1, v + 1]$ . Then we can write

$$t = v + \delta_1, \quad (6.2)$$

$$t\alpha = u + \delta_2, \quad \text{with } u \in \mathbb{Z} \text{ odd and } \delta_1, \delta_2 \in [-1, 1]. \quad (6.3)$$

In particular,

$$h(t) = \left| 2 + e^{i\pi(v+\delta_1)} + e^{i\pi(u+\delta_2)} \right| = |e^{i\pi\delta_1} + e^{i\pi\delta_2} - 2|.$$

By continuity,

$$\inf\{h(t) : t \text{ satisfies (6.2) and (6.3) with } \max\{|\delta_1|, |\delta_2|\} \geq 1/(16\pi)\} =: c_1 > 0.$$

Therefore, if  $\pi |\delta_1| \geq 1/16$  or  $\pi |\delta_2| \geq 1/16$ , then

$$c_1 \left( \min_{u \text{ odd}} |v\alpha - u| \right)^2 \leq h(t)$$

due to the fact that  $\min_{u \text{ odd}} |v\alpha - u| \leq 1$ .

On the other hand, if  $\pi |\delta_1|, \pi |\delta_2| < 1/16$ , then set  $\delta'_i := \pi \delta_i$  for  $i \in \{1, 2\}$ . For  $|\delta| < 1/16$  we have

$$|e^{i\delta} - (1 + i\delta - \delta^2/2)| \leq \delta^2 |\delta| e^{|\delta|} \leq \delta^2 \frac{1}{16} e^{1/16} \leq \frac{3}{16} \delta^2,$$

and so

$$\begin{aligned} h(t) &\geq \left| i(\delta'_1 + \delta'_2) - \frac{1}{2}(\delta_1'^2 + \delta_2'^2) \right| - \frac{3}{16}(\delta_1'^2 + \delta_2'^2) \\ &= \sqrt{(\delta'_1 + \delta'_2)^2 + \frac{1}{4}(\delta_1'^2 + \delta_2'^2)^2} - \frac{3}{16}(\delta_1'^2 + \delta_2'^2) \\ &\geq \frac{1}{2} |\delta'_1 + \delta'_2| + \frac{1}{4}(\delta_1'^2 + \delta_2'^2) - \frac{3}{16}(\delta_1'^2 + \delta_2'^2) \\ &\geq \frac{1}{16} \max \{ |\delta'_1 + \delta'_2|, \delta_1'^2 + \delta_2'^2 \}. \end{aligned}$$

Then by Lemma 6.3 there exists  $c_2 > 0$  such that

$$h(t) \geq \frac{1}{16} \max \{ |\delta'_1 + \delta'_2|, \delta_1'^2 + \delta_2'^2 \} \geq \frac{c_2}{16} |\delta'_2 - \alpha \delta_1'|^2 = c_3 |\delta_2 - \alpha \delta_1|^2;$$

where  $c_3 := \pi c_2/16$ . Substituting (6.2) into (6.3), it follows that

$$h(t) \geq c_3 |\delta_2 - \alpha \delta_1|^2 = c_3 |v\alpha - u|^2.$$

Finally, setting  $c := \min\{c_1, c_3\}$  yields the desired bound.  $\square$

*Proof of Theorem 6.1.* This is the content of Lemmata 6.2 and 6.4.  $\square$

## 7. PROOF OF THEOREM 5.3

We recall  $T_{t,\alpha}$  from (5.2). Due to Corollary 4.5, both assertions require us to estimate  $\|T_{t,\alpha}^{-1}\|$ . To this end – since  $\|T_{t,\alpha}\|$  is uniformly bounded in  $t \in \mathbb{R}$  – by Cramer's rule, it suffices to bound  $1/\det(T_{t,\alpha})$ . However, in order to not overcomplicate the proofs, instead of  $1/\det(T_{t,\alpha})$  we consider  $\det(T_{t,\alpha})$  directly in the following. Therefore, in order to obtain the second assertion in Theorem 5.3, the question becomes when does

$$g(t) := \left| 1 + \frac{1}{2} (e^{it} + e^{it\alpha}) \right|$$

get small for particular values of  $t$  as  $t \rightarrow \infty$ . Note that this expression makes sense for any  $\alpha \in \mathbb{R}$  and that  $g(t) = \frac{1}{2}h(t/\pi)$  with  $h$  from (6.1). The next lemma gives the answer and also, in particular, implies Proposition 5.2.

