

Efficient Ranking, Order Statistics, and Sorting under CKKS

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Abstract

Fully Homomorphic Encryption (FHE) enables operations on encrypted data, making it extremely useful for privacy-preserving applications, especially in cloud computing environments. In such contexts, operations like ranking, order statistics, and sorting are fundamental functionalities often required for database queries or as building blocks of larger protocols. However, the high computational overhead and limited native operations of FHE pose significant challenges for an efficient implementation of these tasks. These challenges are exacerbated by the fact that all these functionalities are based on comparing elements, which is a severely expensive operation under encryption.

Previous solutions have typically based their designs on swap-based techniques, where two elements are conditionally swapped based on the results of their comparison. These methods aim to reduce the primary computational bottleneck: the *comparison depth*, which is the number of non-parallelizable homomorphic comparisons. The current state of the art solution for sorting by Lu et al. (IEEE S&P'21), for instance, achieves a comparison depth of $O(\log^2 N)$.

In this paper, we address the challenge of reducing the comparison depth by shifting away from the swap-based paradigm. We present solutions for ranking, order statistics, and sorting, that all achieve a comparison depth of $O(1)$, making our approach highly parallelizable and suitable for hardware acceleration. Leveraging the SIMD capabilities of the CKKS FHE scheme, our approach re-encodes the input vector under encryption to allow for simultaneous comparisons of all elements with each other. The homomorphic re-encoding incurs a minimal computational overhead of $O(\log N)$ rotations. Experimental results show that our approach ranks a 128-element vector in approximately 2.64s, computes its argmin/argmax in 14.18s, and sorts it in 21.10s, even with a textbook implementation of the comparison function.

1 Introduction

Fully Homomorphic Encryption (FHE) is a cryptographic primitive that enables performing unbounded operations on encrypted data, without decrypting them first. It is a fundamental building block for designing non-interactive protocols in privacy-preserving applications, and can be used to maintain the confidentiality of data stored in the cloud, while enabling outsourced computations on it. Despite the effort of recent developments to make this technology more efficient and closer to be usable in real-world applications [5–7, 25], computing under FHE is still problematic due to both its serious computational overhead and limited native operations. In particular, it is challenging to realize even fundamental functions efficiently, like *ranking*, computing *order statistics*, and *sorting*, which are frequently required database operations. These functionalities also find applications in diverse fields, for instance in privacy-preserving machine learning, where they can be used to evaluate max-pooling layers and the argmax output layer in neural networks [19, 27], or to perform private inference of decision trees [24, 26].

Several works in the literature have attempted to implement these functionalities efficiently (see Table 1). Many studies have focused on sorting under both the Smart-Vercauteren (SV) scheme [25] and the Cheon-Kim-Kim-Song scheme (CKKS), which is particularly relevant as it enables floating-point arithmetic on vectors of data in a Single Instruction Multiple Data (SIMD) fashion. These approaches typically implement swap-based sorting methods, where at each round two elements are compared and conditionally swapped [8, 9, 16, 18, 22]. Similarly, other works have focused on computing the argmax of a vector of elements encrypted in a CKKS ciphertext, also relying on swap-based techniques [19, 27]. However, comparing two values under encryption is significantly expensive, resulting in the bot-

¹ Precisely, in terms of multiplicative depth, ranking requires up to $d_C + 4$ levels, while the extraction of k -statistics and sorting require up to $d_C + d_I + 4$ and $d_C + d_I + 6$ levels, respectively. Here, d_C and d_I represent the multiplicative depth of the comparison and indicator circuits (see Section 3).

Paper	Function	Comp. Depth	FHE Scheme	Remarks
Chatterjee et al. [8] (Indocrypt 2013)	Bubble Sort, Insertion Sort	$O(N^2)$ $O(N^2)$	SV SV	Tested up to 40 elements, for which it runs in around 359.42 minutes (Bubble Sort) and 362.62 minutes (Insertion Sort).
Chatterjee et al. [9] (IEEE TSC 2017)	Quick Sort	$O(N^2)$	SV	Tested up to 40 elements, for which it runs in around 779.28 minutes.
Emmadi et al. [16] (ICCCRI 2015)	Bitonic Sort, Odd-Even Merge Sort	$O(\log^2 N)$ $O(\log^2 N)$	SV SV	Tested up to 64 elements, for which it runs in around 52.63 minutes (Bitonic Sort) and 42.65 minutes (Odd-Even Merge Sort).
Lu et al. [22] (IEEE S&P 2021)	Bitonic Sort (switching to FHEW)	$O(\log^2 N)$	CKKS	Tested up to 64 elements, for which it runs in 409.09s in a 4-thread setting (without taking into account the scheme switching cost).
Hong et al. [18] (IEEE TIFS 2021)	k-Way Sorting Networks	$O(k \log_k^2 N)$	CKKS	It takes around 2 hours to sort 625 elements.
Jovanovic et al. [19] (ACM CCS 2022)	Argmax	$O(N)$	CKKS	It computes the argmax of 128 elements in approximately 366 seconds.
Zhang et al. [27] (preprint, 2024)	Argmax	$O(\log N)$	CKKS	It computes the argmax of 128 elements in approximately 28 seconds.
Our work	Ranking k-Statistics (incl. Argmax) Sorting	$O(1)$ ¹ $O(1)$ ¹ $O(1)$ ¹	CKKS CKKS CKKS	Tested up to 1024 elements. It ranks 128 elements in 2.64s, computes their argmin/argmax in 14.18s, and sorts them in 21.10s.

Table 1: Summary of related work.

tleneck of these methods being the *comparison depth*. The comparison depth refers to the number of comparisons that must be executed in series, and hence cannot be parallelized. Consequently, **the main problem lies in designing algorithms capable of achieving a low comparison depth**. This task is made particularly challenging by the fact that any swap-based algorithm will have worst case complexity under encryption [16].

In this paper, we design and implement novel algorithms for the aforementioned functionalities which, for the first time, require a comparison depth of 1 only. To overcome the limitations of previous solutions, we avoid relying on the strategy of swapping elements under encryption. Our approach heavily exploits the SIMD capabilities of CKKS. The core idea is to use suitable homomorphic rotations and masking to re-encode the encrypted elements in such a way that allows us to compare all elements against each other simultaneously. By aggregating the outcome of this comparison, we compute the rank of each element within the vector. Then, by leveraging appropriate indicator functions, it is possible to use the computed ranks to extract any order statistic and their position, including minimum, maximum, and median (or any percentile). Finally, we show how to parallelize this extraction process to implement a sorting functionality.

By employing only recursive rotation-based operations, we make sure that the re-encoding under encryption requires only $O(\log N)$ rotations, where N is the vector length, thus causing minimal computational overhead. Moreover, the low comparison depth makes our solution highly parallelizable, thus suitable for potential hardware acceleration, such as on GPUs. While we leave this direction to future research, in our present work we show how to further optimize our solution algorithmically in multithreaded environments. Importantly, our approach is agnostic to the specific implementation of the comparison function. Even with a textbook implementation, our approach can rank a vector of 128 elements in approximately 2.64s, compute its argmin/argmax in 14.18s, and sort it in 21.10s.

2 Preliminaries

We provide background information and notation regarding CKKS, along with an overview of the homomorphic functionalities upon which our design is constructed.

2.1 CKKS Scheme

CKKS [11] is a fully homomorphic encryption scheme based on the RLWE problem. It works with residual polynomial rings of the form $R_q = \mathbb{Z}_q[x]/(x^n + 1)$, where the ring dimension n is a power of two. Messages from $\mathbb{C}^{n/2}$ are encoded into plaintexts, which can embed vectors of up to $n/2$ slots. The scheme operates with floating-point values. The encryption and homomorphic operations (see below) introduce noise

on the underlying plaintexts, which is blended with the noise inherent in floating-point arithmetic, making the scheme intrinsically approximate.

CKKS natively supports three homomorphic operations on ciphertexts:

- addition ($X + Y$) corresponds to the component-wise addition of the underlying plaintext vectors;
- multiplication ($X \cdot Y$) corresponds to the component-wise multiplication of the underlying plaintext vectors;
- rotation ($X \ll k$) and ($X \gg k$) correspond to the left and right rotations of the underlying plaintext vector by a plaintext index k .

Moreover, additions and multiplications can also be performed between a ciphertext and a plaintext. Among all these operations, ciphertext-ciphertext multiplications and rotations are computationally the most expensive.

