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On linearization and uniqueness of preduals

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Abstract

We study strong linearizations and the uniqueness of preduals of locally convex Hausdorff spaces of scalar-valued functions. Strong linearizations are special preduals. A locally convex Hausdorff space $\mathcal{F}(\Omega)$ of scalar-valued functions on a nonempty set Ω is said to admit a *strong linearization* if there are a locally convex Hausdorff space Y, a map $\delta : \Omega \to Y$, and a topological isomorphism $T : \mathcal{F}(\Omega) \to Y_b'$ such that $T(f)\circ\delta = f$ for all $f \in \mathcal{F}(\Omega)$. We give sufficient conditions that allow us to lift strong linearizations from the scalar-valued to the vector-valued case, covering many previous results on linearizations, and use them to characterize the bornological spaces $\mathcal{F}(\Omega)$ with (strongly) unique predual in certain classes of locally convex Hausdorff spaces.

KEYWORDS

dual space, linearization, predual, uniqueness

1 | INTRODUCTION

The purpose of this paper is twofold. First, we present a general mechanism how to transfer a strong linearization of a locally convex Hausdorff space $\mathcal{F}(\Omega)$ of scalar-valued functions on a nonempty set Ω to a vector-valued counterpart. Second, we characterize those spaces among the spaces $\mathcal{F}(\Omega)$ that have a (strongly) unique predual in certain classes of locally convex Hausdorff spaces. Strong linearizations are special preduals. Recall that a locally convex Hausdorff space X is called a *dual space* and a tuple (Y, φ) a *predual* of X if the tuple consists of a locally convex Hausdorff space Y and a topological isomorphism $\varphi : X \to Y_b'$ where Y_b' is the strong dual of Y. In [25], we derived necessary and sufficient conditions for the existence of preduals and strong linearizations of bornological spaces such that the predual has certain properties like being complete and barrelled, a DF-space, a Fréchet space, or completely normable (cf. Proposition 3.5 and Corollary 3.6). A *strong linearization* of a locally convex Hausdorff space $\mathcal{F}(\Omega)$ of $\mathbb{K}$-valued functions on a nonempty set Ω where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a triple (δ, Y, T) of a locally convex Hausdorff space Y over the field $\mathbb{K}$, a map $\delta : \Omega \to Y$ and a topological isomorphism $T : \mathcal{F}(\Omega) \to Y_b'$ if $T(f)\circ\delta = f$ for all $f \in \mathcal{F}(\Omega)$ (see [10, p. 683], [17, pp. 181, 184], and [25, Proposition 2.6, p. 1595]).

We show that one can lift a strong linearization (δ, Y, T) of the scalar-valued case $\mathcal{F}(\Omega)$ to the vector-valued case in Theorem 4.5 and unify preceding results on strong vector-valued linearizations from Aron, Dimant, García-Lirola, and Maestre [1], Bonet, Domański, and Lindström [6], Gupta and Baweja [16], Jordá [19], Laitila and Tylli [26], Mujica [29], and Quang [33], where $\mathcal{F}(\Omega)$ is a weighted Banach space of holomorphic or harmonic functions, and from Beltrán [2, 3], Bierstedt, Bonet, and Galbis [5], Bonet and Friz [8], and Galindo, García, and Maestre [11], where $\mathcal{F}(\Omega)$ is a weighted bornological space of holomorphic functions, and Grothendieck [15] where $\mathcal{F}(\Omega)$ is the space of continuous bilinear forms on the product $\Omega := F \times G$ of two locally convex Hausdorff spaces F and G. Further, our results on strong linearizations augment results on continuous linearizations, where δ is continuous but T need not be continuous, by Carando and Zalduendo [10] and Jaramillo, Prieto, and Zalduendo [17]. Linearizations are a useful tool since they identify (usually) nonlinear functions f with (continuous) linear operators $T(f)$ and thus allow to apply linear functional analysis to nonlinear functions. They are often used to transfer results that are known for scalar-valued functions to vector-valued functions, see, for example, [6, 8, 15, 17, 19, 26]. We give an example of such an application to an extension problem in the case that $\mathcal{F}(\Omega)$ is a complete bornological DF-space in Theorem 4.8.

However, our main motivation in considering strong linearizations in this article does not stem from transferring results from the scalar-valued to the vector-valued case. We use strong linearizations to study the question whether a predual is *unique* up to identification via topological isomorphisms. This question is usually only treated in the case of Banach spaces and even then mostly in the isometric setting, that is, in the case of Banach spaces X having a predual (Y, φ) such that Y is a Banach space and φ an isometric isomorphism. We refer the reader to the thorough survey of Godefroy [14] in the isometric setting and to the paper [9] of Brown and Ito in the nonisometric Banach setting. Since we are interested in the general setting of locally convex Hausdorff spaces, our isomorphisms φ are in general only topological and we also have to choose a class $\mathcal{C}$ of locally convex Hausdorff spaces in which we strive for uniqueness of the predual. Such a class $\mathcal{C}$ has to be closed under topological isomorphisms because otherwise there is no hope for uniqueness of the predual up to identification via topological isomorphisms. Using such a class $\mathcal{C}$, it is possible to introduce two notions of uniqueness of a predual that are already known in the Banach setting. First, we say that a dual space X has a *unique $\mathcal{C}$ predual* if for all preduals (Y, φ) and (Z, ψ) of X such that $Y, Z \in \mathcal{C}$, there is a topological isomorphism $\lambda : Y \to Z$. Second, we say that a dual space X has a *strongly unique $\mathcal{C}$ predual* if for all preduals (Y, φ) and (Z, ψ) such that $Y, Z \in \mathcal{C}$ and all topological isomorphisms $\alpha : Z'_b \to Y'_b$, there is a topological isomorphism $\lambda : Y \to Z$ such that $\lambda^t = \alpha$. The second definition might seem a bit strange at first but we will see that it takes the topological isomorphisms of a predual into account, so it does not forget the additional structure, and it fits quite well to strong linearizations. Strong uniqueness of a predual in $\mathcal{C}$ means that all preduals of X are *equivalent* in the sense that for all preduals (Y, φ) and (Z, ψ) of X such that $Y, Z \in \mathcal{C}$, there is a topological isomorphism $\lambda : Y \to Z$ such that $\lambda^t = \varphi \circ \psi^{-1}$, see Proposition 2.7. We note that this definition of the equivalence of preduals is an adaptation of a corresponding definition in the Banach setting by Gardella and Thiel [13]. The notion of equivalence of preduals allows us on the one hand to show in Corollary 5.9 how our construction of a strong linearization is related to the continuous linearization of Carando and Zalduendo, and on the other hand to characterize the bornological spaces $\mathcal{F}(\Omega)$ of $\mathbb{K}$-valued functions, which have a (strongly) unique $\mathcal{C}$ predual, see Corollary 5.10 and Corollary 5.11. We refer the reader who is also interested in the corresponding results of this paper in the isometric Banach setting to [24].

2 | NOTIONS AND PRELIMINARIES

In this short section, we recall some basic notions from the theory of locally convex spaces and present some preliminary results on dual spaces and their preduals (cf. [25, Section 2]). For a locally convex Hausdorff space X over the field $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, we denote by X' the topological linear dual space and by U° the *polar set* of a subset $U \subset X$. If we want to emphasize the dependency on the locally convex Hausdorff topology τ of X, we write (X, τ) and $(X, \tau)'$ instead of just X and X', respectively. We denote by $\sigma(X', X)$ the topology on X' of uniform convergence on finite subsets of X and by $\beta(X', X)$ the topology on X' of uniform convergence on bounded subsets of X. Further, we set $X'_b := (X', \beta(X', X))$. For a continuous linear map $T : X \to Y$ between two locally convex Hausdorff spaces X and Y, we denote by $T^t : Y' \to X'$, $y' \mapsto y' \circ T$, the *dual map* of T and write $T^{tt} := (T^t)^t$. Furthermore, we say that a linear map $T : X \to Y$ between two locally convex Hausdorff spaces X and Y is *(locally) bounded* if it maps bounded sets to bounded sets. Moreover, for two locally convex Hausdorff topologies τ_0 and τ_1 on X, we write $\tau_0 \leq \tau_1$ if τ_0 is coarser than τ_1. For a normed space $(X, \|\cdot\|)$, we denote by $B_{\|\cdot\|} := \{x \in X \mid \|x\| \leq 1\}$ the $\|\cdot\|$-closed unit ball of X. Further, we write τ_{co} for the *compact-open topology*, that is, the topology of uniform convergence on compact subsets of Ω, on the space $\mathcal{C}(\Omega)$ of $\mathbb{K}$-valued continuous functions on a topological Hausdorff space Ω. In addition, we write τ_{p} for the *topology of pointwise convergence* on the space $\mathbb{K}^\Omega$ of $\mathbb{K}$-valued functions on a set Ω. By a slight abuse of notation, we also use the symbols τ_{co} and τ_{p} for the relative compact-open topology and the relative topology of pointwise convergence on topological subspaces of $\mathcal{C}(\Omega)$ and $\mathbb{K}^\Omega$, respectively. For further unexplained notions on the theory of locally convex Hausdorff spaces, we refer the reader to [18, 27, 31].

Definition 2.1. Let X be a locally convex Hausdorff space.

(a) We call X a *dual space* if there are a locally convex Hausdorff space Y and a topological isomorphism $\varphi : X \to Y'_b$. The tuple (Y, φ) is called a *predual* of X.
(b) Let X be a dual space. We say that two preduals (Y, φ) and (Z, ψ) of X are *equivalent* and write $(Z, \psi) \sim (Y, \varphi)$ if there is a topological isomorphism $\lambda : Y \to Z$ such that $\lambda^t = \varphi \circ \psi^{-1}$.

The preceding definition of a predual is already given in [25, Definition 2.1, p. 1593] and for Banach spaces, for example, in [9, p. 321]. In the setting of Banach spaces, the definition of the equivalence of preduals is given in [13, Definition 2.1]. It is also easily checked that $\sim$ actually defines an equivalence relation on the family of all preduals of a dual space. Further, if we have a predual (Y, φ) of a dual space X and another locally convex Hausdorff space Z that is topologically isomorphic to Y, we can always augment Z to a predual, which is equivalent to (Y, φ).

Remark 2.2. Let X be a dual space with predual (Y, φ) and Z a locally convex Hausdorff space. If there is a topological isomorphism $\lambda : Y \to Z$, then the map $\varphi_\lambda : X \to Z'_b, \varphi_\lambda := (\lambda^{-1})^t \circ \varphi$, is a topological isomorphism with $\varphi_\lambda^{-1}(z') = \varphi^{-1}(z' \circ \lambda)$ for all $z' \in Z'$ and $\lambda^t = \varphi \circ \varphi_\lambda^{-1}$. In particular, (Z, φ_λ) is a predual of X and $(Z, \varphi_\lambda) \sim (Y, \varphi)$.

If X is a dual space with a quasi-barrelled predual, we may consider this predual as a topological subspace of the strong dual of X.

Proposition 2.3 [25, Proposition 2.2, p. 1593]. *Let X be a dual space with quasi-barrelled predual (Y, φ). Then the map*

$$\Phi_\varphi : Y \to X'_b,\ y \longmapsto [x \mapsto \varphi(x)(y)],$$

is a topological isomorphism into, that is, a topological isomorphism to its range.

Now, if we want to study whether a dual space has a *unique* predual by identification via topological isomorphisms, we have to restrict the range of preduals we consider because even a Banach space may have a predual which is also a Banach space, and another predual which is not a Banach space (see [25, Example 3.15, p. 1601]). Two such preduals cannot be topologically isomorphic.

Definition 2.4. Let X be a dual space and C a class of locally convex Hausdorff spaces that is *closed under topological isomorphisms*, that is, if $Y \in C$ and Z is a locally convex Hausdorff space, which is topologically isomorphic to Y, then $Z \in C$.

(a) We say that X has a *unique C predual* if for all preduals (Y, φ) and (Z, ψ) of X such that $Y, Z \in C$ there is a topological isomorphism $\lambda : Y \to Z$.
(b) We say that X has a *strongly unique C predual* if for all preduals (Y, φ) and (Z, ψ) such that $Y, Z \in C$ and all topological isomorphisms $\alpha : Z'_b \to Y'_b$, there is a topological isomorphism $\lambda : Y \to Z$ such that $\lambda^t = \alpha$.

In the context of dual Banach spaces, where C is the class of Banach spaces, Definition 2.4(a) is already given in, for example, [9, p. 321] (to be more precise, in the setting of dual Banach spaces, C in Definition 2.4 is not the class of Banach spaces but of completely normable spaces since Z in the definition of closedness under topological isomorphisms is a locally convex Hausdorff space, which need not be a normed space initially). Definition 2.4(b) is inspired by a similar definition of a strongly unique *isometric* Banach predual of a Banach space in, for example, [14, p. 134] and [37, p. 469]. We frequently consider four classes C of preduals in this paper: the class of complete barrelled locally convex Hausdorff spaces, the class of complete barrelled DF-spaces, the class of Fréchet spaces, and the class of completely normable spaces.

Remark 2.5. Let X be a dual space, C_1 and C_2 be classes of locally convex Hausdorff spaces that are closed under topological isomorphisms, and C_1 be contained in C_2. If X has a (strongly) unique C_2 predual and there is a predual (Y, φ) of X such that $Y \in C_1$, then X has also a (strongly) unique C_1 predual.

Proposition 2.6. *Let X be a dual space and $\mathcal{C}$ a class of locally convex Hausdorff spaces that is closed under topological isomorphisms. Then the following assertions are equivalent.*

(a) X has a unique $\mathcal{C}$ predual.
(b) For all preduals (Y,φ) and (Z,ψ) of X such that $Y,Z\in\mathcal{C}$, there is a topological isomorphism $\mu : X\to Z_b'$ such that $(Z,\mu)\sim(Y,\varphi)$.

Proof. Let (Y,φ) and (Z,ψ) be preduals of X such that $Y,Z\in\mathcal{C}$.

(a)⇒(b) Since X has a unique predual, there is a topological isomorphism $\lambda : Y\to Z$. Now, statement (b) follows from Remark 2.2 with $\mu := \varphi_\lambda$.

(b)⇒(a) Let $\mu : X\to Z_b'$ be a topological isomorphism such that $(Z,\mu)\sim(Y,\varphi)$. Then there is a topological isomorphism $\lambda : Y\to Z$ such that $\lambda^t=\varphi\circ\mu^{-1}$. Thus, X has a unique $\mathcal{C}$ predual. □

Proposition 2.7. *Let X be a dual space and $\mathcal{C}$ a class of locally convex Hausdorff spaces that is closed under topological isomorphisms. Then the following assertions are equivalent.*

(a) X has a strongly unique $\mathcal{C}$ predual.
(b) All preduals of X in $\mathcal{C}$ are equivalent (in the sense of Definition 2.1 (b)).