**Lemma 7.1.** *Let  $\alpha \in \mathbb{R}$  be irrational. Then there is  $C > 0$  such that for any  $t_0 \in \mathbb{R}$  we find  $t \geq t_0$  with*

$$g(t) \leq \frac{C}{t^2}.$$

The proof of the lemma is based on the fact that any irrational number can be approximated quadratically well by fractions of the shape odd/odd (Lemma A.6). Recall from the previous section that the behaviour of the function  $h$  (and, thus, of  $g$ ) depends on the exact approximation properties of  $\alpha$  by fractions of the shape odd/odd. A more detailed view of this is described in the next lemma. Lemma 7.1 follows immediately from Lemma 7.2 by setting  $\psi(t) = 1/t$  and using Lemma A.6.

**Lemma 7.2.** *Let  $\alpha \in \mathbb{R}$  be irrational and  $\psi: (0, \infty) \rightarrow (0, \infty)$  decreasing. Assume that there exist infinitely many pairs of odd integers  $u, v$  such that*

$$\left| \alpha - \frac{u}{v} \right| \leq \frac{\psi(v)}{v}.$$

Then there exists  $C > 0$  such that for any  $t_0 \in \mathbb{R}$  we find  $t \geq t_0$  with

$$g(t) \leq C \left( \psi\left(\frac{t}{\pi} - 1\right) \right)^2.$$

*Proof.* First, note that we may assume that  $\psi(t) \rightarrow 0$  for  $t \rightarrow \infty$  because otherwise, the statement is trivial. By assumption we have

$$|v\alpha - u|^2 \leq (\psi(v))^2$$

for infinitely many odd integers  $u, v$ . In particular, this means that we have infinitely many odd integers  $1 \leq v_1 < v_2 < \dots$  which satisfy the above inequality for some odd  $u_n$ 's. Recall the function  $h$  from (6.1). Compactness of the interval  $[v_n - 1, v_n + 1]$  yields  $t_n \in [v_n - 1, v_n + 1]$  such that

$$\begin{aligned} h(t_n) &= \inf_{t \in [v_n - 1, v_n + 1]} h(t) \leq C \left( \min_{u \text{ odd}} |v_n \alpha - u| \right)^2 \\ &\leq C(\psi(v_n))^2 \leq C(\psi(t_n - 1))^2; \end{aligned}$$

where the first inequality is the upper bound from Theorem 6.1. Since  $t_n \rightarrow \infty$  for  $n \rightarrow \infty$ , reformulating the statement in terms of  $g$ , the assertion follows.  $\square$

Next, we want to prove a lower bound for  $g(t)$ , which holds for all sufficiently large  $t$ . It will again depend on the approximation properties of  $\alpha$ .

**Lemma 7.3.** *Let  $\alpha \in \mathbb{R}$  be irrational and  $\psi: (0, \infty) \rightarrow (0, \infty)$  decreasing such that the inequality*

$$\left| \alpha - \frac{u}{v} \right| < \frac{\psi(v)}{v}$$

*has finitely many solutions in odd integers  $u, v$ . Then there exists  $c > 0$  such that*

$$c\left(\psi\left(\frac{t}{\pi} + 1\right)\right)^2 \leq g(t) \quad (t \geq 0). \quad (7.1)$$

Because of Lemma A.5, the assumptions of Lemma 7.3 are satisfied, in particular, for  $\psi(t) = 1/(t(\log t)^{1+\varepsilon})$  and for almost all  $\alpha$ .

*Proof of Lemma 7.3.* By assumption, we can find a constant  $c_1 > 0$  such that

$$|v\alpha - u| \geq c_1 \psi(v)$$

for all odd integers  $u$  and  $v$ . For any  $t \geq 0$ , let us denote by  $v_t$ , the closest odd integer to  $t$ . Then by Theorem 6.1 – using  $h$  as in (6.1) – we can find  $c_2 > 0$  such that

$$\begin{aligned} h(t) &\geq \inf_{s \in [v_t - 1, v_t + 1]} h(s) \geq c_2 \left( \min_{u \text{ odd}} |v_t \alpha - u| \right)^2 \\ &\geq c_2 c_1^2 (\psi(v_t))^2 \geq c_2 c_1^2 (\psi(t + 1))^2. \end{aligned}$$

Reformulation in terms of  $g$  instead of  $h$  yields the result.  $\square$

Setting  $\psi(t) := 1/(t(\log t)^{1+\frac{\varepsilon'}{2}})$  for some  $0 < \varepsilon' < \varepsilon$  and using Lemma A.5, we immediately get the following metric result for the lower bound.

