Exact expected delay and distribution for the fixed-cycle traffic-light model and similar systems in explicit form

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Abstract

In this paper we present a new method on how to compute the expectation and distribution of the queue length for a particular class of systems. We apply this method and prove its time-efficiency for models in road traffic such as the fixed-cycle traffic light (FCTL) model and for the bulk-service queue model. We give several generalizations of the FCTL queue, which model right-turns, disruptions of the traffic and uncertainty in departure times. We also consider different ways of green time allocation, which are based on either minimizing the maximum expected delay per vehicle or minimizing the total expected queue length. We compare these methods to proportional allocation.

Keywords: Fixed-cycle traffic light model, bulk service queue, roots, contour integration

Introduction

The fixed-cycle traffic-light (FCTL) queue is the basic and very important model in the traffic research. The analysis of this system was first based on simulations and approximations, but, in 1964, Darroch [3] proposed a method to find the probability generating function (pgf) for the FCTL queue length. His method resembles the one proposed by Bailey [2] for the bulk-service queue. In both cases the pgf of the queue length is a rational function with several unknown probabilities. In these papers, it is proposed to find these probabilities using the analyticity of the pgf function inside the unit disk. This implies that each zero of the denominator inside the closed unit disk is also a zero of the numerator. Since zeros are roots of the corresponding equation, in what follows we will use these terms interchangeably. In case of stable systems, it can be shown, see [1], that the number of zeros of the denominator coincides with the number of

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the unknown probabilities. Thus, if one knows the roots, one can construct a system of equations for the unknown probabilities by equating the numerator to zero at the zeros of the denominator. In both cases it is a linear system of equations, one of which is trivial due to the fact that 1 is always a zero of both the denominator and the numerator. However, the fact that the pgf is equal to 1 at point 1 yields one more equation. Thus, in case of distinct zeros it is possible to find all the required probabilities. The main drawback of this approach is the problem of finding the roots, since, in general, no closed-form formula exists.

The numerical problem of determining the zeros is now a classical problem in queueing theory. There are different ways how to find these roots in some special cases. For example, for Poisson arrivals there are analytic formulas for the roots, see, e.g., [4]. However, these formulas include infinite summations and the rate of convergence of these summations is not clear. Moreover, the precision of the acquired zeros influences the resulting unknown probabilities.

Some authors decided to derive approximations and bounds for the mean queue length. For the FCTL model the most famous approximation is due to Webster [11], which appeared earlier than the Darroch solution. In this paper, Webster assumed Poisson arrival process and proposed a semi-empiric formula for the mean delay and an empiric formula for the optimal cycle length. The formula for the mean delay is based on an analytical model and simulation results. Later Miller [6] proposed an approximation formula for the mean delay with arbitrary arrivals. Some approximations are difficult to use. For example, in [7] the approximation formula contains an integration. In [5], the bulk service queue was used as an upper bound for the FCTL model and the solution for bulk service contains a double infinite summation. The most simple approximation for the mean delay in the case of an arbitrary arrival process is due to Van den Broek et al. [8].

In this paper, we present a method to calculate the queue length distribution and its mean without finding the zeros of the denominator explicitly. The key point of the method is that the unknown probabilities depend on the denominator zeros in a symmetric way, and it can be shown that the actual values of the zeros are not important. In this paper, we show that it is possible to apply the residue theorem, i.e., to find these unknown probabilities by using contour integrations. The mean queue length is, in fact, a function of only one such contour integral. This makes our method computationally efficient in comparison with standard root-finding method. The proposed method is also general in the sense that it can be applied for an arbitrary arrival process with a pgf analytic in an open neighbourhood of the unit disk. Moreover, it is possible to use our method not only for the FCTL and bulk-service models but also for other simple cases. As an example, we give several generalizations of the FCTL model. Namely, we consider the difference between straight and turning flows, different types of traffic disruptions and uncertain departure times. Additionally, we investigate the impact of the variability of the arrival process on the expected queue length and compare different green time allocation policies.

The paper is structured as follows. In section 1, we give an overview of several models, for which our method is applicable. The method in general form is presented in section 2. In section 3, we give computational recommendations for method parameters. Exact formulas of the average delay for the described models are given in section 4. In section 5, we discuss possible generalizations of FCTL model. Numerical examples are given in section 6. In section 7, we present the final conclusions and discuss future research directions.

Models

In this section, we discuss several discrete-time models. In each model the pgfs of an important performance indicator is a rational function, i.e., it is represented as a fraction. The denominator of the fraction has a general form $z^g - A(z)$, where $A(z)$ is a pgf of a known discrete random variable. The numerator has a "nice" form with g unknown parameters. What is meant by "nice" we explain in section 2. Since the pgf should be analytic inside the unit disc and continuous up to the unit circle, all zeros of $z^g - A(z)$ inside and on the unit circle should be also zeros of the numerator with the same or a higher multiplicity. It turns out that equation $z^g = A(z)$ has exactly g roots inside and on the unit circle (see, e.g., [1]). The common approach to find the unknown variables includes finding these zeros and solving a system of equations.

FCTL model

Consider a fixed-cycle traffic-light. It is a basic model in traffic and was extensively studied before, see, for example, [3], [9]. Since we give several generalizations of this model later, we explain it here in detail. We focus on one approaching lane and consider the queue length on this lane. The same analysis can be repeated to other lanes to find, for example, the average delay of a vehicle on the intersection. Suppose that each delayed vehicle needs the same time τ to depart from the intersection. Thus, if there is a long queue in the beginning of the green time, we see a departure each τ seconds. In what follows, we suppose that time is split in time-intervals, each one of which consists of τ seconds. We suppose that the effective green time consists of $g \in \mathbb{N}$ time-intervals and the effective red time of $r \in \mathbb{N}$ time-intervals. Thus, not more than g delayed vehicles can depart during the green time. We suppose that each cycle starts with g green time-intervals and then switches to r red time-intervals. Together this gives $c = g + r$ time-intervals in a cycle.

Denote as $X_{n,m}(z)$ the pgf of the queue length in the beginning of the n^{th} time-interval during the m^{th} cycle, where $n = 0, \dots, c-1$, $m \in \mathbb{N}$. Let $Y_{n,m}$ be the arrivals during the n^{th} time-interval of the m^{th} cycle with pgf $Y_{n,m}(z)$. We assume that:

Assumption 1 (Independence assumption)**.** *The arrivals $Y_{n,m}$ are identical and independent of each other and of n, m.*

Thus, we can denote $Y_{n,m}$ simply as Y and $Y_{n,m}(z)$ as $Y(z)$. This assumption is realistic for an intersections that lies far enough from another signal-controlled intersections, for example, if the intersection is isolated. If the distance is small, then the vehicles arrive in platoons and this assumption does not hold. Following [9], we add the so called FCTL assumption:

Assumption 2 (FCTL assumption)**.** *If a vehicle arrives during the green time and finds an empty queue, then it proceeds without delay.*

Assumption 2 means that if the queue is cleared before the end of the green time, the vehicles that will arrive during the remaining green time will immediately depart. Therefore, the queue length will be zero till the end of the green time. This assumption is quite realistic for straight-going flow since vehicles that find no queue will proceed without stopping and, thus, are able to depart at the free-flow speed. For turning flows this assumption is not that realistic, especially, for right-turn (or left-turn in the case of left-hand traffic). Thus, we also consider the FCTL model with the one-vehicle assumption in the next subsection.

Denote by $q_{n,m}$ the probability to have an empty queue in the beginning of the n^{th} time-interval during the m^{th} cycle, i.e., $q_{n,m} = X_{n,m}(0)$. Under Assumptions 1 and 2, we get

$$\begin{aligned} X_{n+1,m}(z) &= \frac{X_{n,m}(z) - q_{n,m}}{z} Y(z) + q_{n,m} && \text{for } n = 0, \dots, g-1, \\ X_{n+1,m}(z) &= X_{n,m}(z) Y(z) && \text{for } n = g, \dots, c-2, \\ X_{0,m+1}(z) &= X_{c-1,m}(z) Y(z). \end{aligned} \tag{1}$$

The first equation in (1) is based on the fact that if there is at least one vehicle in the queue in the beginning of the green time-interval, then one vehicle departs and Y vehicles arrive. In terms of pgfs this means multiplying by $\frac{Y(z)}{z}$. However, if the queue is empty, it remains empty during the time-interval. Thus, $q_{n,m}$ should not be multiplied by any additional function. During the red time, each time-interval Y vehicles arrive and none departs. Therefore, we only multiply by $Y(z)$ in the last two equations of (1).