2.2 Evaluating Non-Polynomial Functions

By combining additions and multiplications it is possible to evaluate any polynomial under encryption. Evaluating non-polynomial functions, such as the comparison operation, is not trivial under CKKS. The usual solution consists of approximating the function by a polynomial. However, a good approximation usually requires a high-degree polynomial, which leads to a deep multiplicative circuit to be evaluated and high computational cost. Thus, this paper focuses on designing algorithms in which the number of non-polynomial evaluations is minimized.

In our design, we use two non-polynomial functions: the comparison function and the indicator function, defined as

$$\text{Cmp}(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0.5 & \text{if } x = y \\ 0 & \text{if } x < y \end{cases}, \quad (1)$$

$$\text{Ind}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}. \quad (2)$$

We approximate both using Chebyshev polynomials, which assure uniform convergence to the original function. The evaluation of the polynomials is then performed using the Paterson-Stockmeyer algorithm adapted to Chebyshev basis [10], which requires only $O(\sqrt{d})$ homomorphic multiplications to evaluate a degree- d polynomial. We denote an approximation of degree d of these functions with $\text{Cmp}(\cdot, \cdot; d)$ and $\text{Ind}_A(\cdot; d)$, respectively.

Specific to the comparison function, the higher the degree of the approximation the better it can correctly compare values that are close to each other. For a discussion on this topic and for different implementations of the compare function, we refer to [21].

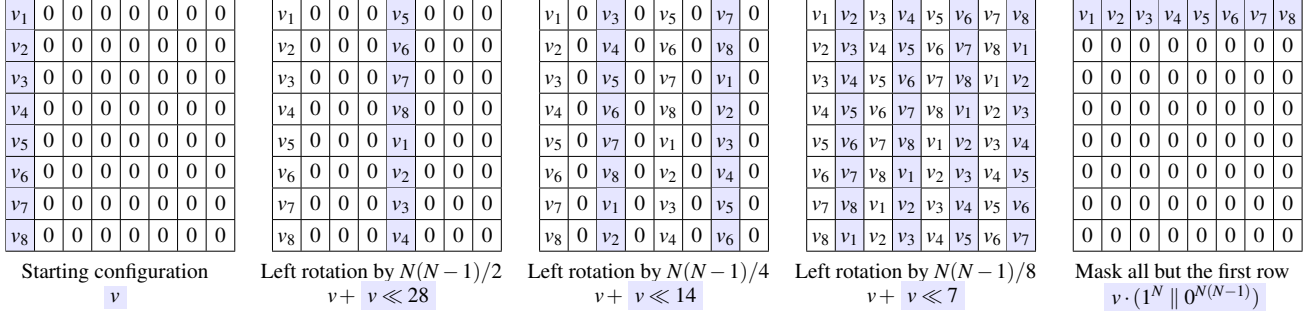


Figure 1: Transposing a vector of length $N = 8$ from column to row (TransC).

2.3 Matrix Encoding and Operations

Our approach relies on matrices for intermediate computations. To encode a matrix into a ciphertext we adopt the row-by-row approach [20], which consists of concatenating each row into a single vector and then encrypting it. For a square matrix of size N , we have the requirement that $N^2 \leq n/2$, where n is the ring dimension, otherwise multiple ciphertexts are needed to store the entire matrix.

We describe some basic functionalities useful for working with encrypted matrices. These functionalities will be the building blocks of our approach. Given a matrix encoded into a ciphertext X :

- $\text{MaskR}(X, k)$ extracts row k by masking everything else, i.e., setting everything else to zero;
- $\text{SumR}(X)$ sums all the rows together component-wise and stores the result in the first row;
- $\text{RepR}(X)$ assumes a matrix with only the first row non-zero and replicates it all over by copying its values into the other rows;
- $\text{TransR}(X)$ assumes a square matrix with only the first row non-zero and transposes it (i.e., moving it into the first column).

Similarly for the columns, we have MaskC , SumC , RepC , TransC . Masking works by multiplying the encrypted matrix by an appropriate plaintext bit mask. For sum and replication there are well-known algorithms in the literature that work recursively, and only require $\log(N)$ rotations, where N is the number of rows/columns of the matrix [17]. For these to work, N must be a power of two, thus the matrix might require padding. We provide their pseudocode in Appendix A. As for transposition, to the best of our knowledge no algorithm that works in $\log(N)$ is available in the literature. Hence, we propose Algorithm 1 and Algorithm 2, which can be of independent interest. Figure 1 shows an example of the TransC algorithm for a generic vector of length $N = 8$.

Algorithm 1 TransR

Input: X encryption of a vector x encoded as a row.

Output: X encryption of the vector x encoded as a column.

- 1: **for** $i = 1, \dots, \lceil \log N \rceil$ **do**
 - 2: $X \leftarrow X + (X \gg N(N-1)/2^i)$
 - 3: **end for**
 - 4: $X \leftarrow \text{MaskC}(X, 0)$
 - 5: **return** X
-

Algorithm 2 TransC

Input: X encryption of a vector x encoded as a column.

Output: X encryption of the vector x encoded as a row.

- 1: **for** $i = 1, \dots, \lceil \log N \rceil$ **do**
 - 2: $X \leftarrow X + (X \ll N(N-1)/2^i)$
 - 3: **end for**
 - 4: $X \leftarrow \text{MaskR}(X, 0)$
 - 5: **return** X
-

3 Our Design for Ranking, Order statistics, and Sorting

The core idea of our design is to manipulate the encrypted vector in such a way that **only a single evaluation of the comparison function is needed**. For instance, given a vector $v = (v_1, v_2, v_3)$, we produce

$$v_R = (v_1, v_2, v_3, v_1, v_2, v_3, v_1, v_2, v_3),$$

$$v_C = (v_1, v_1, v_1, v_2, v_2, v_2, v_3, v_3, v_3).$$

The comparison $\text{Cmp}(v_R, v_C)$ contains information about $v_i < v_j$ for all pairs (v_i, v_j) . It is easier to visualize this by seeing v_R, v_C as square matrices that have been encoded row-by-row into vectors:

$$v_R = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \end{pmatrix}, \quad v_C = \begin{pmatrix} v_1 & v_1 & v_1 \\ v_2 & v_2 & v_2 \\ v_3 & v_3 & v_3 \end{pmatrix}.$$

Turning v into v_R and v_C homomorphically can be done by composing the matrix operations mentioned above, as will be

described in detail later. Note that we are implicitly assuming we can fit N^2 elements in a ciphertext, where the vector length N is 3 in the example above. If this assumption does not hold, we require multiple ciphertexts, which we discuss in Section 5.

3.1 Ranking

Ranking associates the elements of a vector to their rank, that is the position they would have if the vector was sorted, starting from 1. In case of ties, we adopt fractional ranking, for which ties receive a rank equal to the average of the ranks they span. For instance, the ranking of $(50, 10, 20, 20, 40)$ is $(5, 1, 2.5, 2.5, 4)$.

Given an input vector v encrypted as V , we think of it as the first row of a null matrix. The encoding v_R is produced by simply applying ReplR , while v_C is produced by first transposing the initial vector to a column with TransR and then replicating it with ReplC . The component-wise comparison of $v_R > v_C$ is ideally a matrix with values in $\{0, 0.5, 1\}$, whose each column j contains information about the position of v_j in the sorted array:

- a number of ones equal to the number of elements smaller than v_j , and
- a number of 0.5 equal to the number of elements equal to v_j .

Thus, summing the elements in the column gives the fractional rank of v_j . A pseudocode is provided in Algorithm 3, while a schematic with an example can be found in Figure 2.

Algorithm 3 Rank

Input: V encryption of $v = (v_1, \dots, v_N) \in \mathbb{R}^N$, approximation degree $d \in \mathbb{N}$.

Output: R encryption of a vector in \mathbb{R}^N representing the (fractional) ranking of v .

- 1: $V_R \leftarrow \text{ReplR}(V)$
 - 2: $V_C \leftarrow \text{ReplC}(\text{TransR}(V))$
 - 3: $C \leftarrow \text{Cmp}(V_R, V_C; d)$
 - 4: $R \leftarrow \text{SumR}(C) + (0.5, \dots, 0.5)$
 - 5: **return** R
-

We assume the vector size is a power of 2 to work with recursive operations on matrices. If it is not, we can pad it to the next power of 2 and perform an additional masking after the comparison to remove the excess information.

The cost of Algorithm 3 is $4 \lceil \log N \rceil$ rotations, which can be reduced to $3 \lceil \log N \rceil$ by parallelizing ReplR and ReplC , and \sqrt{d} ciphertext-ciphertext multiplications, with a multiplicative depth of $\sim \lceil \log d \rceil$.