**Lemma 7.4.** *For Lebesgue a.e.  $\alpha \in \mathbb{R}$  and all  $\varepsilon > 0$ , there is  $t_0 > 0$  such that*

$$\frac{1}{t^2(\log t)^{2+\varepsilon}} \leq g(t) \quad (t \geq t_0).$$

*Proof of Theorem 5.3.* Keeping the discussion at the beginning of the section in mind, both assertions follow by appealing to Lemmata 7.1 and 7.4.  $\square$

## 8. PROOF OF THEOREM 5.6

The result in Theorem 5.3 only qualifies for Lebesgue-almost every  $\alpha \in \mathbb{R}$ . In fact, one can construct specific  $\alpha$ 's such that  $g(t)$  decays arbitrarily quickly for the “worst”  $t$ 's. This is the content of the present section. Recall that

$$g(t) = \left| 1 + \frac{1}{2}(e^{it} + e^{it\alpha}) \right| \quad (t \in \mathbb{R}).$$

**Lemma 8.1.** *Let  $f: (0, \infty) \rightarrow (0, \infty)$  be decreasing. There exists an irrational  $\alpha > 0$  such that the following assertions hold for some  $c, C > 0$ .*

(a) *For all  $t_0 > \pi$ , there is  $t \geq t_0$  with*

$$g(t) \leq Cf(t - \pi), \quad (8.1)$$

(b) *If  $\lim_{t \rightarrow \infty} f(t)t^2 = 0$ , then there is  $t_1 > 0$  such that for all  $t \geq t_1$  we have*

$$cf(t + \pi) \leq g(t). \quad (8.2)$$

The proof of Lemma 8.1 is based on continued fraction expansions for real numbers. In a particular sense, the continued fraction expansion gives the optimal approximations of irrational numbers by rationals. Some results of the theory of Diophantine approximations are gathered in Appendix A.

**Definition 8.2.** The *continued fraction* of an  $\alpha \in \mathbb{R}$  is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}} \quad \text{or} \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}}$$

where  $a_0, a_1, a_2, \dots$  are obtained from the *continued fraction algorithm*: Set  $a_0 := \lfloor \alpha \rfloor$ . If  $a_0 \neq \alpha$ , we write  $\alpha = a_0 + 1/\alpha_1$ , where  $\alpha_1 > 1$  and set  $a_1 := \lfloor \alpha_1 \rfloor$ . Next, if  $a_1 \neq \alpha_1$ , we write  $\alpha_1 = a_1 + 1/\alpha_2$ , where  $\alpha_2 > 1$  and set  $a_2 := \lfloor \alpha_2 \rfloor$ . We continue this process until  $a_n = \alpha_n$  for some  $n$ . If  $\alpha$  is rational, the process terminates and we obtain a finite continued fraction, which is written as  $[a_0; a_1, a_2, \dots, a_n]$ . If  $\alpha$  is irrational, then the process does not terminate and we obtain an infinite continued fraction, written as  $[a_0; a_1, a_2, \dots]$ . The rationals  $\frac{p_j}{q_j} = [a_0; a_1, a_2, \dots, a_j]$ , where  $p_j, q_j$  denote relatively prime integers with  $q_j > 0$ , are called the *convergents* to  $\alpha$ . Note that for irrational  $\alpha$  the convergents indeed converge to  $\alpha$ , and there is a one-to-one correspondence between the irrationals and the infinite continued fraction expansions.

*Proof of Lemma 8.1.* We set  $\alpha = [1; a_1, a_2, \dots]$  by recursively defining

$$a_n := 2 \left\lceil \frac{1}{\sqrt{f(\pi q_{n-1})} q_{n-1}} \right\rceil \quad \text{for } n \geq 1,$$

where  $q_n = a_n q_{n-1} + q_{n-2}$  for  $n \geq 2$  and  $q_0 = 1$  and  $q_1 = a_1$ , as in Lemma A.1. We turn to (a) first and show (8.1) for infinitely many, arbitrarily large  $t$ . By Lemma A.2, we have for all convergents that

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2 a_{n+1}} < \frac{\sqrt{f(\pi q_n)}}{q_n}. \quad (8.3)$$

Moreover, since  $a_n$  is even for  $n \geq 1$ , one can check that by Lemma A.1 the convergents are of the following shapes: if  $n$  is even, then  $p_n/q_n = \text{odd}/\text{odd}$ ; if  $n$  is

odd, then  $p_n/q_n = \text{odd}/\text{even}$ . Taking only the convergents of the shape odd/odd, we get infinitely many fractions  $u/v = \text{odd}/\text{odd}$  with

$$\left| \alpha - \frac{u}{v} \right| < \frac{\sqrt{f(\pi v)}}{v}.$$

Then Lemma 7.2 immediately implies that there exists  $C > 0$  such that for any  $t_0 > \pi$  there exists  $t \geq t_0$  with

$$g(t) \leq Cf(t - \pi),$$

which is (a).