Denote by $X_n(z)$ the pgf of the queue length in the stationary state in the beginning of the n^{th} time-interval during an arbitrary cycle. It exists in case of stable system, i.e., when the possible amount of departures is bigger than the expected amount of arrivals:

$$g > cY'(1).$$

Let q_n be the stationary probability to have an empty queue in the beginning of the n^{th} time-interval during a cycle, i.e., $q_n = X_n(0)$. Then from (1) it follows that

$$\begin{aligned} X_{n+1}(z) &= \frac{X_n(z) - q_n}{z} Y(z) + q_n && \text{for } n = 0, \dots, g-1, \\ X_{n+1}(z) &= X_n(z) Y(z) && \text{for } n = g, \dots, c-2, \\ X_0(z) &= X_{c-1}(z) Y(z). \end{aligned} \tag{2}$$

This gives us the pgf of *the overflow queue*, defined as the queue length in the beginning of the red time:

$$X_g(z) = \frac{\sum_{k=0}^{g-1} q_k z^k (Y(z))^{g-1-k}}{z^g - (Y(z))^c} (z - Y(z)). \tag{3}$$

FCTL model with the one-vehicle assumption

For some cases it may turn out that the FCTL assumption does not hold. For example, in case of a right turn on the intersection (or left turn in the case of left-hand traffic) vehicles that find the queue empty need to decelerate almost to the speed of a delayed vehicle. Thus, instead of the FCTL assumption it is logical to consider the following assumption:

Assumption 3 (One-vehicle assumption)**.** *If a set of vehicles arrives during the green time-interval and finds the queue empty, then only one of the vehicles proceeds without delay, and the others form a queue.*

This assumption means that even if the queue was cleared during the previous green time-intervals, the overflow queue may be not empty. The stability condition in this model coincides with the stability condition of the FCTL model with the FCTL assumption. Equations (2) will change to

$$\begin{aligned} X_{n+1}(z) &= X_n(z)\frac{Y(z)}{z} + q_n\left(1-\frac{1}{z}\right)Y(0) \quad \text{for } n=0,\dots,g-1,\\ X_{n+1}(z) &= X_n(z)Y(z) \qquad\qquad\qquad\qquad\quad \text{for } n=g,\dots,c-2,\\ X_0(z) &= X_{c-1}(z)Y(z). \end{aligned} \tag{4}$$

Indeed, if there are no vehicles, then $\max\{Y-1,0\}$ arrive, with pgf $\frac{Y(z)-Y(0)}{z}+Y(0)$. Therefore, we get for $n<g$ that $X_{n+1}(z)=(X_n(z)-q_n)\frac{Y(z)}{z}+q_n\left(\frac{Y(z)-Y(0)}{z}+Y(0)\right)$. This gives us the above result after a small rearrangement. Hence, the pgf of the overflow queue is

$$X_g(z) = \frac{\sum_{k=0}^{g-1} q_k z^k (Y(z))^{g-1-k}}{z^g-(Y(z))^c}(z-1)Y(0). \tag{5}$$

Note that in case of Bernoulli arrivals Assumptions 2 and 3 coincide. In this case $Y(z)=(1-Y(0))z+Y(0)$, and, thus, $z-Y(z)=Y(0)(z-1)$. Note also that even though the equation for roots is the same, generally q_k may have different values then in case of the FCTL assumption. This happens because one of the roots is 1 and plugging it in the numerator gives 0 without giving an additional equation to find q_k. The required g^{th} equation comes from the fact that $X_g(1)=1$. This normalization equation will have different forms for Assumptions 2 and 3. Under Assumption 2, we have that

$$\sum_{k=0}^{g-1} q_k(1-Y'(1)) = g-cY'(1).$$

However, under Assumption 3, the normalization equation has the following form:

$$\sum_{k=0}^{g-1} q_k Y(0) = g-cY'(1).$$

We see that these models coincide only if $1-Y(0)=Y'(1)$. This happens only in case of Bernoulli arrivals.

Discrete bulk-service model

Consider a discrete-time queue where each time-unit the server serves g customers and during this time A_b new customers arrive with a pgf $A_b(z)$. The server serves only those customers that are present in the queue in the beginning of the time-unit. If there are less than g customers in the queue, all of them are served. Denote as $X_b(z)$ the pgf of the queue length at the beginning of a time-unit in the stationary state. Let $\mathfrak{q}_k$ be the stationary probability to have k customers in the queue in the beginning of a time-unit. Then, (see e.g., [2]) we have that

$$X_b(z) = \frac{\sum_{k=0}^{g-1} \mathfrak{q}_k (z^g - z^k)}{z^g - A_b(z)} A_b(z). \tag{6}$$

The stability condition in this model is $g > A_b'(1)$.

Method description

In this section, we propose a generic method to analyse different discrete-time queuing systems such as those of section 1. Let us consider the function $X(z)$ with the following form:

$$X(z) = \frac{\sum_{k=0}^{g-1} x_k z^k (B(z))^{g-1-k}}{z^g - A(z)} f(z), \tag{7}$$

where x_k are unknown variables that we want to determine, $A(z)$ and $B(z)$ are pgfs, $f(z)$ is a function such that $f(1) = 0$ and for each zero $z_l \neq 1$ of the denominator $f(z_l) \neq 0$, and $X(1) = 1$. Note, we do not assume that $X(z)$ is a pgf. The functions $A(z), B(z)$ and the coefficients x_k satisfy the following conditions:

Assumption 4 (Analyticity assumption)**.** *For some $\varepsilon > 0$, the functions $A(z)$ and $B(z)$ are analytic in the disk $D_{1+\varepsilon} = \{z\colon |z| < 1 + \varepsilon\}$. The function $X(z)$ is analytic inside the unit disk and continuous up to the unit circle.*

Assumption 5 (Stability assumption)**.** *The functions $A(z)$ and $B(z)$ satisfy $g > A'(1)$ and $B'(1) < 1$.*

Under the stability and analyticity assumptions, there exist exactly g roots $z_0 = 1, z_1, \dots, z_{g-1}$ of the equation

$$z^g = A(z) \tag{8}$$

inside and on the unit circle. Note that 1 is always a simple root, since according to the stability assumption $(z^g - A(z))'|_{z=1} = g - A'(1) \neq 0$.

Coefficients in terms of roots

In this subsection, we discuss how x_k, $k = 0, \dots, g-1$, depend on the roots of (8). First, we state the following theorem, its detailed proof is given in the Appendix.

Theorem 1. *If $z \neq w$ and $z, w \in \bar{D}_1 = \{z\colon |z| \leqslant 1\}$, then*

$$zB(w) \neq wB(z).$$

Corollary. *Equation $z = B(z)$ has only one root in $\bar{D}_1$, namely* 1.

Remark 1. Without loss of generality, we may assume that $A(0) \neq 0$. To see this, consider the case when $A(0) = 0$. In this case, we can reduce the complexity in the following way. Note that $A(z) = z^n A_0(z)$ for some $n \in \mathbb{N}$ and $A_0(z)$ that is analytic in the same disk $D_{1+\varepsilon}$ as $A(z)$, $A_0(0) \neq 0$. Then the function $A_0(z)$ is a pgf. According to the stability assumption, $A'(1) = n + A_0'(1) < g$, and, hence, $n < g$. Therefore, the multiplicity of $z_1 = 0$ as a root of the equation (8) is equal to n. Using the analyticity of $X(z)$ we conclude that 0 is also a zero of multiplicity n of the numerator (7). Since $f(0) = f(z_1) \neq 0$, point 0 is a zero of multiplicity n of the function $\sum_{k=0}^{g-1} x_k z^k (B(z))^{g-1-k}$.

By induction, one can check that the coefficients x_k, $k = 0, \dots, n-1$, are equal to zero. We explain the basis of the induction, i.e., $x_0 = 0$. Indeed, suppose $x_0 \neq 0$, then the series representation of the function $\sum_{k=0}^{n-1} x_k z^k (B(z))^{g-1-k}$

starts with power z^0 with coefficient $x_0 B(0)^{g-1} \neq 0$. Hence, the numerator is not equal to zero in 0. Here we used that $B(0) \neq 0$. This follows from the second part of the stability assumption, i.e., $B'(1) < 1$. Similarly, using the fact that $x_j = 0$ for all $j = 0, \dots, n-1$ one can prove that $x_k = 0$. Therefore, whenever $A(z) = 0$, we can reduce the complexity of the system changing g to $g - n$, x_{k-n} to x_k for $k \geqslant n$ and $A(z)$ to $A_0(z)$. Note that all assumptions, namely 4 and 5, still hold. Thus, we assume from now on that $A(0) \neq 0$.