Proof. Let (Y,φ) and (Z,ψ) be preduals of X such that $Y,Z\in\mathcal{C}$.

(a)⇒(b) The map $\alpha : Z_b'\to Y_b'$, $\alpha := \varphi\circ\psi^{-1}$, is a topological isomorphism. Since X has a strongly unique predual, there is a topological isomorphism $\lambda : Y\to Z$ such that $\lambda^t=\alpha=\varphi\circ\psi^{-1}$. Thus, we have $(Z,\psi)\sim(Y,\varphi)$.

(b)⇒(a) Let $\alpha : Z_b'\to Y_b'$ be a topological isomorphism. The tuple $(Y,\alpha\circ\psi)$ is also a predual of X. Since all preduals are equivalent, there is a topological isomorphism $\lambda : Y\to Z$ such that $\lambda^t=(\alpha\circ\psi)\circ\psi^{-1}=\alpha$. Thus, X has a strongly unique $\mathcal{C}$ predual. □

In the context of dual Banach spaces, Proposition 2.7(b) is used in [13, Definition 2.7] to give an equivalent definition of a dual Banach space having a strongly unique Banach predual. Clearly, reflexive locally convex Hausdorff spaces are dual spaces. The bornological ones among them also have a strongly unique complete barrelled predual.

Example 2.8. Let X be a reflexive bornological locally convex Hausdorff space. Then X has a strongly unique complete barrelled predual. In particular, every complete barrelled predual of X is reflexive.

Proof. Since X is reflexive and bornological, the strong dual X_b' is a reflexive, hence barrelled, predual of X (equipped with the canonical evaluation map $\mathcal{J}_X : X\to (X_b')_b'$, $x\longmapsto [x'\mapsto x'(x)]$), and also complete by [20, Section 39, 6.(4), p. 143]. Let (Y,φ) be another complete barrelled predual of the reflexive space X. Then Y_b' is reflexive and thus $(Y_b')_b'$ as well. Since Y is complete and barrelled, it is a closed subspace of $(Y_b')_b'$ via the map $\mathcal{J}_Y$ by [18, 11.2.2 Proposition, p. 222], implying that Y is reflexive by [18, 11.5.5 Proposition (a), p. 228]. Therefore, the map $\lambda : X_b'\to Y$, $\lambda := \mathcal{J}_Y^{-1}\circ(\mathcal{J}_X\circ\varphi^{-1})^t\circ\mathcal{J}_{X_b'}$, is a topological isomorphism and

$$\begin{aligned}\lambda^t &= \mathcal{J}_{X_b'}^t\circ(\mathcal{J}_X\circ\varphi^{-1})^{tt}\circ(\mathcal{J}_Y^{-1})^t = \mathcal{J}_{(X_b')_b'}^{-1}\circ(\mathcal{J}_X\circ\varphi^{-1})^{tt}\circ(\mathcal{J}_Y^t)^{-1}\\ &= \mathcal{J}_{(X_b')_b'}^{-1}\circ(\mathcal{J}_X\circ\varphi^{-1})^{tt}\circ\mathcal{J}_{Y_b'} = \mathcal{J}_{(X_b')_b'}^{-1}\circ\mathcal{J}_{(X_b')_b'}\circ(\mathcal{J}_X\circ\varphi^{-1}) = \mathcal{J}_X\circ\varphi^{-1}.\end{aligned}$$

We conclude that $(Y,\varphi)\sim(X_b',\mathcal{J}_X)$. Due to Proposition 2.7, this means that X has a strongly unique complete barrelled predual. □

If X is a reflexive Fréchet space, then X_b' is a complete reflexive DF-space by [18, 12.4.5 Theorem, p. 260] and so X has a strongly unique complete barrelled DF-predual by Remark 2.5. Similarly, if X is a reflexive bornological DF-space, then X_b' is a complete reflexive Fréchet space by [18, 12.4.2 Theorem, p. 258] and so X has a strongly unique Fréchet predual. If X is a reflexive Banach space, then X_b' is a completely normable reflexive space and so X has a strongly unique Banach predual. Let us turn to linearizations of function spaces whose definition is motivated by the notion of strong Banach linearizations [17, p. 184, 187].

3 LINEARIZATION AND UNIQUENESS

We begin this section with the definition of a linearization.

Definition 3.1 [25, Definition 2.3, p. 1593]. Let $\mathcal{F}(\Omega)$ be a linear space of $\mathbb{K}$-valued functions on a nonempty set Ω.

(a) We call a triple (δ, Y, T) of a locally convex Hausdorff space Y over the field $\mathbb{K}$, a map $\delta : \Omega \to Y$, and an algebraic isomorphism $T : \mathcal{F}(\Omega) \to Y'$ a *linearization of* $\mathcal{F}(\Omega)$ if $T(f)\circ\delta = f$ for all $f \in \mathcal{F}(\Omega)$.
(b) Let Ω be a topological Hausdorff space. We call a linearization (δ, Y, T) of $\mathcal{F}(\Omega)$ *continuous* if δ is continuous.
(c) Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space. We call a linearization (δ, Y, T) of $\mathcal{F}(\Omega)$ *strong* if $T : \mathcal{F}(\Omega) \to Y_b'$ is a topological isomorphism.
(d) We call a (strong) linearization (δ, Y, T) of $\mathcal{F}(\Omega)$ a (strong) *complete barrelled (Fréchet, DF-, Banach) linearization* if Y is a complete barrelled (Fréchet, DF-, completely normable) space.
(e) We say that $\mathcal{F}(\Omega)$ *admits a (continuous, strong, complete barrelled, Fréchet, DF-, Banach) linearization* if there exists a (continuous, strong, complete barrelled, Fréchet, DF-, Banach) linearisation (δ, Y, T) of $\mathcal{F}(\Omega)$.

Clearly, the tuple (Y, T) of a linearisation (δ, Y, T) is a predual of $\mathcal{F}(\Omega)$. If we have two strong linearisations of a common function space $\mathcal{F}(\Omega)$ such that the corresponding preduals are equivalent, then we have the following relation between the δ-maps.

Proposition 3.2. *Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω and (δ, Y, T) and $(\widetilde{\delta}, Z, \varphi)$ strong linearizations of $\mathcal{F}(\Omega)$. If there is a topological isomorphism $\lambda : Y \to Z$ such that $\lambda^t = T\circ\varphi^{-1}$, then it holds $\lambda(\delta(x)) = \widetilde{\delta}(x)$ for all $x \in \Omega$.*

Proof. We note that

$$z'(\lambda(\delta(x))) = \lambda^t(z')(\delta(x)) = (T\circ\varphi^{-1})(z')(\delta(x)) = T(\varphi^{-1}(z'))(\delta(x))$$
$$= \varphi^{-1}(z')(x) = \varphi(\varphi^{-1}(z'))(\widetilde{\delta}(x)) = z'(\widetilde{\delta}(x))$$

for all $z' \in Z'$ and $x \in \Omega$. Since Z is Hausdorff, our statement follows from the Hahn–Banach theorem. □

Further, we observe that the map δ which makes a predual a strong linearization is unique.

Remark 3.3. If $\mathcal{F}(\Omega)$ is a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω and (δ, Y, T) and $(\widetilde{\delta}, Y, T)$ are strong linearizations of $\mathcal{F}(\Omega)$, then it holds $\delta = \widetilde{\delta}$. Indeed, this follows from Proposition 3.2 with $Z := Y$, $\varphi := T$, and $\lambda := \mathrm{id}$.

Now, we recall two conditions, named (BBC) and (CNC), that we need to guarantee the existence of a strong complete barrelled linearization of a bornological function space (see [4, p. 114], [4, 2. Corollary, p. 115], and [28, Theorem 1, p. 320–321]).

Definition 3.4 [25, Definition 3.1, pp. 1596–1597]. Let (X, τ) be a locally convex Hausdorff space.

(a) We say that (X, τ) satisfies condition (BBC) if there exists a locally convex Hausdorff topology $\widetilde{\tau}$ on X such that every τ-bounded subset of X is contained in an absolutely convex τ-bounded $\widetilde{\tau}$-compact set.
(b) We say that (X, τ) satisfies condition (CNC) if there exists a locally convex Hausdorff topology $\widetilde{\tau}$ on X such that τ has a 0-neighborhood basis $\mathcal{U}_0$ of absolutely convex $\widetilde{\tau}$-closed sets.

If we want to emphasize the dependency on $\widetilde{\tau}$, we say that (X,τ) satisfies (BBC) resp. (CNC) for $\widetilde{\tau}$. We say (X,τ) satisfies (BBC) and (CNC) for $\widetilde{\tau}$ if it satisfies both conditions for the same $\widetilde{\tau}$.

In order to obtain a candidate for the predual (Y,T) of a given function space $\mathcal{F}(\Omega)$, we recall the following space of linear functionals from [4, 25, 28, 30]. Let (X,τ) be a locally convex Hausdorff space, $\mathcal{B}$ the family of τ-bounded sets, and $\widetilde{\tau}$ another locally convex Hausdorff topology on X. We denote by $X^{\star}$ the algebraic dual space of X and define

$$X'_{\mathcal{B},\widetilde{\tau}} := \{x^{\star} \in X^{\star} \mid x^{\star}_{|B} \text{ is } \widetilde{\tau}\text{-continuous for all } B \in \mathcal{B}\}$$

and observe that $(X,\widetilde{\tau})' \subset X'_{\mathcal{B},\widetilde{\tau}}$ as linear spaces. We equip $X'_{\mathcal{B},\widetilde{\tau}}$ with the topology $\beta := \beta(X'_{\mathcal{B},\widetilde{\tau}},(X,\tau))$ of uniform convergence on the τ-bounded subsets of X.

Proposition 3.5 [25, Proposition 3.7, p. 1599]. *Let (X,τ) be a bornological locally convex Hausdorff space satisfying* (BBC) *for some $\widetilde{\tau}$ and $\mathcal{B}$ the family of τ-bounded sets. Then the following assertions hold.*

(a) $X'_{\mathcal{B},\widetilde{\tau}}$ is a closed subspace of the complete space $(X,\tau)'_b$. In particular, $(X'_{\mathcal{B},\widetilde{\tau}},\beta)$ is complete.
(b) If (X,τ) is a DF-space, then $(X'_{\mathcal{B},\widetilde{\tau}},\beta)$ is a Fréchet space.
(c) If (X,τ) is normable, then $(X'_{\mathcal{B},\widetilde{\tau}},\beta)$ is completely normable.

We recall from [25, pp. 1611–1612] that a topological space Ω is said to be a *$gk_{\mathbb{R}}$-space* if for any completely regular space Y and any map $f : \Omega \to Y$, whose restriction to each compact $K \subset \Omega$ is continuous, the map is already continuous on Ω. Examples of Hausdorff $gk_{\mathbb{R}}$-spaces are Hausdorff k-spaces, metrizable spaces, locally compact Hausdorff spaces, and strong duals of Fréchet–Montel spaces (*DFM-spaces*). Further, for a locally convex Hausdorff space $\mathcal{F}(\Omega)$ of $\mathbb{K}$-valued functions on a nonempty set Ω, we define $\delta_x : \mathcal{F}(\Omega) \to \mathbb{K}$, $\delta_x(f) := f(x)$, for $x \in \Omega$.

Corollary 3.6 [25, Corollary 4.1, Remark 4.3, Theorem 4.10, pp. 1609, 1612]. *Let $(\mathcal{F}(\Omega),\tau)$ be a bornological locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω satisfying* (BBC) *and* (CNC) *for some $\widetilde{\tau} \leq \tau_{\mathrm{p}}$ and $\mathcal{B}$ the family of τ-bounded sets. Then the following assertions hold.*

(a) $\Delta(x) := \delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ for all $x \in \Omega$ and $(\Delta, \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$ is a strong complete barrelled linearization of $\mathcal{F}(\Omega)$ where

$$\mathcal{I} : (\mathcal{F}(\Omega),\tau) \to (\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}},\beta)'_b,\ f \longmapsto [f' \mapsto f'(f)].$$

(b) If Ω is a $gk_{\mathbb{R}}$-space and $\mathcal{F}(\Omega)$ a space of continuous functions, then the map $\Delta : \Omega \to (\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}},\beta)$ is continuous.

We note that the conditions (BBC) and (CNC) for some $\widetilde{\tau} \leq \tau_{\mathrm{p}}$ are also necessary for the existence of a strong complete barrelled linearization of a bornological function space $\mathcal{F}(\Omega)$ (see [25, Theorem 4.5, p. 1609]).

Let us turn to some examples from [25]. Let Ω be a nonempty topological Hausdorff space. We call $\mathcal{V}$ a *directed family of continuous weights* if $\mathcal{V}$ is a family of continuous functions $v : \Omega \to [0,\infty)$ such that for every $v_1, v_2 \in \mathcal{V}$, there are $C \geq 0$ and $v_0 \in \mathcal{V}$ with $\max(v_1,v_2) \leq Cv_0$ on Ω. We call a directed family of continuous weights $\mathcal{V}$ *point-detecting* if for every $x \in \Omega$, there is $v \in \mathcal{V}$ such that $v(x) > 0$. For an open set $\Omega \subset \mathbb{R}^d$, we denote by $\mathcal{C}^{\infty}(\Omega)$ the space of $\mathbb{K}$-valued infinitely continuously partially differentiable functions on Ω. The next examples are slight generalizations of [8, p. 34] and [4, 3. Examples B, pp. 125–126] where the weighted spaces $\mathcal{HV}(\Omega)$ and $\mathcal{VH}(\Omega)$ of holomorphic functions on an open connected set $\Omega \subset \mathbb{C}^d$ are considered and $\mathcal{V}$ is a point-detecting Nachbin family of continuous weights. Using Proposition 3.5 and Corollary 3.6, we get the following examples of continuous strong complete barrelled (Fréchet, DF-, Banach) linearizations.

Example 3.7 [25, Examples 3.3, 3.25, 4.11, pp. 1597–1598, 1607–1608, 1613]. Let $\Omega \subset \mathbb{R}^d$ be open and $P(\partial)$ a hypoelliptic linear partial differential operator on $\mathcal{C}^{\infty}(\Omega)$.

