

Asymptotic period of an aperiodic Markov chain and the strong ratio limit property

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Abstract. We introduce the concept of *asymptotic period* for an irreducible and aperiodic discrete-time Markov chain on a countable state space. If the chain is transient its asymptotic period may be larger than one. We present some sufficient conditions and, in the more restricted setting of birth-death processes, a necessary and sufficient condition for *asymptotic aperiodicity*. It is subsequently shown that a birth-death process has the *strong ratio limit property* if a related birth-death process is asymptotically aperiodic. In the general setting a similar statement is not true, but validity of the converse implication is posed as a conjecture.

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1 Introduction

Let $P := (P(i, j), i, j \in S)$ be the matrix of one-step transition probabilities of a homogeneous, discrete-time Markov chain $\mathcal{X} := \{X(n), n = 0, 1, \dots\}$ on a countably infinite state space S , so that the matrix $P^{(n)} := (P^{(n)}(i, j), i, j \in S)$ of n -step transition probabilities

$$P^{(n)}(i, j) := \Pr\{X(m+n) = j \mid X(m) = i\}, \quad i, j \in S, \quad m, n = 0, 1, \dots,$$

is given by

$$P^{(n)} = P^n, \quad n = 0, 1, \dots$$

We will assume throughout that the Markov chain \mathcal{X} is stochastic, irreducible, and aperiodic.

Although \mathcal{X} is aperiodic, it may happen that, in the long run, the chain will move cyclically through a finite number of sets constituting a partition of S . This phenomenon occurs for instance if \mathcal{X} is a *transient* birth-death process on the nonnegative integers with only a finite number of positive self-transition probabilities, since then the chain will eventually move cyclically between the even-numbered and odd-numbered states. It then seems natural to say that the *asymptotic period* of \mathcal{X} equals two or, possibly, a multiple of two. In the general setting the *asymptotic period* of \mathcal{X} may be defined as the maximum number of sets involved in the type of cyclic behaviour described above. In this paper we will formalize these ideas, and investigate some of their consequences.

After discussing preliminary concepts and results in Section 2 we will, in Section 3, formally define the *asymptotic period* for Markov chains that are, in a sense to be defined, *simple*. We will subsequently derive some sufficient conditions for *asymptotic aperiodicity*. The framework developed in Section 2 draws heavily on the work of Blackwell [2] on transient Markov chains, while our definition of asymptotic period resembles in some aspects the definition of *period of an irreducible positive operator* by Moy [14], and is directly related to the definition of *asymptotic period of a tail sequence of subsets of S* , proposed by Abrahamse [1] in a setting that is more general than ours. Actually, Abrahamse introduces the concept of asymptotic period while generalizing Blackwell's results. Our further elaboration of the concept in a more restricted setting makes it more convenient for us to build on the foundations laid down by Blackwell.

In Section 4 we will investigate the occurrence of asymptotic periodicity in a birth-death process on the nonnegative integers, and establish a necessary and sufficient condition for asymptotic aperiodicity in terms of the one-step transition probabilities of the process.

In Section 5 we investigate the relation between asymptotic aperiodicity and the *strong ratio limit property*, which is said to prevail if there exist positive

constants R , $\mu(i)$, $i \in S$, and $f(i)$, $i \in S$, such that

$$\lim_{n \rightarrow \infty} \frac{P^{(n+m)}(i, j)}{P^{(n)}(k, l)} = R^{-m} \frac{f(i)\mu(j)}{f(k)\mu(l)}, \quad i, j, k, l \in S, \quad m \in \mathbb{Z}. \quad (1)$$

The strong ratio limit property was enunciated in the setting of recurrent Markov chains by Orey [15], and introduced in the more general setting at hand by Pruitt [17]. We will display how the necessary and sufficient condition for asymptotic aperiodicity of a birth-death process established in Section 4 leads to a sufficient condition for a birth-death process to have the strong ratio limit property. This result is suggestive of a sufficient condition for the strong ratio limit property in our general setting, which, however, is not correct. A necessary condition for the strong ratio limit property of a Markov chain involving asymptotic aperiodicity of two related chains will be posed as a conjecture.

We end this introduction with some notation. We recall that a nonzero measure μ on S is called an *x-invariant measure* for P (or, for \mathcal{X}) if, letting $\mu(i) \equiv \mu(\{i\})$,

$$\mu P(i) := \sum_{j \in S} \mu(j)P(j, i) = x\mu(i), \quad i \in S, \quad (2)$$

and a nonzero function f on S is called an *x-harmonic function* (or *x-invariant vector*) for P (or, for \mathcal{X}) if

$$Pf(i) := \sum_{j \in S} P(i, j)f(j) = xf(i), \quad i \in S. \quad (3)$$

Our notation is meant to suggest that the numbers $\mu(i)$ and $f(i)$ appearing in (1) will usually satisfy (2) and (3) for a particular value of x .

Finally, when \mathcal{X} is a discrete-time birth-death process on the nonnegative integers – a process often encountered in what follows – we write

$$p_i := P(i, i+1), \quad q_{i+1} := P(i+1, i) \quad \text{and} \quad r_i := P(i, i), \quad i = 0, 1, \dots, \quad (4)$$

for the birth, death and self-transition probabilities, respectively. It will be convenient to define $q_0 := 0$. Since \mathcal{X} is stochastic, irreducible and aperiodic, we have $p_i > 0$, $q_{i+1} > 0$, and $r_i \geq 0$ for $i \geq 0$, with $r_i > 0$ for at least one state i , while $p_i + q_i + r_i = 1$ for $i \geq 0$. In what follows a *birth-death process* will always refer to a discrete-time birth-death process on the nonnegative integers.

2 Preliminaries

We start off by introducing some further notation and terminology related to the Markov chain $\mathcal{X} = \{X(n), n = 0, 1, \dots\}$. By \mathbb{P} we denote the probability

measure on the set of sample paths induced by P and the (unspecified) initial distribution.

For $C \subset S$ we define the events

$$U(C) := \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} \{X(k) \in C\} \quad \text{and} \quad L(C) := \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} \{X(k) \in C\},$$

and we let

$$\mathcal{T} := \{C \subset S \mid U(C) \stackrel{\text{a.s.}}{=} \emptyset\},$$

that is, $C \in \mathcal{T}$ if $\mathbb{P}(X(n) \in C \text{ infinitely often}) = 0$, and

$$\mathcal{R} := \{C \subset S \mid U(C) \stackrel{\text{a.s.}}{=} L(C)\},$$

that is, $C \in \mathcal{R}$ if $\mathbb{P}(X(n) \in C \text{ infinitely often} \Rightarrow X(n) \in C \text{ for } n \text{ sufficiently large}) = 1$. In the terminology of Revuz [18, Sect. 2.3] \mathcal{T} is the collection of *transient* sets and \mathcal{R} is the collection of *regular* sets. Evidently, $\mathcal{T} \subset \mathcal{R}$, while it is not difficult to see that \mathcal{R} is closed under finite union and complementation, and hence a field. Note that \mathcal{T} and \mathcal{R} are independent of the initial distribution, since, by the irreducibility of \mathcal{X} , $\mathbb{P}(U(C))$ and $\mathbb{P}(U(C) \setminus L(C))$ are zero or positive for all initial states (and hence all initial distributions) simultaneously.

We will say that two regular sets C_1 and C_2 are *equivalent* if their symmetric difference $C_1 \Delta C_2 := (C_1 \cup C_2) \setminus (C_1 \cap C_2)$ is transient, and *almost disjoint* if their intersection $C_1 \cap C_2$ is transient. Following Blackwell [2] (see also Chung [3, Section I.17]), we call a subset $C \subset S$ *almost closed* if $C \notin \mathcal{T}$ and $C \in \mathcal{R}$. An almost closed set C is said to be *atomic* if C does not contain two disjoint almost closed subsets. The relevance of these concepts comes to light in the next theorem.

Theorem 1. (Blackwell [2]) *There is a finite or countable collection $\{C_1, C_2, \dots\}$ of disjoint almost closed sets, which is unique up to equivalence and such that*

- (i) *every C_i , except at most one, is atomic;*
- (ii) *the nonatomic C_i , if present, contains no atomic subsets and consists of transient states;*
- (iii) $\sum_i \mathbb{P}(X(n) \in C_i \text{ for } n \text{ sufficiently large}) = 1$.

A collection of sets $\{C_1, C_2, \dots\}$ satisfying the conditions in the theorem will be called a *Blackwell decomposition* (of S) for \mathcal{X} . A set $C \subset S$ is a *Blackwell component* (of S) for \mathcal{X} if there exists a Blackwell decomposition for \mathcal{X} such that C is one of the almost closed sets in the decomposition. The uniqueness up to equivalence of the Blackwell decomposition for \mathcal{X} means that if C_1 and C_2 are Blackwell components, then they are either equivalent or almost disjoint. The number of almost closed sets in the Blackwell decomposition for \mathcal{X} will be denoted by $\beta(\mathcal{X})$. If $\beta(\mathcal{X}) = 1$ then \mathcal{X} is called *simple*, and a simple process is called *atomic* or *nonatomic* according to the type of its state space. Evidently, if \mathcal{X} is simple and nonatomic then S does not contain atomic subsets, but infinitely many disjoint almost closed subsets. It will be useful to observe the following.