Let $y_k = \frac{B(z_k)}{z_k}$ for $k = 0, \dots, g-1$. Since we know that $A(0) \neq 0$, then $z_k \neq 0$ for each $k = 0, \dots, g-1$. According to Theorem 1, $y_k \neq y_l$ if $z_k \neq z_l$. Rewrite $X(z)$ in the following way:

$$X(z) = \frac{\sum_{k=0}^{g-1} x_k \left(\frac{B(z)}{z}\right)^{g-1-k}}{z^g - A(z)} f(z) z^{g-1}.$$

As we know $z_k \neq 0$ and $f(z_k) \neq 0$ for $k = 1, \dots, g-1$. The numerator is also equal to 0 for $z = z_k$, $k = 1, \dots, g-1$. Thus,

$$\sum_{l=0}^{g-1} x_l y_k^{g-1-l} = 0,$$

where $k = 1, \dots, g-1$. Consider the following polynomial

$$h(y) = \sum_{l=0}^{g-1} x_l y^{g-1-l} = \sum_{l=0}^{g-1} x_{g-1-l} y^l.$$

The function $h(y)$ is a polynomial of degree $g-1$ and y_k for $k = 1, \dots, g-1$ are zeros of this polynomial. Note that whenever z_k is a multiple root of the equation $z^g = A(z)$, y_k is a multiple root of the polynomial $h(y)$ with the same multiplicity. Thus, $h(y)$ can be written as

$$h(y) = x_0 \prod_{k=1}^{g-1} (y - y_k). \tag{9}$$

By applying Vieta's formulas (see [10]) to the polynomial $h(y)$, we get that

$$\frac{x_k}{x_0} = (-1)^k \sigma_k(y_1, \dots, y_{g-1}), \tag{10}$$

where, for $k = 1, \dots, g-1$,

$$\sigma_k(y_1, \dots, y_{g-1}) = \sigma_k = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant g-1} y_{i_1} \dots y_{i_k}$$

are elementary symmetric polynomials. Here for notational convenience we assume

$$\sigma_0(y_1, \dots, y_{g-1}) = \sigma_0 = 1.$$

This gives us x_k up to a normalization constant, which we derive from the assumption that $X(1) = 1$:

$$X(1) = \frac{h(1)}{g - A'(1)} f'(1) = \frac{\sum_{k=0}^{g-1} x_k}{g - A'(1)} f'(1) = 1. \tag{11}$$

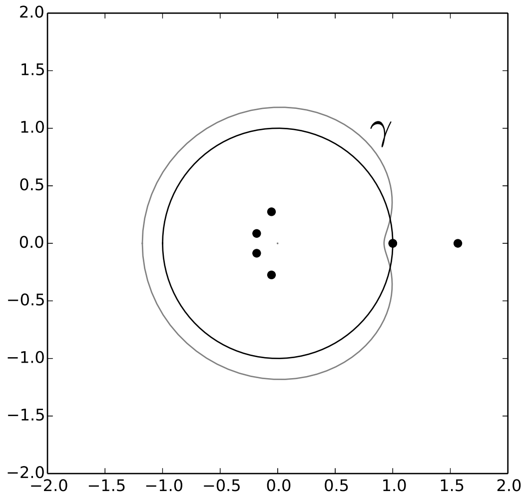


Figure 1: An example of curve γ. The roots of the equation $z^5 2^9 = (z+1)^9$ are represented as bold points.

Coefficients in terms of contour integrations

Using (10), we can find the coefficients x_k if we know the values of elementary symmetric polynomials σ_k. We will represent these symmetric polynomials as functions of the following symmetric polynomials:

$$\eta_k = \eta_k(y_1, \dots, y_{g-1}) = \sum_{l=1}^{g-1} y_l^k$$

for $k = 1, \dots, g-1$. To find η_k, consider a curve γ that embraces all zeros z_k, $k = 1, \dots, g-1$ but does not embrace any other roots of (8). For example, the curve that embraces the unit disk without 1, see Figure 1. In section 3.1, we elaborate on how to choose this curve. Note that the following lemma holds:

Lemma 1. *Consider a function $g(z)$ that is analytic in a neighbourhood of z_k, where $k = 1, \dots, g-1$. Then the residue of function $\frac{gz^{g-1} - A'(z)}{z^g - A(z)} g(z)$ at z_k is equal to $m_{z_k} g(z_k)$, where m_{z_k} is the multiplicity of the zero z_k.*

Using this lemma and the residue theorem, we get that

$$\eta_k = \frac{1}{2\pi i} \oint_\gamma \frac{gz^{g-1} - A'(z)}{z^g - A(z)} \left(\frac{B(z)}{z} \right)^k dz - \frac{1}{2\pi i} \oint_{S_\delta} \frac{gz^{g-1} - A'(z)}{z^g - A(z)} \left(\frac{B(z)}{z} \right)^k dz, \tag{12}$$

where S_δ is the circle with radius $\delta > 0$ around the origin. The parameter δ should be chosen in such way that there are no roots of (8) in $\bar{D}_\delta = \{z \colon |z| \leqslant \delta\}$.

The following lemma provides a way how to find σ_k, $k = 1, \dots, g-1$, using η_k, $k = 1, \dots, g-1$. We omit the proof of this lemma since it requires only careful computation of the monomials' coefficients in the right side of the equation (13).

Lemma 2. *The following recurrence equation holds:*

$$\sigma_k = \frac{1}{k}\sum_{l=1}^{k}(-1)^{l+1}\sigma_{k-l}\eta_l \tag{13}$$

for each $k = 1, \dots, g-1$.

Thus, we can use formulas (10), (11) and (12) together with Lemma 2 to find the coefficients x_k, $k = 0, \dots, g-1$.

Expectation in terms of the roots

Now consider $X'(1)$. In case of FCTL model it represents the expected overflow queue.

$$X'(1) = \left(\frac{\sum_{k=0}^{g-1} x_k z^k (B(z))^{g-1-k}}{z^g - A(z)} f(z)\right)'\Bigg|_{z=1} =$$

$$= \frac{f'(1)}{g - A'(1)} \cdot \left(h\left(\frac{B(z)}{z}\right) z^{g-1}\right)'\Bigg|_{z=1} + \; h(1) \cdot \left(\frac{f(z)}{z^g - A(z)}\right)'\Bigg|_{z=1} =$$

$$= \frac{f'(1)}{g - A'(1)}\Big(h'(1)(B'(1)-1) + (g-1)\,h(1)\Big) + h(1)\frac{f''(1)(g - A'(1)) - f'(1)(g(g-1) - A''(1))}{2(g - A'(1))^2}.$$

Now we can plug in $h(1)$ from (11).

$$X'(1) = (B'(1)-1)\frac{h'(1)}{h(1)} + g - 1 + \frac{f''(1)(g - A'(1)) - f'(1)(g(g-1) - A''(1))}{2(g - A'(1))f'(1)}. \tag{14}$$

The remaining term we need to find here is $\frac{h'(1)}{h(1)}$. By using representation (9), we get that

$$\frac{h'(1)}{h(1)} = \frac{\left(\prod_{k=1}^{g-1}(y - y_k)\right)'}{\prod_{k=1}^{g-1}(y - y_k)}\Bigg|_{y=1} = \sum_{k=1}^{g-1}\frac{1}{1 - y_k}.$$

Thus, in terms of roots we have that

$$X'(1) = (B'(1)-1)\sum_{k=1}^{g-1}\frac{z_k}{z_k - B(z_k)} + g - 1 + \frac{f''(1)}{2f'(1)} - \frac{g(g-1) - A''(1)}{2(g - A'(1))}. \tag{15}$$

Expectation in terms of a contour integration

We can use a contour integral to find $\sum_{k=1}^{g-1}\frac{z_k}{z_k - B(z_k)}$. Let γ be as above, then

$$\sum_{k=1}^{g-1}\frac{z_k}{z_k - B(z_k)} = \frac{1}{2\pi i}\oint_\gamma \frac{g z^{g-1} - A'(z)}{z^g - A(z)}\,\frac{z}{z - B(z)}dz. \tag{16}$$

By the Corollary, we know that there are no roots of the equation $z = B(z)$ inside $\bar{D}_1$ except 1. Thus, we do not need to subtract any other residues of the function inside the integral.