(i) Let $\mathcal{V}$ be a point-detecting directed family of continuous weights. We define the space

$$\mathcal{C}_P\mathcal{V}(\Omega) := \{f \in \mathcal{C}_P(\Omega) \mid \forall\, v \in \mathcal{V}:\ \|f\|_v := \sup_{x\in\Omega} |f(x)|v(x) < \infty\},$$

where $\mathcal{C}_P(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) \mid f \in \ker P(\partial)\}$, and equip $\mathcal{C}_P\mathcal{V}(\Omega)$ with the locally convex Hausdorff topology $\tau_{\mathcal{V}}$ induced by the seminorms $(\|\cdot\|_v)_{v\in\mathcal{V}}$. If the space $(\mathcal{C}_P\mathcal{V}(\Omega), \tau_{\mathcal{V}})$ is bornological, then $(\Delta, \mathcal{C}_P\mathcal{V}(\Omega)'_{\mathcal{B},\tau_{\text{co}}}, \mathcal{I})$ is a continuous strong complete barrelled linearization of $\mathcal{C}_P\mathcal{V}(\Omega)$. If $\mathcal{V}$ is countable and *increasing*, that is, $v_n \leq v_{n+1}$ for all $n \in \mathbb{N}$, then $(\Delta, \mathcal{C}_P\mathcal{V}(\Omega)'_{\mathcal{B},\tau_{\text{co}}}, \mathcal{I})$ is a continuous strong complete barrelled DF-linearization of $\mathcal{C}_P\mathcal{V}(\Omega)$, and if $\mathcal{V} = \{v\}$, then $(\Delta, \mathcal{C}_Pv(\Omega)'_{\mathcal{B},\tau_{\text{co}}}, \mathcal{I})$ is a continuous strong Banach linearization of $\mathcal{C}_Pv(\Omega)$.

(ii) Let $\mathcal{V} := (v_n)_{n\in\mathbb{N}}$ be a *decreasing*, that is, $v_{n+1} \leq v_n$ for all $n \in \mathbb{N}$, family of continuous functions $v_n : \Omega \to (0, \infty)$. In addition, let $\mathcal{V}$ be *regularly decreasing*, that is, for every $n \in \mathbb{N}$, there is $m \geq n$ such that for every $U \subset \Omega$ with $\inf_{x\in U} v_m(x)/v_n(x) > 0$, we also have $\inf_{x\in U} v_k(x)/v_n(x) > 0$ for all $k \geq m + 1$. We define the inductive limit

$$\mathcal{V}\mathcal{C}_P(\Omega) := \varinjlim_{n\in\mathbb{N}} \mathcal{C}_Pv_n(\Omega)$$

of the Banach spaces $(\mathcal{C}_Pv_n(\Omega), \|\cdot\|_{v_n})$, and equip $\mathcal{V}\mathcal{C}_P(\Omega)$ with its locally convex inductive limit topology ${}_{\mathcal{V}}\tau$. Then $(\mathcal{V}\mathcal{C}_P(\Omega), {}_{\mathcal{V}}\tau)$ is a (ultra)bornological Hausdorff DF-space and $(\Delta, \mathcal{V}\mathcal{C}_P(\Omega)'_{\mathcal{B},\tau_{\text{co}}}, \mathcal{I})$ is a continuous strong Fréchet linearization of $\mathcal{V}\mathcal{C}_P(\Omega)$.

Now, we turn to the relation between strong linearizations and (strong) uniqueness of preduals. Let X be a dual space with predual (Y, φ). Considering X as the dual space of Y, we define the system of seminorms

$$p_N(x) := \sup_{y\in N} |\varphi(x)(y)|, \quad x \in X,$$

for finite sets $N \subset Y$, which induces a locally convex Hausdorff topology on X w.r.t. the dual paring $\langle X, Y, \varphi\rangle$ and we denote this topology by $\sigma_\varphi(X, Y)$.

Proposition 3.8. *Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω, $\mathcal{C}$ be a subclass of the class of complete barrelled locally convex Hausdorff spaces such that $\mathcal{C}$ is closed under topological isomorphisms, and (δ, Y, T) a strong linearization of $\mathcal{F}(\Omega)$ such that $Y \in \mathcal{C}$. Consider the following assertions.*

(a) $\mathcal{F}(\Omega)$ has a strongly unique $\mathcal{C}$ predual.
(b) For every predual (Z, φ) of $\mathcal{F}(\Omega)$ such that $Z \in \mathcal{C}$ and every $x \in \Omega$, there is a (unique) $z_x \in Z$ with $T(\cdot)(\delta(x)) = \varphi(\cdot)(z_x)$.
(c) For every predual (Z, φ) of $\mathcal{F}(\Omega)$ such that $Z \in \mathcal{C}$ and every $x \in \Omega$, it holds $T(\cdot)(\delta(x)) \in (\mathcal{F}(\Omega), \sigma_\varphi(\mathcal{F}(\Omega), Z))'$.
(d) For every predual (Z, φ) of $\mathcal{F}(\Omega)$ such that $Z \in \mathcal{C}$, there is a (unique) map $\widetilde{\delta} : \Omega \to Z$ such that $(\widetilde{\delta}, Z, \varphi)$ is a strong linearization of $\mathcal{F}(\Omega)$.

Then it holds (a)⇒(b)⇔(c)⇔(d). If the family $(\delta(x))_{x\in\Omega}$ is linearly independent, then it holds (b)⇒(a).

Proof. (b)⇔ (c) This equivalence follows directly from [36, Chapter IV, Section 1, 1.2, p. 124].

(a)⇒(b) Let (Z, φ) be predual of $\mathcal{F}(\Omega)$ such that $Z \in \mathcal{C}$. Since $\mathcal{F}(\Omega)$ has a strongly unique $\mathcal{C}$ predual, there is a topological isomorphism $\lambda : Y \to Z$ such that $\lambda^t = T\circ\varphi^{-1}$ by Proposition 2.7. For $x \in \Omega$, we set $z_x := \lambda(\delta(x)) \in Z$ and observe that

$$\varphi(f)(z_x) = \varphi(f)(\lambda(\delta(x))) = \lambda^t(\varphi(f))(\delta(x)) = (T\circ\varphi^{-1})(\varphi(f))(\delta(x)) = T(f)(\delta(x))$$

for all $f \in \mathcal{F}(\Omega)$. The existence of such a z_x already implies that it is unique by [36, Chapter IV, Section 1, 1.2, p. 124].

(b)⇒(d) Let (Z, φ) be a predual of $\mathcal{F}(\Omega)$ such that $Z \in \mathcal{C}$. We set $\widetilde{\delta} : \Omega \to Z$, $\widetilde{\delta}(x) := z_x$. Then we have $\varphi(f)(\widetilde{\delta}(x)) = T(f)(\delta(x)) = f(x)$ for all $f \in \mathcal{F}(\Omega)$, which implies that $(\widetilde{\delta}, Z, \varphi)$ is a strong linearization of $\mathcal{F}(\Omega)$. The uniqueness of $\widetilde{\delta}$ follows from Remark 3.3.

(d)⇒(b) This statement follows from the choice $z_x := \widetilde{\delta}(x) \in Z$ for $x \in \Omega$.

(b)⇒(a) if the family $(\delta(x))_{x\in\Omega}$ is linearly independent. Let (Z,φ) be a predual of $\mathcal{F}(\Omega)$ such that $Z \in \mathcal{C}$. We set $Y_0 := \operatorname{span}\{\delta(x) \mid x \in \Omega\}$ and $Z_0 := \operatorname{span}\{z_x \mid x \in \Omega\}$ and note that the linear map $\lambda_0 : Y_0 \to Z_0$ induced by $\lambda_0(\delta(x)) := z_x$ for $x \in \Omega$ is well defined since $(\delta(x))_{x\in\Omega}$ is linearly independent. Clearly, λ_0 is surjective. Next, we show that it is also injective. Let $y \in Y_0$ such that $\lambda_0(y) = 0$. Then y can be represented as $y = \sum_{x\in\Omega} a_x\delta(x)$ with finitely many nonzero $a_x \in \mathbb{K}$. We note that

$$0 = \lambda_0(y) = \lambda_0\left(\sum_{x\in\Omega} a_x\delta(x)\right) = \sum_{x\in\Omega} a_x z_x$$

and

$$0 = \varphi(f)\left(\sum_{x\in\Omega} a_x z_x\right) = \sum_{x\in\Omega} a_x\varphi(f)(z_x) = T(f)\left(\sum_{x\in\Omega} a_x\delta(x)\right) = \Phi_T(y)(f)$$

for all $f \in \mathcal{F}(\Omega)$. Due to the injectivity of Φ_T by Proposition 2.3, we get that $y = 0$. Thus, λ_0 is injective.

We claim that λ_0 is a topological isomorphism. Indeed, for every $x \in \Omega$, we have

$$\Phi_\varphi(z_x) = \varphi(\cdot)(z_x) = T(\cdot)(\delta(x)) = \Phi_T(\delta(x))$$

and hence $\lambda_0(\delta(x)) = (\Phi_\varphi^{-1}\circ\Phi_T)(\delta(x))$, implying our claim as $(\Phi_\varphi^{-1}\circ\Phi_T)_{|Y_0} : Y_0 \to Z_0$ is a topological isomorphism by Proposition 2.3. Due to [18, 3.4.2 Theorem, pp. 61–62] and the density of Y_0 in the complete space Y by [25, Proposition 2.7, p. 1596], we can uniquely extend λ_0 to a topological isomorphism $\lambda : Y \to \overline{Z_0}$ where $\overline{Z_0}$ denotes the closure of Z_0 in the complete space Z.

Next, we show that $\overline{Z_0} = Z$. Suppose there is some $z \in Z$ such that $z \notin \overline{Z_0}$. Since $\overline{Z_0}$ is closed in Z, there is $z' \in Z'$ such that $z'_{|\overline{Z_0}} = 0$ and $z'(z) = 1$ by the Hahn–Banach theorem. It follows that $0 \neq (T\circ\varphi^{-1})(z') \in Y'$ and

$$(T\circ\varphi^{-1})(z')(\delta(x)) = T(\varphi^{-1}(z'))(\delta(x)) = \varphi(\varphi^{-1}(z'))(z_x) = z'(z_x) = 0$$

for all $x \in \Omega$. The density of Y_0 in Y implies that $(T\circ\varphi^{-1})(z') = 0$, which is a contradiction.

Therefore, $\lambda : Y \to Z$ is a topological isomorphism. Further, we have for every $z' \in Z'$ with $y' := (T\circ\varphi^{-1})(z') \in Y'$ that

$$\begin{aligned}\lambda^t(z')(\delta(x)) &= z'(\lambda(\delta(x))) = z'(\lambda_0(\delta(x))) = z'(z_x) = (\varphi\circ T^{-1})(y')(z_x)\\ &= \varphi(T^{-1}(y'))(z_x) = T(T^{-1}(y'))(\delta(x)) = y'(\delta(x))\\ &= (T\circ\varphi^{-1})(z')(\delta(x))\end{aligned}$$

for all $x \in \Omega$. Again, the density of Y_0 in Y implies that $\lambda^t = T\circ\varphi^{-1}$, which means that $(Z,\varphi) \sim (Y,T)$. We deduce from Proposition 2.7 that $\mathcal{F}(\Omega)$ has a strongly unique $\mathcal{C}$ predual. □

Corollary 3.9. *Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω, $\mathcal{C}$ be a subclass of the class of complete barrelled locally convex Hausdorff spaces such that $\mathcal{C}$ is closed under topological isomorphisms, and (δ, Y, T) a strong linearization of $\mathcal{F}(\Omega)$ such that $Y \in \mathcal{C}$. Consider the following assertions.*

(a) $\mathcal{F}(\Omega)$ has a unique $\mathcal{C}$ predual.
(b) For every predual (Z,φ) of $\mathcal{F}(\Omega)$ such that $Z \in \mathcal{C}$, there is a topological isomorphism $\psi : \mathcal{F}(\Omega) \to Z_b'$ such that for every $x \in \Omega$ there is a (unique) $z_x \in Z$ with $T(\cdot)(\delta(x)) = \psi(\cdot)(z_x)$.
(c) For every predual (Z,φ) of $\mathcal{F}(\Omega)$ such that $Z \in \mathcal{C}$, there is a topological isomorphism $\psi : \mathcal{F}(\Omega) \to Z_b'$ such that for every $x \in \Omega$, it holds $T(\cdot)(\delta(x)) \in (\mathcal{F}(\Omega), \sigma_\psi(\mathcal{F}(\Omega), Z))'$.
(d) For every predual (Z,φ) of $\mathcal{F}(\Omega)$ such that $Z \in \mathcal{C}$, there is a topological isomorphism $\psi : \mathcal{F}(\Omega) \to Z_b'$ and a (unique) map $\widetilde{\delta} : \Omega \to Z$ such that $(\widetilde{\delta}, Z, \psi)$ is a strong linearization of $\mathcal{F}(\Omega)$.

Then it holds (a)⇒(b)⇔(c)⇔(d). If the family $(\delta(x))_{x\in\Omega}$ is linearly independent, then it holds (b)⇒(a).

Proof. This statement follows from the proof of Proposition 3.8 with φ replaced by ψ and Proposition 2.7 replaced by Proposition 2.6. □

Looking at statement (b) of Proposition 3.8 and Corollary 3.9, we also see that continuous point evaluation functionals naturally appear in the context of strong complete barrelled linearizations.

Remark 3.10. Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω, (δ, Y, T) a strong linearization of $\mathcal{F}(\Omega)$ such that Y is quasi-barrelled. Then we have $\delta_x = T(\cdot)(\delta(x))$ and $\delta_x \in \mathcal{F}(\Omega)'$ for every $x \in \Omega$ by Proposition 2.3 since $T(\cdot)(\delta(x)) = \Phi_T(\delta(x))$.

In our last section, we will show in Corollary 5.10 and Corollary 5.11 how the condition of linear independence for the implication (b)⇒(a) in Proposition 3.8 and Corollary 3.9 can be avoided.

Remark 3.11. Let $\mathcal{F}(\Omega)$ be a linear space of $\mathbb{K}$-valued functions on a nonempty set Ω such that for every finite set $A \subset \Omega$ and $x \in A$, there is $f \in \mathcal{F}(\Omega)$ such that $f(x) \neq 0$ and $f(z) = 0$ for all $z \in A \setminus \{x\}$. If (δ, Y, T) is a linearization of $\mathcal{F}(\Omega)$, then the family $(\delta(x))_{x\in\Omega}$ is linearly independent. Indeed, for a family $(a_x)_{x\in\Omega}$ in $\mathbb{K}$ with finitely many nonzero a_x, we have

$$T(f)\left(\sum_{x\in\Omega} a_x \delta(x)\right) = \sum_{x\in\Omega} a_x f(x)$$

for all $f \in \mathcal{F}(\Omega)$, which implies our claim.

The condition on $\mathcal{F}(\Omega)$ in Remark 3.11 is, for example, fulfilled if $\Omega \subset \mathbb{C}$ and $\mathcal{F}(\Omega)$ contains all polynomials of degree less than $|\Omega|$ where $|\Omega|$ denotes the cardinality of Ω.