Lemma 1. *Let $S = \{0, 1, \dots\}$ and \mathcal{X} have jumps that are uniformly bounded by M . Then $\beta(\mathcal{X}) \leq M$ and every Blackwell component for \mathcal{X} is atomic.*

Proof. Let C be an almost closed set for \mathcal{X} and let $s_1 < s_2 < \dots$ denote the states of C . We claim that there exists a constant N such that for every $n \geq N$ we have $s_{n+1} \leq s_n + M$. Indeed, if $s_{n+1} > s_n + M$, then the process will leave C when it leaves the set $\{s_1, s_2, \dots, s_n\}$. The irreducibility of S insures that a visit to this finite set of states will almost surely be followed by a departure from the set. So if, for each N , there is an integer $n \geq N$ such that $s_{n+1} > s_n + M$, then each entrance in C is almost surely followed by a departure from C , and hence $\mathbb{P}(L(C)) = 0$, contradicting the fact that C is almost closed.

Next, let C_1, C_2, \dots, C_β , with $\beta \equiv \beta(\mathcal{X})$, be the Blackwell components for \mathcal{X} , $s_1^{(i)} < s_2^{(i)} < \dots$ the states of C_i , and N_i such that for every $n \geq N_i$ we have $s_{n+1}^{(i)} \leq s_n^{(i)} + M$. If $\beta > M$, then, choosing

$$s = \max_{1 \leq i \leq M+1} \{s_{N_i}^{(i)}\},$$

the set $\{s + 1, s + 2, \dots, s + M\}$ must have a nonempty intersection with each of the disjoint sets C_1, C_2, \dots, C_{M+1} , which is clearly impossible. Hence, $\beta \leq M$.

Finally, let C be a Blackwell component for \mathcal{X} and suppose C is nonatomic. Then C contains infinitely many disjoint almost closed subsets, so we can choose $M + 1$ disjoint almost closed subsets C_1, C_2, \dots, C_{M+1} of C . By the same argument as before there must be a state s in C such that each of the disjoint sets C_1, C_2, \dots, C_{M+1} shares a state with the set $\{s + 1, s + 2, \dots, s + M\}$. This is impossible, so C must be atomic. \square

A criterion for deciding whether a process is simple and atomic is given in the next theorem.

Theorem 2. (Blackwell [2]) *The process \mathcal{X} is simple and atomic if and only if the only bounded nonnegative 1-harmonic function for \mathcal{X} is the constant function.*

Actually, Blackwell states the result without the adjective “nonnegative”, but by slightly adapting Blackwell’s proof one obtains the result of Theorem 2, which suits our needs better.

As an aside we note that when \mathcal{X} is transient – the setting of primary interest to us – and the constant function is the only bounded nonnegative 1-harmonic function, then there is precisely one escape route to infinity, or, in the terminology of Hou and Guo [7] (see, in particular, Sections 7.13 and 7.16), the *exit space* of \mathcal{X} contains exactly one *atomic exit point*. Of course, the existence, up to a multiplicative constant, of a unique *bounded* nonnegative 1-harmonic function does not, in general, preclude the existence of an *unbounded* nonnegative 1-harmonic function, but when \mathcal{X} is recurrent the constant function happens to be the only nonnegative 1-harmonic function (see, for example, Chung [3, Theorem I.7.6]).

A function f on the space $\Omega := \{(\omega_0, \omega_1, \dots) \mid \omega_i \in S, i = 0, 1, \dots\}$ will be called m -invariant if, for every $\omega := (\omega_0, \omega_1, \dots) \in \Omega$, $f(\omega) = f(\theta^m \omega)$, where θ is the shift operator $\theta(\omega_0, \omega_1, \dots) = (\omega_1, \omega_2, \dots)$, and $\theta^m \omega = \theta(\theta^{m-1} \omega)$. We also use the notation $\theta^m E := \{\theta^m \omega \mid \omega \in E\}$, for $E \subset \Omega$. An event is called m -invariant if its indicator function is m -invariant. A 1-invariant function or event is simply referred to as *invariant*. Evidently, the collection of invariant events constitutes a σ -field. We shall need another of Blackwell's results, involving invariant events (see [1, Theorem 5] for a generalization).

Theorem 3. (Blackwell [2]) *For any invariant event E there is a $C \in \mathcal{R}$ such that $E \stackrel{\text{a.s.}}{=} U(C)$.*

Note that the event $U(C)$ is actually invariant for *any* subset C of S , so for every $C \subset S$ there must be a *regular* set \tilde{C} such that $U(C) \stackrel{\text{a.s.}}{=} U(\tilde{C})$. It follows in particular that every invariant event has probability zero or one if \mathcal{X} is simple and atomic.

The regular set corresponding to an invariant event is unique up to equivalence. For if C_1 and C_2 are regular sets satisfying $U(C_1) \stackrel{\text{a.s.}}{=} U(C_2)$, then

$$U(C_1 \setminus C_2) \subset U(C_1) \setminus L(C_2) \stackrel{\text{a.s.}}{=} U(C_2) \setminus L(C_2) \stackrel{\text{a.s.}}{=} \emptyset,$$

and similarly with C_1 and C_2 interchanged. Since $U(C_1 \Delta C_2) \subset U(C_1 \setminus C_2) \cup U(C_2 \setminus C_1)$, it follows that $C_1 \Delta C_2$ must be transient. So, up to events of probability zero, the σ -field of invariant events is identical with the σ -field of events of the form $U(C)$ with $C \in \mathcal{R}$.

Theorem 3 plays a crucial role in the proof of Proposition 1, which involves $\mathcal{X}^{(m)} := \{X^{(m)}(n) \equiv X(nm), n = 0, 1, \dots\}$, the m -step Markov chain associated with \mathcal{X} , and is instrumental in our definition of asymptotic period. For $C \subset S$ we let

$$U^{(m)}(C) := \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} \{X(mk) \in C\} \text{ and } L^{(m)}(C) := \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} \{X(mk) \in C\},$$

so that $U^{(1)}(C) = U(C)$ and $L^{(1)}(C) = L(C)$. Note that $E = \theta^m E$ if and only if E is m -invariant, so in particular we have $\theta^m U^{(m)}(C) = U^{(m)}(C)$. The following simple observation will prove useful.

Lemma 2. *Let E be an m -invariant event for some $m \geq 1$. Then, for all $i \geq 1$,*

$$E \stackrel{\text{a.s.}}{=} \emptyset \iff \theta^i E \stackrel{\text{a.s.}}{=} \emptyset,$$

Proof. If $\mathbb{P}(E) > 0$ there must be a state s , say, such that $\mathbb{P}(E \mid X(0) = s) > 0$. Moreover, aperiodicity and irreducibility of the chain imply that there is an integer k such that $\mathbb{P}(X(km - i) = s) > 0$. Since, by Theorem 3, $E \stackrel{\text{a.s.}}{=} U^{(m)}(C)$ for some set C , we obviously have $\mathbb{P}(\theta^i E \mid X(km - i) = s) = \mathbb{P}(E \mid X(0) = s)$. Hence, if $\mathbb{P}(E) > 0$, then

$$\begin{aligned} \mathbb{P}(\theta^i E) &\geq \mathbb{P}(\theta^i E \mid X(km - i) = s) \mathbb{P}(X(km - i) = s) \\ &= \mathbb{P}(E \mid X(0) = s) \mathbb{P}(X(km - i) = s) > 0. \end{aligned}$$

The same argument with E and $\theta^i E$ interchanged and $km - i$ replaced by $km + i$ yields the converse. \square

Before stating and proving Proposition 1 we establish some additional auxiliary lemmas. In what follows we write $E \overset{\text{a.s.}}{\subset} F$ for $E \setminus F \overset{\text{a.s.}}{=} \emptyset$.

Lemma 3. *Let E_1 and E_2 be m -invariant events for some $m \geq 1$. Then, for all $i \geq 0$ and $j \geq 0$,*

- (i) $E_1 \overset{\text{a.s.}}{\subset} \theta^j E_2 \iff \theta^i E_1 \overset{\text{a.s.}}{\subset} \theta^{i+j} E_2,$
- (ii) $E_1 \overset{\text{a.s.}}{=} \theta^j E_2 \iff \theta^i E_1 \overset{\text{a.s.}}{=} \theta^{i+j} E_2.$

Proof. The event $E_1 \setminus \theta^j E_2$ is m -invariant, so, by Lemma 2, we have

$$E_1 \setminus \theta^j E_2 \overset{\text{a.s.}}{=} \emptyset \iff \theta^i (E_1 \setminus \theta^j E_2) \overset{\text{a.s.}}{=} \emptyset,$$

which implies the first statement. Moreover, the first statement remains valid, by a similar argument, if we interchange the sets E_1 and $\theta^j E_2$. Combining both results yields the second statement. \square

Note that the second statement of this lemma generalizes Lemma 2. The next auxiliary result is a straightforward corollary to the previous lemma.

Lemma 4. *Let E be an m -invariant event for some $m \geq 1$. Then, for all $j \geq 0$ and $k_2 \geq k_1 \geq 0$,*

- (i) $E \overset{\text{a.s.}}{\subset} \theta^j E \implies \theta^{k_1 j} E \overset{\text{a.s.}}{\subset} \theta^{k_2 j} E,$
- (ii) $E \overset{\text{a.s.}}{=} \theta^j E \implies \theta^{k_1 j} E \overset{\text{a.s.}}{=} \theta^{k_2 j} E.$

Our final preparatory lemma is the following.