(b) We want to employ Lemma 7.3. For this purpose, we show that the inequality

$$\left| \alpha - \frac{u}{v} \right| < \frac{\sqrt{f(\pi v)}/4}{v} \quad (8.4)$$

has only finitely many solutions in odd integers  $u, v$ .

First, note that since  $f(t)t^2 \rightarrow 0$  for  $t \rightarrow \infty$ , the right-hand side of (8.4) becomes less than  $1/(2v^2)$  for sufficiently large  $v$ . Thus, Lemma A.4 implies that every solution  $u/v$  to (8.4), with  $v$  sufficiently large, has to be a convergent to  $\alpha$ . Therefore, it suffices to check that only finitely many convergents  $u/v = p_n/q_n$  satisfy (8.4). The lower bound for approximations by convergents from Lemma A.2 tells us that

$$\frac{1}{(a_{n+1} + 2)q_n^2} < \left| \alpha - \frac{p_n}{q_n} \right|. \quad (8.5)$$

Note that

$$(a_{n+1} + 2)q_n^2 = \left( 2 \left\lceil \frac{1}{q_n \sqrt{f(\pi q_n)}} \right\rceil + 2 \right) q_n^2 \leq 3 \frac{q_n}{\sqrt{f(\pi q_n)}}$$

for sufficiently large  $q_n$ . Thus, (8.5) in conjunction with (8.3) gives

$$\frac{\sqrt{f(\pi q_n)}}{3q_n} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{\sqrt{f(\pi q_n)}}{4q_n},$$

which is clearly impossible. Therefore, inequality (8.4) only has finitely many solutions  $u/v$ , and Lemma 7.3 implies that there is  $c_1 > 0$  and  $t_1 > 0$  such that

$$\frac{c_1}{16} f(t + \pi) \leq g(t) \quad (t \geq t_1),$$

which is (b). □

*Proof of Theorem 5.6.* Recalling the discussion at the beginning of Section 7 and setting  $f = \frac{1}{\gamma}$ , the statement is a direct consequence of Lemma 8.1. □

**Example 8.3.** For  $f(t) = e^{-t}$ , we can construct  $\alpha$  by setting  $\alpha = [1; a_1, a_2, \dots]$  where

$$a_n := 2 \left\lceil \frac{1}{e^{-\pi q_{n-1}/2} q_{n-1}} \right\rceil \quad \text{for } n \geq 1.$$

Then we have

$$ce^{-t} \leq g(t) \leq Ce^{-t},$$

where the left inequality holds for all  $t$  sufficiently large, and the right inequality is satisfied for infinitely many  $t$  accumulating at  $\infty$ .



## 9. PROOFS OF THEOREMS 5.4 AND 5.5

For convenience, let us consider

$$\tilde{m}_\alpha(\eta) := m_\alpha(\pi\eta)/2 = \sup_{t \in [-\eta, \eta]} h(t)^{-1};$$

where  $h(t) := |2 + e^{i\pi t} + e^{i\pi\alpha t}|$ . Clearly,  $\tilde{m}_\alpha(\eta)$  grows quadratically if and only if  $m_\alpha(\eta)$  grows quadratically. Recall that  $\alpha \in (0, \infty)$  is badly approximable, if there is  $c > 0$  such that for all  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ , we have

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^2}.$$

It is a basic fact (which follows from Lemma A.2) that  $\alpha = [a_0; a_1, a_2, \dots]$  is badly approximable if and only if  $(a_n)_n$  is bounded.

The decisive information establishing Theorem 5.4 is the following.