4 | LINEARIZATION OF VECTOR-VALUED FUNCTIONS

In this section, we derive a strong linearization of weak vector-valued functions from a strong linearization of scalar-valued functions. Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω with fundamental system of seminorms $\Gamma_{\mathcal{F}(\Omega)}$. For a locally convex Hausdorff space E over the field $\mathbb{K}$ with fundamental system of seminorms Γ_E, we define the *space of weak E-valued $\mathcal{F}$-functions* by

$$\mathcal{F}(\Omega, E)_\sigma := \{f : \Omega \to E \mid \forall\, e' \in E' :\ e' \circ f \in \mathcal{F}(\Omega)\}.$$

For $p \in \Gamma_E$ we set $U_p := \{x \in E \mid p(x) < 1\}$ and recall that U_p° denotes the polar set of U_p w.r.t. the dual pairing $\langle E, E'\rangle$. We define the linear subspace of $\mathcal{F}(\Omega, E)_\sigma$ given by

$$\mathcal{F}(\Omega, E)_{\sigma,b} := \{f \in \mathcal{F}(\Omega, E)_\sigma \mid \forall\, q \in \Gamma_{\mathcal{F}(\Omega)},\ p \in \Gamma_E :\ \|f\|_{\sigma,q,p} := \sup_{e'\in U_p^\circ} q(e' \circ f) < \infty\}.$$

$\mathcal{F}(\Omega, E)_{\sigma,b}$ equipped with the system of seminorms $(\|\cdot\|_{\sigma,q,p})_{q\in\Gamma_{\mathcal{F}(\Omega)}, p\in\Gamma_E}$ becomes a locally convex Hausdorff space. We note that $\mathcal{F}(\Omega, \mathbb{K})_{\sigma,b} = \mathcal{F}(\Omega, \mathbb{K})_\sigma = \mathcal{F}(\Omega)$. Further, for a large class of spaces $\mathcal{F}(\Omega)$, namely, BC-spaces such that the point evaluation functionals belong to the dual, the spaces $\mathcal{F}(\Omega, E)_\sigma$ and $\mathcal{F}(\Omega, E)_{\sigma,b}$ actually coincide for any E. Let us recall the definition of a BC-space from [32, p. 395]. A locally convex Hausdorff space X is called a *BC-space* if for every Banach space Y and every linear map $f : Y \to X$ with closed graph in $Y \times X$, one has that f is continuous. A characterization of BC-spaces is given in [32, 6.1 Corollary, pp. 400–401]. Since every Banach space is ultrabornological and barrelled, the [27, Closed graph theorem 24.31, p. 289] of de Wilde and the Pták–Kōmura–Adasch–Valdivia closed graph theorem [20, §34, 9.(7), p. 46] imply that webbed spaces and B_r-complete spaces are BC-spaces. In particular, the Fréchet space $C_P\mathcal{V}(\Omega)$ for a countable point-detecting increasing family $\mathcal{V}$ and the complete DF-space $\mathcal{V}C_P(\Omega)$ for a countable decreasing family $\mathcal{V}$ from Example 3.7 are webbed and thus BC-spaces.

Remark 4.1. If $\mathcal{F}(\Omega)$ is a BC-space of $\mathbb{K}$-valued functions on a nonempty set Ω such that $\delta_x \in \mathcal{F}(\Omega)'$ for all $x \in \Omega$ and E is a locally convex Hausdorff space over the field $\mathbb{K}$, then $\mathcal{F}(\Omega, E)_\sigma = \mathcal{F}(\Omega, E)_{\sigma,b}$ by [22, 3.18 Proposition, p. 319].

Remark 4.2. Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω, Y a locally convex Hausdorff space over the field $\mathbb{K}$ and $\delta : \Omega \to Y$. Then condition (i) of [25, Proposition 2.5, p. 1595] is equivalent to $\delta \in \mathcal{F}(\Omega, Y)_\sigma$.

For two locally convex Hausdorff spaces Y and E over the field $\mathbb{K}$, we denote by $L(Y, E)$ the space of continuous linear maps from Y to E. Moreover, we write $L_b(Y, E)$ if the space $L(Y, E)$ is equipped with the topology of uniform convergence on bounded subsets of Y.

Proposition 4.3. *Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω, (δ, Y, T) a strong linearization of $\mathcal{F}(\Omega)$ and E a locally convex Hausdorff space over the field $\mathbb{K}$ with fundamental system of seminorms Γ_E. Then the following assertions hold.*

(a) The map

$$\chi : L_b(Y, E) \to \mathcal{F}(\Omega, E)_{\sigma,b}, \ \chi(u) := u \circ \delta,$$

is well defined, linear, continuous, and injective.

(b) If $(\mathcal{F}(\Omega), \|\cdot\|)$ is a Banach space such that $\delta_x \in (\mathcal{F}(\Omega), \|\cdot\|)'$ for all $x \in \Omega$, $(Y, \|\cdot\|_Y)$ a normed space, and the map $T : (\mathcal{F}(\Omega), \|\cdot\|) \to ((Y, \|\cdot\|_Y)', \|\cdot\|_{Y'})$ an isometry, then $\mathcal{F}(\Omega, E)_\sigma = \mathcal{F}(\Omega, E)_{\sigma,b}$ and χ is a topological isomorphism into and for all $p \in \Gamma_E$

$$\|\chi(u)\|_{\sigma,p} = \sup_{\substack{y \in Y \\ \|y\|_Y \le 1}} p(u(y)), \quad u \in L(Y, E).$$

Proof.

(a) We note that $e' \circ u \in Y'$ for all $e' \in E'$ and $u \in L(Y, E)$. Therefore, $e' \circ \chi(u) = (e' \circ u) \circ \delta \in \mathcal{F}(\Omega)$ for all $e' \in E'$ and $u \in L(Y, E)$, which yields that $\chi(u) \in \mathcal{F}(\Omega, E)_\sigma$.

Let $q \in \Gamma_{\mathcal{F}(\Omega)}$ and $p \in \Gamma_E$ where $\Gamma_{\mathcal{F}(\Omega)}$ is a fundamental system of seminorms of $\mathcal{F}(\Omega)$. For all $u \in L(Y, E)$ and $e' \in E'$, there is $f_{e' \circ u} \in \mathcal{F}(\Omega)$ such that $e' \circ u = T(f_{e' \circ u})$ and $T(f_{e' \circ u}) \circ \delta = f_{e' \circ u}$ because (δ, Y, T) is a linearization of $\mathcal{F}(\Omega)$ and $e' \circ u \in Y'$. Due to the continuity of $T^{-1} : Y'_b \to \mathcal{F}(\Omega)$, there are $C \ge 0$ and a bounded set $B \subset Y$ such that

$$\begin{aligned} \|\chi(u)\|_{\sigma,q,p} &= \sup_{e' \in U_p^\circ} q(e' \circ u \circ \delta) = \sup_{e' \in U_p^\circ} q(T(f_{e' \circ u}) \circ \delta) = \sup_{e' \in U_p^\circ} q(f_{e' \circ u}) \\ &= \sup_{e' \in U_p^\circ} q(T^{-1}(T(f_{e' \circ u}))) \le C \sup_{e' \in U_p^\circ} \sup_{y \in B} |T(f_{e' \circ u})(y)| \\ &= C \sup_{e' \in U_p^\circ} \sup_{y \in B} |(e' \circ u)(y)| = C \sup_{y \in B} p(u(y)) < \infty \end{aligned} \tag{1}$$

for all $u \in L(Y, E)$, where we used [27, Proposition 22.14, p. 256] for the last equation. Therefore, $\chi(u) \in \mathcal{F}(\Omega, E)_{\sigma,b}$, which means that the map χ is well defined. The map χ is also linear and thus the estimate above implies that it is continuous. The map χ is also injective as the span of $\{\delta(x) \mid x \in \Omega\}$ is dense in Y by [25, Proposition 2.7, p. 1596].

(b) We have $\mathcal{F}(\Omega, E)_\sigma = \mathcal{F}(\Omega, E)_{\sigma,b}$ by Remark 4.1 since Banach spaces are webbed and so BC-spaces. The rest of part (b) follows from part (a) since we have equality in (1) with $q = \|\cdot\|$, $C = 1$ and $B = B_{\|\cdot\|_Y}$. □

Remark 4.4. Let the assumptions of Proposition 4.3(b) be fulfilled. If $(E, \|\cdot\|_E)$ is also a normed space and $L(Y, E)$ equipped with operator norm given by $\|u\|_{L(Y,E)} := \sup\{\|u(y)\|_E \mid y \in Y, \ \|y\|_Y \le 1\}$, $u \in L(Y, E)$, then the map χ is an isometry by choosing $\Gamma_E := \{\|\cdot\|_E\}$.

Next, we prove the surjectivity of the map χ if E is complete. The proof is quite similar to the one given in [21, Theorem 14(i), p. 1524]. We identify E with a linear subspace of the algebraic dual $E'^{\star}$ of E' by the canonical linear injection $x \longmapsto [e' \mapsto e'(x)] =: \langle x, e'\rangle$.

Theorem 4.5. *Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω, (δ, Y, T) a strong linearization of $\mathcal{F}(\Omega)$ such that Y is quasi-barrelled, and E a complete locally convex Hausdorff space over the field $\mathbb{K}$. Then the map*

$$\chi : L_b(Y,E) \to \mathcal{F}(\Omega,E)_{\sigma,b},\ \chi(u) := u\circ\delta,$$

is a topological isomorphism and its inverse fulfils

$$\langle \chi^{-1}(f)(y), e'\rangle = T(e'\circ f)(y), \quad f \in \mathcal{F}(\Omega,E)_{\sigma,b},\ y \in Y,\ e' \in E'.$$

In particular, $\mathcal{F}(\Omega,E)_{\sigma,b}$ is quasi-complete. If Y is even bornological, then the space $\mathcal{F}(\Omega,E)_{\sigma,b}$ is complete.

Proof. Due to Proposition 4.3(a), we only need to show that χ is surjective and its inverse continuous. Fix $f \in \mathcal{F}(\Omega,E)_{\sigma,b}$. For all $y \in Y$, the map $\Psi(f)(y) : E' \to \mathbb{K}$, $\langle \Psi(f)(y), e'\rangle := T(e'\circ f)(y)$, is clearly linear, thus $\Psi(f)(y) \in E'^{\star}$. Let Γ_E denote a fundamental system of seminorms of E. We set for $p \in \Gamma_E$

$$|z|_{U_p^\circ} := \sup_{e'\in U_p^\circ} |z(e')| \le \infty, \quad z \in {E'}^{\star},$$

and note that $p(x) = |\langle\cdot, x\rangle|_{U_p^\circ}$ for every $x \in E$. Further, we define $R_f(e') := e'\circ f$ for $e' \in E'$ and denote by Γ_Y a fundamental system of seminorms of Y. We observe that $R_f(U_p^\circ)$ is a bounded set in $\mathcal{F}(\Omega)$ and that there are $\widetilde{C} \ge 0$ and $\widetilde{q} \in \Gamma_Y$ such that

$$|\Psi(f)(y)|_{U_p^\circ} = \sup_{h\in R_f(U_p^\circ)} |T(h)(y)| = \sup_{h\in R_f(U_p^\circ)} |\mathcal{J}_Y(y)(T(h))| \le \widetilde{C}\widetilde{q}(y) \tag{2}$$

for all $y \in Y$ since Y is a quasi-barrelled space and thus the canonical evaluation map $\mathcal{J}_Y : Y \to (Y_b')_b'$ continuous by [18, 11.2.2. Proposition, p. 222]. Next, we show that $\Psi(f)(y) \in E$ for all $y \in Y$. First, we remark that

$$\langle \Psi(f)(\delta(x)), e'\rangle = T(e'\circ f)(\delta(x)) = (e'\circ f)(x) = \langle f(x), e'\rangle, \quad x \in \Omega,$$

for every $e' \in E'$, yielding $\Psi(f)(\delta(x)) \in E$ for every $x \in \Omega$. By [25, Proposition 2.7, p. 1596], the span of $\{\delta(x) \mid x \in \Omega\}$ is dense in Y. Thus, for every $y \in Y$, there is a net $(y_\iota)_{\iota\in I}$ in this span such that it converges to y and $\Psi(f)(y_\iota) \in E$ for each $\iota \in I$. For every $p \in \Gamma_E$, we have

$$|\Psi(f)(y_\iota) - \Psi(f)(y)|_{U_p^\circ} \underset{(2)}{\le} \widetilde{C}\widetilde{q}(y_\iota - y) \to 0.$$

Hence $(\Psi(f)(y_\iota))_{\iota\in I}$ is a Cauchy net in the complete space E with a limit $g \in E$. For every $p \in \Gamma_E$, we get

$$\begin{aligned} |g - \Psi(f)(y)|_{U_p^\circ} &\le |g - \Psi(f)(y_\iota)|_{U_p^\circ} + |\Psi(f)(y_\iota) - \Psi(f)(y)|_{U_p^\circ} \\ &\underset{(2)}{\le} |g - \Psi(f)(y_\iota)|_{U_p^\circ} + \widetilde{C}\widetilde{q}(y_\iota - y) \end{aligned}$$

and deduce that $\Psi(f)(y) = g \in E$. In combination with (2), we derive that $\Psi(f) \in L(Y,E)$. Finally, we note that

$$\chi(\Psi(f))(x) = \Psi(f)(\delta(x)) = f(x), \quad x \in \Omega,$$

implying the surjectivity of χ.

Let $\Gamma_{\mathcal{F}(\Omega)}$ denote a fundamental system of seminorms of $\mathcal{F}(\Omega)$. Let $B \subset Y$ be bounded and $p \in \Gamma_E$. Due to the continuity of $T : \mathcal{F}(\Omega) \to Y_b'$, there are $C \geq 0$ and $q \in \Gamma_{\mathcal{F}(\Omega)}$ such that

$$\sup_{y\in B} p(\Psi(f)(y)) = \sup_{y\in B} \sup_{e'\in U_p^\circ} |e'(\Psi(f)(y))| = \sup_{e'\in U_p^\circ} \sup_{y\in B} |T(e'\circ f)(y)| \leq C \sup_{e'\in U_p^\circ} q(e'\circ f)$$

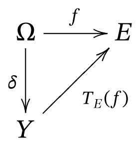
$$= C\|f\|_{\sigma,q,p},$$

where we used [27, Proposition 22.14, p. 256] for the first equation, which implies the continuity of $\Psi = \chi^{-1}$.