Lemma 5. *Let C_1 and C_2 be subsets of S that are regular with respect to $\mathcal{X}^{(m)}$ for some $m \geq 1$. Then*

$$U^{(m)}(C_1 \cap C_2) \overset{\text{a.s.}}{=} U^{(m)}(C_1) \cap U^{(m)}(C_2).$$

Proof. We clearly have

$$U^{(m)}(C_1 \cap C_2) \subset U^{(m)}(C_1) \cap U^{(m)}(C_2) \overset{\text{a.s.}}{=} L^{(m)}(C_1) \cap L^{(m)}(C_2).$$

Since

$$L^{(m)}(C_1) \cap L^{(m)}(C_2) = L^{(m)}(C_1 \cap C_2) \subset U^{(m)}(C_1 \cap C_2),$$

the result follows. \square

Proposition 1. *If \mathcal{X} is simple and atomic and $m > 1$, then $\beta \equiv \beta(\mathcal{X}^{(m)})$ is a divisor of m and the Blackwell decomposition for $\mathcal{X}^{(m)}$ consists of a collection $\{C_0, C_1, \dots, C_{\beta-1}\}$ of disjoint atomic almost closed sets, which can be chosen such that, for $i = 0, 1, \dots, \beta - 1$,*

$$\mathbb{P}([X(k) \in C_i \Rightarrow X(k+1) \in C_{i+1 \pmod{\beta}}] \text{ for } k \text{ sufficiently large}) = 1. \quad (5)$$

If \mathcal{X} is simple and nonatomic, then $\mathcal{X}^{(m)}$ is simple and nonatomic for all $m \geq 1$.

Proof. First suppose \mathcal{X} is simple and atomic. Let C_0 be a Blackwell component for $\mathcal{X}^{(m)}$ and assume, for the time being, that C_0 is atomic. Since $\theta^i U^{(m)}(C_0)$ is m -invariant for all i , we can apply Theorem 3 to $\mathcal{X}^{(m)}$ and conclude that there is a sequence C_1, C_2, \dots of regular sets (with respect to $\mathcal{X}^{(m)}$) such that

$$\theta^i U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_i), \quad i = 1, 2, \dots \quad (6)$$

By Lemma 2 the sets C_i are almost closed, since C_0 is almost closed. Also, by Lemma 3, we have

$$U^{(m)}(C_{i+1}) \stackrel{\text{a.s.}}{=} \theta^{i+1} U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} \theta U^{(m)}(C_i),$$

so that $L^{(m)}(C_{i+1}) \stackrel{\text{a.s.}}{=} \theta U^{(m)}(C_i)$, and hence

$$\mathbb{P}([X(k) \in C_i \Rightarrow X(k+1) \in C_{i+1}] \text{ for } k \text{ sufficiently large}) = 1. \quad (7)$$

Next defining

$$b := \min\{i \geq 1 \mid \theta^i U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_0)\}, \quad (8)$$

we have $b \leq m$ since $U^{(m)}(C_0)$ is m -invariant. Also, b must be a divisor of m , for otherwise, by Lemma 4, we would have

$$U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} \theta^{\ell b} U^{(m)}(C_0) = \theta^{m+i} U^{(m)}(C_0) = \theta^i U^{(m)}(C_0),$$

with $\ell = \min\{k \in \mathbb{N} \mid kb > m\}$ and $i = \ell b - m < b$, contradicting (8). For $i \geq b$ we have, by Lemma 3,

$$U^{(m)}(C_i) \stackrel{\text{a.s.}}{=} \theta^i U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} \theta^{i-b} U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_{i-b}),$$

so that C_i and C_{i-b} are equivalent (with respect to $\mathcal{X}^{(m)}$). We can therefore replace (7) by

$$\mathbb{P}([X(k) \in C_i \Rightarrow X(k+1) \in C_{i+1 \pmod{b}}] \text{ for } k \text{ sufficiently large}) = 1. \quad (9)$$

Our next step will be to prove that the sets C_0, C_1, \dots, C_{b-1} are almost disjoint. Since the collection of sets that are regular with respect to $\mathcal{X}^{(m)}$ constitutes a field, the sets $C_0 \setminus C_i$ and $C_0 \cap C_i$, with $0 < i < b$, are regular. But C_0 , being an

atomic Blackwell component for $\mathcal{X}^{(m)}$, cannot contain two almost closed subsets, so that either $C_0 \setminus C_i$ or $C_0 \cap C_i$ must be transient. If $C_0 \setminus C_i$ is transient, then

$$U^{(m)}(C_0) \setminus U^{(m)}(C_i) \subset U^{(m)}(C_0 \setminus C_i) \stackrel{\text{a.s.}}{=} \emptyset,$$

which implies

$$U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_0) \cap U^{(m)}(C_i) \subset U^{(m)}(C_i) \stackrel{\text{a.s.}}{=} \theta^i U^{(m)}(C_0),$$

that is, $U^{(m)}(C_0) \stackrel{\text{a.s.}}{\subset} \theta^i U^{(m)}(C_0)$. But then, by Lemma 4,

$$\theta^i U^{(m)}(C_0) \stackrel{\text{a.s.}}{\subset} \theta^{bi} U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_0),$$

so that $U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} \theta^i U^{(m)}(C_0)$, contradicting (8). So we conclude, for $0 < i < b$, that $C_0 \cap C_i$ is transient, and hence that C_0 and C_i , are almost disjoint. It subsequently follows that C_i and C_j , with $0 \leq i < j < b$, are also almost disjoint. Indeed, C_0 and C_{j-i} being almost disjoint, we have, by Lemma 5,

$$U^{(m)}(C_0) \cap \theta^{j-i} U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_0) \cap U^{(m)}(C_{j-i}) \stackrel{\text{a.s.}}{=} U^{(m)}(C_0 \cap C_{j-i}) \stackrel{\text{a.s.}}{=} \emptyset.$$

Hence, by Lemma 2 and Lemma 5,

$$U^{(m)}(C_i \cap C_j) \stackrel{\text{a.s.}}{=} U^{(m)}(C_i) \cap U^{(m)}(C_j) \stackrel{\text{a.s.}}{=} \theta^i (U^{(m)}(C_0) \cap \theta^{j-i} U^{(m)}(C_0)) \stackrel{\text{a.s.}}{=} \emptyset,$$

establishing our claim. It is no restriction of generality to assume that the sets C_0, C_1, \dots, C_{b-1} are actually *disjoint* (rather than *almost disjoint*), since replacing C_i by the equivalent set C'_i , where $C'_0 = C_0$ and $C'_i = C_i \setminus \cup_{j < i} C_j$, $i = 1, \dots, b-1$, does not disturb the validity of (6).

Our next step will be to show that $\{C_0, C_1, \dots, C_{b-1}\}$ constitutes a Blackwell decomposition for $\mathcal{X}^{(m)}$, still assuming the Blackwell component C_0 to be atomic. First note that, by (9), $\cup_{i=0}^{b-1} C_i$ is regular, while

$$\mathbb{P}(U(\cup_{i=0}^{b-1} C_i)) \geq \mathbb{P}(U^{(m)}(\cup_{i=0}^{b-1} C_i)) \geq \mathbb{P}(U^{(m)}(C_0)) > 0.$$

So $\cup_{i=0}^{b-1} C_i$ is in fact almost closed, and it follows, \mathcal{X} being simple and atomic, that $\cup_{i=0}^{b-1} C_i$ and S are equivalent with respect to \mathcal{X} . As a consequence

$$\sum_{i=0}^{b-1} \mathbb{P}(X(mk) \in C_i \text{ for } k \text{ sufficiently large}) = 1.$$

If $b = 1$ then C_0 and S are equivalent with respect to \mathcal{X} , and hence with respect to $\mathcal{X}^{(m)}$, so that $\beta(\mathcal{X}^{(m)}) = 1$, and we are done. So suppose $b > 1$ and let Γ be an arbitrary almost closed subset of C_i , $0 < i < b$. Since $\theta^{b-i} U^{(m)}(\Gamma)$ is invariant with respect to $\mathcal{X}^{(m)}$, there exists, by Theorem 3, a regular set Γ_0 such that $\theta^{b-i} U^{(m)}(\Gamma) \stackrel{\text{a.s.}}{=} U^{(m)}(\Gamma_0)$. Lemma 2 implies that Γ_0 is almost closed, while,

by (9), $U^{(m)}(\Gamma_0) \stackrel{\text{a.s.}}{\subset} U^{(m)}(C_0)$. But since C_0 is atomic, we must actually have $U^{(m)}(\Gamma_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_0)$. Hence, by Lemma 3,

$$U^{(m)}(\Gamma) = \theta^i(\theta^{b-i}U^{(m)}(\Gamma)) \stackrel{\text{a.s.}}{=} \theta^i U^{(m)}(\Gamma_0) \stackrel{\text{a.s.}}{=} \theta^i U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_i),$$

so that Γ and C_i are equivalent. Hence C_i is atomic. So we conclude that if C_0 is atomic then $\{C_0, C_1, \dots, C_{b-1}\}$ constitutes a Blackwell decomposition for $\mathcal{X}^{(m)}$ (with atomic components) and hence $\beta(\mathcal{X}^{(m)}) = b$, a divisor of m .