It is also possible to find the variance of the queue length in the same way, but since it is a lengthy expression, we omit it here.

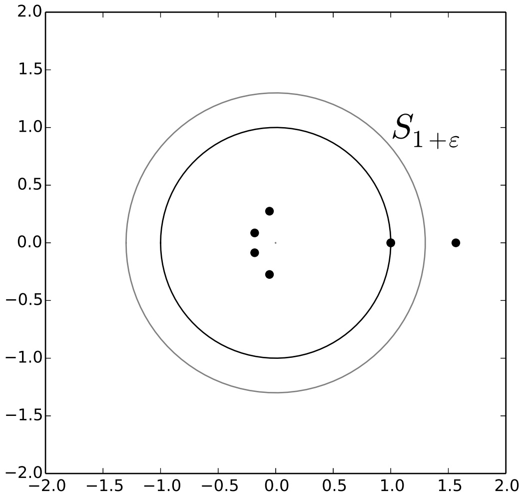


Figure 2: Circle $S_{1+\varepsilon}$ such that there are no roots of the equation (8) in $\bar{D}_{1+\varepsilon} \setminus \bar{D}_1$. The roots of the equation $z^5 2^9 = (z+1)^9$ are represented as bold points.

Computational remarks

In this section, we give several computational remarks and present the algorithms.

Choice of curve γ

The main computational issue in our method is how to choose γ. The curve γ that we suggested before is computationally inconvenient. Hence, we suggest to take integrals over circle $S_{1+\varepsilon}$, where $\varepsilon > 0$. We want ε to be small such that there are no roots of the equation (8) in $\bar{D}_{1+\varepsilon} \setminus \bar{D}_1$, see Figure 2. This circle is easily parametrizable, but 1 is inside the circle. Therefore, the integrals (12) and (16) along γ and along circle $S_{1+\varepsilon}$ differs by the residue at 1. In case of the expectation, this residue is

$$res_1 = \frac{1}{1-B'(1)}\left(1 + \frac{B''(1)}{2(1-B'(1))} + \frac{g(g-1)-A''(1)}{2(g-A'(1))}\right). \tag{17}$$

and in case of η_k the residue is equal to 1.

We parametrize $S_{1+\varepsilon}$ by $z(\varphi) = (1+\varepsilon)e^{i\varphi}$. Then $dz = izd\varphi$ and

$$\oint_{S_{1+\varepsilon}} g(z)dz = i\int_{-\pi}^{\pi} g(z(\varphi))z(\varphi)d\varphi. \tag{18}$$

From a complex integral to real integrals

In this subsection, we give a suggestion how to compute the required complex integrals. We need to compute a complex integral, so we need to compute the imaginary and real parts of the integral (18). It is equivalent to computing the following integrals:

$$\int_{-\pi}^{\pi} Re(g(z(\varphi))z(\varphi))d\varphi, \quad \int_{-\pi}^{\pi} Im(g(z(\varphi))z(\varphi))d\varphi.$$

However, in the applications the performance indicators are real values. Note that in this case the imaginary part vanishes, and we need to compute only one integral.

Choice of δ

In (12) we use the parameter δ such that there are no roots of the equation (8) in $\bar{D}_\delta = \{z\colon |z| \leqslant \delta\}$. Let us find an analytic value for it. Consider $A(z) = \sum_{j=0}^{\infty} a_j z^j$, since $a_j \geqslant 0$ and $\sum_{j=0}^{\infty} a_j = 1$

$$|A(z) - A(0)| = \left|\sum_{j=1}^{\infty} a_j z^j\right| \leqslant \sum_{j=1}^{\infty} a_j |z|^j \leqslant (1 - A(0))|z|.$$

Note that $a_0 = A(0) > 0$, since $A(0) \neq 0$. Therefore, for z such that $|z|(2 - A(0)) < A(0)$ we get

$$A(z) \geqslant A(0) - |A(z) - A(0)| \geqslant A(0) - (1 - A(0))|z| > |z| \geqslant |z|^g.$$

Thus, there are no roots of the equation (8) in $\bar{D}_\delta$ for $\delta < \frac{A(0)}{2-A(0)}$.

Note that if $g > 1$ and $A(0) \neq 1$, $\frac{A(0)}{2-A(0)} < 1$, and, thus, for each z with $|z|(2 - A(0)) = A(0)$ we have $|z| > |z|^g$. Hence, there are no roots of (8) in $\bar{D}_\delta$ for $\delta = \frac{A(0)}{2-A(0)}$. So, in almost all cases $\delta = \frac{A(0)}{2-A(0)}$ is an appropriate value. However, such δ can be too small, and, thus, computational errors will arise. In case of very small analytical value δ, it may be better to use the following method.

Consider the integral

$$\oint_{S_\delta} \frac{gz^{g-1} - A'(z)}{z^g - A(z)} dz = i \int_{-\pi}^{\pi} \frac{gz^g - zA'(z)}{z^g - A(z)} d\varphi, \tag{19}$$

where z in the last integral stands for $z(\varphi) = \delta e^{i\varphi}$. By Lemma 1 and the residue theorem, the value of this integral is equal to the amount of roots of the equation (8) inside the disk $\bar{D}_\delta$ multiplied by 2π. Therefore, if this integral is equal to zero, δ is correct.

Choice of ε

The choice of ε may be done in the same way as the choice of δ by computing contour integrals. However, one can prove using Rouche's theorem that there are no roots of equation (8) in $\bar{D}_{1+\varepsilon} \setminus \bar{D}_1$ for $1 + \varepsilon < z_{-1}$, where z_{-1} is the (only) root of equation (8) on the open ray $(1, \infty)$. Thus, if a lower bound on z_{-1} is known or it is easy to compute numerically, one can use any point on $(0, z_{-1} - 1)$ as ε. However, the smaller the ε the bigger the computational error for the integral, so we suggest to take $\varepsilon = \frac{z_{-1}-1}{2}$. In this way, we stay far enough from the roots of the equation (8).

Zero-free case

From Theorem 1, it follows that the function $B(z)$ has at most one zero in D_1. It is possible that the function $B(z)$ has no zero in D_1. For example, if $B(z)$ is the pgf of a Poisson random variable. We will show here how to simplify computation in this case. Then, we will give a simple test how to check if $B(z)$ is zero-free in D_1.

Consider $\tilde{y}_k = \frac{z_k}{B(z_k)}$. In the same manner as before, we can get formulas for $\tilde{\eta}_k = \sum_{j=1}^{g-1} \tilde{y}_j^k$ that do not contain any integrals along S_δ. More specifically, we consider

$$\tilde{h}(y) = \sum_{k=0}^{g-1} x_k y^k.$$

Thus, $\tilde{h}\left(\frac{z}{B(z)}\right)(B(z))^{g-1} = h\left(\frac{B(z)}{z}\right) z^{g-1}$. We conclude that $\tilde{y}_k$ are zeros of $\tilde{h}(y)$, and, thus,

$$\tilde{h}(y) = x_{g-1} \prod_{k=1}^{g-1} (y - \tilde{y}_k).$$

Using the same argument as in subsection 2.1, we get

$$\frac{x_{g-1-k}}{x_{g-1}} = (-1)^k \sigma_k(\tilde{y}_1, \ldots, \tilde{y}_{g-1}) = (-1)^k \tilde{\sigma}_k. \tag{20}$$

Note that, if $B(z)$ does not have zeros in $\bar{D}_1$, then

$$\tilde{\eta}_k = \frac{1}{2\pi i} \oint_\gamma \frac{g z^{g-1} - A'(z)}{z^g - A(z)} \left(\frac{z}{B(z)}\right)^k dz. \tag{21}$$

So, we do not need to compute 2 integrals. As before, we can change integral to integral along the circle $S_{1+\varepsilon}$. Then, we need to subtract the residue. It will be equal to 1 in this case.

The necessary and sufficient condition for $B(z)$ to be zero-free in D_1 is given by the following lemma.

Lemma 3. *The pgf $B(z)$ is zero-free in D_1 if and only if $B(-1) > 0$.*

Proof. Suppose that $B(-1) > 0$ and there is a zero t of $B(z)$ in D_1. Since $B(z)$ is a real function, this zero is on the real line. Also t should be a zero of multiplicity at least 2. Therefore, $B(t) = B'(t) = tB'(t) = 0$. Consider function $C(z) = B(z) - zB'(z)$ and the Taylor expansion $\sum_{j=0}^\infty b_j z^j$ of the function B at zero. Note that $b_j \geqslant 0$ for all j. Hence, $C(t) = b_0 - \sum_{j=2}^\infty (j-1) b_j t^j \geqslant b_0 - \sum_{j=2}^\infty (j-1) b_j = C(1) = 1 - B'(1) > 0$, and t is not a multiple zero of B.