The space $L_b(Y,E)$ is quasi-complete, thus $\mathcal{F}(\Omega,E)_{\sigma,b}$ as well, by [20, Section 39, 6.(5), p. 144] since Y is quasi-barrelled and E complete. If Y is even bornological, then $L_b(Y,E)$ is complete by [20, Section 39, 6.(4), p. 143] and hence $\mathcal{F}(\Omega,E)_{\sigma,b}$ as well. □

Due to weak–strong principles for holomorphic, harmonic, and Lipschitz continuous functions special cases of Theorem 4.5 and Remark 4.4 for Banach spaces E over $\mathbb{C}$ were already obtained before, for instance, in [29, Theorem 2.1, p. 869] for the Banach space $\mathcal{F}(\Omega) = \mathcal{H}^\infty(\Omega)$ of bounded holomorphic functions on an open subset Ω of a Banach space, in [6, Lemma 10, p. 243] for a Banach space $\mathcal{F}(\mathbb{D})$ of holomorphic functions on the open unit disc $\mathbb{D} \subset \mathbb{C}$ such that its closed unit ball is τ_{co}-compact, in [26, Lemma 5.2, p. 14] for a Banach space $\mathcal{F}(\mathbb{D})$ of harmonic functions on $\mathbb{D}$ such that its closed unit ball is τ_{co} -compact, in [19, Proposition 6, p. 3], which also covers [16, Theorem 3.1 (Linearization Theorem), p. 128] and [1, Proposition 2.3 (c), p. 3029], for a closed subspace $\mathcal{F}(\Omega)$ of the Banach space $\mathcal{H}v(\Omega)$ such that its closed unit ball is τ_{co}-compact where Ω is an open connected subset of a Banach space and $v : \Omega \to (0,\infty)$ a continuous function, and in [11, Theorem 1, pp. 239–240] where $\mathcal{F}(\Omega) = \mathcal{H}_b(\Omega)$ is the space of holomorphic functions defined on a balanced open subset Ω of a complex normed space, which are bounded on Ω-bounded sets. $\mathcal{H}_b(\Omega)$ is a Fréchet space if equipped with the topology of uniform convergence on Ω-bounded sets, and we note that its predual $Y := \mathfrak{P}_b(\Omega)$ in [11, pp. 238–239] is an inductive limit of a sequence of Banach spaces and thus ultrabornological, in particular barrelled, by construction. For Banach spaces E over $\mathbb{C}$ Theorem 4.5 and Remark 4.4 also generalise [3, Lemma 2.3.1, p. 67] where $(\delta, Y, T) := (\Delta, G, J)$ is a strong Banach linearization of a Banach space $\mathcal{F}(\Omega)$ of $\mathbb{C}$-valued functions on a nonempty set Ω with Banach predual (G,J) such that $\delta_x \in G$ for all $x \in \Omega$, as well as [3, Corollary 2.3.4, p. 68] where $\mathcal{F}(\Omega) = \mathcal{H}\mathcal{V}(\Omega)$ is the weighted Fréchet space of holomorphic functions on a complex Banach space Ω and $\mathcal{V} := (v_n)_{n\in\mathbb{N}}$ an increasing sequence of continuous functions $v_n : \Omega \to (0,\infty)$. Further, Theorem 4.5 for Banach spaces E over $\mathbb{C}$ also covers [2, Theorem 2, p. 283] where $\mathcal{F}(\Omega) = \mathcal{V}\mathcal{H}(\Omega)$ is the inductive limit of the weighted Banach spaces $\mathcal{H}v_n(\Omega)$ of holomorphic functions on a complex Banach space Ω and $\mathcal{V} := (v_n)_{n\in\mathbb{N}}$ a decreasing sequence of continuous functions $v_n : \Omega \to (0,\infty)$ if $\mathcal{V}\mathcal{H}(\Omega)$ satisfies (CNC) for τ_{co} or $(\mathcal{V}\mathcal{H}(\Omega)_{B,\tau_{\mathrm{co}}}, \beta)$ is distinguished by [25, Corollary 4.7, pp. 1610–1611]. For complete locally convex Hausdorff spaces E over $\mathbb{C}$, Theorem 4.5 also generalizes [5, 3.7 Proposition, p. 292] combined with [5, 1.5 Theorem (e), pp. 277–278] where $\mathcal{F}(\Omega) = \mathcal{H}\mathcal{V}(\Omega)$ is the weighted space of holomorphic functions on a balanced open subset of $\Omega \subset \mathbb{C}^d$ and $\mathcal{V}$ a family of radial upper semicontinuous functions $v : \Omega \to [0,\infty)$ such that the weighted topology $\tau_{\mathcal{V}}$ induced by $\mathcal{V}$ fulfils $\tau_{\mathrm{co}} \leq \tau_{\mathcal{V}}$ (see [5, pp. 272, 274]), $\mathcal{H}\mathcal{V}(\Omega)$ is bornological and the polynomials are contained in $\mathcal{H}\mathcal{V}_0(\Omega) := \{f \in \mathcal{H}\mathcal{V}(\Omega) \mid \forall\, v \in \mathcal{V} : \; fv \text{ vanishes at } \infty\}$. However, we note that in contrast to Theorem 4.5, quasi-complete E are allowed in [5, 3.7 Proposition, p. 292]. Theorem 4.5 for complete locally convex Hausdorff spaces E over $\mathbb{C}$ also improves [8, Theorem 3.3, p. 35] where $\mathcal{F}(\Omega) = \mathcal{H}\mathcal{V}(\Omega)$ is bornological, Ω an open connected subset of $\mathbb{C}^d$, and $\mathcal{V}$ a point-detecting Nachbin family of continuous nonnegative functions on Ω. If E is a complete locally convex Hausdorff space and $\mathcal{F}(\Omega) = \mathscr{L}(F\times G,\mathbb{K})$ the space of continuous bilinear forms on the product $\Omega := F \times G$ of two locally convex Hausdorff spaces F and G, then Theorem 4.5 also covers [15, Chapter I, Section 1, n°1, Proposition 2, pp. 30–31] (cf. [18, 15.1.2 Theorem, p. 325]) if the projective tensor product $Y := F \otimes_\pi G$ is quasi-barrelled (instead of $F \otimes_\pi G$, one may also use its completion $F\widehat{\otimes}_\pi G$ by [18, 3.4.2 Theorem, pp. 61–62]). For instance, $F \otimes_\pi G$ is (quasi-)barrelled (and $F\widehat{\otimes}_\pi G$ barrelled by [18, 11.3.1 Proposition (e), p. 223]) if F and G are metrizable and barrelled by [18, 15.6.6 Proposition, p. 337], or if F and G are quasi-barrelled DF-spaces by [18, 15.6.8 Proposition, p. 338]. However, we emphasize that [15, Chapter I, Section 1, n°1, Proposition 2, pp. 30–31] in contrast to Theorem 4.5 does not need the restrictions that E is complete and $F \otimes_\pi G$ quasi-barrelled. Moreover, Theorem 4.5 also complements [10, Theorem 3, p. 690] where a special continuous linearization $(e, \mathcal{F}_*(\Omega), L)$ of a linear space $\mathcal{F}(\Omega)$ of scalar-valued continuous functions on a nonempty topological Hausdorff space Ω instead of a strong linearization is considered and it is shown that the map $L(\mathcal{F}_*(\Omega),E) \to \mathcal{F}(\Omega,E)_\sigma \cap \mathcal{C}(\Omega,E)$, $u \mapsto u\circ e$, is an algebraic isomorphism if E is a complete locally convex Hausdorff space. Here, $\mathcal{C}(\Omega,E)$ denotes the space of continuous maps from Ω to E. We refer the reader to [17, pp. 182–184] for a summary of the construction of $(e, \mathcal{F}_*(\Omega), L)$.

Remark 4.6. Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω, Y a quasi-barrelled locally convex Hausdorff space over the field $\mathbb{K}$ and (δ, Y, T) a strong linearization of $\mathcal{F}(\Omega)$. For any complete locally convex Hausdorff space E over the field $\mathbb{K}$, we have $u \circ \delta \in \mathcal{F}(\Omega, E)_{\sigma,b}$ for all $u \in L_b(Y, E)$ and there is a topological isomorphism $T_E : \mathcal{F}(\Omega, E)_{\sigma,b} \to L_b(Y, E)$, namely $T_E := \chi^{-1} = \Psi$, with $T_E(f) \circ \delta = f$ for all $f \in \mathcal{F}(\Omega, E)_{\sigma,b}$ by Theorem 4.5. Looking at Definition 3.1, we may consider $(\delta, L_b(Y, E), T_E)$ as a strong linearization of $\mathcal{F}(\Omega, E)_{\sigma,b}$. In particular, the following diagram commutes:



If Y is complete, then Y is also a *free object generated by* Ω *in the category* $\mathcal{C}_2$ of complete locally convex Hausdorff spaces with continuous linear operators as morphisms in the sense of [12, p.] once the spaces Ω belong to a category $\mathcal{C}_1$ such that δ and any $f \in \mathcal{F}(\Omega, E)_{\sigma,b}$ is a morphism of $\mathcal{C}_1$ and there is a "forgetful" functor from $\mathcal{C}_2$ to $\mathcal{C}_1$.

Next, we generalize the extension result [19, Theorem 10, p. 5]. For this purpose, we need to recall some definitions. Let E be a locally convex Hausdorff space. A linear subspace $G \subset E'$ is said to *determine boundedness* of E if every $\sigma(E, G)$ -bounded set $B \subset E$ is already bounded in E (see [7, p. 231]). In particular, such a G is $\sigma(E', E)$-dense in E'. Further, by Mackey's theorem $G := E'$ determines boundedness. Another example is the following one.

Remark 4.7. Let (E, τ) be a bornological locally convex Hausdorff space, $\mathcal{B}$ the family of τ-bounded sets, $\widetilde{\tau}$ a locally convex Hausdorff topology on E, and $\widetilde{\gamma}$ denote the finest locally convex Hausdorff topology on E, which coincides with $\widetilde{\tau}$ on τ-bounded sets. Suppose that a subset of E is τ-bounded if and only if it is $\widetilde{\gamma}$-bounded (see [25, Definition 3.12, p. 1600]). Then $(E, \widetilde{\gamma})'$ determines boundedness of (E, τ). Indeed, $(E, \widetilde{\gamma})' \subset (E, \tau)'$ by [25, Remark 3.13(c), pp. 1600–1601]. Let $B \subset E$ be $\sigma(E, (E, \widetilde{\gamma})')$-bounded. Then B is $\widetilde{\gamma}$-bounded by Mackey's theorem. It follows that B is τ-bounded by assumption.

Moreover, let $(\mathcal{F}(\Omega), \tau)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω. A set $U \subset \Omega$ is called a *set of uniqueness* for $\mathcal{F}(\Omega)$ if for each $f \in \mathcal{F}(\Omega)$ the validity of $f(x) = 0$ for all $x \in U$ implies $f = 0$ on Ω (see [19, p. 3]). If $(\mathcal{F}(\Omega), \tau)$ is bornological and satisfies (BBC) and (CNC) for some $\widetilde{\tau}$ such that $\tau_{\mathrm{p}} \leq \widetilde{\tau}$, then $U \subset \Omega$ is a set of uniqueness for $\mathcal{F}(\Omega)$ if and only if the span of $\{\delta_x \mid x \in U\}$ is $\sigma(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{F}(\Omega))$-dense. For instance, a sequence $U := (z_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ of distinct elements is a set of uniqueness for $\mathcal{F}(\mathbb{D}) = \mathcal{H}^\infty(\mathbb{D})$ if and only if it satisfies the Blaschke condition $\sum_{n\in\mathbb{N}}(1 - |z_n|) = \infty$ (see, e.g., [35, 15.23 Theorem, p. 303]). Further examples of sets of uniqueness for the spaces $\mathcal{C}\mathcal{P}v(\Omega)$ are given in [23, pp. 102–103]. Now, we only need to adapt the proof of [19, Theorem 10, p. 5] a bit to get the following theorem.

Theorem 4.8. *Let* $(\mathcal{F}(\Omega), \tau)$ *be a complete bornological DF-space of* $\mathbb{K}$*-valued functions on a nonempty set* Ω *satisfying* (BBC) *and* (CNC) *for some* $\widetilde{\tau}$ *such that* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$, $U \subset \Omega$ *a set of uniqueness for* $\mathcal{F}(\Omega)$, E *a complete locally convex Hausdorff space, and* $G \subset E'$ *determine boundedness. If* $f : U \to E$ *is a function such that* $e' \circ f$ *admits an extension* $f_{e'} \in \mathcal{F}(\Omega, E)_\sigma$ *for each* $e' \in G$*, then there exists a unique extension* $F \in \mathcal{F}(\Omega, E)_\sigma$ *of* f*.*

Proof. Let Y_U denote the span of $\{\delta_x \mid x \in U\}$. Since U is a set of uniqueness for $\mathcal{F}(\Omega)$ and $\tau_{\mathrm{p}} \leq \widetilde{\tau}$, Y_U is $\sigma(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{F}(\Omega))$ -dense and thus also β -dense by [18, 8.2.5 Proposition, p. 149] as $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \sigma(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{F}(\Omega)))' = \mathcal{F}(\Omega) = (\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \beta)'$ (as linear spaces). We note that the linear map $A : Y_U \to E$ determined by $A(\delta_x) := f(x)$ for $x \in U$ is well defined because G is $\sigma(E', E)$ -dense. Let $B \subset \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ be β -bounded, $x' \in B \cap Y_U$ and $e' \in G$. Then x' can be represented as $x' = \sum_{x\in U} a_x \delta_x$ with finitely many nonzero $a_x \in \mathbb{K}$ and

$$|e'(A(x'))| = \left|\sum_{x\in U} a_x e'(f(x))\right| = \left|\sum_{x\in U} a_x f_{e'}(x)\right| = \left|\left(\sum_{x\in U} a_x \delta_x\right)(f_{e'})\right| = |x'(f_{e'})|.$$

Hence, by the β -boundedness of B, there is some $C \geq 0$ such that $|e'(A(x'))| \leq C$ for all $x' \in B \cap Y_U$. We deduce that the set $A(B \cap Y_U)$ is $\sigma(E, G)$ -bounded and thus bounded in E because G determines boundedness. We observe that

$(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is a Fréchet space by Proposition 3.5(b) and so its linear subspace Y_U is metrizable. We conclude that $A : (Y_U, \beta_{|Y_U}) \to E$ is continuous by [27, Proposition 24.10, p. 282] as the metrizable space $(Y_U, \beta_{|Y_U})$ is bornological by [27, Proposition 24.13, p. 283]. Since Y_U is β -dense in $\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$, there is a unique continuous linear extension $\widetilde{A} : \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}} \to E$ of A by [18, 3.4.2 Theorem, pp. 61–62]. Setting $F := \chi \circ \widetilde{A} \in \mathcal{F}(\Omega, E)_\sigma$ by Corollary 3.6(a) and Theorem 4.5, we observe that

$$F(x) = (\chi \circ \widetilde{A})(x) = \widetilde{A}(\delta_x) = A(\delta_x) = f(x)$$

for all $x \in U$, which proves the existence of the extension of f. The uniqueness of the extension follows from U being a set of uniqueness for $\mathcal{F}(\Omega)$ and G being $\sigma(E', E)$ -dense. □

In particular, Theorem 4.8 is applicable to the spaces $\mathcal{F}(\Omega) = \mathcal{VC}_P(\Omega)$ from Example 3.7(ii) by [25, Corollary 4.7, pp. 1610–1611].