We will now show that, in fact, each component in the Blackwell decomposition for $\mathcal{X}^{(m)}$ has to be atomic if \mathcal{X} is simple and atomic. If $\beta(\mathcal{X}^{(m)}) > 1$, we could replace C_0 in the preceding argument by an atomic Blackwell component for $\mathcal{X}^{(m)}$, and subsequently reach a contradiction, since all the components in the Blackwell decomposition for $\mathcal{X}^{(m)}$ have to be atomic if C_0 is atomic. So it remains to consider the case $\beta(\mathcal{X}^{(m)}) = 1$. Assuming S to be nonatomic with respect to $\mathcal{X}^{(m)}$, there are almost closed sets that are *not* equivalent to S . Let Γ_0 be such a set. Then, by Theorem 3, there are sets Γ_i , regular with respect to $\mathcal{X}^{(m)}$ and unique up to equivalence, such that

$$\theta^i U^{(m)}(\Gamma_0) \stackrel{\text{a.s.}}{=} U^{(m)}(\Gamma_i), \quad i = 1, 2, \dots$$

Copying the argument following (6) up to and including (9) with C_i replaced by Γ_i , we conclude from the analogue of (9) that $\cup_{i=0}^{b-1} \Gamma_i$ is regular with respect to \mathcal{X} , while

$$\mathbb{P}(U(\cup_{i=0}^{b-1} \Gamma_i)) \geq \mathbb{P}(U^{(m)}(\cup_{i=0}^{b-1} \Gamma_i)) \geq \mathbb{P}(U^{(m)}(\Gamma_0)) > 0.$$

So $\cup_{i=0}^{b-1} \Gamma_i$ is in fact almost closed, and it follows, \mathcal{X} being simple and atomic, that $\cup_{i=0}^{b-1} \Gamma_i$ and S are equivalent with respect to \mathcal{X} .

It is no restriction of generality to assume that the sets $\Gamma_0, \Gamma_1, \dots, \Gamma_{b-1}$ are disjoint. Indeed, $\Gamma_0 \setminus \Gamma_i$ cannot be transient, by the same argument we have used earlier for $C_0 \setminus C_i$. Hence, the collection of regular sets constituting a field, $\Gamma_0 \setminus \Gamma_i$ must be almost closed with respect to $\mathcal{X}^{(m)}$. So, if $\Gamma_0 \cap \Gamma_i$ is *not* transient, we may replace Γ_0 by $\Gamma_0 \setminus \Gamma_i$ in the preceding argument and end up with new sets $\Gamma_0, \Gamma_1, \dots, \Gamma_{b-1}$ such that $\Gamma_0 \cap \Gamma_i$ is transient. Repeating the procedure if necessary, we reach, after less than b steps, a situation in which $\Gamma_0 \cap \Gamma_i$ is transient for each $i < b$. It follows, by the same argument we have used before for the C_i 's, that *all* Γ_i 's are almost disjoint and by a similar adaptation as before for the C_i 's we can actually make them disjoint without essentially changing the situation. But if $\Gamma_0, \Gamma_1, \dots, \Gamma_{b-1}$ are disjoint almost closed sets such that the analogue of (9) is satisfied, and $\cup_{i=0}^{b-1} \Gamma_i$ and S are equivalent with respect to \mathcal{X} , then $\{\Gamma_0, \Gamma_1, \dots, \Gamma_{b-1}\}$ constitutes a Blackwell decomposition for $\mathcal{X}^{(m)}$, which, since $\beta(\mathcal{X}^{(m)}) = 1$, implies $b = 1$, and hence that Γ_0 and S are equivalent, contradicting our assumption on Γ_0 . So if \mathcal{X} is simple and atomic and $\beta(\mathcal{X}^{(m)}) = 1$, then S has to be atomic. Summarizing we conclude that every component in the Blackwell decomposition of S for $\mathcal{X}^{(m)}$ must be atomic if \mathcal{X} is simple and atomic.

Finally, suppose \mathcal{X} is simple and nonatomic. Evidently, each subset of S that is almost closed with respect to \mathcal{X} contains a subset that is almost closed with respect to $\mathcal{X}^{(m)}$, and it follows that a nonatomic almost closed set with respect to \mathcal{X} must contain a nonatomic almost closed set with respect to $\mathcal{X}^{(m)}$. So S must contain a nonatomic almost closed set with respect to $\mathcal{X}^{(m)}$. We have seen that all components in the Blackwell decomposition of S for $\mathcal{X}^{(m)}$ must be atomic if $\beta(\mathcal{X}^{(m)}) > 1$, so the only remaining possibility is that $\mathcal{X}^{(m)}$ is simple and nonatomic. \square

Proposition 1 provides the framework for the formal definition of the asymptotic period of a simple Markov chain in the next section. We conclude this section with a series of lemmas and corollaries, which supply further information on $\beta(\mathcal{X}^{(m)})$. In what follows we will refer to a Blackwell decomposition of S for $\mathcal{X}^{(m)}$ satisfying (5) as a *cyclic* decomposition.

Lemma 6. *Let \mathcal{X} be simple and atomic, and $m \geq 1$. Then a Blackwell component for $\mathcal{X}^{(m)}$ is almost closed with respect to $\mathcal{X}^{(k\beta)}$ for all $k \geq 1$, where $\beta \equiv \beta(\mathcal{X}^{(m)})$. Also, $\beta(\mathcal{X}^{(\beta)}) = \beta$.*

Proof. Let C be a Blackwell component for $\mathcal{X}^{(m)}$. As a consequence of (5) we have $U^{(\beta)}(C) \stackrel{\text{a.s.}}{=} L^{(\beta)}(C)$, and hence $U^{(k\beta)}(C) \stackrel{\text{a.s.}}{=} L^{(k\beta)}(C)$ for any $k \geq 1$. Also,

$$\mathbb{P}(L^{(k\beta)}(C)) \geq \mathbb{P}(L^{(\beta)}(C)) = \mathbb{P}(U^{(\beta)}(C)) \geq \mathbb{P}(U^{(m)}(C)) > 0,$$

since β is a divisor of m . So we conclude that C is almost closed with respect to $\mathcal{X}^{(k\beta)}$. It follows in particular that a Blackwell component for $\mathcal{X}^{(m)}$ must contain a Blackwell component for $\mathcal{X}^{(\beta)}$. Hence $\beta(\mathcal{X}^{(\beta)}) \geq \beta$, and so $\beta(\mathcal{X}^{(\beta)}) = \beta$, since $\beta(\mathcal{X}^{(\beta)})$ is a divisor of β . \square

The following corollary is immediate.

Corollary 1. *Let \mathcal{X} be simple. If $\beta(\mathcal{X}^{(m)}) < m$ for all $m > 1$, then $\beta(\mathcal{X}^{(m)}) = 1$ for all m .*

Lemma 7. *Let \mathcal{X} be simple, $k \geq 1$ and $\ell \geq 1$. Then $\beta(\mathcal{X}^{(k\ell)}) = \kappa\beta(\mathcal{X}^{(\ell)})$, where κ is a divisor of $\beta(\mathcal{X}^{(k)})$.*

Proof. If \mathcal{X} is nonatomic then, by Proposition 1, $\beta(\mathcal{X}^{(m)}) = 1$ for all m , so that the statement is trivially true. So let us assume that \mathcal{X} is simple and atomic. We write $\beta_\ell \equiv \beta(\mathcal{X}^{(\ell)})$, and denote the (atomic) Blackwell components for $\mathcal{X}^{(\ell)}$ by $B_0, B_1, \dots, B_{\beta_\ell}$. By the previous lemma these sets are almost closed with respect to $\mathcal{X}^{(k\ell)}$, so each B_i must contain at least one Blackwell component for $\mathcal{X}^{(k\ell)}$. Let $C_0 \subset B_0$ be such a Blackwell component and consider the sets C_i defined in the proof of Proposition 1 in terms of C_0 and $m = k\ell$. We let

$$\kappa := \min\{k \geq 1 \mid \theta^{k\beta_\ell} U^{(k\ell)}(C_0) \stackrel{\text{a.s.}}{=} U^{(k\ell)}(C_0)\},$$

and claim that $\kappa\beta_\ell = \beta(\mathcal{X}^{(k\ell)})$.

To prove the claim we first note that part of the proof of Proposition 1 can be copied to show that the sets $C_0, C_{\beta_\ell}, \dots, C_{(\kappa-1)\beta_\ell}$ are almost disjoint, while, for $i \geq \kappa$, the sets $C_{i\beta_\ell}$ and $C_{(i-\kappa)\beta_\ell}$ are equivalent with respect to $\mathcal{X}^{(m)}$. Since B_0 is a Blackwell component for $\mathcal{X}^{(\ell)}$ and $C_0 \subset B_0$, we have $\cup_{i=0}^{\kappa-1} C_{i\beta_\ell} \subset B_0$. But, again in analogy with part of the proof of Proposition 1, it is easily seen that $\cup_{i=0}^{\kappa-1} C_{i\beta_\ell}$ is almost closed with respect to $\mathcal{X}^{(\ell)}$, so, B_0 being atomic, we actually have $\cup_{i=0}^{\kappa-1} C_{i\beta_\ell} \stackrel{\text{a.s.}}{=} B_0$. As in the proof of Proposition 1 it is no restriction to assume that the sets $C_0, C_{\beta_\ell}, \dots, C_{(\kappa-1)\beta_\ell}$ are disjoint rather than almost disjoint.