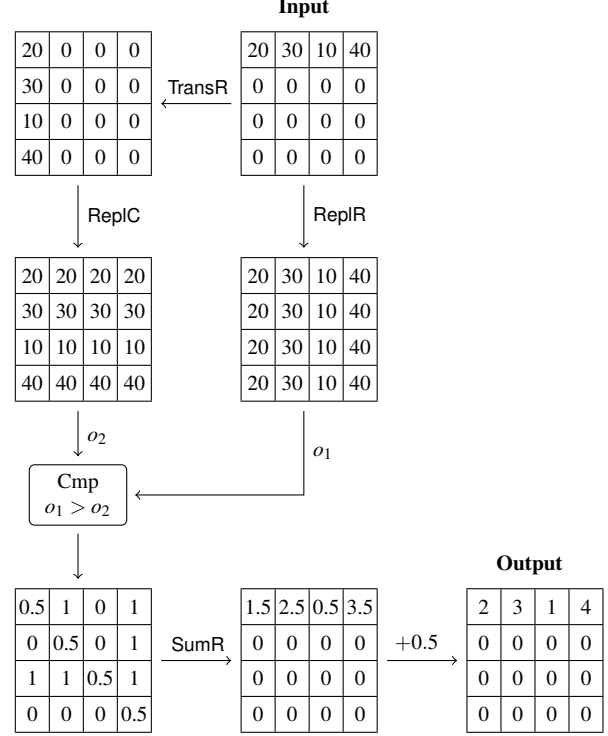


Figure 2: Schematic example of ranking a 4-element vector.

Correctness Proof Assuming the correctness of the building blocks and the ideal functionality of the algorithm, namely no noise from the scheme and no approximation error for the comparison function, we prove that the output R produced by Algorithm 3 on input the encryption of a vector $v = (v_1, \dots, v_N)$ is actually the encryption of the fractional ranking of v . Let $r = (r_1, \dots, r_N)$ be the decryption of R . Let $j \in \{1, \dots, N\}$, we prove that r_j is the rank of v_j . Let v_R and v_C be the decryption as matrices of V_R and V_C , respectively. By the correctness of ReplR , $v_{R;i,j} = v_j$ for all $i \in \{1, \dots, N\}$, where $v_{R;i,j}$ is the element at row i and column j of v_R . Similarly, by the correctness of ReplC and TransR , $v_{C;i,j} = v_i$ for all $i \in \{1, \dots, N\}$. Let c be the decryption as matrix of C , then $c_{i,j} = \text{Cmp}(v_j, v_i)$, where Cmp is defined in Equation (1). Then $r_j = \sum_{i=1}^N c_{i,j} + 0.5$. We split the sum over the following partition of $\{1, \dots, N\}$:

$$\begin{aligned} \mathcal{L}_j &:= \{i \in \{1, \dots, N\} : v_i < v_j\} \\ \mathcal{E}_j &:= \{i \in \{1, \dots, N\} : v_i = v_j\} \\ \mathcal{G}_j &:= \{i \in \{1, \dots, N\} : v_i > v_j\} \end{aligned}$$

and get

$$\begin{aligned}
r_j &= \sum_{\mathcal{L}_j} c_{i,j} + \sum_{\mathcal{E}_j} c_{i,j} + \sum_{\mathcal{G}_j} c_{i,j} + 0.5 \\
&= \sum_{\mathcal{L}_j} 1 + \sum_{\mathcal{E}_j} 0.5 + \sum_{\mathcal{G}_j} 0 + 0.5 \\
&= |\mathcal{L}_j| + 0.5 \cdot |\mathcal{E}_j| + 0.5.
\end{aligned}$$

The elements equal to v_j span a rank from $|\mathcal{L}_j| + 1$ to $|\mathcal{L}_j| + |\mathcal{E}_j|$, thus the fractional rank of v_j is their average, namely

$$\begin{aligned}
\frac{1}{|\mathcal{E}_j|} \sum_{k=1}^{|\mathcal{E}_j|} (|\mathcal{L}_j| + k) &= \frac{1}{|\mathcal{E}_j|} (|\mathcal{E}_j| \cdot |\mathcal{L}_j| + 0.5 \cdot |\mathcal{E}_j| \cdot (|\mathcal{E}_j| + 1)) \\
&= |\mathcal{L}_j| + 0.5 \cdot (|\mathcal{E}_j| + 1) = r_j.
\end{aligned}$$

□

3.2 Order Statistics

The k -th order statistic (or k -statistic) of a vector is its k -th smallest value, that is the value of rank k if such rank exists. A value of given rank might not exist if there are ties in the vector. Here, we will show how to work around this issue in the specific case of the first and last order statistics (i.e., min and max). While we will provide a general-purpose solution to it in Section 4.

We can determine the k -th order statistics of a vector v by first computing its ranking and then applying to it an indicator function “around k ”, namely for the interval $[k - 0.5, k + 0.5]$. It will output a bit mask whose ones correspond to the positions of the elements with rank k , if they exist (see Algorithm 4).

Algorithm 4 Order Statistic

Input: V encryption of $v = (v_1, \dots, v_N) \in \mathbb{R}^N$, approximation degrees $d_C, d_I \in \mathbb{N}$, index $k \in \{1, \dots, N\}$.

Output: O encryption of a Boolean vector in $\{0, 1\}^N$ that has value 1 in position i if and only if v_i has rank k .

- 1: $R \leftarrow \text{Rank}(V; d_C)$
 - 2: $O \leftarrow \text{Ind}_k(R; d_I)$
 - 3: **return** O
-

One can then retrieve the actual value of the statistic by computing the inner product $\langle o, V \rangle = \text{SumC}(o \cdot V)$ and dividing it by the L1 norm of the mask $\text{SumC}(o)$. This can be done under encryption by approximating $1/x$ in the range $[0.5, N + 0.5]$, or by using Goldschmidt’s division algorithm [13]. The outcome will be zero if no element of rank k exists.

Correctness Proof Assuming the correctness of the building blocks and Algorithm 3, and the ideal functionality of the algorithm, we prove that Algorithm 4 is correct. The input is

the encryption of a vector $v = (v_1, \dots, v_N)$, together with an index $k \in \mathbb{N}$. Let r, o be the decryption of R, O respectively. For $i \in \{1, \dots, N\}$,

$$o_i = \text{Ind}_k(r_i) = \begin{cases} 1 & \text{if Rank}(v_i) = k \\ 0 & \text{if Rank}(v_i) \neq k \end{cases}.$$

□

Min and Max We can ensure that we can always compute minimum and maximum (first and last order statistic) by employing a slightly different definition of the comparison function. By using

$$\text{Cmp}_G(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

all the minimal elements are assigned to the first rank, thus the minimum can be retrieved with Ind_1 . Similarly, by using

$$\text{Cmp}_{GE}(x, y) = \begin{cases} 1 & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$$

all the maximal elements are assigned to the last rank, thus the maximum can be retrieved with Ind_N .

3.3 Sorting

We sort a vector by extracting all its order statistics simultaneously, parallelizing the strategy presented in Algorithm 4. For now, we assume that all elements in the vector are distinct, ensuring that there is exactly one element for each rank $k \in \{1, \dots, N\}$, and allowing us to extract any order statistic. We will show how to remove this assumption in Section 4.

To extract all the order statistics in one go, the idea is to compute the ranking of v , replicate it over the rows, and extract a different k -statistic for each row k . Normally, this would require applying N different indicator functions. To avoid this, we shift each row’s domain by subtracting the entire row by (k, \dots, k) . Applying an indicator function around zero then produces a one-hot encoding of the position of the k -statistic for each row k . Next, we perform an inner-product with the original vector: we replicate the vector over the rows, multiply it by the previously-computed mask, and sum the result over the columns. This way, the first element of each row k contains the corresponding k -statistic of v . The pseudocode is provided in Algorithm 5, while a schematic with an example can be found in Figure 3.

Note that the quantity $\text{RepIR}(V)$ is already computed within the ranking algorithm, thus the extra cost is given by only $3 \lceil \log N \rceil$ rotations, the evaluation of the indicator function, and the multiplication $M \cdot V_R$. As an additional optimization, we can avoid the final transposition and save $\lceil \log N \rceil$ rotations with a slight tweak in the ranking algorithm. By inverting the order of the operands in the Cmp and replacing the

Algorithm 5 Sorting

Input: V encryption of $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ with distinct elements, approximation degrees $d_C, d_I \in \mathbb{N}$.

Output: S encryption of the sorted form of v .