5 | EQUIVALENCE AND UNIQUENESS OF PREDUALS

Having two strong linearizations (δ, Y, T) and $(\widetilde{\delta}, Z, \widetilde{T})$ of a locally convex Hausdorff space $\mathcal{F}(\Omega)$ of $\mathbb{K}$-valued functions on a nonempty set Ω, one might suspect that the preduals (Y, T) and $(Z, \widetilde{T})$ are equivalent. This section is dedicated to deriving necessary and sufficient conditions for this to happen. Our candidate for the topological isomorphism $\lambda : Y \to Z$ such that $\lambda^t = T \circ \widetilde{T}^{-1}$ is the map $T_Z(\widetilde{\delta})$ with T_Z from Remark 4.6 for complete Z if $\widetilde{\delta} \in \mathcal{F}(\Omega, Z)_{\sigma,b}$.

Proposition 5.1. *Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω and (δ, Y, T) a strong linearization of $\mathcal{F}(\Omega)$ such that Y is quasi-barrelled. Let $(\widetilde{\delta}, Z, \widetilde{T})$ be a linearization of $\mathcal{F}(\Omega)$ such that Z is complete and $\widetilde{\delta} \in \mathcal{F}(\Omega, Z)_{\sigma,b}$. Then the following assertions hold.*

(a) The map $T_Z(\widetilde{\delta}) : Y \to Z$ is linear, continuous, injective, and has dense range, $\widetilde{T}^{-1} : Z'_b \to \mathcal{F}(\Omega)$ is continuous and $T_Z(\widetilde{\delta})^t = T \circ \widetilde{T}^{-1}$.
(b) If
  (i) Z'_b is webbed and $\mathcal{F}(\Omega)$ ultrabornological, or
  (ii) Z'_b is B_r-complete and $\mathcal{F}(\Omega)$ barrelled,
  then $\widetilde{T} : \mathcal{F}(\Omega) \to Z'_b$ is a topological isomorphism.
(c) If Y is complete, Z barrelled, and $\widetilde{T} : \mathcal{F}(\Omega) \to Z'_b$ continuous, then $T_Z(\widetilde{\delta})$ is surjective.
(d) If $\widetilde{T} : \mathcal{F}(\Omega) \to Z'_b$ is continuous and
  (i) Y is complete and webbed and Z ultrabornological, or
  (ii) Y is B_r-complete and Z barrelled,
  then $T_Z(\widetilde{\delta})$ is a topological isomorphism.

Proof.

(a) We note that $T_Z(\widetilde{\delta})$ is a continuous linear map by Theorem 4.5 and thus $T_Z(\widetilde{\delta})^t : Z'_b \to Y'_b$ as well. We observe that

$$T_Z(\widetilde{\delta})^t(z')(y) = \langle z', T_Z(\widetilde{\delta})(y)\rangle = \langle \chi^{-1}(\widetilde{\delta})(y), z'\rangle = T(z' \circ \widetilde{\delta})(y)$$
$$= T(\widetilde{T}(\widetilde{T}^{-1}(z')) \circ \widetilde{\delta})(y) = T(\widetilde{T}^{-1}(z'))(y)$$

for all $z' \in Z'$ and $y \in Y$ by Theorem 4.5 and using that $T_Z(\widetilde{\delta}) = \chi^{-1}(\widetilde{\delta})$. In particular, $T_Z(\widetilde{\delta})^t : Z'_b \to Y'_b$ is a continuous algebraic isomorphism. Since $\widetilde{T}^{-1} = T^{-1} \circ T_Z(\widetilde{\delta})^t$, we get that $\widetilde{T}^{-1} : Z'_b \to \mathcal{F}(\Omega)$ is continuous.

Further, let $y \in Y$ such that $T_Z(\widetilde{\delta})(y) = 0$. For every $y' \in Y'$, there is $z' \in Z'$ with $T_Z(\widetilde{\delta})^t(z') = y'$ by the surjectivity of $T_Z(\widetilde{\delta})^t$, which implies

$$y'(y) = T_Z(\widetilde{\delta})^t(z')(y) = z'(T_Z(\widetilde{\delta})(y)) = z'(0) = 0$$

for all $y' \in Y'$. Hence $y = 0$ by the Hahn–Banach theorem and $T_Z(\widetilde{\delta})$ is injective.

Moreover, (the restriction of) $T_Z(\widetilde{\delta})$ is a bijective map from the span of $\{\delta(x) \mid x \in \Omega\}$ to the span of $\{\widetilde{\delta}(x) \mid x \in \Omega\}$, which is dense in Z by [25, Proposition 2.7, p. 1596], because $T_Z(\widetilde{\delta})(\delta(x)) = \widetilde{\delta}(x)$ for all $x \in \Omega$. Thus, $T_Z(\widetilde{\delta})$ has dense range.

(b) This statement follows from part (a) and [27, Open mapping theorem 24.30, p. 289] in case (i) and [18, 11.1.7 Theorem (b), p. 221] in case (ii).

(c) We denote by $j_Y : Y \to (Y_b')'$ and $j_Z : Z \to (Z_b')'$ the canonical linear injections and observe that the map $T_Z(\widetilde{\delta})^{tt} : (Y_b')_b' \to (Z_b')_b'$ is linear, continuous, and bijective, and its inverse fulfils

$$(T_Z(\widetilde{\delta})^{tt})^{-1} = ((T_Z(\widetilde{\delta})^t)^{-1})^t = (\widetilde{T}\circ T^{-1})^t$$

by part (a). We note that $j_Y : Y \to (Y_b')_b'$ and $j_Z : Z \to (Z_b')_b'$ are topological isomorphisms into by [18, 11.2.2 Proposition, p. 222] as Y and Z are (quasi-)barrelled. Since $\widetilde{T} : \mathcal{F}(\Omega) \to Z_b'$ is continuous by assumption, we get that $(T_Z(\widetilde{\delta})^{tt})^{-1}$ is also continuous and hence $T_Z(\widetilde{\delta})^{tt}$ a topological isomorphism.

Let $z \in Z$. Since the span of $\{\widetilde{\delta}(x) \mid x \in \Omega\}$ is dense in Z by [25, Proposition 2.7, p. 1596], there is a net $(z_\iota)_{\iota\in I}$ converging to z and where all z_ι can be represented as $z_\iota = \sum_{x\in\Omega} a_{x,\iota}\widetilde{\delta}(x)$ with finitely many nonzero $a_{x,\iota} \in \mathbb{K}$. Using that $T_Z(\widetilde{\delta})''\circ j_Y = j_Z\circ T_Z(\widetilde{\delta})$ and setting $y_\iota := \sum_{x\in\Omega} a_{x,\iota}\delta(x) \in Y$ for $\iota \in I$, we get

$$T_Z(\widetilde{\delta})^{tt}(j_Y(y_\iota)) = (j_Z\circ T_Z(\widetilde{\delta}))(y_\iota) = j_Z\left(\sum_{x\in\Omega} a_{x,\iota}\widetilde{\delta}(x)\right) = j_Z(z_\iota)$$

and so

$$j_Y(y_\iota) = (T_Z(\widetilde{\delta})^{tt})^{-1}(j_Z(z_\iota))$$

for all $\iota \in I$, which implies that the net $(j_Y(y_\iota))_{\iota\in I}$ converges to $(T_Z(\widetilde{\delta})^{tt})^{-1}(j_Z(z))$ in $(Y_b')_b'$ since $T_Z(\widetilde{\delta})^{tt}$ is a topological isomorphism and $(j_Z(z_\iota))_{\iota\in I}$ converges to $j_Z(z)$ due to the barrelledness of Z. The quasi-barrelledness of Y implies that $(y_\iota)_{\iota\in I}$ is a Cauchy net in Y. From the completeness of Y, we deduce that $(y_\iota)_{\iota\in I}$ converges to some $y \in Y$, yielding

$$(j_Z\circ T_Z(\widetilde{\delta}))(y) = (T_Z(\widetilde{\delta})^{tt}\circ j_Y)(y) = j_Z(z)$$

and thus $T_Z(\widetilde{\delta})(y) = z$ by the injectivity of j_Z, which means that $T_Z(\widetilde{\delta})$ is surjective.

(d) The statement follows from part (a) and (c) and [27, Open mapping theorem 24.30, p. 289] in case (i) and [18, 11.1.7 Theorem (b), p. 221] in case (ii) combined with the observations that ultrabornological spaces are barrelled, and B_r-complete spaces are complete by [18, 9.5.1 Proposition (b), p. 183].

□

Proposition 5.1 complements [10, Corollary 2, p. 695] where for a special continuous linearization $(e, \mathcal{F}_*(\Omega), L)$ of $\mathcal{F}(\Omega)$ instead of a strong linearization, it is shown that $\mathcal{F}_*(\Omega)$ is topologically isomorphic to Z for any other continuous linearization $(\widetilde{e}, Z, \widetilde{T})$ of $\mathcal{F}(\Omega)$ such that Z is a Fréchet space. Looking at part (b), we note that Z_b' is a Fréchet space, so B_r-complete and webbed, by [18, 9.5.2 Krein–Šmulian Theorem, p. 184] and [18, 12.4.2 Theorem, p. 258] if Z is a complete gDF-space. Further, Z_b' is a complete DF-space, so webbed, by [18, 12.4.5 Theorem, p. 260] and [18, 12.4.6 Proposition, p. 260] if Z is a Fréchet space.

Proposition 5.2. *Let $(\mathcal{F}(\Omega), \|\cdot\|)$ be a Banach space of $\mathbb{K}$-valued functions on a nonempty set Ω such that $\delta_x \in (\mathcal{F}(\Omega), \|\cdot\|)'$ for all $x \in \Omega$, (δ, Y, T) a linearization of $\mathcal{F}(\Omega)$ such that $(Y, \|\cdot\|_Y)$ is a Banach space and $T : (\mathcal{F}(\Omega), \|\cdot\|) \to ((Y, \|\cdot\|_Y)', \|\cdot\|_{Y'})$ an isometry. If $(\widetilde{\delta}, Z, \widetilde{T})$ is a linearization of $\mathcal{F}(\Omega)$ such that $(Z, \|\cdot\|_Z)$ is a Banach space and the map $\widetilde{T} : (\mathcal{F}(\Omega), \|\cdot\|) \to ((Z, \|\cdot\|_Z)', \|\cdot\|_{Z'})$ an isometry, then the map $T_Z(\widetilde{\delta}) : (Y, \|\cdot\|_Y) \to (Z, \|\cdot\|_Z)$ is an isometric isomorphism.*

Proof. First, we note that $\mathcal{F}(\Omega, Z)_{\sigma,b} = \mathcal{F}(\Omega, Z)_\sigma$ by Remark 4.1 because the Banach space $(\mathcal{F}(\Omega), \|\cdot\|)$ is a BC-space. Due to Proposition 5.1(a) and (d), we know that $T_Z(\widetilde{\delta})$ is a topological isomorphism and $T_Z(\widetilde{\delta})^t = T\circ\widetilde{T}^{-1}$. Thus $T_Z(\widetilde{\delta})^t : ((Z, \|\cdot\|_Z)', \|\cdot\|_{Z'}) \to ((Y, \|\cdot\|_Y)', \|\cdot\|_{Y'})$ is an isometric isomorphism since T and $\widetilde{T}$ are isometries. Hence we have

$T_Z(\widetilde{\delta})^t(B_{\|\cdot\|_{Z'}}) = B_{\|\cdot\|_{Y'}}$ and

$$\|T_Z(\widetilde{\delta})(y)\|_Z = \sup_{z' \in B_{\|\cdot\|_{Z'}}} |z'(T_Z(\widetilde{\delta})(y))| = \sup_{z' \in B_{\|\cdot\|_{Z'}}} |T_Z(\widetilde{\delta})^t(z')(y)| = \sup_{y' \in B_{\|\cdot\|_{Y'}}} |y'(y)|$$
$$= \|y\|_Y$$

for all $y \in Y$. Thus $T_Z(\widetilde{\delta})$ is an isometry. □

Proposition 5.1(d) combined with Proposition 5.2 improves [1, Proposition 2.3 (c), p. 3029], [16, Theorem 3.1 (Linearization Theorem), p. 128], [29, Theorem 2.1, p. 869], and [33, Theorem 3.5, p. 19] since it shows that the Banach space Y of a strong linearization (δ, Y, T) of a Banach space $\mathcal{F}(\Omega)$ such that T is an isometry is uniquely determined up to an isometric isomorphism without the need of existence of isometric isomorphisms $\widetilde{T}_E : \mathcal{F}(\Omega, E)_\sigma \to L(Z, E)$ such that $\widetilde{T}_E(f) \circ \widetilde{\delta} = f$ for all $f \in \mathcal{F}(\Omega)$ for every Banach space E over $\mathbb{K}$. Further, Proposition 5.1(d) combined with Proposition 5.2 also implies the corresponding result for the (completion of the) projective tensor product of two Banach spaces given in [34, Proposition 1.5 (Uniqueness of the Tensor Product), p. 6] and [34, Theorem 2.9, p. 22].

Proposition 5.3. *Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω, (δ, Y, T) a strong linearization of $\mathcal{F}(\Omega)$ and Z a locally convex Hausdorff space. If there exists a topological isomorphism $\varphi : \mathcal{F}(\Omega) \to Z_b'$ such that $(Z, \varphi) \sim (Y, T)$, then there exists $\widetilde{\delta} : \Omega \to Z$ such that $(\widetilde{\delta}, Z, \varphi)$ is a strong linearization of $\mathcal{F}(\Omega)$.*

Proof. Since $(Z, \varphi) \sim (Y, T)$, there exists a topological isomorphism $\lambda : Y \to Z$ such that $\lambda^t = T \circ \varphi^{-1}$. We set $\widetilde{\delta} : \Omega \to Z$, $\widetilde{\delta}(x) := \lambda(\delta(x))$, and note that

$$(\varphi(f) \circ \widetilde{\delta})(x) = \varphi(f)(\lambda(\delta(x))) = \lambda^t(\varphi(f))(\delta(x)) = (T \circ \varphi^{-1})(\varphi(f))(\delta(x))$$
$$= T(f)(\delta(x)) = f(x)$$

for all $f \in \mathcal{F}(\Omega)$ and $x \in \Omega$. Hence $(\widetilde{\delta}, Z, \varphi)$ is a strong linearization of $\mathcal{F}(\Omega)$. □

Theorem 5.4. *Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a nonempty set Ω, (δ, Y, T) a strong linearization of $\mathcal{F}(\Omega)$, $(Z, \widetilde{T})$ a predual of $\mathcal{F}(\Omega)$ such that Z is complete, and let*

(i) Y be complete, barrelled and webbed and Z ultrabornological, or
(ii) Y be B_r-complete and barrelled and Z barrelled.