Assuming that the Blackwell components for $\mathcal{X}^{(\ell)}$ are suitably numbered, we have $C_1 \subset B_1$ and the preceding argument can be repeated to show that the sets $C_1, C_{\beta_\ell+1}, \dots, C_{(\kappa-1)\beta_\ell+1}$ are disjoint, while $\cup_{i=0}^{\kappa-1} C_{i\beta_\ell+1} \stackrel{\text{a.s.}}{=} B_1$. Thus proceeding it follows eventually that $\{C_0, C_1, \dots, C_{\kappa\beta_\ell-1}\}$ constitutes a Blackwell decomposition of S for $\mathcal{X}^{(m)}$, so that $\beta(\mathcal{X}^{(m)}) = \kappa\beta_\ell$, as claimed.

We finally observe that the κ sets $\cup_{i=0}^{\beta_\ell-1} C_{i\kappa+j}$, $j = 0, 1, \dots, \kappa - 1$, are almost closed with respect to $\mathcal{X}^{(k)}$, so that κ must be a divisor of $\beta(\mathcal{X}^{(k)})$. \square

This lemma has some interesting and useful corollaries, of which the first is immediate.

Corollary 2. *Let \mathcal{X} be simple and $m > 1$. If ℓ is a divisor of m , then $\beta(\mathcal{X}^{(\ell)})$ is a divisor of $\beta(\mathcal{X}^{(m)})$.*

Corollary 3. *Let \mathcal{X} be simple and $m > 1$. If $\beta(\mathcal{X}^{(m)}) = m$, then $\beta(\mathcal{X}^{(\ell)}) = \ell$ for all divisors ℓ of m .*

Proof. Let $m = k\ell$. Then, by Lemma 7,

$$\beta(\mathcal{X}^{(m)}) = k\ell = \kappa\beta(\mathcal{X}^{(\ell)}),$$

with κ a divisor of $\beta(\mathcal{X}^{(k)})$, and, hence, by Proposition 1, of k . Since $\beta(\mathcal{X}^{(\ell)})$ is a divisor of ℓ we must have $\kappa = k$ and $\beta(\mathcal{X}^{(\ell)}) = \ell$. \square

Corollary 4. *Let \mathcal{X} be simple, $k \geq 1$ and $\ell \geq 1$. Then $\beta(\mathcal{X}^{(k\ell)}) = \beta(\mathcal{X}^{(k)})\beta(\mathcal{X}^{(\ell)})$ if $\beta(\mathcal{X}^{(k)})$ and $\beta(\mathcal{X}^{(\ell)})$ are relatively prime.*

Proof. By Lemma 7 we have $\beta(\mathcal{X}^{(k\ell)}) = \kappa\beta(\mathcal{X}^{(\ell)}) = \lambda\beta(\mathcal{X}^{(k)})$, with κ a divisor of $\beta(\mathcal{X}^{(k)})$ and λ a divisor of $\beta(\mathcal{X}^{(\ell)})$. But if $\beta(\mathcal{X}^{(k)})$ and $\beta(\mathcal{X}^{(\ell)})$ are relatively prime this is possible only if $\kappa = \beta(\mathcal{X}^{(k)})$ and $\lambda = \beta(\mathcal{X}^{(\ell)})$. \square

3 Asymptotic period

We are now ready to formally define the *asymptotic period* of a *simple* Markov chain. As in the previous section, \mathcal{X} denotes the Markov chain of the Introduction, and is, accordingly, stochastic, irreducible, and aperiodic.

Definition Let the Markov chain \mathcal{X} be simple. The *asymptotic period* of \mathcal{X} is given by

$$d(\mathcal{X}) := \sup\{m \geq 1 \mid \beta(\mathcal{X}^{(m)}) = m\}; \quad (10)$$

\mathcal{X} is *asymptotically aperiodic* if $d(\mathcal{X}) = 1$, otherwise \mathcal{X} is *asymptotically periodic* with asymptotic period $d(\mathcal{X}) > 1$.

If, for some m , we would have $\beta \equiv \beta(\mathcal{X}^{(m)}) > d(\mathcal{X})$, then, by Lemma 6, $\beta(\mathcal{X}^{(\beta)}) = \beta > d(\mathcal{X})$, which is a contradiction. So we actually have the following result, which formalizes the intuitive concept of asymptotic period put forward in the Introduction.

Theorem 4. *The asymptotic period of a simple Markov chain \mathcal{X} satisfies*

$$d(\mathcal{X}) = \sup\{\beta(\mathcal{X}^{(m)}) \mid m \geq 1\}. \quad (11)$$

From Proposition 1 we immediately conclude the following.

Theorem 5. *If \mathcal{X} is simple and nonatomic then \mathcal{X} is asymptotically aperiodic.*

It is not difficult to see that the aperiodic chain \mathcal{X} is also asymptotically aperiodic if it is recurrent, so the new concepts are relevant in particular for transient Markov chains. An example of a chain with an asymptotic period greater than 1 is obtained by letting \mathcal{X} be a transient birth-death process (as defined in the Introduction) with self-transition probabilities $r_i = 0$ except $r_0 = 1 - p_0 > 0$. Clearly, \mathcal{X} is irreducible and aperiodic, while Lemma 1 implies that \mathcal{X} is simple (and atomic). But it is readily seen that $\beta(\mathcal{X}^{(2)}) = 2$, so that $d(\mathcal{X}) > 1$. (We will see in the next section that, actually, $d(\mathcal{X}) = 2$.)

It is possible for the asymptotic period of a Markov chain to be infinity. Indeed, let us assume that the birth probabilities p_i in a birth-death process are such that $\prod_{i=0}^{\infty} p_i > 0$. Then there is a probability $\prod_{i=j}^{\infty} p_i \geq \prod_{i=0}^{\infty} p_i$ that a visit to state j is followed solely by jumps to the right. Hence, with probability one, the process will make only a finite number of self-transitions or jumps to the left. It follows that the sets $C_i := \{i, n + i, 2n + i, \dots\}$, $i = 0, 1, \dots, n - 1$, are (disjoint) atomic almost closed sets of $\mathcal{X}^{(n)}$, so that $\beta(\mathcal{X}^{(n)}) = n$ for all n and, hence, $d(\mathcal{X}) = \infty$.

Some further conditions for a simple Markov chain to be asymptotically aperiodic are given next.

Theorem 6. *Let \mathcal{X} be a simple Markov chain. Then the following are equivalent:*

- (i) \mathcal{X} is asymptotically aperiodic;
- (ii) $\mathcal{X}^{(m)}$ is simple for all $m > 1$;
- (iii) $\mathcal{X}^{(m)}$ is simple for all prime numbers m .

Proof. By Corollary 1 the first statement implies the second. Evidently, the second statement implies the third. To show that the third statement implies the first, suppose $\beta(\mathcal{X}^{(m)}) = 1$ for all primes m . If $d \equiv d(\mathcal{X}) > 1$, then $\beta(\mathcal{X}^{(d)}) = d$ and d must have a prime factor $p > 1$. But then, by Corollary 4, $\beta(\mathcal{X}^{(p)}) = p$, which is impossible. \square

It may be desirable to have an upper bound on the asymptotic period of a Markov chain. The next theorem provides a criterion which may be used for this purpose.

Theorem 7. *If the simple Markov chain \mathcal{X} is such that for some $n \in \mathbb{N}$*

$$\begin{aligned} & \text{there exists a constant } \delta > 0 \text{ such that } P^{(n)}(i, i) \geq \delta \text{ for all but} \\ & \text{finitely many states } i \in S, \end{aligned} \tag{12}$$

then $d(\mathcal{X})$ is a divisor of n .

Proof. Suppose that $\beta(\mathcal{X}^{(m)}) = m$ for some $m \geq 1$, and let $\{C_0, C_1, \dots, C_{m-1}\}$ be a cyclic Blackwell decomposition for $\mathcal{X}^{(m)}$. We then have, for all n ,

$$\mathbb{P}([X(k) \in C_0 \Rightarrow X(k+n) \in C_{n \pmod{m}}] \text{ for } k \text{ sufficiently large}) = 1.$$

But, if n is such that (12) is satisfied, this is possible only if $C_{n \pmod{m}} = C_0$, that is, if m is a divisor of n . The result follows by definition of $d(\mathcal{X})$. \square

As an example, consider the birth-death process again. If there exists a $\delta > 0$ such that $r_i > \delta$ for all but finitely many states i , then condition (12) is satisfied for $n = 1$, so that the process must be asymptotically aperiodic. If the birth, death and self-transition probabilities are such that $P^{(2)}(i, i) = r_i^2 + p_i q_{i+1} + q_i p_{i-1} > \delta$ for some $\delta > 0$ and all but finitely many states i , then either the process is asymptotically aperiodic, or it has asymptotic period two.

Theorem 7 has some interesting consequences.