The rest of the proof follows from the fact that $B(z)$ is a continuous function and $B(1) > 0$. Indeed, if $B(-1) \leqslant 0$, then the intermediate value theorem states that there is a zero of function B in the segment $[-1, 1]$. □

Algorithms

In this subsection, we give our full algorithms to compute $X'(1)$ and the coefficients x_k, respectively. First, let us consider $X'(1)$.

Algorithm 1 (Computation of $X'(1)$).

1. *Using one of two ways described in subsection 3.4, find such $\varepsilon > 0$ that there are only g roots of the equation (8) in the disk $\bar{D}_{1+\varepsilon}$.*

2. *Compute (the real part of) the integral*

$$I = \int_{-\pi}^{\pi} \frac{gz^g - zA'(z)}{z^g - A(z)} \frac{z}{z - B(z)} d\varphi,$$

where $z(\varphi) = (1+\varepsilon)e^{i\varphi}$.

3. *Compute $X'(1)$ by*

$$X'(1) = \frac{(B'(1)-1)}{2\pi} I + \frac{B''(1)}{2(1-B'(1))} + g + \frac{f''(1)}{2f'(1)}. \tag{22}$$

Note that (22) differs from (15) and (16) due to the residue (17) at one. Now we give an algorithm to find the unknown variables x_k:

Algorithm 2 (Computations of x_k, $k = 0, \ldots, g-1$).

1. *Using one of two ways described in subsection 3.4, find such $\varepsilon > 0$ that there are only g roots of the equation (8) in the disk $\bar{D}_{1+\varepsilon}$.*

2. *Check if there is a zero of $B(z)$ in $\bar{D}_1 = \{z\colon |z| \leqslant 1\}$. If there is a zero, use one of two ways described in subsection 3.3 to find such $\delta > 0$ that there are no roots of the equation (8) in the disk $\bar{D}_\delta$.*

3. *Compute η_k (or $\tilde{\eta}_k$ if there is no root of $B(z)$ in $\bar{D}_1$) using*

$$\eta_k = -1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{gz^g - zA'(z)}{z^g - A(z)} \left(\frac{B(z)}{z}\right)^k d\varphi - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{gw^g - wA'(w)}{w^g - A(w)} \left(\frac{B(w)}{w}\right)^k d\varphi$$

$$\left(\tilde{\eta}_k = -1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{gz^g - zA'(z)}{z^g - A(z)} \left(\frac{z}{B(z)}\right)^k d\varphi\right),$$

where for convenience $z = z(\varphi) = (1+\varepsilon)e^{i\varphi}$, $w = w(\varphi) = \delta e^{i\varphi}$.

4. *Compute σ_k (or $\tilde{\sigma}_k$) by (13) using η_k (or $\tilde{\eta}_k$).*

5. *Compute x_k, $k = 0, \ldots, g-1$, using (10) (or (20)) together with (11).*

Our method applied to the models

In this section, we apply our method to the models of section 1. For these models we first specify $A(z)$, $B(z)$, $f(z)$ and x_k. Then, we check that Assumptions 4 and 5 hold. Finally, we apply the proposed method and present formulas for the queue length expectation.

FCTL model

In the FCTL model, we take $A(z) = (Y(z))^c$, $B(z) = Y(z)$, $f(z) = z - Y(z)$ and $x_k = q_k$. Note that by the Corollary $f(z) = 0$ inside and on the unit circle only for $z = 1$. Thus, $f(z_k) \neq 0$ for each root z_k of (8), $k = 1, \ldots, g-1$. We need to assume that $Y(z)$ is an analytic function in the disk $D_{1+\varepsilon}$ for some $\varepsilon > 0$. Then the analyticity assumption holds. Since we are considering only stable systems, we assume that $g > cY'(1) = A'(1)$, and, thus, $B'(1) = Y'(1) < 1$, so the stability assumption holds.

One can prove, see, e.g., [3], that the expected overflow queue $\mathbb{E}X_g = X_g'(1)$ and the expected queue length at an arbitrary point during the cycle are connected in the following way:

$$\mathbb{E}L_{\text{FCTL}} = \frac{r}{c(1-Y'(1))}X_g'(1) + \frac{r^2Y'(1)}{2c(1-Y'(1))} + \frac{r(Y''(1)+Y'(1)-(Y'(1))^2)}{2c(1-Y'(1))^2}. \tag{23}$$

In this case, we can rewrite (22) as

$$X_g'(1) = \frac{(Y'(1)-1)}{2\pi}I + g, \tag{24}$$

where

$$I = \int_{-\pi}^{\pi} \frac{gz^g - zc(Y(z))^{c-1}Y'(z)}{z^g - (Y(z))^c}\frac{z}{z - Y(z)}d\varphi \tag{25}$$

and $z = z(\varphi) = (1+\varepsilon)e^{i\varphi}$.

Plugging (24) in (23) gives after rearrangement:

$$\mathbb{E}L_{\text{FCTL}} = -\frac{r}{2\pi c}I + \frac{gr}{c} + \frac{r(c+g+1)Y'(1)}{2c(1-Y'(1))} + \frac{rY''(1)}{2c(1-Y'(1))^2}. \tag{26}$$

FCTL model with the one-vehicle assumption

In the case of the one-vehicle assumption, we take $A(z) = (Y(z))^c$, $B(z) = Y(z)$, $f(z) = (z-1)Y(0)$ and $x_k = q_k$. Here, since $f(z)$ is a polynomial of degree 1, it has only one root, namely 1. Using the same arguments as in the previous subsection, we get that if our system is stable, then the stability assumption holds. Thus, we can use our method for a stable system with a function $Y(z)$ that is analytic in some disk $D_{1+\varepsilon}$.

In this model, one can check that the expected queue length at an arbitrary point during the cycle is given by the following formula:

$$\mathbb{E}L_{1v} = \frac{r}{c(1-Y'(1))}X_g'(1) + \frac{r(r+1)}{2c}\frac{Y'(1)}{1-Y'(1)} + \frac{Y''(1)}{2(1-Y'(1))},$$

where $X_g'(1)$ is the expected overflow queue.

Now consider $X_g'(1)$. From (22) we get

$$X_g'(1) = \frac{(Y'(1)-1)}{2\pi}I + g + \frac{Y''(1)}{2(1-Y'(1))}, \tag{27}$$

where I is given in (25). Finally, we get

$$\mathbb{E}L_{1v} = -\frac{r}{2\pi c}I + \frac{r(r+1)}{2c}\frac{Y'(1)}{1-Y'(1)} + \frac{gr}{c(1-Y'(1))} + \frac{rY''(1)}{2c(1-Y'(1))^2} + \frac{Y''(1)}{2(1-Y'(1))}. \tag{28}$$

By comparing (27) with (24) and (28) with (26), we see that the difference between straight and turning directions queues is just $\frac{Y''(1)}{2(1-Y'(1))}$ for both the overflow queue and the expected queue length during the cycle. This is not a coincidence. In fact, for each $k = 0, \ldots, c-1$:

$$X_{k,1v}(z) = X_{k,fctl}(z)X_{dif}(z),$$

where

$$X_{dif}(z) = \frac{(1 - Y'(1))(z - 1)}{z - Y(z)},$$

$X_{k,fctl}(z)$ and $X_{k,1v}(z)$ are pgfs in the beginning of the k^{th} time-interval for the FCTL and one-vehicle models. This equation is a simple corollary of the equations (2), (3), (4), (5) and the fact that the zeros of the numerator are the same for both models. Note that $X_{dif}(1) = 1$ and $X'_{dif}(1) = \frac{Y''(1)}{2(1-Y'(1))}$. Note also that $X_{dif}(z)$ is independent of g and c.

Consider the case $g = c = 1$. In the case of the FCTL assumption the queue is always empty after it becomes empty once since there is no red time, during which vehicles can form a queue. Therefore, in case of stability, i.e., $Y'(1) < 1$, we get $X_{0,fctl}(z) = 1$. Thus, we get $X_{0,1v} = X_{dif}(z)$, i.e., $X_{dif}(z)$ is the pgf of the queue with $g = c = 1$ and arrivals $Y(z)$. Hence, we get the following decomposition result:

Theorem 2. *For arbitrary $g \leqslant c$ the queue with the one-vehicle assumption can be considered as a sum of two independent queues: the FCTL queue with the same g, c and the one-vehicle queue with $g = c = 1$.*

Note that $X_{dif}(z)$ can be also viewed as a pgf of the queue length in a bottleneck, e.g., when part of the road has small speed limit and all the vehicles decelerate before this place.