Consider the following assertions.

(a) There exists $\widetilde{\delta} \in \mathcal{F}(\Omega, Z)_{\sigma,b}$ such that $(\widetilde{\delta}, Z, \widetilde{T})$ is a strong linearization of $\mathcal{F}(\Omega)$.
(b) It holds $(Z, \widetilde{T}) \sim (Y, T)$.

Then it holds (a)⇒(b). If $\mathcal{F}(\Omega)$ is a BC-space such that $\delta_x \in \mathcal{F}(\Omega)'$ for all $x \in \Omega$, then it holds (b)⇒(a).

Proof. (a)⇒(b) Since $(\widetilde{\delta}, Z, \widetilde{T})$ is a strong linearization of $\mathcal{F}(\Omega)$, the topological isomorphism $\widetilde{T} : \mathcal{F}(\Omega) \to Z_b'$ fulfils $\widetilde{T}(f) \circ \widetilde{\delta} = f$ for all $f \in \mathcal{F}(\Omega)$. We conclude statement (b) from Proposition 5.1(a) and (d).

(b)⇒(a) This implication follows from Proposition 5.3 with $\varphi := \widetilde{T}$ and Remark 4.1. □

Corollary 5.5. *Let $(\mathcal{F}(\Omega), \tau)$ be a bornological BC-space of $\mathbb{K}$-valued functions on a nonempty set Ω satisfying* (BBC) *and* (CNC) *for some $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ and $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \beta)$ webbed, where $\mathcal{B}$ is the family of τ-bounded sets, and $(Z, \widetilde{T})$ a predual of $(\mathcal{F}(\Omega), \tau)$ such that Z is complete and ultrabornological. Then the following assertions are equivalent.*

(a) There exists $\widetilde{\delta} : \Omega \to Z$ *such that* $(\widetilde{\delta}, Z, \widetilde{T})$ *is a strong linearization of* $\mathcal{F}(\Omega)$.
(b) It holds $(Z, \widetilde{T}) \sim (\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$.

Proof. Due to Corollary 3.6(a) $(\Delta, \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$ is a strong complete barrelled linearization of $\mathcal{F}(\Omega)$. Hence our statement follows from Theorem 5.4(i) since $\Delta(x) = \delta_x \in (\mathcal{F}(\Omega), \tau)'$ for all $x \in \Omega$ by Proposition 3.5(a). □

In particular, $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is a complete DF-space (see [25, Corollary 3.23, p. 1606]) and thus webbed by [18, 12.4.6 Proposition, p. 260] if $(\mathcal{F}(\Omega), \tau)$ is a Fréchet space.

Corollary 5.6. *Let* $(\mathcal{F}(\Omega), \tau)$ *be a complete bornological DF-space of* $\mathbb{K}$*-valued functions on a nonempty set* Ω *satisfying* (BBC) *and* (CNC) *for some* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ *and* $\mathcal{B}$ *the family of* τ*-bounded sets as well as* $(Z, \widetilde{T})$ *a predual of* $(\mathcal{F}(\Omega), \tau)$ *such that* Z *is complete and barrelled. Then the following assertions are equivalent.*

(a) There exists $\widetilde{\delta} : \Omega \to Z$ *such that* $(\widetilde{\delta}, Z, \widetilde{T})$ *is a strong linearization of* $\mathcal{F}(\Omega)$.
(b) It holds $(Z, \widetilde{T}) \sim (\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$.

Proof. Due to Proposition 3.5(b) and Corollary 3.6(a) $(\Delta, \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$ is a strong Fréchet linearization of $\mathcal{F}(\Omega)$. By Proposition 3.5(a), it holds $\Delta(x) = \delta_x \in (\mathcal{F}(\Omega), \tau)'$ for all $x \in \Omega$. Since the Fréchet space $(\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \beta)$ is B_r-complete by [18, 9.5.2 Krein–Sm̌ulian Theorem, p. 184] and the complete DF-space $(\mathcal{F}(\Omega), \tau)$ webbed by [18, 12.4.6 Proposition, p. 260], so a BC-space, our statement follows from Theorem 5.4(ii). □

Using [25, Propositions 3.16, 3.17, p. 1602], Proposition 3.5(c) and Corollary 3.6(a), we get the following corollary for completely normable spaces $\mathcal{F}(\Omega)$.

Corollary 5.7. *Let* $(\mathcal{F}(\Omega), \tau)$ *be a completely normable space of* $\mathbb{K}$*-valued functions on a nonempty set* Ω *satisfying* (BBC) *for some* $\tau_{\mathrm{p}} \leq \widetilde{\tau}$ *and* $(Z, \widetilde{T})$ *a predual of* $(\mathcal{F}(\Omega), \tau)$ *such that* Z *is complete and barrelled. Then the following assertions are equivalent.*

(a) There exists $\widetilde{\delta} : \Omega \to Z$ *such that* $(\widetilde{\delta}, Z, \widetilde{T})$ *is a strong linearization of* $\mathcal{F}(\Omega)$.
(b) It holds $(Z, \tilde{T}) \sim ((\mathcal{F}(\Omega), \tilde{\gamma})', \mathcal{I})$.

Let $(\delta, Y, T_{\mathbb{K}})$ be a strong linearization of $\mathcal{F}(\Omega)$ such that Y is quasi-barrelled. If we not only have a linearization $(\widetilde{\delta}, Z, \widetilde{T}_{\mathbb{K}})$ of $\mathcal{F}(\Omega)$ in the scalar-valued case as in the results above but also of $\mathcal{F}(\Omega, Y)_\sigma$ in the vector-valued case, we may get rid of some of the assumptions in Proposition 5.1(d) on $\widetilde{T}_{\mathbb{K}}$, Y, and Z.

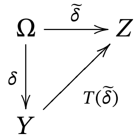
Proposition 5.8. *Let* $\mathcal{F}(\Omega)$ *be a locally convex Hausdorff space of* $\mathbb{K}$*-valued functions on a nonempty set* Ω*,* $(\delta, Y, T_{\mathbb{K}})$ *a strong linearization of* $\mathcal{F}(\Omega)$ *such that* Y *is quasi-barrelled, and* $\delta \in \mathcal{F}(\Omega, Y)_{\sigma,b}$*. If* Z *is a complete locally convex Hausdorff space and* $\widetilde{\delta} \in \mathcal{F}(\Omega, Z)_{\sigma,b}$ *such that for all* $E \in \{\mathbb{K}, Y\}$*, there is an algebraic isomorphism* $\widetilde{T}_E : \mathcal{F}(\Omega, E)_{\sigma,b} \to L(Z, E)$ *with* $\widetilde{T}_E(f) \circ \widetilde{\delta} = f$ *for all* $f \in \mathcal{F}(\Omega, E)_{\sigma,b}$*, then the map* $T_Z(\widetilde{\delta}) : Y \to Z$ *is a topological isomorphism with inverse* $T_Z(\widetilde{\delta})^{-1} = \widetilde{T}_Y(\delta)$*, the map* $\widetilde{T}_{\mathbb{K}}$ *is a topological isomorphism and* $T_Z(\widetilde{\delta})^t = T_{\mathbb{K}} \circ \widetilde{T}_{\mathbb{K}}^{-1}$*. In particular, it holds* $(Z, \widetilde{T}_{\mathbb{K}}) \sim (Y, T_{\mathbb{K}})$.

Proof. First, we prove that $T_Z(\widetilde{\delta}) : Y \to Z$ is a topological isomorphism. Due to Proposition 5.1(a), we only need to show that $T_Y(\widetilde{\delta})$ is surjective and its inverse continuous. We do this by proving that $T_Z(\widetilde{\delta})^{-1} = \widetilde{T}_Y(\delta)$ holds.

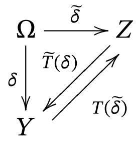
There is an algebraic isomorphism $\widetilde{T} : \mathcal{F}(\Omega, Y)_{\sigma,b} \to L(Z, Y)$ with $\widetilde{T}(f) \circ \widetilde{\delta} = f$ for all $f \in \mathcal{F}(\Omega, Y)_\sigma$ where we set $\widetilde{T} := \widetilde{T}_Y$. Since $\delta \in \mathcal{F}(\Omega, Y)_{\sigma,b}$ by assumption, we get the commuting diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{\ \delta\ } & Y \\ {\scriptstyle \widetilde{\delta}}\downarrow & \nearrow {\scriptstyle \widetilde{T}(\delta)} & \\ Z & & \end{array}$$

Furthermore, there is a topological isomorphism $T := T_Z : \mathcal{F}(\Omega, Z)_{\sigma,b} \to L_b(Y, Z)$ with $T(f)\circ\delta = f$ for all $f \in \mathcal{F}(\Omega, Z)_{\sigma,b}$ by Remark 4.6. Since $\widetilde{\delta} \in \mathcal{F}(\Omega, Z)_{\sigma,b}$ by assumption, we obtain the commuting diagram:



Combining both diagrams, we get the commuting diagram:



Now, we note that $T(\widetilde{\delta})\circ\widetilde{T}(\delta) \in L(Z)$, $\widetilde{T}(\delta)\circ T(\widetilde{\delta}) \in L(Y)$, and $(T(\widetilde{\delta})\circ\widetilde{T}(\delta))(\widetilde{\delta}(x)) = T(\widetilde{\delta})(\delta(x)) = \widetilde{\delta}(x)$ as well as $(\widetilde{T}(\delta)\circ T(\widetilde{\delta}))(\delta(x)) = \widetilde{T}(\delta)(\widetilde{\delta}(x)) = \delta(x)$ for all $x \in \Omega$. Since the span of $\{\widetilde{\delta}(x) \mid x \in \Omega\}$ is dense in Z and the span of $\{\delta(x) \mid x \in \Omega\}$ dense in Y by [25, Proposition 2.7, p. 1596], we obtain that $T(\widetilde{\delta})$ and $\widetilde{T}(\delta)$ are topological isomorphisms with $T(\widetilde{\delta}) = \widetilde{T}(\delta)^{-1}$.

Second, we know by Proposition 5.1(a) that $\widetilde{T}_{\mathbb{K}}^{-1}$ is continuous and $T_Z(\widetilde{\delta})^t = T_{\mathbb{K}}\circ\widetilde{T}_{\mathbb{K}}^{-1}$. Since

$$\tilde{T}_{\mathbb{K}} = (T_Z(\tilde{\delta})^t)^{-1} \circ T_{\mathbb{K}} = (T_Z(\tilde{\delta})^{-1})^t \circ T_{\mathbb{K}}$$

and $T_{\mathbb{K}}$ as well as $T_Z(\widetilde{\delta})$ are topological isomorphisms, we get that $\widetilde{T}_{\mathbb{K}}$ is a topological isomorphism, too. This implies that $(Z, \widetilde{T}_{\mathbb{K}}) \sim (Y, T_{\mathbb{K}})$. □

The proof of $T_Z(\widetilde{\delta})^{-1} = \widetilde{T}_Y(\delta)$ in Proposition 5.8 is an adaptation of parts of the proof of [10, Corollary 1, p. 691]. Further, Proposition 5.8 complements [10, Corollary 1, p. 691] where a continuous linearization $(e, \mathcal{F}_*(\Omega), L)$ of $\mathcal{F}(\Omega)$ instead of a strong linearization is considered. In comparison to [10, Corollary 1, p. 691], we do not need that $\widetilde{T}_E$ is an algebraic isomorphism for all complete locally convex Hausdorff spaces E over $\mathbb{K}$ (even though an inspection of its proof tells us that this is not needed there either) and we get more importantly that $(Z, \widetilde{T}_{\mathbb{K}}) \sim (Y, T_{\mathbb{K}})$. The latter equivalence also allows us to show how the continuous linearization $(e, \mathcal{F}_*(\Omega), L)$ is related to our strong linearizations from Section 3 in many cases.

Corollary 5.9. *Let $(\mathcal{F}(\Omega), \tau)$ be a bornological BC-space of $\mathbb{K}$-valued continuous functions on a nonempty Hausdorff $gk_{\mathbb{R}}$-space Ω satisfying* (BBC) *and* (CNC) *for some $\tau_{\mathrm{p}} \leq \widetilde{\tau}$. Then the maps $\mathcal{I}_Z(e) : \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}} \to \mathcal{F}_*(\Omega)$ and L are topological isomorphism such that $\mathcal{I}_Z(e)^t = \mathcal{I}\circ L^{-1}$, the triple $(e, \mathcal{F}_*(\Omega), L)$ is a continuous strong complete barrelled linearization of $\mathcal{F}(\Omega)$ and $(\mathcal{F}_*(\Omega), L) \sim (\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$.*

Proof. The triple $(\delta, Y, T_{\mathbb{K}}) := (\Delta, \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$ is a continuous strong complete barrelled linearization of $\mathcal{F}(\Omega)$ by Corollary 3.6, and $\Delta(x) = \delta_x \in (\mathcal{F}(\Omega), \tau)'$ for all $x \in \Omega$ by Proposition 3.5(a). Furthermore, since $(\mathcal{F}(\Omega), \tau)$ is a BC-space and Δ continuous, we have

$$\mathcal{F}(\Omega, E)_{\sigma,b} = \mathcal{F}(\Omega, E)_{\sigma} = (\mathcal{F}(\Omega, E)_{\sigma} \cap C(\Omega, E)) =: \omega\mathcal{F}(\Omega, E)$$

for any complete locally convex Hausdorff space E by Remark 4.1 and Theorem 4.5. Due to [10, Theorems 2, 3, pp. 689–690] $Z := \mathcal{F}_*(\Omega)$ is complete, $e \in \omega\mathcal{F}(\Omega, \mathcal{F}_*(\Omega))$, and for any locally convex Hausdorff space E, there is an algebraic isomorphism $L_E : \omega\mathcal{F}(\Omega, E) \to L(\mathcal{F}_*(\Omega), E)$ with $L_E(f)\circ e = f$ for all $f \in \omega\mathcal{F}(\Omega, E)$. Hence it follows from Proposition 5.8 with $\widetilde{\delta} := e$ and $\widetilde{T}_E := L_E$ that the maps $\mathcal{I}_Z(e)$ and L are topological isomorphism with $\mathcal{I}_Z(e)^t = \mathcal{I}\circ L^{-1}$, the triple $(e, \mathcal{F}_*(\Omega), L)$ is a continuous strong complete barrelled linearization of $\mathcal{F}(\Omega)$, and $(\mathcal{F}_*(\Omega), L) \sim (\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$. □

In the case that $(\mathcal{F}(\Omega),\tau)$ is a Banach space of continuous $\mathbb{K}$-valued functions on a topological Hausdorff space Ω satisfying (BBC) for some $\tau_{\mathrm{p}} \leq \widetilde{\tau}$, it is already known that $(e, \mathcal{F}_*(\Omega), L)$ is a continuous strong Banach linearization of $\mathcal{F}(\Omega)$ by [17, Theorem 2.2, p. 188].