Corollary 5. *If, for some n , the simple Markov chain \mathcal{X} satisfies condition (12) while $\mathcal{X}^{(n)}$ is simple, then \mathcal{X} is asymptotically aperiodic.*

Proof. If \mathcal{X} satisfies (12), then, by Theorem 7, $d \equiv d(\mathcal{X})$ is a divisor of n , so that, by Corollary 2, $\beta(\mathcal{X}^{(d)})$ is a divisor of $\beta(\mathcal{X}^{(n)})$. Hence we must have $d = \beta(\mathcal{X}^{(d)}) = 1$ if $\beta(\mathcal{X}^{(n)}) = 1$. \square

A somewhat subtler criterion, relevant for an application we will discuss in Section 5, is the following.

Corollary 6. *If the simple Markov chain \mathcal{X} is such that*

$$\begin{aligned} & \text{there exists a constant } n_0, \text{ and for every } n > n_0 \text{ there exist} \\ & \text{an integer } m \equiv m(n) \text{ and a constant } \delta \equiv \delta(n, m) > 0, \text{ such} \\ & \text{that } P^{(n+m)}(i, j) \geq \delta P^{(m)}(i, j) \text{ for all } i, j \in S, \end{aligned} \tag{13}$$

then \mathcal{X} is asymptotically aperiodic.

Proof. If \mathcal{X} satisfies (13), then there exist two values of n satisfying (12) that are relatively prime. Since, by Theorem 7, $d(\mathcal{X})$ is a divisor of both values we must have $d(\mathcal{X}) = 1$. \square

4 Birth-death processes

Let \mathcal{X} be a stochastic and irreducible birth-death process (on the nonnegative integers) with at least one positive self-transition probability, so that \mathcal{X} is aperiodic. As usual P denotes the matrix of one-step transition probabilities of \mathcal{X} , and we use the notation (4). Note that, by Lemma 1, \mathcal{X} is simple and atomic.

Theorem 8. *The asymptotic period $d(\mathcal{X})$ of the birth-death process \mathcal{X} equals 1, 2, or ∞ . Moreover $d(\mathcal{X}) = \infty$ if and only if $\prod_{i=0}^{\infty} p_i > 0$.*

Proof. Suppose $2 < d \equiv d(\mathcal{X}) < \infty$, and let C_0, C_1, \dots, C_{d-1} be a cyclic Blackwell decomposition of S for $\mathcal{X}^{(d)}$.

By (5) we have, for $\ell = 0, 1, \dots$ and k sufficiently large, $X(k + \ell) \in C_{\ell \pmod{d}}$ if $X(k) \in C_0$, and in particular $X(k + 1) \in C_1$. Since C_0 and C_1 are disjoint, $X(k + 1) = X(k)$ is impossible, but also $X(k + 1) = X(k) - 1$ leads to a contradiction. Indeed, if $X(k) \in C_0$ and $X(k + 1) = X(k) - 1 \in C_1$ then $X(k + 2) \in C_2$ and hence $X(k + 2) = X(k) - 2$, since the other options would contradict the fact that C_0, C_1 and C_2 are disjoint. Thus continuing we eventually find that $X(k + X(k) - 1) = 1 \in C_{X(k)-1 \pmod{d}}$ and $X(k + X(k)) = 0 \in C_{X(k) \pmod{d}}$. But this would imply $X(k + X(k) + 1) = 0$ or $X(k + X(k) + 1) = 1$, which is impossible since $C_{X(k)-1 \pmod{d}}, C_{X(k) \pmod{d}}$ and $C_{X(k)+1 \pmod{d}}$ are disjoint.

So, assuming k sufficiently large and $X(k) \in C_0$, we must have $X(k + 1) = X(k) + 1 \in C_1$. Repeating the argument leads to the conclusion that for k sufficiently large, $X(k) \in C_0$ implies $X(k + \ell) = s + \ell \in C_{\ell \pmod{d}}$ for all $\ell = 0, 1, \dots$. We conclude that in the long run \mathcal{X} will solely make jumps to the right, that is, the number of self-transitions or jumps to the left will be finite. But then, as we have observed in Section 3, $\beta(\mathcal{X}^{(n)}) = n$ for all n , since the sets $C'_i := \{i, n + i, 2n + i, \dots\}$, $i = 0, 1, \dots, n - 1$, are (disjoint) atomic almost closed sets for $\mathcal{X}^{(n)}$. Hence $d(\mathcal{X}) = \infty$, contradicting our assumption $d(\mathcal{X}) < \infty$.

In Section 3 we showed already that $d(\mathcal{X}) = \infty$ if $\prod_{i=0}^{\infty} p_i > 0$, so it remains to prove the converse. So let $d(\mathcal{X}) = \infty$ and $d > 2$ be such that $\beta(\mathcal{X}^{(d)}) = d$. The argument used to prove the first part of the theorem can be copied to conclude that, with probability one, \mathcal{X} will, in the long run, solely make jumps to the right, but this obviously implies $\prod_{i=0}^{\infty} p_i > 0$. \square

So if \mathcal{X} is asymptotically aperiodic, then $\beta(\mathcal{X}^{(2)}) < 2$, and hence $\beta(\mathcal{X}^{(2)}) = 1$, that is, $\mathcal{X}^{(2)}$ is simple. On the other hand, if \mathcal{X} is *not* asymptotically aperiodic then $d(\mathcal{X}) = 2$ or $d(\mathcal{X}) = \infty$, which both imply $\beta(\mathcal{X}^{(2)}) = 2$, that is, $\mathcal{X}^{(2)}$ is *not* simple. So we have the following corollary.

Corollary 7. *The birth-death process \mathcal{X} is asymptotically aperiodic if and only if $\mathcal{X}^{(2)}$ is simple.*

Note that, by Lemma 1 again, $\mathcal{X}^{(2)}$ will be atomic if it is simple.

To obtain a necessary and sufficient condition for \mathcal{X} to be asymptotically aperiodic directly in terms of the transition probabilities we define the polynomials Q_i by the recurrence relation

$$\begin{aligned} p_i Q_{i+1}(x) &= (x - r_i)Q_i(x) - q_i Q_{i-1}(x), \quad i > 0, \\ p_0 Q_1(x) &= x - r_0, \quad Q_0(x) = 1, \end{aligned} \tag{14}$$

or equivalently, writing $Q(x) := (Q_0(x), Q_1(x), \dots)^T$ (where superscript T denotes transposition), by

$$PQ(x) = xQ(x). \tag{15}$$

Observe that $Q_i(1) = 1$ for all i , while an x -harmonic function f for \mathcal{X} has to satisfy $f(i) = cQ_i(x)$, $i \geq 0$, for some constant c . Since

$$P^2 Q(x) = x^2 Q(x), \tag{16}$$

the vectors $Q(1)$ and $Q(-1)$ are two distinct solutions of the system of equations

$$P^2 y = y. \tag{17}$$

Moreover, P^2 being a pentadiagonal matrix, any solution to (17) must be a linear combination of $Q(1)$ and $Q(-1)$. It follows that the constant function is the only bounded nonnegative 1-harmonic function for P^2 if and only if $|Q_i(-1)|$ is unbounded. Since $|Q_i(-1)|$ is increasing (see Karlin and McGregor [8, p. 76]), Theorem 2 leads to the following result.

Corollary 8. *The birth-death process \mathcal{X} is asymptotically aperiodic if and only if $|Q_i(-1)| \rightarrow \infty$ as $i \rightarrow \infty$.*

5 Strong ratio limit property

5.1 Introduction

We return to the general setting of the Introduction and recall that the Markov chain \mathcal{X} with transition matrix P has the strong ratio limit property (SRLP) if and only if there exist positive constants R , $\mu(i)$, $i \in S$, and $f(i)$, $i \in S$, satisfying (1), or, equivalently, satisfying both

$$\lim_{n \rightarrow \infty} \frac{P^{(n+1)}(i, j)}{P^{(n)}(i, j)} = \frac{1}{R}, \quad i, j \in S, \tag{18}$$

and

$$\lim_{n \rightarrow \infty} \frac{P^{(n)}(i, j)}{P^{(n)}(k, l)} = \frac{f(i)\mu(j)}{f(k)\mu(l)}, \quad i, j, k, l \in S. \quad (19)$$

Kingman [12] established the classical result that there exists a real number $\rho \equiv \rho(P)$ such that $0 < \rho \leq 1$ and

$$\lim_{n \rightarrow \infty} (P^{(n)}(i, j))^{1/n} = \rho, \quad i, j \in S. \quad (20)$$

It follows that the limits in (18), if they exist, must be equal to ρ , so that

$$R = \frac{1}{\rho}. \quad (21)$$

Evidently, $R = \rho = 1$ if \mathcal{X} is positive recurrent. Kendall [9] has shown that the same conclusion can be drawn if \mathcal{X} is null-recurrent.

We note that existence of the limits in (19) would be sufficient for the existence of the limits in (18) if the interchange of limit and summation in

$$\lim_{n \rightarrow \infty} \sum_{k \in S} \frac{P^{(n)}(i, k)}{P^{(n)}(i, j)} P(k, j) \quad \text{or} \quad \lim_{n \rightarrow \infty} \sum_{k \in S} P(i, k) \frac{P^{(n)}(k, j)}{P^{(n)}(i, j)}$$

would be allowed, which is not true a priori. Evidently, if \mathcal{X} is a Markov chain on the nonnegative integers with uniformly bounded jumps – for example a birth-death process – then the interchange is justified, so to prove the SRLP in this case it suffices to establish (19).