**Lemma 9.1.** *Let  $\alpha \in \mathbb{R}$  be badly approximable and let  $(v_n)$  be a strictly increasing sequence such that*

$$\{v_n : n \in \mathbb{N}\} = \{v \in \mathbb{N} : v \text{ odd and there is odd } u \in \mathbb{N} \text{ with } |v\alpha - u| \leq 2/v\}.$$

*Then there exists  $C \geq 0$  such that  $v_{n+1} \leq Cv_n$  for all  $n \in \mathbb{N}$ .*

*Proof.* As  $\alpha = [a_0; a_1, a_2, \dots]$  is badly approximable,  $(a_n)$  is bounded by, say  $C_1 > 0$ . Fix an integer  $n \geq 2$  and denote by  $(q_k)_k$  the sequence of denominators of the convergents of  $\alpha$ . Since  $q_k \rightarrow \infty$ , we find  $k \in \mathbb{N}$  such that  $q_{k-1} \leq v_n < q_k$ . By Lemma A.8, there exists  $u, v \in \mathbb{N}$  odd such that  $q_k \leq v \leq q_{k+2}$  and  $|v\alpha - u| \leq 2/v$ . Thus,  $v > v_n$ . As  $(v_n)$  is strictly increasing, we get  $v_{n+1} \leq v \leq q_{k+2}$ .

By the recurrence relation in Lemma A.1 and the fact that  $(q_k)_k$  is strictly increasing, we get

$$q_{k+2} = a_{k+2}q_{k+1} + q_k < (C_1 + 1)q_{k+1}$$

A recursive application of the above inequality implies

$$q_{k+2} \leq (C_1 + 1)^3 q_{k-1}.$$

Combining the observations in the previous two paragraphs gives  $v_{n+1} \leq q_{k+2} \leq (C_1 + 1)^3 q_{k-1} \leq (C_1 + 1)^3 v_n$ , establishing the assertion.  $\square$

*Proof of Theorem 5.4.* We want to show that there exists  $c, C, \eta_0 > 0$  such that

$$c\eta^2 \leq \tilde{m}_\alpha(\eta) \leq C\eta^2$$

for all  $\eta \geq \eta_0$ . As  $\alpha$  is badly approximable, the assumptions of Lemma 7.3 are fulfilled with  $\psi(v) = \tilde{c}v^{-1}$  for some  $\tilde{c} > 0$ . The required upper bound can therefore be obtained by Lemma 7.3, again owing to the discussion at the beginning of Section 7.

We now prove the lower bound. Since  $\alpha$  is irrational, by Lemma A.9, we find a strictly increasing sequence  $(v_n)_n$  of odd integers such that for each  $n \in \mathbb{N}$ , there exists an odd  $u_n \in \mathbb{N}$  with

$$|v_n\alpha - u_n| \leq \frac{2}{v_n}.$$

As  $\alpha = [a_0; a_1, a_2, \dots]$  is badly approximable, by Lemma 9.1, we find  $C \geq 0$  such that  $v_{n+1} \leq Cv_n$  for all  $n \in \mathbb{N}$ .

Next, let  $\eta \geq 2$  and  $\ell \in \mathbb{N}$  be maximal with  $v_\ell \leq \eta - 1$ . It follows from our choice of  $(v_n)$  and  $(u_n)$  and Theorem 6.1 that there is  $C_2 \geq 0$  such that

$$\begin{aligned} \inf_{t \in [-\eta, \eta]} h(t) &\leq C_2 \left( \min_{\substack{v \in [-(\eta-1), \eta-1] \text{ odd} \\ u \text{ odd}}} |v\alpha - u| \right)^2 \\ &\leq C_2 |v_\ell \alpha - u_\ell|^2 \leq \frac{2C_2}{v_\ell^2}. \end{aligned} \quad (9.1)$$

Moreover, our choice of  $\ell, C$  and that  $(v_n)_n$  is strictly increasing ensure that  $\eta - 1 < v_{\ell+1} \leq C v_\ell$ . This implies  $v_\ell \geq c_1 \eta$ , for  $c_1 := (2C)^{-1}$ . Combining this with (9.1), we infer

$$\inf_{t \in [-\eta, \eta]} h(t) \leq \frac{2C_2}{(c_1 \eta)^2} = \frac{C_3}{\eta^2},$$

where  $C_3 = 2C_2 c_1^{-2}$ . This settles the lower bound for  $\tilde{m}_\alpha(\eta)$ .  $\square$

For  $\alpha \in \mathbb{R}$ , we say that  $\alpha = [a_0; a_1, a_2, \dots]$  has *large odd/odd gaps*, if for any  $C > 0$  there exists  $n \in \mathbb{N}$  such that the  $n$ -th convergent is of the shape  $p_n/q_n = \text{odd/odd}$  and  $a_{n+1} \geq C$ . As we shall see below, this property is generic in the Lebesgue measure sense (Lemma 9.3). First, we confirm that such  $\alpha$  do not yield  $m_\alpha$  of positive increase.