- 1: $R \leftarrow \text{Rank}(V; d_C)$
 - 2: $R_R \leftarrow \text{ReplR}(R)$
 - 3: $M \leftarrow \text{Ind}_0(R_R - (1^N \parallel \dots \parallel N^N); d_I)$
 - 4: $V_R \leftarrow \text{ReplR}(V)$
 - 5: $S \leftarrow \text{TransC}(\text{SumC}(M \cdot V_R))$
 - 6: **return** S
-

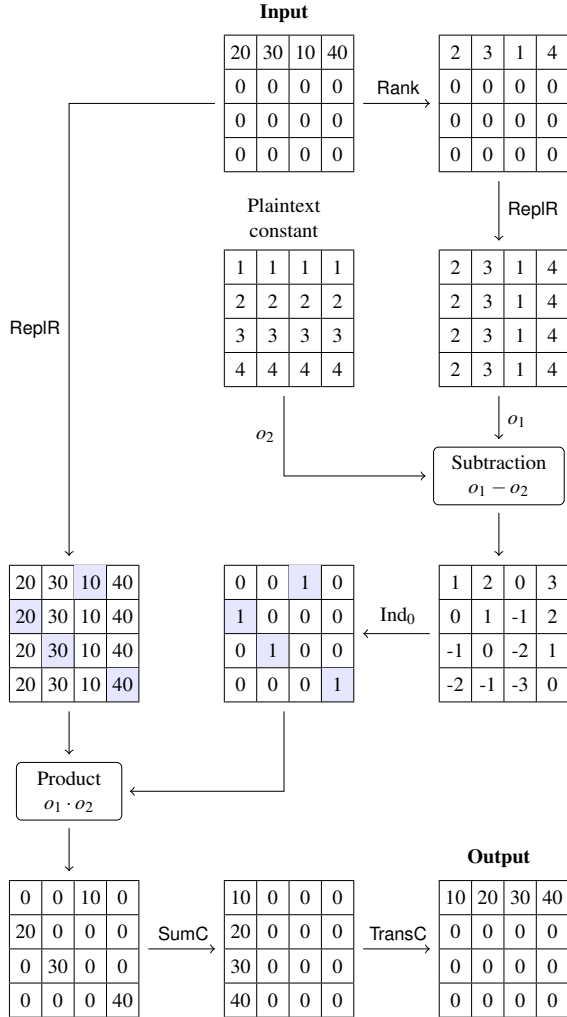


Figure 3: Schematic example of sorting a 4-element vector.

SumR with a SumC, the output of the ranking algorithm will be in column-form instead of row-form. This entails swapping row- and column-operations in the sorting algorithm, and replacing $\text{ReplR}(V)$ with $\text{ReplC}(\text{TransR}(V))$, which is also a quantity computed within the ranking algorithm. The (parallel) cost of the sorting algorithm is then:

- $5 \lceil \log N \rceil$ rotations, and
- $\sqrt{d_C} + \sqrt{d_I} + 1$ ciphertext-ciphertext multiplications, with a multiplicative depth of $\sim \lceil \log d_C \rceil + \lceil \log d_I \rceil + 1$.

Correctness Proof Assuming the correctness of the building blocks and Algorithm 3, and the ideal functionality of the algorithm, we prove that the output S produced by Algorithm 5 on input the encryption of a vector with distinct elements $v = (v_1, \dots, v_N)$ is actually the encryption of the sorted form of v . Let r_R, m, v_R, s be the decryption of R_R, M, V_R, S respectively. As the elements v_i are all distinct, the ranking of v is a permutation of $\{1, \dots, N\}$. Let $i \in \{1, \dots, N\}$, we thus have to prove that $\text{Rank}(s_i) = i$. By the correctness of ReplR, we have

$$m_{i,j} = \text{Ind}_0(r_{R,i,j} - i) = \text{Ind}_0(r_j - i) = \begin{cases} 1 & \text{if } r_j = i \\ 0 & \text{if } r_j \neq i \end{cases}$$

and by the correctness of SumC, TransC, and ReplR, we have

$$s_i = \sum_{j=1}^N m_{i,j} \cdot v_{R,i,j} = \sum_{j=1}^N m_{i,j} \cdot v_j.$$

Since r is a permutation of $\{1, \dots, N\}$, there exists one and only one index k such that $\text{Rank}(v_k) = i$. Hence, $s_i = v_k$ and $\text{Rank}(s_i) = i$. \square

4 Tie-Correction Offset

If two or more elements are in a *tie*, namely share the same value, they receive the same rank. As noted in Section 3, this causes the ranking function to become non-surjective, which hinders the extraction of certain order statistics and, consequently, prevents a correct sorting of the input vector. For example, the (fractional) ranking of the input vector $v = [10, 20, 20, 40]$ is $r = [1, 2.5, 2.5, 4]$. If we now want to extract the second or third order statistics (which should both correspond to the value 20), we should apply an indicator around rank 2 and 3, respectively, which would miss the actual rank value 2.5. To fix this issue, we build an offset vector, which redistribute the fractional ranking of all elements in a tie over the ranks they span. In our example, the offset vector would be $f = [0, -0.5, 0.5, 0]$, which corrects the fractional ranking to

$$r + f = [1, 2, 3, 4]$$

allowing us to correctly extract all four order statistics. As follows, we explain how to build this tie-correction offset vector under encryption with small computational overhead.

To build this offset, we need to evaluate the equality operator (Eq) among all pairs of elements in the input vector. Similarly to Cmp, the output of this function is a square matrix e of size $N \times N$ such that $e_{i,j} = 1$ if $v_i = v_j$, and 0 otherwise. The equality can be evaluated as an indicator function around

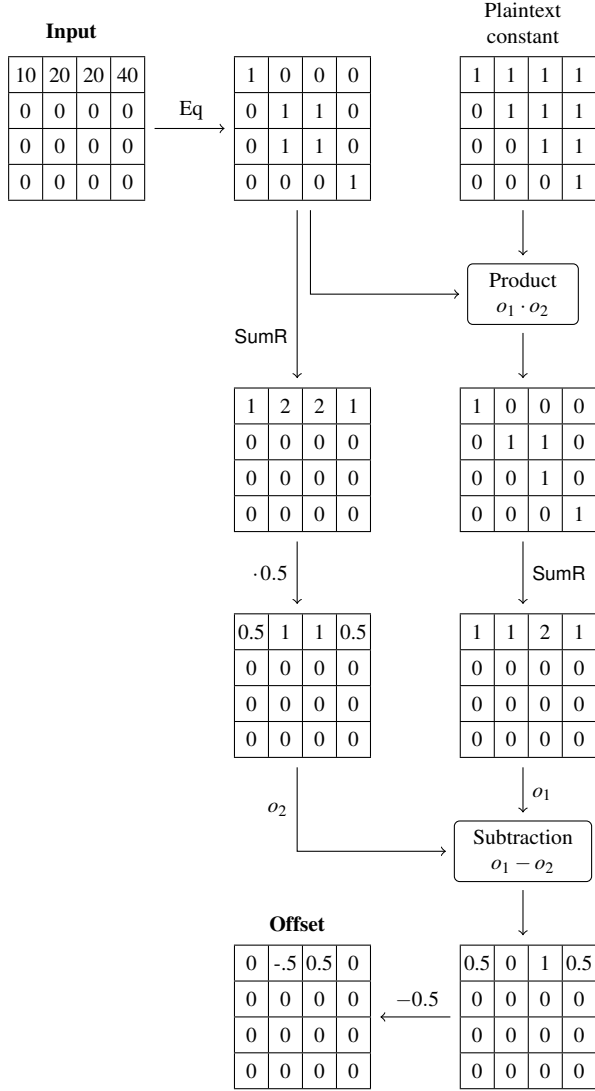


Figure 4: Schematic example of computing the tie-correction offset for a 4-element vector.

0, which can be done in parallel to the greater-than Cmp in the ranking. However, we note that the information needed to compute the equality matrix e is already contained in the comparison matrix c . We can reuse it by computing the equality as

$$e = 4 \cdot c \cdot (1 - c)$$

which maps both zeros and ones of c to 0, and the values 0.5 to 1. In our implementation we mainly use the latter option, which comes with an overhead of just two multiplications.