As announced in the Introduction we will show in the next subsection that the necessary and sufficient condition for asymptotic aperiodicity of a birth-death process established in Section 4, leads to a sufficient condition for a birth-death process to have the SRLP. In the Subsections 5.3 and 5.4 this result is shown to be suggestive of a sufficient condition for the SRLP in our general setting, which, however, is not correct. A necessary condition for the SRLP, which, when applied in a birth-death setting, amounts to the sufficient condition of the next subsection being also necessary, is posed as a conjecture.

5.2 Birth-death processes

If \mathcal{X} is a birth-death process on the nonnegative integers with $\rho = 1$, then, by [4, Theorems 3.1 and 3.2], the limits (19) exist – and hence the SRLP prevails – if $|Q_i(-1)| \rightarrow \infty$ as $i \rightarrow \infty$. Since it was shown in [4] (and claimed already by Papangelou [16]) that a birth-death process possesses the SRLP if and only if

$$\lim_{n \rightarrow \infty} \frac{P^{(n+1)}(0, 0)}{P^{(n)}(0, 0)} \quad (22)$$

exists (in which case the limit must be ρ), we conclude from Corollary 8 that a birth-death process with $\rho = 1$ has the SRLP if it is asymptotically aperiodic. The constants $f(i)$ and $\mu(i)$ of (19) are in this case given by

$$f(i) = c_1 \quad \text{and} \quad \mu(i) = c_2 \pi_i, \quad i \geq 0, \quad (23)$$

for some positive constants c_1 and c_2 , and

$$\pi_0 := 1, \quad \pi_i := \frac{p_0 p_1 \cdots p_{i-1}}{q_1 q_2 \cdots q_i}, \quad i > 0. \quad (24)$$

More generally, if P corresponds to a birth-death process with an unspecified value of ρ , we can link occurrence of the SRLP to asymptotic aperiodicity of an associated birth-death process. Indeed, defining $\tilde{q}_0 = 0$ and

$$\tilde{p}_i := \frac{Q_{i+1}(\rho) p_i}{Q_i(\rho) \rho}, \quad \tilde{r}_i := \frac{r_i}{\rho}, \quad \tilde{q}_{i+1} := \frac{Q_i(\rho) q_{i+1}}{Q_{i+1}(\rho) \rho}, \quad i \geq 0, \quad (25)$$

the parameters \tilde{p}_i , \tilde{q}_i and \tilde{r}_i may be interpreted as the birth, death, and self-transition probabilities of an irreducible, stochastic, and aperiodic birth-death process $\tilde{\mathcal{X}}$ with one-step transition matrix \tilde{P} and $\rho(\tilde{P}) = 1$. (A more general result is Lemma 8 in Subsection 5.4.) Moreover, defining the polynomials \tilde{Q}_i in analogy with the polynomials Q_i of (14), it follows readily that

$$\tilde{Q}_i(x) = \frac{Q_i(\rho x)}{Q_i(\rho)}, \quad i \geq 0, \quad x \in \mathbb{R}. \quad (26)$$

So, by Corollary 8, $\tilde{\mathcal{X}}$ is asymptotically aperiodic if and only if $|Q_i(-\rho)/Q_i(\rho)| \rightarrow \infty$ as $i \rightarrow \infty$. Subsequently applying [4, Theorems 3.1 and 3.2] again we obtain the following theorem.

Theorem 9. *Let \mathcal{X} be a birth-death process and $\tilde{\mathcal{X}}$ the associated birth-death process defined by the probabilities (25). If $\tilde{\mathcal{X}}$ is asymptotically aperiodic then \mathcal{X} possesses the SRLP.*

We note that validity of [4, Conjecture 3.1] would imply validity of the converse implication in Theorem 9, but the conjecture is still open. It is encompassed by the conjecture posed at the end of Subsection 5.4.

Generalizing (23), the numbers $f(i)$ and $\mu(i)$ of (19) are now given by

$$f(i) = c_1 Q_i(\rho) \quad \text{and} \quad \mu(i) = c_2 \pi_i Q_i(\rho), \quad i \geq 0, \quad (27)$$

for some positive constants c_1 and c_2 . As observed in Section 4, f defined by (27) constitutes a ρ -harmonic function on the nonnegative integers. Moreover, it is easy to see that the $\mu(i)$ defined by (27) determine a ρ -invariant measure μ on the nonnegative integers. It should be noted that for a birth-death process a ρ -harmonic function and ρ -invariant measure always exist and are unique up to multiplicative constants, as a consequence of the tridiagonal structure of P .

5.3 The general setting: preliminaries

Now turning to the SRLP in our general setting we assume, as usual, that the Markov chain \mathcal{X} on S , with matrix P of one-step transition probabilities, is stochastic, irreducible and aperiodic. We start off by mentioning two important results from the literature. First, Kesten [10] has shown that the existence of the limits in (18) is assured if

$$\begin{aligned} &\text{there exists a constant } n_0, \text{ and for every } n > n_0 \text{ there exists} \\ &\text{a constant } \delta \equiv \delta(n) > 0 \text{ such that } P^{(n)}(i, i) \geq \delta \text{ for all } i \in S. \end{aligned} \quad (28)$$

(The present formulation is taken from the proof of Kesten's Lemma 4, where it is shown to be equivalent to Kesten's condition (1.5).) Secondly, assuming (28), Handelman [6] has shown (actually allowing P to be any irreducible nonnegative matrix) that the limits in (19) exist if and only if there exist (up to multiplicative constants) a unique nonnegative ρ -invariant measure μ and a unique nonnegative ρ -harmonic function f for P , in which case the limits are determined by μ and f as in (19). Handelman [6, p. 105] remarks that his conclusions would remain valid under an assumption weaker than (28) – and equivalent to (13) – if this assumption would guarantee the existence of the limits in (18) (which he conjectures to be true). One could surmise that the existence of the limits in (18) per se would be enough for Handelman's conclusions, but this is not generally true. Papangelou [16] gives examples of transition matrices P , not only satisfying (18) but having the full SRLP, such that the quantities $\mu(i)$ appearing in (19) fail to satisfy one of the equations (2).

Assuming (28), Kesten [10] has obtained sufficient conditions for the existence of a unique nonnegative ρ -invariant measure and ρ -harmonic function for P , and hence, by Handelman's results, for P to possess the SRLP. In the next subsection 5.4 we take the opposite position by assuming that there exist a unique nonnegative ρ -invariant measure and ρ -harmonic function for P , and investigating under which additional conditions P possesses the SRLP. It is a challenge in particular to weaken Kesten's condition (28) for the existence of the limits in (18).

We continue with some further information on circumstances under which the SRLP is known to prevail, but first recall (see Vere-Jones [19] and, for a comprehensive generalisation, Vere-Jones [20]) that the power series

$$P_{ij}(z) := \sum_{n=0}^{\infty} P^{(n)}(i, j) z^n, \quad i, j \in S, \quad (29)$$

have a common radius of convergence R , and converge or diverge together. Evidently, R can be identified with $R = \rho^{-1}$ in (18) if the limit exists. If $P_{ij}(R) < \infty$, then P (and \mathcal{X}) is called *R-transient*, while it is called *R-recurrent* otherwise. If P is *R-recurrent* then either $\lim_{n \rightarrow \infty} R^n P^{(n)}(i, j) = 0$ for all $i, j \in S$, or

$\lim_{n \rightarrow \infty} R^n P^{(n)}(i, j) > 0$ for all $i, j \in S$. P (and \mathcal{X}) is said to be R -null in the former case and R -positive in the latter. Interestingly, by [19, Theorem II] and [20, Theorem 4.1], R -recurrence of P implies the existence of a unique non-negative ρ -invariant measure and ρ -harmonic function (up to a multiplicative factor).

Pruitt [17] has shown that, if \mathcal{X} is R -recurrent, the existence of the limits in (18) is necessary and sufficient for the SRLP. Actually, Pruitt's necessary and sufficient condition for the SRLP if \mathcal{X} is R -recurrent is somewhat weaker than (18), namely,

$$\limsup_{n \rightarrow \infty} \frac{P^{((n+1)m)}(i, i)}{P^{(nm)}(i, i)} \leq R^{-m} \quad \text{for some } m \in \mathbb{N} \text{ and } i \in S. \quad (30)$$

Pruitt [17] also shows that (30) is satisfied if the Markov chain is *symmetrizable* (called *reversible* by Pruitt), that is, if there are positive numbers $r(i)$, $i \in S$, such that

$$r(i)P(i, j) = r(j)P(j, i), \quad i, j \in S.$$

Note that a birth-death is always symmetrizable, since we can choose $r(i) = \pi_i$ the constants defined in (24). We conclude that R -recurrence of \mathcal{X} implies the SRLP when \mathcal{X} is also symmetrizable, so in particular when \mathcal{X} is a birth-death process. It may be shown, however, that R -recurrence is not necessary for a birth-death process to have the SRLP. (Research into the SRLP in an R -transient setting was initiated by Kijima [11].)