**Lemma 9.2.** *If  $\alpha \in \mathbb{R}$  has large odd/odd gaps, then  $m_\alpha$  is not of positive increase.*

*Proof.* Again, for simplicity, we consider

$$\tilde{m}_\alpha(\eta) := m_\alpha(\pi\eta)/2 = \sup_{t \in [-\eta, \eta]} h(t)^{-1},$$

as clearly  $m_\alpha(\eta)$  is of positive increase if and only if  $\tilde{m}_\alpha(\eta)$  is.

Assume towards a contradiction that  $\tilde{m}_\alpha(\eta)$  is of positive increase, i.e., there exist  $\beta > 0, c \in (0, 1]$ , and  $\eta_0 > 0$  such that for all  $\lambda \geq 1$  and  $\eta \geq \eta_0$ , we have

$$\tilde{m}_\alpha(\lambda\eta) \geq c\lambda^\beta \tilde{m}_\alpha(\eta). \quad (9.2)$$

By Theorem 6.1, we find  $c_1, C_1 > 0$  such that for all odd integers  $v$  we have

$$c_1 \left( \min_{u \text{ odd}} |v\alpha - u| \right)^2 \leq \inf_{t \in [v-1, v+1]} h(t) \leq C_1 \left( \min_{u \text{ odd}} |v\alpha - u| \right)^2.$$

We choose  $\lambda > 1$  such that

$$c\lambda^\beta > C_1/c_1. \quad (9.3)$$

Since  $\alpha$  has large odd/odd gaps, there is  $n \in \mathbb{N}$  such that the  $n$ -th convergent is of the shape  $p_n/q_n = \text{odd/odd}$  and  $a_{n+1} > 2\lambda$ . Without loss of generality, assume  $q_n \geq \eta_0$ , otherwise choose a larger  $n$ , and set  $\eta := q_n + 1$ . Then by Theorem 6.1 we have

$$\inf_{t \in [-\eta, \eta]} h(t) \leq \inf_{t \in [q_n-1, q_n+1]} h(t) \leq C_1 |q_n \alpha - p_n|^2,$$

and thus

$$\tilde{m}_\alpha(\eta) = \sup_{t \in [-\eta, \eta]} h(t)^{-1} \geq C_1^{-1} |q_n \alpha - p_n|^{-2}. \quad (9.4)$$

On the other hand, note that by Lemma A.3 for all integers  $u, v$  with  $0 < |v| < q_{n+1}$  we have  $|\alpha v - u| \geq |q_n \alpha - p_n|$ . By Lemma A.1, we have  $q_{n+1} = a_{n+1} q_n + q_{n-1} > 2\lambda q_n + 1 \geq \lambda(q_n + 1) + 1 = \lambda\eta + 1$  and in turn,

$$|\alpha v - u| \geq |q_n \alpha - p_n| \quad \text{for all } |v| \leq \lambda\eta + 1.$$

Thus, Theorem 6.1 implies

$$\inf_{t \in [-\lambda\eta, \lambda\eta]} h(t) \geq c_1 |q_n \alpha - p_n|^2,$$

and so

$$\tilde{m}_\alpha(\lambda\eta) \leq c_1^{-1} |q_n\alpha - p_n|^{-2}. \quad (9.5)$$

Combining (9.5), (9.4), and (9.3), we obtain

$$\tilde{m}_\alpha(\lambda\eta) \leq c_1^{-1} |q_n\alpha - p_n|^{-2} \leq c_1^{-1} C_1 \tilde{m}_\alpha(\eta) < c\lambda^\beta \tilde{m}_\alpha(\eta),$$