Note that each column j contains a number of ones equal to the number of elements that are in a tie with v_j . This includes the trivial equality $v_j = v_j$ on the main diagonal. By masking out the lower triangle of e and summing over the rows, we

count the non-trivial identities only once, namely

$$u_j = |\{i \in \{1, \dots, j\} : v_i = v_j\}|.$$

For instance, if the first four elements of v are in a tie, then the corresponding values in u are 1, 2, 3, 4. We can use u to offset the ranking. But, since we are using fractional ranking, we first need to shift it by half of the tie size, that is the range the elements in the tie span. To do this, we sum directly over the rows of e , without masking it, and get

$$t_j = |\{i \in \{1, \dots, N\} : v_i = v_j\}|.$$

Now, the correction offset for the ranking can be computed as

$$f_j = u_j - 0.5 \cdot t_j - 0.5$$

where the last -0.5 makes the offset start from zero, and it nicely cancels out with the $+0.5$ in the last line of the ranking algorithm.

The pseudocode to compute the tie-correction offset is presented in Algorithm 6. This runs at the end of the ranking algorithm (Algorithm 3), and the offset can just be added to the fractional ranking to make it suitable for order statistics extraction and sorting.

Algorithm 6 Tie-Correction Offset

Input: V encryption of $v = (v_1, \dots, v_N) \in \mathbb{R}^N$.

Output: F encryption of a vector in \mathbb{R}^N representing the tie-correction offset vector of v .

- 1: $E \leftarrow \text{Eq}(V)$
 - 2: $\text{mask} \leftarrow \delta_{j \geq i}$
 - 3: $U \leftarrow \text{SumR}(E \cdot \text{mask})$
 - 4: $T \leftarrow \text{SumR}(E)$
 - 5: $F \leftarrow U - 0.5 \cdot T - (0.5, \dots, 0.5)$
 - 6: **return** F
-

As a particular case, we can modify Algorithm 4 to compute the *median* by extracting the $(N+1)/2$ statistic if N is odd, or both the $N/2$ and $(N/2)+1$ statistics if N is even. In the latter case, an additional plaintext multiplication by 0.5 is needed after the inner product. In a similar way, one can compute any percentile of the given vector.

Correctness Proof Given a vector $v \in \mathbb{R}^N$, let V be its encryption. Let R and F be the output of Algorithm 3 and Algorithm 6 on input V , respectively. And let r and f be the corresponding decryption. Assuming the correctness of the building blocks and Algorithm 3, and the ideal functionality of the algorithms, we prove that (1) $k := r + f$ is a permutation of $(1, \dots, N)$, and (2) v_j is the k_j -th order statistic of v , for all $j \in \{1, \dots, N\}$.

Let e , u , and t be the decryption of the intermediate computations E , U , and T in Algorithm 6, respectively. Let

$j \in \{1, \dots, N\}$, and let us define the following subsets of $\{1, \dots, N\}$:

$$\begin{aligned} \mathcal{L}_j &:= \{i \in \{1, \dots, N\} : v_i < v_j\} \\ \mathcal{E}_j &:= \{i \in \{1, \dots, N\} : v_i = v_j\} \\ \mathcal{U}_j &:= \{i \in \{1, \dots, j\} : v_i = v_j\} . \end{aligned}$$

By the correctness of SumR, we have that

$$\begin{aligned} u_j &= \sum_{i=1}^N e_{ij} \delta_{j \geq i} = |\mathcal{U}_j| \\ t_j &= \sum_{i=1}^N e_{ij} = |\mathcal{E}_j| \end{aligned}$$

and thus

$$\begin{aligned} f_j &= u_j - 0.5 \cdot t_j - 0.5 \\ &= |\mathcal{U}_j| - 0.5 \cdot |\mathcal{E}_j| - 0.5 . \end{aligned}$$

On the other hand, by the correctness of Algorithm 3, we know that

$$r_j = |\mathcal{L}_j| + 0.5 \cdot (|\mathcal{E}_j| + 1) .$$

Combining these two identities we get

$$\begin{aligned} k_j &:= r_j + f_j \\ &= |\mathcal{U}_j| - 0.5 \cdot |\mathcal{E}_j| - 0.5 + |\mathcal{L}_j| + 0.5 \cdot (|\mathcal{E}_j| + 1) \\ &= |\mathcal{U}_j| + |\mathcal{L}_j| . \end{aligned}$$

Let w be the sorted array, then we note that $w_i = v_j$ for all $i \in \{|\mathcal{L}_j| + 1, \dots, |\mathcal{L}_j| + |\mathcal{E}_j|\}$. Since $|\mathcal{U}_j| \geq 1$ (as it contains at least the trivial identity), and $|\mathcal{U}_j| \leq |\mathcal{E}_j|$ (as $\mathcal{U}_j \subseteq \mathcal{E}_j$), we have that $|\mathcal{L}_j| + 1 \leq k \leq |\mathcal{L}_j| + |\mathcal{E}_j|$. Hence, $w_{k_j} = v_j$, proving point (2).

Now, we prove that k is a permutation of $(1, \dots, N)$. Let $j \in \{1, \dots, N\}$, then

- $k_j \in \mathbb{N}$: this is trivial, since both $|\mathcal{U}_j|, |\mathcal{L}_j| \in \mathbb{N}$;
- $k_j \geq 1$: since $v_j = v_j$, we have that $k_j \geq |\mathcal{U}_j| \geq 1$;
- $k_j \leq N$: this is true since \mathcal{U}_j and \mathcal{L}_j are non-overlapping subsets of $\{1, \dots, N\}$;
- for all $j' \neq j, k_{j'} \neq k_j$: we consider three cases
 1. if $v_j = v_{j'}$, then $\mathcal{L}_j = \mathcal{L}_{j'}$; without loss of generality, we assume $j' > j$, thus $|\mathcal{U}_{j'}| > |\mathcal{U}_j|$, hence $k_{j'} > k_j$;
 2. if $v_j < v_{j'}$, then $\mathcal{L}_j \cup \mathcal{U}_j \subseteq \mathcal{L}_{j'} \cup \mathcal{E}_{j'} \subseteq \mathcal{L}_{j'}$, thus $k_j = |\mathcal{L}_j| + |\mathcal{U}_j| < |\mathcal{L}_{j'}| + 1 \leq k_{j'}$;
 3. if $v_j > v_{j'}$, the proof is symmetric to the previous case.

This proves point (1). \square

		v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_N
		V_1				V_2				...	V_L	
v_1	V_1											
v_2		$V_1 > V_1$										
v_3			$V_1 > V_2$...		$V_1 > V_L$
v_4												
v_5												
v_6	V_2		$V_2 > V_1$									
v_7				$V_2 > V_2$...		$V_2 > V_L$
v_8												
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
\vdots	V_L		$V_L > V_1$...		$V_L > V_L$
v_N												

Figure 5: Comparison with a multi-ciphertext encoding. The vector v is split into blocks of size $B = 4$. The upper (or lower) triangle contains information about the comparison between all pairs $v_i > v_j$.

5 Multiple Ciphertexts Encoding

When a vector is too long and does not fit in a ciphertext we can split it into multiple blocks. This happens when $N^2 > n/2$, where n is the ring dimension and N is the vector length. In this case, we divide the vector into L blocks V_1, \dots, V_L of size $B = 2^{\lceil \log \sqrt{n/2} \rceil}$, which can be done under encryption by suitable rotations and masking. When it comes to comparisons, we also have to consider the comparisons between different blocks, that is, computing

$$C_{i,j} = (V_i > V_j) := \text{Cmp}(\text{ReplR}(V_i), \text{ReplC}(\text{TransR}(V_j)))$$

for all $i, j \in \{1, \dots, L\}$. The results can then be aggregated row-wise and block-wise to compute the ranking

$$R_i = \sum_{j=1}^L \text{SumR}(C_{i,j}) + 0.5$$

for $i \in \{1, \dots, L\}$. Note that we can also compute the block-sum first, as $\sum_{j=1}^L \text{SumR}(C_{i,j}) = \text{SumR}(\sum_{j=1}^L C_{i,j})$, allowing for evaluating SumR only once per block. The split output R_1, \dots, R_L can then be merged back into a single ciphertext if needed and if it fits.