We can as well change the model to allow more than one vehicle, but not all, to pass the junction if the queue is empty. Then the changes will be only in the function $f(z)$, and we can easily use our method if Assumptions 4 and 5 hold. For the same reason, there would be a similar decomposition result. The expected queue length in this new case will be less than the expected queue length in case of the one-vehicle assumption but more than the expected queue length in case of the FCTL assumption.

Discrete bulk-service model

From the first glance the pgf of the queue length for this system (6) does not have our kind of form, but we can rearrange it as follows:

$$X_b(z) = \frac{\sum_{k=0}^{g-1} \mathfrak{q}_k(z^g - z^k)}{z^g - A_b(z)} A_b(z) = \frac{\sum_{l=0}^{g-1} \left(\sum_{k=0}^{l} \mathfrak{q}_k\right) z^l}{z^g - A_b(z)} (z - 1)A_b(z).$$

Hence, we take $A(z) = A_b(z)$, $B(z) = 1$, $f(z) = (z-1)A_b(z)$ and $x_k = \sum_{l=0}^{k} \mathfrak{q}_l$. Note that $\mathfrak{q}_k = x_k - x_{k-1}$, $k = 2, \ldots, g-1$, and $\mathfrak{q}_1 = x_1$. In case $A_b(0) = 0$, we can reduce the complexity as before, since $\mathfrak{q}_l$, as the probability to have l customers in the queue, is zero for $l = 0, \ldots, n-1$, where n is the least possible amount of customers that arrive in one time slot. Thus, we suppose that $A_b(0) \neq 0$. Hence, $z_k \neq 0$ for every $k = 1, \dots, g-1$ and

$$f(z_k) = A_b(z_k)(z_k - 1) = z_k^g(z_k - 1) \neq 0.$$

If the system is stable, then $A'(1) = A_b'(1) < g$. Since $B'(1) = 0 < 1$, the stability assumption holds. Therefore, we can use our method if $A_b(z)$ is analytic in the disk $D_{1+\varepsilon}$ for some $\varepsilon > 0$.

From (22), we get that the expected queue length is given by the following formula:

$$X_b'(1) = -\frac{I}{2\pi} + g + A_b'(1), \tag{29}$$

where I is given by

$$I = \int_{-\pi}^{\pi} \frac{gz^g - zA_b'(z)}{z^g - A_b(z)} \frac{z}{z-1} d\varphi \tag{30}$$

and $z = z(\varphi) = (1+\varepsilon)e^{i\varphi}$. Note that (29) and (30) are equivalent to the following formula:

$$X_b'(1) = A_b'(1) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{gz^g - zA_b'(z)}{z^g - A_b(z)} \frac{1}{z-1} d\varphi.$$

Generalizations of the FCTL model

In this section, we consider several possible realistic situations that are not included in the FCTL model:

1. pedestrian and/or bicycle traffic lights have an actuated control;
2. just after intersection there is a bridge or a railway and a part of the green times may be lost;
3. another lane on the intersection does not have a fixed length of the green time due to an actuated control;
4. times between departures differ due to driver distraction and/or vehicle condition, length;
5. the arrivals during the cycle are heterogeneous.

Even though the first three situations differ a lot they can be modelled in the same way. Namely, by assuming that the green and the red times consist of a random number of time-intervals. Thus, we first consider this model and these three cases. Then, we discuss a way to model the distraction of the drivers. In the end, we present the case for which we can use our method even though the arrivals during different time-intervals are not identically distributed.

Random green and red times

Suppose we know that with probability $\theta_{r,g}$ the red time consists of r time-intervals and the green time after this red time consists of g time-intervals, $r, g \in \mathbb{N} \cup \{0\}$, $r + g \neq 0$. Thus, cycles start with a red time, followed by a green time. The length of the red time and the green time are independent of the lengths of red and green times before this cycle. We assume that the length of the green time is at most N, i.e., $\sum_{k=0}^{\infty} \sum_{l=N+1}^{\infty} \theta_{k,l} = 0$. For simplicity, we consider the smallest such N, so $\sum_{k=0}^{\infty} \theta_{k,N} > 0$. As before in each time-interval Y vehicles arrive with pgf $Y(z)$. We also use the FCTL assumption. One can easily alter the formulas in case of the one-vehicle assumption.

Let $X(z)$ be the pgf of the overflow queue. With probability $\theta_{r,g}$ the pgf of the queue-length in the beginning of the next red time will be

$$X(z)\frac{(Y(z))^{g+r}}{z^g} + \left(1 - \frac{Y(z)}{z}\right)\sum_{k=0}^{g-1} p_{k,r,g}\left(\frac{Y(z)}{z}\right)^{g-1-k},$$

where $p_{k,r,g}$ is the probability to have an empty queue in the beginning of the k^{th} green time-interval in case of a cycle with r red and g green time-intervals. Therefore, the pgf of the overflow queue satisfies the following equation:

$$X(z) = \sum_{r,g} \theta_{r,g}\left(X(z)\frac{(Y(z))^{g+r}}{z^g} + \left(1 - \frac{Y(z)}{z}\right)\sum_{k=0}^{g-1} p_{k,r,g}\left(\frac{Y(z)}{z}\right)^{g-1-k}\right).$$

Gathering all uses of the function $X(z)$ in the left side gives us

$$X(z) = \frac{\sum_{r,g} \theta_{r,g}\left(1 - \frac{Y(z)}{z}\right)\sum_{k=0}^{g-1} p_{k,r,g}\left(\frac{Y(z)}{z}\right)^{g-1-k}}{1 - \sum_{r,g} \theta_{r,g}\frac{(Y(z))^{g+r}}{z^g}}.$$

After multiplying both numerator and denominator by z^N and making some rearrangement, we get

$$X(z) = \frac{\sum_{k=0}^{N-1} q_k (Y(z))^k z^{N-1-k}}{z^N - A(z)}(z - Y(z)),$$

where $q_k = \sum_{r,g} \theta_{r,g} p_{g-1-k,r,g}$ and $A(z) = \sum_{r,g} \theta_{r,g}(Y(z))^{g+r} z^{N-g}$. As we see we can apply our method if $A(z)$ and $Y(z)$ are analytic in some disk $D_{1+\varepsilon}$, $A'(1) < N$ and $Y'(1) < 1$.

Note that if the summation $\Omega_1(z) = \sum_{r,g} \theta_{r,g}(g + r)(Y(z))^{g+r-1} z^{N-g}$ converges at some point $a > 1$, then $\Omega_1(z)$, $\Omega_1(z)Y'(z)$, $A(z)$ and

$$\Omega_2(z) = \sum_{r,g} \theta_{r,g}(Y(z))^{g+r}(N - g)z^{N-g-1}$$

converge absolutely in D_a. Then, if the function $Y(z)$ is analytic in D_a, the function $A(z)$ is also analytic in D_a and $A'(z) = \Omega_1(z)Y'(z) + \Omega_2(z)$. For applications, $A(z)$ is always a finite sum, so it is analytic if $Y(z)$ is analytic. Note that

$$A'(1) = \sum_{r,g} \theta_{r,g}((g + r)Y'(1) + (N - g)) = \mathbb{E}(G + R)Y'(1) + N - \mathbb{E}G,$$

where G and R are random variables of the length of the green and red times during a cycle. Thus, if $A'(1) < N$, we get $\mathbb{E}(G+R)Y'(1) < \mathbb{E}G$. It immediately follows that $Y'(1) < 1$. We see once again that the stability assumption holds in the case of a stable system.

Now we give three examples of possible applications of this model.

Interruption by pedestrians and/or cyclists

Let us focus on one line and, for simplicity, on one source causing interruptions. Suppose that with probability p there is an arrival of a cyclist during a cycle. Suppose that we need t_c green time-intervals for switching on and off the cyclists green time. The green time for pedestrians and cyclists can be given from green time g of this lane or can be added as extra time for cycle time $c = g + r$. In the first case, we have $\theta_{r,g} = 1 - p$, $\theta_{r+t_c,g-t_c} = p$, and in the second case, we have $\theta_{r,g} = 1 - p$, $\theta_{r+t_c,g} = p$.