contradicting (9.2).  $\square$

**Lemma 9.3.** *For Lebesgue a.e.  $\alpha \in \mathbb{R}$ ,  $\alpha$  has large odd/odd gaps.*

*Proof.* By [7, Theorem 9.2], for Lebesgue a.e.  $\alpha \in \mathbb{R}$ ,  $\alpha$  has a *normal continued fraction* – in particular, every block of positive integers  $d_1, \dots, d_k$  occurs in the continued fraction expansion  $[a_0; a_1, a_2, \dots]$ . Let  $\alpha \in \mathbb{R}$  have a normal continued fraction. Then for any odd  $C \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that  $a_{n+1} = a_{n+2} = a_{n+3} = C$ . So, at least one of the convergents  $p_n/q_n, p_{n+1}/q_{n+1}, p_{n+2}/q_{n+2}$  must be of the shape odd/odd. Indeed, assume by contradiction that none of them are of the shape odd/odd. Then, by Lemma A.7, both consecutive denominators as well as consecutive numerators need to change parity. Thus, only the two cases,  $p_n/q_n$  odd/even,  $p_{n+1}/q_{n+1}$  even/odd, and  $p_{n+2}/q_{n+2}$  odd/even along with  $p_n/q_n$  even/odd,  $p_{n+1}/q_{n+1}$  odd/even, and  $p_{n+2}/q_{n+2}$  even/odd need to be rendered impossible. Recalling Lemma A.1, we obtain the two equalities

$$q_{n+2} = a_{n+2}q_{n+1} + q_n \quad \text{and} \quad p_{n+2} = a_{n+2}p_{n+1} + p_n.$$

The first equality, as  $a_{n+2}$  is odd, yields if  $q_n$  and  $q_{n+2}$  both are even, then so is  $q_{n+1}$ , excluding the first case. Whereas the second equality necessitates  $p_{n+1}$  to be even, if both  $p_{n+2}$  and  $p_n$  are, eliminating the second case as well. Hence, we find  $\ell \in \{n, n+1, n+2\}$  such that  $p_\ell/q_\ell$  is of the shape odd/odd. Since  $a_{\ell+1} = C$  can be chosen arbitrarily large, we eventually establish the assertion.  $\square$

*Proof of Theorem 5.5.* The statement is a direct consequence of Lemma 9.2 and Lemma 9.3.  $\square$

**Remark 9.4.** Note that the  $\alpha$  from Example 8.3 clearly satisfies the assumption of Lemma 9.2 as well because  $a_n \rightarrow \infty$  for  $n \rightarrow \infty$ , and every other convergent is odd/odd.

## 10. CONCLUSION

We have revisited the stability problem for port-Hamiltonian systems and provided an estimate of the resolvent growth function in terms of the matrix norm of a suitable inverse of a derived quantity with which it was possible to characterise exponential stability and asymptotic stability in earlier findings. The precise estimate led to the consideration and analysis of a rather elementary port-Hamiltonian system, where many different examples for possible decay of the port-Hamiltonian semigroup can be provided. The core observation was that the resolvent growth can be reformulated in terms of Diophantine approximation results, that is, how well irrational numbers can be approximated by rationals.

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## APPENDIX A. BACKGROUND ON DIOPHANTINE APPROXIMATION

In this section, we summarise some background information on Diophantine approximations that are used in the main body of the text. Recall the continued fraction expansion from Section 8.

**Lemma A.1** (See e.g. [3, pg 45]). *The convergents  $p_n/q_n = [a_0; a_1, \dots, a_n]$  to a real number  $\alpha = [a_0; a_1, a_2, \dots]$  are given by the recurrence  $p_0 = a_0$ ,  $q_0 = 1$ ,  $p_1 = a_0a_1 + 1$ ,  $q_1 = a_1$  and*

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}, \quad \text{for } n \geq 2.$$

Moreover, they satisfy the formula

$$p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}.$$

**Lemma A.2** (See e.g. [3, pg 47]). *The convergents  $p_n/q_n$  to  $\alpha = [a_0; a_1, a_2, \dots] \in \mathbb{R}$  satisfy*

$$\frac{1}{(a_{n+1} + 2)q_n^2} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}.$$

**Lemma A.3** (See e.g. [3, pg 47]). *Let  $p_1/q_1, p_2/q_2, \dots$  be the convergents to a real number  $\alpha$ . Then for any integers  $p, q$  with  $0 < q < q_{n+1}$ , we have*

$$|q\alpha - p| \geq |q_n\alpha - p_n|.$$

**Lemma A.4** (Legendre, 1798). *For  $\alpha \in \mathbb{R}$  and coprime integers  $p, q$  with  $q > 0$  and*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{2q^2},$$

*we have  $p/q$  is a convergent to  $\alpha$ .*

**Lemma A.5** (Corollary of Khintchin's approximation theorem [12, Satz II]). *Let  $\varepsilon > 0$ . Then for Lebesgue a.e.  $\alpha \in \mathbb{R}$ , the inequality*

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{q^2(\log q)^{1+\varepsilon}}$$

*has only finitely many solutions  $(p, q) \in \mathbb{Z}^2$ .*

It was proven in [14, Theorem II] that there always exist quadratically good approximations  $u/v$  even if we restrict  $u$  and  $v$  to a certain parity.