All the comparisons can be computed in parallel, making this approach suitable for a multi-threaded environment. To reduce the computational burden, we notice that not all the comparisons $C_{i,j}$ are actually needed, as the information in $C_{i,j}$ is already contained in $C_{j,i}$ for all i, j (see Figure 5). In

particular, we have that

$$C_{i,j} = (1 - C_{j,i})^\top. \quad (3)$$

Proof of Equation (3) For the sake of notation, let $A := C_{i,j}$ and $B := C_{j,i}$. Then $A_{m,n} = \text{Cmp}(V_{i;n}, V_{j;m})$, where $V_{x;y}$ denotes the y -th element of block x . On the other hand, $B_{n,m} = \text{Cmp}(V_{j;m}, V_{i;n}) = 1 - \text{Cmp}(V_{i;n}, V_{j;m})$. \square

Hence, we compute $C_{i,j}$ only for $j \geq i$ and use Equation (3) for $j < i$. To avoid the transposition of $1 - C_{j,i}$ as a whole matrix, which is expensive, we operate on it column-wise, and only transpose it in the end, after summing it up to a vector:

$$R_i = \text{TransC}\left(\text{SumC}\left(\sum_{j=1}^{i-1} (1 - C_{j,i})\right)\right) + \text{SumR}\left(\sum_{j=i}^L C_{i,j}\right).$$

This optimization makes us save $L(L-1)/2$ comparisons. Algorithm 7 describes the full pseudocode for multi-ciphertext ranking. The correctness can be proven similarly as for Algorithm 3, by exploiting Equation (3).

Algorithm 7 Multi-Ciphertext Ranking

Input: V_1, \dots, V_L multi-ciphertext encryption of $v = (v_1, \dots, v_N) \in \mathbb{R}^N$, approximation degree $d \in \mathbb{N}$.

Output: R_1, \dots, R_L multi-ciphertext encryption of a vector in \mathbb{R}^N representing the (fractional) ranking of v .

```

1: parallel for  $i = 1, \dots, L$  do
2:    $V_{R;i} \leftarrow \text{ReplR}(V_i)$ 
3:    $V_{C;i} \leftarrow \text{ReplC}(\text{TransR}(V_i))$ 
4: end for
5: parallel for  $i = 1, \dots, L$  do
6:   parallel for  $j = i, \dots, L$  do
7:      $C_{i,j} \leftarrow \text{Cmp}(V_{R;i}, V_{C;j}; d)$ 
8:   end for
9: end for
10: parallel for  $i = 1, \dots, L$  do
11:    $R_i \leftarrow \text{TransC}(\text{SumC}(\sum_{j=1}^{i-1} (1 - C_{j,i}))) +$ 
       $\text{SumR}(\sum_{j=i}^L C_{i,j}) + (0.5, \dots, 0.5)$ 
12: end for
13: return  $R_1, \dots, R_L$ 

```

The algorithm in case of ranking with tie-correction is similar. We omit its description in the multi-ciphertext pseudocode for the sake of clarity.

We proceed on the same line to adapt sorting to the multi-ciphertext setting. First, a multi-ciphertext ranking is computed. As we have to extract N order statistics, and each one could be in any of the ciphertext blocks, the ranking blocks are replicated both row-wise and block-wise. Then each row of each block is shifted by a constant going from 1 to N , as in Algorithm 5, although this time it spans over multiple instances of the same ranking block. We conclude by applying the indicator function to each instance of each ranking block

Algorithm 8 Multi-Ciphertext Sorting

Input: V_1, \dots, V_L multi-ciphertext encryption of $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ with distinct elements, approximation degrees $d_C, d_I \in \mathbb{N}$.

Output: S_1, \dots, S_L multi-ciphertext encryption of the sorted form of v .

```

1:  $R_1, \dots, R_L \leftarrow \text{Rank}(V_1, \dots, V_L; d_C)$ 
2: parallel for  $i = 1, \dots, L$  do
3:    $R_{R;i} \leftarrow \text{ReplR}(R_i)$ 
4: end for
5: parallel for  $i = 1, \dots, L$  do
6:    $V_{R;i} \leftarrow \text{ReplR}(V_i)$ 
7:    $S_i \leftarrow 0$ 
8:   parallel for  $j = 1, \dots, L$  do
9:      $M_{i,j} \leftarrow \text{Ind}_0(R_{R;j} - ((B(i-1) + 1)^N \parallel \dots \parallel$ 
       $(Bi)^N); d_I)$ 
10:     $S_i \leftarrow S_i + M_{i,j} \cdot V_{R;j}$ 
11:   end for
12:    $S_i \leftarrow \text{TransC}(\text{SumC}(S_i))$ 
13: end for
14: return  $S_1, \dots, S_L$ 

```

and summing up the results, both row- and block-wise. A detailed description is offered in Algorithm 8.

6 Experimental Evaluation

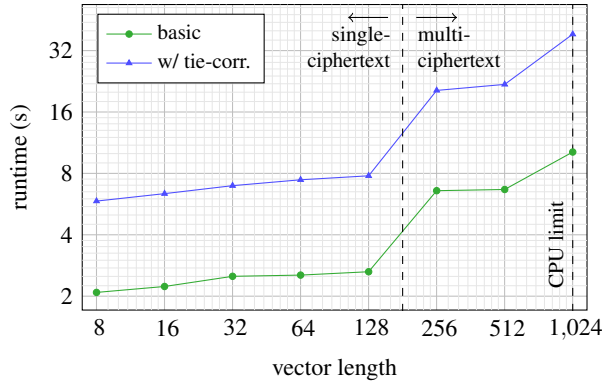
We evaluate the performance of our designs for different vector sizes.

6.1 Experimental Setup

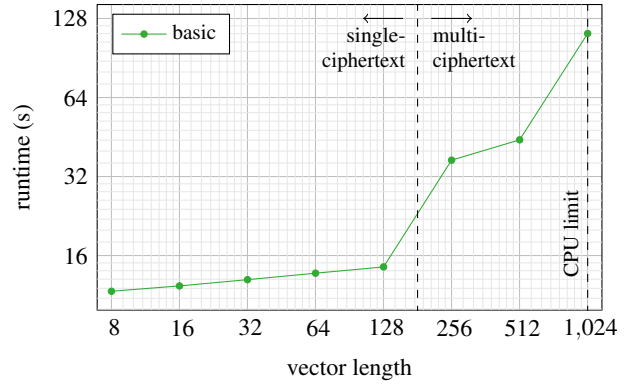
We use the CKKS implementation provided by the OpenFHE library [1],² with 12-bit integral precision and 48-bit decimal precision (scaling factor). The ring dimension is 2^{15} for ranking, and 2^{16} ranking with tie-correction, order statistics, and sorting, to accommodate the higher multiplicative depth. The parameters are chosen in accordance with the Homomorphic Encryption Standard to assure 128-bit security [2, 3]. Our code is available at <https://github.com/FedericoMazzone/openfhe-statistics>.

We present the runtime of ranking, computing the minimum, the median, and sorting elements that are generated uniformly at random in a bounded interval. The metrics are reported as an average over 10 runs of the algorithm. For minimum and median, the approximation degree of Ind is $\sim 2^8$, and for sorting is $\sim 2^9$. While the approximation degree of Cmp is $\sim 2^8$ across all the experiments. See Appendix B for a discussion on how different degrees influence the algorithm's performance. As the multiplicative depth of our circuit is relatively small (22 in the worst case), bootstrapping is not

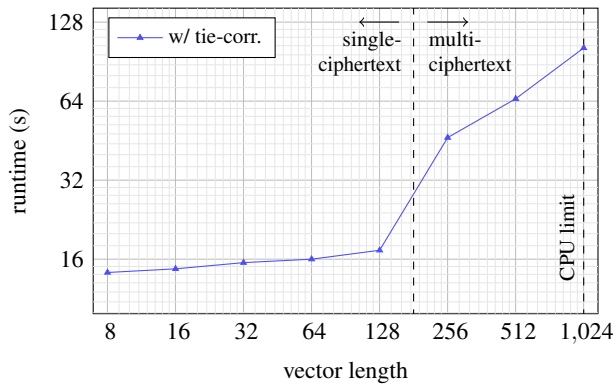
²<https://github.com/openfheorg/openfhe-development>



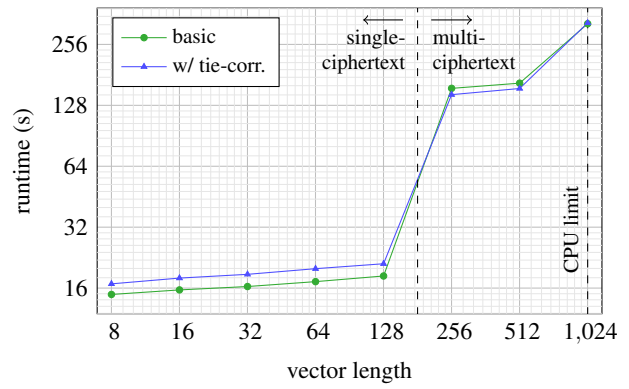
(a) Ranking $d \approx 2^8$



(b) Minimum $d_C \approx 2^8, d_I \approx 2^8$



(c) Median $d_C \approx 2^8, d_I \approx 2^8$



(d) Sorting $d_C \approx 2^8, d_I \approx 2^9$

Figure 6: Run time of ranking, minimum, median, and sorting for increasing vector size. Both axes are in logarithmic scale.

needed, and the scheme is used as a leveled homomorphic encryption. All the experiments are performed on a Linux machine with Intel Xeon Platinum 8358 running at 2.60 GHz, with 32 cores, and 512 GB RAM.