Remark. The fact that an R -recurrent birth-death process \mathcal{X} possesses the SRLP may also be established by observing that the associated process $\tilde{\mathcal{X}}$ defined in Subsection 5.2 must be asymptotically aperiodic, and applying Theorem 9. Indeed, it is easily seen that $\tilde{\mathcal{X}}$ is recurrent if (and only if) \mathcal{X} is R -recurrent, while we know that a recurrent process must be asymptotically aperiodic.

If \mathcal{X} is R -recurrent another sufficient condition for (30) – and consequently for the SRLP – is

$$\text{for some } m \in \mathbb{N} \text{ there is a } \delta > 0 \text{ such that } P^{(m)}(i, i) > \delta \text{ for all } i \in S. \quad (31)$$

This result may be obtained by generalizing a result on recurrent chains of Kingman and Orey [13, Theorem 1.1] (see also Freedman [5, Section 2.6]) to R -recurrent chains. (Since R -recurrence of P implies the existence of a unique nonnegative ρ -invariant measure and ρ -harmonic function, application of Kingman and Orey's result to one of the transformed Markov chains (32) of the next subsection readily leads to the required conclusion.) Note that (31) is weaker than Kesten's condition (28), but implies the SRLP if one additionally assumes R -recurrence.

We recall that we would like to replace Kesten's condition (28) by a weaker condition such that Handelman's conclusion – the SRLP holds if and only if a unique nonnegative ρ -invariant measure and ρ -harmonic function exist – remains valid under the weaker condition. But also we do not want to assume R -recurrence, which is sufficient but not necessary for the existence of a unique nonnegative ρ -invariant measure and ρ -harmonic function. While realization of these ambitions proved feasible in the setting of birth-death processes, our more general setting defies a similar approach, as we will show next.

5.4 The general setting: results and a conjecture

From now on we assume that \mathcal{X} has a unique nonnegative ρ -invariant measure μ and ρ -harmonic function f . It can easily be seen that under our irreducibility condition μ and f are in fact positive componentwise.

Our conjecture will involve Markov chains that are defined in terms of P , μ and f . Namely, with μ_D and f_D denoting the diagonal matrices

$$\mu_D := \text{diag}(\mu(i), i \in S) \quad \text{and} \quad f_D := \text{diag}(f(i), i \in S),$$

we define the matrices

$$P_\mu := \frac{1}{\rho} \mu_D^{-1} P^T \mu_D \quad \text{and} \quad P_f := \frac{1}{\rho} f_D^{-1} P f_D, \quad (32)$$

where a superscript T denotes transposition. It is easy to see that P_μ and P_f are nonnegative and stochastic, so that they can be interpreted as matrices of one-step transition probabilities of Markov chains \mathcal{X}_μ and \mathcal{X}_f , respectively. The n -step transition probabilities of these Markov chains are related to those of the original chain \mathcal{X} as

$$P_\mu^{(n)}(i, j) = \frac{1}{\rho^n} \frac{\mu(j)}{\mu(i)} P^{(n)}(j, i) \quad (33)$$

and

$$P_f^{(n)}(i, j) = \frac{1}{\rho^n} \frac{f(j)}{f(i)} P^{(n)}(i, j), \quad (34)$$

respectively, as can easily be verified. It follows immediately that \mathcal{X}_μ and \mathcal{X}_f are positive recurrent (null-recurrent, transient) if and only if \mathcal{X} is R -positive (R -null, R -transient).

Lemma 8. *The Markov chains \mathcal{X}_μ and \mathcal{X}_f are irreducible, aperiodic, simple and atomic. Moreover,*

$$\rho(P_\mu) = \rho(P_f) = 1. \quad (35)$$

Proof. Irreducibility and aperiodicity of \mathcal{X}_μ and \mathcal{X}_f follow immediately from (33), (34) and the fact that \mathcal{X} is irreducible and aperiodic. Next observe that the system of equations

$$P_\mu g = g,$$

can be rewritten as

$$(\mu_D g)P = \rho \mu_D g.$$

But since μ is the unique nonnegative ρ -invariant measure for P , the function g , if nonnegative, must be constant. Hence, by Theorem 2, \mathcal{X}_μ is simple and atomic. In a similar manner one shows that \mathcal{X}_f is simple and atomic. Finally, the last statement follows immediately from (33), (34) and (20). \square

Note that this result and its proof imply that \mathcal{X}_μ and \mathcal{X}_f have a common, unique 1-harmonic function, namely the constant function. Regarding 1-invariant measures for P_μ and P_f we have the following result.

Lemma 9. *The Markov chains \mathcal{X}_μ and \mathcal{X}_f have a common, unique 1-invariant measure ν , given by*

$$\nu(i) = \mu(i)f(i), \quad i \in S.$$

Proof. Since $\rho(P_\mu) = 1$, the system of equations

$$\nu P_\mu = \rho(P_\mu)\nu,$$

can be rewritten as

$$P(\mu_D^{-1}\nu) = \rho\mu_D^{-1}\nu.$$

But since f is the unique nonnegative ρ -harmonic function for P , we must have $\mu_D^{-1}\nu = f$, that is, $\nu = \mu_D f$. In a similar way one shows that the assumption $\nu P_f = \rho(P_f)\nu$ leads to the same conclusion. \square

Note that the measure ν is finite if and only if \mathcal{X}_μ and \mathcal{X}_f are positive recurrent, which, as we have seen, occurs if and only if \mathcal{X} is R -positive (cf. Vere-Jones [20, Criterion III on p. 375]).

The next lemma concerns the special case in which \mathcal{X} is symmetrizable and enables us to establish a link with our results on birth-death processes. As an aside we note that \mathcal{X} is symmetrizable if and only if \mathcal{X}_μ (or \mathcal{X}_f) is symmetrizable, as can easily be verified from (33) and (34).

Lemma 10. *We have $P_\mu = P_f$ if and only if the Markov chain \mathcal{X} is symmetrizable.*

Proof. If $P_\mu = P_f$ then $\mu_D^{-1}P^T\mu_D = f_D^{-1}Pf_D$, whence $f_D\mu_D^{-1}P^T = Pf_D\mu_D^{-1}$, so that P is symmetrizable with respect to $f(i)/\mu(i)$, $i \in S$.

Conversely, suppose P is symmetrizable with respect to $r(i)$, $i \in S$, and let μ be the (unique) nonnegative ρ -invariant measure for P . Then $r(i)P(i, j) = r(j)P(j, i)$ and $\sum_i \mu(i)P(i, j) = \rho\mu(j)$, so that

$$\sum_{i \in S} \frac{\mu(i)}{r(i)} P(j, i) = \rho \frac{\mu(j)}{r(j)}, \quad j \in S.$$

But since P has a unique nonnegative ρ -harmonic function f , we must have $f(i) = \mu(i)/r(i)$, $i \in S$, that is, with obvious notation, $f_D r_D = \mu_D$. Hence,

$$P_\mu = \frac{1}{\rho} \mu_D^{-1} P^T \mu_D = \frac{1}{\rho} f_D^{-1} r_D^{-1} P^T r_D f_D = \frac{1}{\rho} f_D^{-1} P f_D = P_f.$$

proving our claim. \square

In fact, it is not difficult to see that for a birth-death process $P_\mu = P_f = \tilde{P}$, the one-step transition matrix corresponding to the probabilities (25). This fact and Lemma 8 justify the claims made on \tilde{P} in Subsection 5.2.

It is easy to see that \mathcal{X}_μ and \mathcal{X}_f satisfy Handelman's condition (13) if \mathcal{X} satisfies Handelman's condition. So Corollary 6 immediately gives us the following.

Lemma 11. *If \mathcal{X} satisfies Handelman's condition (13) (or, a fortiori, if it satisfies Kesten's condition (28)), then \mathcal{X}_μ and \mathcal{X}_f are asymptotically aperiodic.*

Considering the preceding results and Theorem 9 on birth-death processes it is tempting to believe that asymptotic periodicity of \mathcal{X}_μ and \mathcal{X}_f is a candidate for replacing Kesten's condition (28), and even Handelman's condition (13), for prevalence of the SRLP given the existence of a unique nonnegative ρ -invariant measure and ρ -harmonic function. But this is contradicted by Dyson's example (given by Chung [3, Section I.10]) of a recurrent, aperiodic Markov chain that does *not* have the SRLP, and the next lemma.

Lemma 12. *If \mathcal{X} is R -recurrent then \mathcal{X}_μ and \mathcal{X}_f are asymptotically aperiodic.*

Proof. We have observed already that \mathcal{X}_μ and \mathcal{X}_f are aperiodic since \mathcal{X} is aperiodic, and that \mathcal{X}_μ and \mathcal{X}_f are recurrent if (and only if) \mathcal{X} is R -recurrent. Since the asymptotic period of a recurrent aperiodic Markov chain equals one, \mathcal{X}_μ and \mathcal{X}_f must be asymptotically aperiodic. \square

We venture, however, to suggest the following extension of the conjecture mentioned after Theorem 9 in the setting of birth-death processes.

Conjecture. *Let the Markov chain \mathcal{X} have a unique nonnegative ρ -invariant measure μ and ρ -harmonic function f . If \mathcal{X} has the SRLP then the associated chains \mathcal{X}_μ and \mathcal{X}_f are asymptotically aperiodic.*

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