**Lemma A.6** (odd/odd approximations, see [14, Theorem II]). *For every irrational  $\alpha \in \mathbb{R}$ , there exist infinitely many rationals  $u/v$  of the shape odd/odd such that*

$$\left| \alpha - \frac{u}{v} \right| \leq \frac{1}{v^2}. \quad (\text{A.1})$$

We make use of insights related to the latter result in Section 9. In fact, the results used are gathered in the following two lemmas.

**Lemma A.7.** *Let  $\alpha \in \mathbb{R}$  be irrational. Let  $n \in \mathbb{N}$  and  $\frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}$  be two consecutive convergents to  $\alpha$ . Then  $q_n$  and  $q_{n+1}$  cannot both be even and  $p_n$  and  $p_{n+1}$  cannot both be even.*

*Proof.* Recall that by Lemma A.1 we have  $q_{n+1} = a_{n+1}q_n + q_{n-1}$ . If, by contradiction, both  $q_{n+1}$  and  $q_n$  are even,  $q_{n-1}$  is even as well. By induction, all denominators  $q_{n+1}, q_n, \dots, q_2, q_1, q_0$  are even, which contradicts  $q_0 = 1$ .

Similarly, by Lemma A.1, both  $p_{n+1}$  and  $p_n$  being even implies  $p_{n+1}, p_n, \dots, p_1, p_0$  are even. This is a contradiction because  $p_0 = a_0$  and  $p_1 = a_0a_1 + 1$  cannot both be even.  $\square$

**Lemma A.8.** *Let  $\alpha \in \mathbb{R}$  be irrational. Let  $n \in \mathbb{N}$ ,  $n \geq 2$ , and  $\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}$  be three consecutive convergents to  $\alpha$ . Then there exist odd integers  $u, v$  such that*

$$\left| \alpha - \frac{u}{v} \right| < \frac{2}{v^2}$$

*with  $q_{n-1} \leq v \leq q_{n+1}$ .*

*Proof.* Recall from Lemma A.2 that for every convergent  $p_n/q_n$ , we have

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

Therefore, the claim is evident if either  $p_n/q_n$  or  $p_{n+1}/q_{n+1}$  are of the shape odd/odd. By Lemma A.7, it thus suffices to discuss the following cases: Either  $p_n/q_n = \text{odd/even}$  and  $p_{n+1}/q_{n+1} = \text{even/odd}$  or  $p_n/q_n = \text{even/odd}$  and  $p_{n+1}/q_{n+1} = \text{odd/even}$ .

In either case, set  $u = p_{n+1} - p_n$  and  $v = q_{n+1} - q_n$ . Then  $u$  and  $v$  are odd and, using Lemma A.2 and Lemma A.1, we get

$$\begin{aligned} \left| \alpha - \frac{u}{v} \right| &\leq \left| \alpha - \frac{p_{n+1}}{q_{n+1}} \right| + \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_{n+1} - p_n}{q_{n+1} - q_n} \right| \\ &\leq \frac{1}{q_{n+1}^2} + \frac{|p_{n+1}(q_{n+1} - q_n) - q_{n+1}(p_{n+1} - p_n)|}{q_{n+1}(q_{n+1} - q_n)} \\ &= \frac{1}{q_{n+1}^2} + \frac{|p_n q_{n+1} - p_{n+1} q_n|}{q_{n+1}(q_{n+1} - q_n)} \\ &= \frac{1}{q_{n+1}^2} + \frac{1}{q_{n+1}(q_{n+1} - q_n)} < \frac{2}{(q_{n+1} - q_n)^2} = \frac{2}{v^2}. \end{aligned}$$

The estimate  $q_{n-1} \leq q_{n+1} - q_n$ , which follows from Lemma A.1, and the various choices for  $v$  establish the remaining estimates.  $\square$

With the latter two results, we may also prove a variant of Lemma A.6 as follows.

**Lemma A.9.** *Let  $\alpha \in \mathbb{R}$  be irrational. Then there exist infinitely many pairs of odd integers  $u, v$  such that*

$$\left| \alpha - \frac{u}{v} \right| < \frac{2}{v^2}. \quad (\text{A.2})$$

*Proof.* The claim follows from Lemma A.8 together with the fact that  $q_n \rightarrow \infty$  as  $n \rightarrow \infty$ .  $\square$

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