In our last results of this section, we show how to get rid of the condition that the family $(\delta(x))_{x\in\Omega}$ should be linearly independent for the implication (b)⇒(a) of Proposition 3.8 and Corollary 3.9, at least if it is a family of point evaluation functionals.

Corollary 5.10. *Let $(\mathcal{F}(\Omega),\tau)$ be a bornological BC-space of $\mathbb{K}$-valued functions on a nonempty set Ω satisfying* (BBC) *and* (CNC) *for some $\tau_{\mathrm{p}} \leq \widetilde{\tau}$. Let $\mathcal{C}$ be a subclass of the class of complete barrelled locally convex Hausdorff spaces such that $\mathcal{C}$ is closed under topological isomorphisms and $\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}} \in \mathcal{C}$ where $\mathcal{B}$ is the family of τ-bounded sets. Then the following assertions are equivalent.*

(a) $(\mathcal{F}(\Omega),\tau)$ has a strongly unique $\mathcal{C}$ predual.
(b) For every predual (Z,φ) of $(\mathcal{F}(\Omega),\tau)$ such that $Z \in \mathcal{C}$ and every $x\in\Omega$, there is a (unique) $z_x \in Z$ with $\delta_x = \varphi(\cdot)(z_x)$.

Proof. (a)⇒(b) This part follows from Proposition 3.8 since $(\Delta, \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}, \mathcal{I})$ is a strong complete barrelled linearization by Corollary 3.6(a), $\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}} \in \mathcal{C}$ by assumption and $\mathcal{I}(f)(\Delta(x)) = \mathcal{I}(f)(\delta_x) = \delta_x(f)$ for all $f \in \mathcal{F}(\Omega)$ and $x\in\Omega$.

(b)⇒(a) Let (Z,φ) be a predual of $(\mathcal{F}(\Omega),\tau)$ such that $Z\in\mathcal{C}$. Due to [25, Proposition 3.21 (a), p. 1605] $(\mathcal{F}(\Omega),\tau)$ satisfies (BBC) and (CNC) for $\sigma_\varphi(\mathcal{F}(\Omega),Z)$ because Z is complete and barrelled. By assumption for every $x\in\Omega$, there is $z_x\in Z$ with $\delta_x = \varphi(\cdot)(z_x)$ and thus $\delta_x \in \mathcal{F}(\Omega)'_{\mathcal{B},\sigma_\varphi(\mathcal{F}(\Omega),Z)}$ by [25, Proposition 3.21 (b), (c), p. 1605]. Hence $(\Delta, \mathcal{F}(\Omega)'_{\mathcal{B},\sigma_\varphi(\mathcal{F}(\Omega),Z)}, \mathcal{I}_\varphi)$ is a strong complete barrelled linearization of $\mathcal{F}(\Omega)$ by Corollary 3.6(a). Further, $\mathcal{F}(\Omega)'_{\mathcal{B},\sigma_\varphi(\mathcal{F}(\Omega),Z)} \in \mathcal{C}$ since Z and $\mathcal{F}(\Omega)'_{\mathcal{B},\sigma_\varphi(\mathcal{F}(\Omega),Z)}$ are topologically isomorphic by [25, Proposition 3.21 (b), p. 1605] and $\mathcal{C}$ closed under topological isomorphisms. By Proposition 3.5(a), we know that $\Delta(x) = \delta_x \in (\mathcal{F}(\Omega),\tau)'$ for all $x\in\Omega$. Applying Theorem 4.5 combined with Remark 4.1 to the triple $(\Delta, \mathcal{F}(\Omega)'_{\mathcal{B},\sigma_\varphi(\mathcal{F}(\Omega),Z)}, \mathcal{I}_\varphi)$, where

$$\mathcal{I}_\varphi : (\mathcal{F}(\Omega),\tau) \to (\mathcal{F}(\Omega)'_{\mathcal{B},\sigma_\varphi(\mathcal{F}(\Omega),Z)}, \beta)'_b,\ f \longmapsto [f' \mapsto f'(f)],$$

we get that for all complete locally convex Hausdorff spaces E there is a topological isomorphism $\widetilde{T}_E : \mathcal{F}(\Omega,E)_\sigma \to L_b(\mathcal{F}(\Omega)'_{\mathcal{B},\sigma_\varphi(\mathcal{F}(\Omega),Z)}, E)$ with $\widetilde{T}_E(f)\circ\Delta = f$ for all $f\in\mathcal{F}(\Omega,E)_\sigma$, where $\widetilde{T}_{\mathbb{K}} = \mathcal{I}_\varphi$ by [25, Proposition 3.21 (b), p. 1605]. Thus, it follows from Proposition 5.8 with $\widetilde{\delta} := \Delta$ and $Y := \mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}}$ that $(\mathcal{F}(\Omega)'_{\mathcal{B},\sigma_\varphi(\mathcal{F}(\Omega),Z)}, \mathcal{I}_\varphi) \sim (\mathcal{F}(\Omega)'_{\mathcal{B},\tau}, \mathcal{I})$. We also know that $(\mathcal{F}(\Omega)'_{\mathcal{B},\sigma_\varphi(\mathcal{F}(\Omega),Z)}, \mathcal{I}_\varphi) \sim (Z,\varphi)$ by [25, Proposition 3.21 (b), p. 1605], which implies $(Z,\varphi) \sim (\mathcal{F}(\Omega)'_{\mathcal{B},\tau}, \mathcal{I})$. Therefore, $(\mathcal{F}(\Omega),\tau)$ has a strongly unique $\mathcal{C}$ predual by Proposition 2.7. □

Corollary 5.11. *Let $(\mathcal{F}(\Omega),\tau)$ be a bornological BC-space of $\mathbb{K}$-valued functions on a nonempty set Ω satisfying* (BBC) *and* (CNC) *for some $\tau_{\mathrm{p}} \leq \widetilde{\tau}$. Let $\mathcal{C}$ be a subclass of the class of complete barrelled locally convex Hausdorff spaces such that $\mathcal{C}$ is closed under topological isomorphisms and $\mathcal{F}(\Omega)'_{\mathcal{B},\widetilde{\tau}} \in \mathcal{C}$ where $\mathcal{B}$ is the family of τ-bounded sets. Then the following assertions are equivalent.*

(a) $(\mathcal{F}(\Omega),\tau)$ has a unique $\mathcal{C}$ predual.
(b) For every predual (Z,φ) of $(\mathcal{F}(\Omega),\tau)$ such that $Z\in\mathcal{C}$, there is a topological isomorphism $\psi : (\mathcal{F}(\Omega),\tau) \to Z'_b$ such that for every $x\in\Omega$ there is a (unique) $z_x\in Z$ with $\delta_x = \psi(\cdot)(z_x)$.

Proof. This statement follows from Corollary 3.9 and the proof of Corollary 5.10 with φ replaced by ψ and Proposition 2.6 instead of Proposition 2.7 in the end. □

CONFLICT OF INTEREST STATEMENT

The author has no relevant financial or nonfinancial interests to disclose.

DATA AVAILABILITY STATEMENT

Not applicable.

REFERENCES

[1] R. Aron, V. Dimant, L. C. García-Lirola, and M. Maestre, *Linearization of holomorphic Lipschitz functions*, Math. Nachr. **297** (2024), no. 8, 3024–3051, DOI 10.1002/mana.202300527.

[2] M. J. Beltrán, *Linearization of weighted (LB)-spaces of entire functions on Banach spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **106** (2012), no. 2, 275–286, DOI 10.1007/s13398-011-0049-z.

[3] M. J. Beltrán, *Operators on weighted spaces of holomorphic functions*, Ph.D. thesis, Universitat Politècnica de València, Valencia, 2014, DOI 10.4995/Thesis/10251/36578.

[4] K. D. Bierstedt and J. Bonet, *Biduality in Fréchet and (LB)-spaces*, In K. D. Bierstedt, J. Bonet, J. Horváth, and M. Maestre, eds., Progess in functional analysis (Proc., Peñíscola, 1990), vol. 170, North-Holland Math. Stud. , Amsterdam, North-Holland, 1992, DOI 10.1016/S0304-0208(08)70314-1.

[5] K. D. Bierstedt, J. Bonet, and A. Galbis, *Weighted spaces of holomorphic functions on balanced domains*, Michigan Math. J. **40** (1993), no. 2, 271–297, DOI 10.1307/mmj/1029004753.

[6] J. Bonet, P. Domański, and M. Lindström, *Weakly compact composition operators on analytic vector-valued function spaces*, Ann. Acad. Sci. Fenn. Math. **26** (2001), 233–248, DOI 10.5186/aasfm.00.

[7] J. Bonet, L. Frerick, and E. Jordá, *Extension of vector-valued holomorphic and harmonic functions*, Studia Math. **183** (2007), no. 3, 225–248, DOI 10.4064/sm183-3-2.

[8] J. Bonet and M. Friz, *Weakly compact composition operators on locally convex spaces*, Math. Nachr. **245** (2002), no. 1, 26–44, DOI 10.1002/1522-2616(200211)245:1<26::AID-MANA26>3.0.CO;2-J.

[9] L. Brown and T. Ito, *Some non-quasireflexive spaces having unique isomorphic preduals*, Israel J. Math. **20** (1975), no. 3, 321–325, DOI 10.1007/BF02760336.

[10] D. Carando and I. Zalduendo, *Linearization of functions*, Math. Ann. **328** (2004), no. 4, 683–700, DOI 10.1007/s00208-003-0502-1.

[11] P. Galindo, D. García, and M. Maestre, *Holomorphic mappings of bounded type*, J. Math. Anal. Appl. **166** (1992), no. 1, 236–246, DOI 10.1016/0022-247X(92)90339-F.

[12] E. García-Sánchez, D. de Hevia, and P. Tradacete, *Free objects in Banach space theory*, In H. Herrero, M. C. Navarro, and H. Serrano, eds., Cutting-Edge Mathematics. RSME 2022. RSME Springer Series, vol 13, Springer, Cham, 2024, pp. 100–124, https://doi.org/10.1007/978-3-031-62025-6_6.

[13] E. Gardella and H. Thiel, *Preduals and complementation of spaces of bounded linear operators*, Internat. J. Math. **31** (2020), no. 07, 2050053, DOI 10.1142/S0129167X20500536.

[14] G. Godefroy, *Existence and uniqueness of isometric preduals: a survey*, In B.-L. Lin, ed., Banach space theory (Proc., Iowa City, 1987), vol. 85, Contemp. Math., AMS, Providence, RI, 1989, pp. 131–193, DOI 10.1090/conm/085.

[15] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., vol. 16, 4th ed., AMS, Providence, RI, 1966, DOI 10.1090/memo/0016.

[16] M. Gupta and D. Baweja, *Weighted spaces of holomorphic functions on Banach spaces and the approximation property*, Extracta Math. **31** (2016), no. 2, 123–144.

[17] J. A. Jaramillo, A. Prieto, and I. Zalduendo, *Linearization and compactness*, Studia Math. **191** (2009), no. 2, 181–200, DOI 10.4064/sm191-2-6.

[18] H. Jarchow, *Locally convex spaces*, Math. Leitfäden. Teubner, Stuttgart, 1981, DOI 10.1007/978-3-322-90559-8.

[19] E. Jordá, *Weighted vector-valued holomorphic functions on Banach spaces*, Abstr. Appl. Anal. **2013** (2013), 1–9, DOI 10.1155/2013/50159.

[20] G. Köthe, *Topological vector spaces II*, Grundlehren Math. Wiss., vol. 237, Springer, Berlin, 1979, DOI 10.1007/978-1-4684-9409-9.

[21] K. Kruse, *Weighted spaces of vector-valued functions and the ε-product*, Banach J. Math. Anal. **14** (2020), no. 4, 1509–1531, DOI 10.1007/s43037-020-00072-z.

[22] K. Kruse, *Extension of vector-valued functions and sequence space representation*, Bull. Belg. Math. Soc. Simon Stevin **29** (2022), no. 3, 307–322, DOI 10.36045/j.bbms.211009.

[23] K. Kruse, *On vector-valued functions and the ε-product*, Habilitation thesis, Hamburg University of Technology, 2023, DOI 10.15480/882.4898.

[24] K. Kruse, *On linearisation, existence and uniqueness of preduals: the isometric case*, 2023, arXiv preprint, https://arxiv.org/abs/2307.16299v1.

[25] K. Kruse, *On linearisation and existence of preduals*, Rend. Circ. Mat. Palermo (2) **73** (2024), no. 4, 1591–1615, DOI 10.1007/s12215-024-01004-8.

[26] J. Laitila and H.-O. Tylli, *Composition operators on vector-valued harmonic functions and Cauchy transforms*, Indiana Univ. Math. J. **55** (2006), no. 2, 719–746, DOI 10.1512/iumj.2006.55.2785.

[27] R. Meise and D. Vogt, *Introduction to functional analysis*, Oxf. Grad. Texts Math., vol. 2, Clarendon Press, Oxford, 1997.

[28] J. Mujica, *A completeness criterion for inductive limits of Banach spaces*, In G. I. Zapata, ed., Functional analysis, holomorphy and approximation theory II (Proc., Rio de Janeiro, 1981), vol. 86, North-Holland Math. Stud. , North-Holland, Amsterdam, 1984, pp. 319–329, DOI 10.1016/S0304-0208(08)70834-X.

[29] J. Mujica, *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc. **342** (1991), no. 2, 867–887, DOI 10.1090/S0002-9947-1991-1000146-2.

[30] K. Ng, *On a theorem of Dixmier*, Math. Scand. **29** (1971), 279–280, DOI 10.7146/math.scand.a-11054.

[31] P. Pérez Carreras and J. Bonet, *Barrelled locally convex spaces*, Math. Stud., vol. 131, North-Holland, Amsterdam, 1987.

[32] M. H. Powell, *On Kōmura's closed-graph theorem*, Trans. Am. Math. Soc. **211** (1975), 391–426, DOI 10.1090/S0002-9947-1975-0380339-9.

[33] T. Quang, *Banach-valued Bloch-type functions on the unit ball of a Hilbert space and weak spaces of Bloch-type*, Constr. Math. Anal. **6** (2023), no. 1, 6–21, DOI 10.33205/cma.1243686.

[34] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monogr. Math. Springer, Berlin, 2002, DOI 10.1007/978-1-4471-3903-4.

[35] W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1970.

[36] H. H. Schaefer, *Topological vector spaces*, Grad. Texts in Math., Springer, Berlin, 1971, DOI 10.1007/978-1-4684-9928-5.

[37] N. Weaver, *On the unique predual problem for Lipschitz spaces*, Math. Proc. Camb. Phil. Soc. **165** (2018), no. 3, 467–473, DOI 10.1017/S0305004117000597.

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