Interruption by trains and boats

This case is more or less the same as the previous one. The only difference is in the length of interruption. We assume that interruption happens during a cycle with probability p. We also assume that if it happens, then all green time is effectively red. Therefore, $\theta_{r,g} = 1 - p$, $\theta_{c,0} = p$.

Vehicle-actuated control on another lane

Suppose that another lane has a vehicle-actuated control, and, thus, the length of its green time is not deterministic. In this case, to find an approximation for the queue length on the considered lane we assume that the lengths of red times during different cycles are independent of each other. Then, if we know the distribution of the green time for the other lane, we get a distribution of the red time for the considered lane. Let $\theta_{r,g} = p_r$, where p_r is the distribution of the red time length and g is fixed. In the case of a fixed cycle, we take $\theta_{r,c-r} = p_r$.

Remark 2. The analysis of the actuated-controlled lane is complicated due to the fact that the arrival and service processes are connected. We leave this problem for the future research.

Uncertain departure time

In this subsection, we consider the following extension of the FCTL model. Suppose that during a green time-interval the driver of the departing vehicle may be distracted with fixed probability $p \in [0, 1]$. Therefore, during the green time, even in case of a non-empty queue, vehicles do not depart each time-interval. Then, each driver has a geometrically distributed amount of tries to depart from the queue. This can be modelled using the FCTL model in the following way. Suppose that during the green time there is an extra arrival with probability p. In this way, we compensate for the uncertainty in the departures. The distribution of the queue length in this model will be the same as distribution in our new model. So we consider the case that the amount of the arrivals during the green time has pgf $Y(z)(pz + (1 - p))$. However, during the red time we have still $Y(z)$ arrivals. Hence, the pgf of the overflow queue is given by the following formula:

$$X_g(z) = \frac{\sum_{k=0}^{g-1} q_k z^k (Y(z)(pz+1-p))^{g-1-k}}{z^g - (Y(z))^c (pz+1-p)^g}(z - Y(z)(pz+1-p)).$$

Using the same argumentation as in Subsection 4.1, one can see that the pgf is of the general form, described in Section 2. Thus, it is possible to use the same techniques in case when $Y(x)$ is analytic in some $D_{1+\varepsilon}$, and the system is stable, i.e., $cY'(1) + gp < g$.

Generally, we can consider any arrivals during the red time. Let us introduce the weak-independence assumption:

Assumption 6 (Weak-independence assumption)**.** *The arrivals $Y_{n,m}$ are independent of m and of the arrivals during other cycles. The arrivals $Y_{k,m}$ for $k = 0, \ldots, g-1$ are identically distributed, independent of each other and of the arrivals during the red time. The arrivals during one red time are distributed identically for each cycle.*

Note that the arrivals during the red time may be dependent on each other and have a different distribution than the arrivals during the green time. Denote the pgf of all arrivals during the red time as $A_r(z)$. If Assumption 1 holds, the pgf $A_r(z)$ is just $(Y(z))^r$.

In case of Assumption 6 the pgf of $X_g(z)$ in FCTL model does not change much and has the following form:

$$X_g(z) = \frac{\sum_{k=0}^{g-1} q_k z^k (Y(z))^{g-1-k}}{z^g - A_r(z)(Y(z))^g}(z - Y(z)), \tag{31}$$

where q_k is the probability of an empty queue in the beginning of the k^{th} time-interval in a cycle. These variables q_k, in general, have different value from q_k in (3). In this case the system is stable if $g > gY'(1) + A_r'(1)$.

Note that in case of the one-vehicle assumption, we will get

$$X_g(z) = \frac{\sum_{k=0}^{g-1} q_k z^k (Y(z))^{g-1-k}}{z^g - A_r(z)(Y(z))^g}(z-1)Y(0),$$

with the same stability condition.

One can check that equations (24) and (27) do not change except of I, which will be given by the following formula:

$$I = \int_{-\pi}^{\pi} \frac{gz^g - zg(Y(z))^{g-1}Y'(z)A_r(z) - z(Y(z))^g A_r'(z)}{z^g - (Y(z))^g A_r(z)} \frac{z}{z - Y(z)} d\varphi,$$

where $z = z(\varphi) = (1+\varepsilon)e^{i\varphi}$. We do not give formulas for the expected queue length during the cycle, since they depend on the arrivals during the red time.

Bernoulli case

In this subsection, we discuss the case of non-identically distributed arrivals during the green time. This happens for a tandem of intersections. Suppose that $Y_k(z)$ is the pgf of the arrivals during the k^{th} time-interval, $k = 0, \ldots, c-1$.

We assume that arrivals during different time-intervals are independent of each other. We admit that this assumption is debatable for a tandem of intersections, but we can use it to construct an approximation. One can check that (2) changes to

$$\begin{aligned} X_{n+1}(z) &= \frac{X_n(z) - q_n}{z} Y_n(z) + q_n && \text{for } n = 0, \dots, g-1, \\ X_{n+1}(z) &= X_n(z) Y_n(z) && \text{for } n = g, \dots, c-2, \\ X_0(z) &= X_{c-1}(z) Y_{c-1}(z), \end{aligned} \tag{32}$$

where q_n, as before, denotes the stationary probability to have an empty queue in the beginning of the n^{th} time-interval during a cycle. By putting everything together we get

$$X_g(z) = \frac{\sum_{k=0}^{g-1} q_k z^k \prod_{l=k+1}^{g-1} Y_l(z)(z - Y_k(z))}{z^g - \prod_{l=0}^{c-1} Y_l(z)}. \tag{33}$$

Consider the case of Bernoulli arrivals, i.e., $Y_k(z) = \lambda_k z + (1 - \lambda_k)$. Thus, $z - Y_k(z) = (1 - \lambda_k)(z - 1)$ and we can rewrite (33) as

$$X_g(z) = \frac{\sum_{k=0}^{g-1} q_k z^k \prod_{l=k+1}^{g-1} Y_l(z)(1 - \lambda_k)}{z^g - \prod_{l=0}^{c-1} Y_l(z)} (z - 1). \tag{34}$$

Note that the numerator is a polynomial in z of degree $g - 1$. As in our general case we know that z_k, $k = 1, \dots, g-1$, are zeros of the numerator, and, thus, we can find a relation between q_k and z_k.

Remark 3. Without loss of generality we can assume that $\lambda_k \neq 1$ for all $k = 0, \dots, c-1$. To see this, consider first the case when $\lambda_k = 1$ for some $k \leqslant g - 1$. This means that during this green time-interval there is always an arrival. Thus, if there is a queue in the beginning of this time-interval, it will decrease by 1 and, then, increase by 1, i.e., it will be the same. If there is no queue in the beginning of the green time-interval, there will be no queue in the beginning of the next time-interval. Thus, we see that the queue length does not change during this time-interval. Hence, we can delete this time-interval and consider a system with less amount of the green time-intervals.

Now consider the case when $\lambda_k = 1$ for some $k \geqslant g$. This means that there is always an arrival during the red time. Therefore, during the first green time-interval there is always a queue (thus, $q_0 = 0$) and there is always a departure. Hence, if we take away the k^{th} time-interval and make the first green time-interval red, we will get the same distribution of the queue length after new red time-interval as it was in the original model after the first green time-interval. Therefore, in both cases, such a type of arrival decreases the green time and the complexity of the system.

Note that the above procedures do not influence the overflow queue, but they do influence the average queue length during the cycle. Therefore, we use this reduction only to find this pgf, and after finding it we need to return to the original model to be able to consider the average queue length during the cycle. In what follows we assume that $\lambda_k \neq 1$ for each $k = 0, \dots, c-1$.

Let r_k be the coefficient of z^k in the numerator. Then

$$\sum_{k=0}^{g-1} q_k z^k (1 - \lambda_k) \prod_{l=k+1}^{g-1} Y_l(z) = \sum_{k=0}^{g-1} r_k z^k = r_{g-1} \prod_{k=1}^{g-1} (z - z_k).$$

Table 1: The considered types of the arrival process.