6.2 Empirical Results

As shown in Figure 6, ranking a vector of 128 elements takes around 2.64s, for an amortized cost of 21ms per element. This is not much more than the 2.09s required for a vector of length 8, as a consequence of the logarithmic scalability of the rotations in our approach. Similar behavior can be observed also for the other algorithms. For instance, the runtime goes 11.69s to 14.48s for the minimum computation, and from 14.88s to 18.35s for sorting distinct elements. We compute the median to test the order statistic extraction with the tie-correction enabled, and, as expected, it comes with a little overhead over the basic order statistic algorithm. The runtime goes from 14.23s for 8 elements to 17.29s for 128 elements, that is around 3 seconds more than for the minimum computation. Note that the cost of computing any other order statistic is the same, as it only changes the interval of the indicator function.

In Figure 6a and Figure 6d we can clearly see the computational overhead of the tie-correction for ranking and sorting. For sorting, the overhead is approximately 2 seconds for vectors with up to 128 elements, while it becomes negligible for larger vectors. In contrast, the ranking algorithm exhibits more significant overhead. This notable increase in runtime is due to a switch in ring-dimension from 2^{15} to 2^{16} , which is necessary to handle the additional multiplications required for the given approximation degree.

For the given ring dimension, we split vectors in chunks of $2^7 = 128$ elements. Switching from single- to multi-ciphertext mode causes a steep increase in the runtime, which can be observed in all four cases. This is due to the fact that, when OpenFHE detects loop parallelization at the application level, it automatically disables OMP at library level to avoid nested parallelism. At 1024 elements, we get close to the CPU limits of our machine, which severely impacts the runtime.

6.3 Comparison with Previous Work

Order Statistics For computing order statistics, we compare our approach with Phoenix [19] and NEXUS [27]. Like in our solution, both of these works operate with elements encrypted in a single ciphertext. Table 2 presents a comparison of our approach with these works for computing the argmin/argmax of (a vector encoded in) a ciphertext, focusing on the number of evaluations of the comparison function, homomorphic rotations, and the number of slots required in the ciphertext. Our approach employs significantly fewer evaluations of the comparison function compared to Phoenix and NEXUS, at the expense of increased space complexity. For

	Number of comparisons	Number of rotations	Slots required
Phoenix [19]	$O(N)$	$O(N)$	N
NEXUS [27]	$O(\log N)$	$O(\log N)$	$2N$
Our work	$O(1)$	$O(\log N)$	N^2

Table 2: Evaluation of argmin/argmax on a ciphertext encoding a vector of length N .

instance, in the use case of NEXUS [27], where the argmax is applied for secure transformer inference, particularly for computing the output layer in BERT-based and GPT-2 models with $N = 128$ nodes, we observe the following:

- Phoenix [19] requires 128 comparisons, 128 rotations, resulting in a total runtime of 366 seconds;³
- NEXUS [27] requires 7 comparisons, 7 rotations, with a total runtime of 28 seconds;³
- our approach requires 1 comparison, 28 rotations, resulting in a total runtime of 14 seconds.

This outcome is even more remarkable considering that our measurements are based on a non-optimized Cmp function and were conducted on a less powerful machine than what was used in [19] and [27].

Sorting For sorting vectors, we compare our approach with those proposed by Lu et al. (PEGASUS) [22] and Hong et al. [18], both of which implement sorting in CKKS. Lu et al. [22] employ Bitonic Sort and exploit a scheme switch to FHEW [25] to use lookup tables for efficient comparison. They achieve a comparison depth of $\log^2 N$, and they test sorting for vectors of 32 and 64 elements. In a 4-thread setting, their solution runs in 148.88s and 409.09s, respectively (without taking into account the cost for scheme switching). Our solution in the same setting requires 55.14s and 56.75s, on comparable hardware. Hong et al. [18], although also leveraging the SIMD capabilities of CKKS, maintain a comparison depth of $k \log_k^2 N$ when employing a k-way sorting network, whereas our approach achieves a depth of 1. In their case, sorting a 512-element array takes 150.57, 129.38, and 114.32 minutes for $k = 2, 3$, and 5, respectively. Meanwhile, our method, albeit employing a less optimized comparison function, runs in 2.58 minutes, on comparable hardware.

7 Related Works

A vast body of literature has focused on either sorting elements or computing their maximum value under encryption.

³As reported in [27], where the experiments are conducted on a machine with 3.20GHz Intel Xeon processors and 128 GB RAM.

Sorting under FHE has been studied starting from 2010 under the Smart-Vercauteren (SV) scheme [25] using bitwise encoding and comparison based swaps to implement algorithms like Bubble Sort, Insertion Sort [8], and Quick Sort [9], all of which requiring $O(N^2)$ comparisons. Subsequently, Bitonic Sort and Odd-Even Merge Sort were also implemented, reducing the cost to $O(N \log^2 N)$ comparisons, of which N can be potentially run in parallel, making the comparison depth $O(\log^2 N)$ [16]. In 2021, some works started designing sorting for floating-point values under CKKS. Hong et al. [18] use k -way sorting networks to achieve a $O(k \log_k^2 N)$ comparison depth. While Lu et al. (PEGASUS) [22] also implement Bitonic Sort but performing the comparisons using the efficient look-up tables of FHEW [15] after a scheme switching from CKKS.

An entire line of work has focused specifically on improving on the evaluation of the comparison function itself. Chialva et al. [14] use the identity $\tanh(kx) = (e^{kx} - e^{-kx}) / (e^{kx} + e^{-kx})$ to approximate the sign function for large $k > 0$, while Boura et al. [4] employ an approximation based on Fourier series. The work by Cheon et al. [13] is the first one to study the max function under CKKS, and it is based on an iterative computation of $u^k / (u^k + v^k)$ for large $k > 0$. The same author proposes a new solution in [12], where a composition of 2 polynomials is used to approximate the sign function, proving an optimal asymptotic complexity. This study was then picked up by Lee et al. [21], who generalized the technique to composition of k polynomials, and found the optimal set of polynomials for any given multiplicative depth.

In Phoenix [19], the authors face the problem of computing the argmax in the output layer of a neural network to perform privacy-preserving inference. There, the elements are stored within a single CKKS ciphertext and they propose a method based on rotations to compute the argmax in N comparisons and N rotations. In NEXUS [27], the authors apply the same strategy recursively, exploiting SIMD slot folding, which results in comparing the elements in a binary tree fashion, thus reducing the cost to $\log N$ rotations and $\log N$ comparisons. They use it for secure transformer inference, in particular for computing the argmax output layer in BERT-based and GPT-2 models.

It is also worth mentioning the work by Lu et al. [23], where the authors propose FHE algorithms that compute a variety of descriptive statistics, including percentile. However, their method is limited to ordinal attributes and requires plaintext encoding dependent on value order, whereas our approach addresses numerical attributes, operating with encrypted vectors without specific plaintext encoding, and thus it can be easily integrated in larger (numerical) circuits. Our paper contributes to these lines of work by introducing a novel approach for implementing comparison-based functionalities that achieve a comparison depth of $O(1)$. This represents an important reduction with respect to existing solutions, which have higher comparison depths, as also summarized in Table 1.

8 Conclusions

In this paper, we have presented a novel approach for computing ranking, order statistics, and sorting a vector under CKKS. Our method relies on homomorphic matrix encoding and on the SIMD capabilities of the cryptosystem to compare all elements with each other in one go, reducing the comparison depth of these algorithms to 1. This makes our solution highly parallelizable, opening potential future work in the direction of hardware acceleration. We showed that our approach achieves a remarkable efficiency in terms of runtime, rendering it practical for real-world applications. We consider that the algorithms we designed are practical for a wide range of privacy-preserving scenarios, especially for data outsourcing and secure machine learning, or they can serve as fundamental building blocks for larger protocols.

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Availability

We make our code available at <https://github.com/FedericoMazzone/openfhe-statistics>. In the repository, a demo for ranking, computing the argmin, and sorting can be found.

Ethics Considerations and Compliance with the Open Science Policy

Our work focuses solely on computational methods under encryption, and no real-world data has been used to test our approach. Consequently, we see no privacy concerns, risks of data misuse, or potential harm to individuals or communities arising from our work. The algorithms and protocols we developed are purely theoretical in nature, designed to enhance computational efficiency and security in encrypted environments. They do not interact with human subjects or physical systems in any way that could cause harm or raise ethical concerns. Our methods can be actually put in place to defend sensitive data in specific applications.

In alignment with the principles of Open Science, we have made our codebase freely available as an open-source resource. This decision was taken in a view to transparency, reproducibility, and collaboration within the scientific community. Our open-source code is available for public access under the BSD 2-Clause. This licensing choice reflects our intent to allow broad reuse and adaptation of our work while ensuring

proper attribution. Additionally, comprehensive documentation and examples are provided to facilitate understanding and ease of use by other researchers.

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A Recursive Matrix Operations

We provide the pseudocode for SumR, SumC, ReplR, ReplC for a square matrix with N number of rows/columns. The

matrix is assumed to be padded in such a way that N is a power of 2.

Algorithm 9 SumR

Input: X encryption of a square matrix of size N .

Output: X encryption of a row vector.

```

1: for  $i = 0, \dots, \log N - 1$  do
2:    $X \leftarrow X + (X \ll N \cdot 2^i)$ 
3: end for
4:  $X \leftarrow \text{MaskR}(X, 0)$ 
5: return  $X$ 

```

Algorithm 10 SumC

Input: X encryption of a square matrix of size N .

Output: X encryption of a column vector.

```

1: for  $i = 0, \dots, \log N - 1$  do
2:    $X \leftarrow X + (X \ll 2^i)$ 
3: end for
4:  $X \leftarrow \text{MaskC}(X, 0)$ 
5: return  $X$ 

```

Algorithm 11 ReplR

Input: X encryption of a row vector of size N .

Output: X encryption of a square matrix.

```

1: for  $i = 0, \dots, \log N - 1$  do
2:    $X \leftarrow X + (X \gg N \cdot 2^i)$ 
3: end for
4: return  $X$ 

```

Algorithm 12 ReplC

Input: X encryption of a column vector of size N .

Output: X encryption of a square matrix.

```

1: for  $i = 0, \dots, \log N - 1$  do
2:    $X \leftarrow X + (X \gg 2^i)$ 
3: end for
4: return  $X$ 

```

B Effect of Chebyshev Approximation Degree on Performance

In this work we employ a relatively basic implementation of comparison and indicator functions. There is an entire body of literature that focuses specifically on this topic, which one may consider for real-life application, see for instance [12, 21]. Nonetheless, we still provide some indication on how different approximation degrees influence the run time of our algorithms, in exchange of having a more precise result.

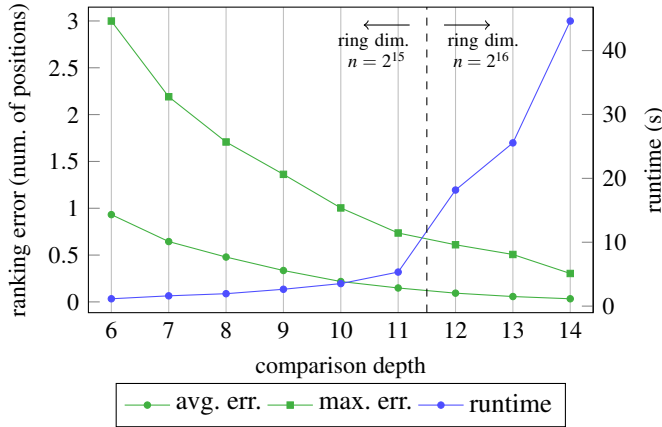


Figure 7: Ranking a vector of 128 elements for different approximation degrees of the comparison function (as multiplicative depth). The ranking error and run time are reported.

We use ranking and minimum computation as case studies for this analysis. Figure 7 shows the impact of the approximation degree of the comparison function, represented in terms of its multiplicative depth, on run time and approximation error in the ranking task. Higher degrees yield lower error but incur longer run times. The reported error is for a vector of 128 elements, indicating how many positions each element is ranked away, on average and in the worst case. The steep increase after depth 11 is due to the fact that the ring dimension must increase to assure 128 bits of security, making the basic homomorphic operations more expensive.

Sweet spots in the trade-off can be noticed at depth 10 and 11, which corresponds to an approximation degree of 2^9 and 2^{10} , respectively. At depth 10, the run time is around 3.52 seconds, while the elements are ranked no more than 1 position away from their actual rank. Notably, this error is proportional to the separation between elements; closely positioned elements are more susceptible to rank swapping. One could willingly decide to use a lower approximation degree and exploit this effect to achieve a form of differential privacy.

Similar considerations can be applied to the minimum computation. Figure 8 reports the error as the L1 distance between the computed and the expected minimum values. We can see that a sensitive improvement in accuracy occurs when transitioning from comparison depth 11 to 12. Moreover, we notice a sweet spot for the run time when the sum of the approximation depths equals 24 (the upper-right diagonal), with the (12, 12) combination yielding the lowest error.

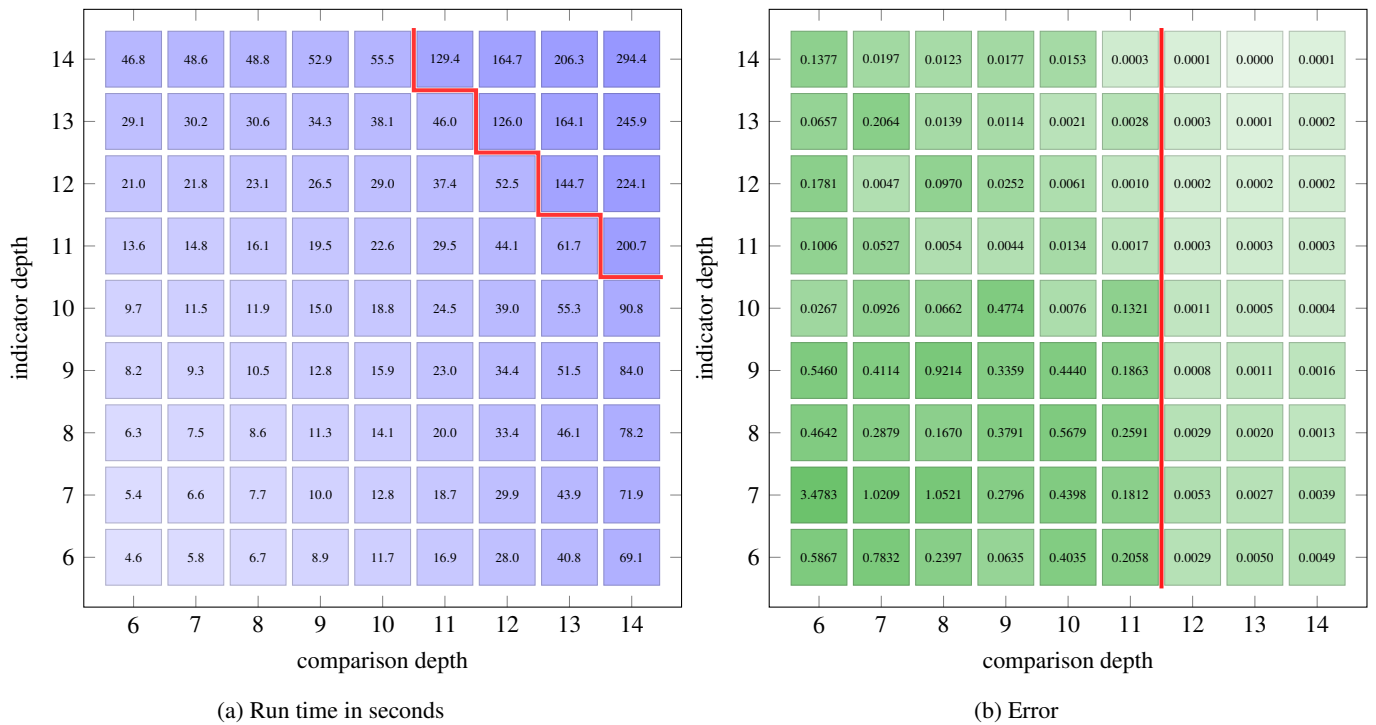


Figure 8: Computing the minimum of a vector of 32 elements for different approximation degrees of the comparison and indicator functions (as multiplicative depth). The run time and L1 error are reported.