Type of arrivals	Pgf $Y(z)$	Variance
Bernoulli	$\lambda z + (1-\lambda)$	$\lambda - \lambda^2$
Binomial	$\left(\frac{\lambda}{n} z + 1 - \frac{\lambda}{n}\right)^n$	$\lambda - \frac{\lambda^2}{n}$
Poisson	$e^{\lambda(z-1)}$	λ
Negative binomial	$\left(\frac{n}{n+\lambda-\lambda z}\right)^n$	$\lambda + \frac{\lambda^2}{n}$

So, as before, the coefficients r_k depends on the roots in a symmetric way:

$$\frac{r_k}{r_{g-1}} = \sigma_{g-1-k}(z_1, \dots, z_{g-1}).$$

The only problem is to find q_k from r_k. One can check, by finding the coefficients of z^n, $n = 0, \dots, g-1$, in each summand of the numerator that

$$r_n = \sum_{k=0}^{n} q_{n-k}(1-\lambda_{n-k})\,\sigma_k\left(\frac{\lambda_{n-k+1}}{1-\lambda_{n-k+1}}, \dots, \frac{\lambda_{g-1}}{1-\lambda_{g-1}}\right)\prod_{l=n-k+1}^{g-1}(1-\lambda_l).$$

By setting $\bar{q}_k = q_k \prod_{l=k}^{g-1}(1-\lambda_l)$ and changing the coefficient of summation from k to $j = n-k$, we can rewrite the later equation in a more compact way

$$r_n = \sum_{j=0}^{n} \bar{q}_j\,\sigma_{n-j}\left(\frac{\lambda_{j+1}}{1-\lambda_{j+1}}, \dots, \frac{\lambda_{g-1}}{1-\lambda_{g-1}}\right).$$

As we see r_n depends only on q_k for $k = 0, \dots, n$. Hence, the dependence matrix is triangular, and it is easy to find q_k from r_n.

Numerical results

In this section, we present numerical results on the following cases. We consider the impact of the variability of the arrival process, the difference between the FCTL assumption and the one-vehicle assumption, the impact of the traffic interruptions, caused by cyclists, trains or uncertain departure time. We end this section by comparing three types of green time allocation.

The variability of the arrival process

In this subsection, we show the impact of the variability of the arrival process on the queue length for FCTL model. In what follows λ denotes the arrival rate per time-interval, i.e., $\lambda = Y'(1)$. In Table 1 we summarize pgfs and variances of the considered arrival processes.

Note that Bernoulli arrivals have the least possible variance for a fixed λ. Indeed, the variance of an arbitrary arrival process with rate λ is equal to $Y''(1) + Y'(1) - (Y'(1))^2 \geqslant \lambda - \lambda^2$. In what follows the parameter n of binomial and negative binomial distribution is set to be equal to 2.

Table 2: The differences (in seconds) in the expected delay for different types of arrival process, $c = 60$ and load $x = 0.98(3)$.

Difference	$g = 5$	$g = 15$	$g = 30$	$g = 40$
$\mathbb{E}D_{negbin} - \mathbb{E}D_{pois}$	29.1472	28.6778	28.1833	27.7916
$\mathbb{E}D_{pois} - \mathbb{E}D_{binom}$	29.1369	28.6156	28.0097	27.5466
$\mathbb{E}D_{binom} - \mathbb{E}D_{bern}$	29.1258	28.5392	27.7332	27.0498

First, we focus on the expected delay per vehicle for different types of arrivals. The expected delay is computed using Little's law $\mathbb{E}D = \frac{\mathbb{E}L}{\lambda}$, where $\mathbb{E}L = \mathbb{E}L_{\text{FCTL}}$ is computed by (26). In Figure 3, the expected delay is plotted as a function of the load $x = \frac{c\lambda}{g}$ for $c = 60$ and $g = 5, 15, 30, 40$. The expected delay is given in seconds instead of time-intervals, one time-interval is set to be equal to 2 seconds.

Our first remark is that for arrivals with higher variability the expected delay is higher, but for each green time g the difference is really important only for a load higher than 0.8. As we can see for high load, the relative difference between expected delays for different types of arrivals is increasing with the green time. However, the absolute difference is decreasing. For load at $x = 0.98(3)$ (the highest load in the figures) the absolute difference is given in Table 2. In the table we use $\mathbb{E}D_{bern}$, $\mathbb{E}D_{binom}$, $\mathbb{E}D_{pois}$, $\mathbb{E}D_{negbin}$ for the expected delay in case of Bernoulli, binomial, Poisson and negative binomial arrivals respectively.

From the table we see that the absolute difference is decreasing with the green time. In many approximations, see [6] and [8], the dependence on the variance is set to be linear for the fixed arrival rate, green and cycle times. However, we see that this difference increases for higher variance.

Next we consider the probabilities for queue to be empty q_k for $k = 0, \dots, g-1$. In Figure 4, for $g = 10$, $c = 20$ and $\lambda = 0.2, 0.3, 0.4, 0.45$ these probabilities are plotted as functions of k. For the fixed g, c and λ the sum of probabilities is the same for different type of arrivals, but the distribution of this sum is different.

As we see for all rates of arrivals, in the beginning of the green time the probability for queue to be empty is lower for the arrival processes that have lower variability and in the end of green time the situation is reverse. For small arrival rate, the graphs are concave, but for high load the graphs are convex. Also for low load the difference between graphs for different arrival types is smaller than for higher load.

Comparing the FCTL and one-vehicle assumptions

Let us first consider $X'_{dif}(1)$, i.e., the expected difference in the queue length between FCTL and one-vehicle models. We plot it as a function of arrival rate $\lambda = Y'(1)$ in Figure 5.

Note that if on an intersection there are at least two conflicting "main" phases, then both of them has less than half of the cycle time. Thus, in case of stable system, the arrival rate is not bigger than 0.5. If we consider, for example, Poisson arrivals, the expected extra delay is $\frac{\lambda^2}{2(1-\lambda)} \leqslant \frac{1}{2(1-\lambda)} \leqslant 1$ time-interval, i.e., not more than 2 seconds. Therefore, for most of the cases the extra delay will be very small.

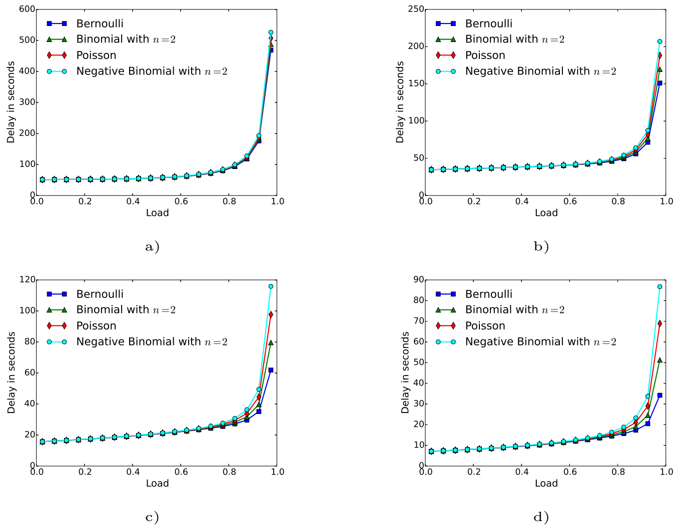


Figure 3: The expected delay as a function of load for Bernoulli, binomial, Poisson and negative binomial arrivals with $c = 60$, $n = 2$. For: a) $g = 5$ b) $g = 15$ c) $g = 30$ d) $g = 40$.

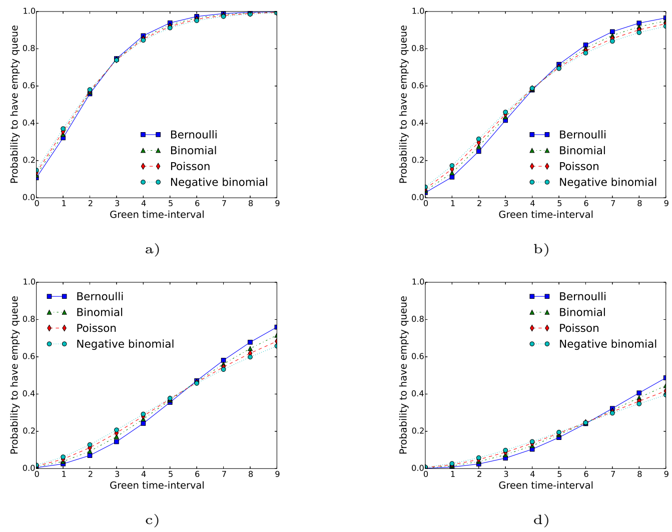


Figure 4: The probabilities q_k for queue to be empty, $k = 0, \ldots, g-1$, for $g = 10$, $c = 20$, $n = 2$. For: a) $\lambda = 0.2$, b) $\lambda = 0.3$, c) $\lambda = 0.4$, d) $\lambda = 0.45$.

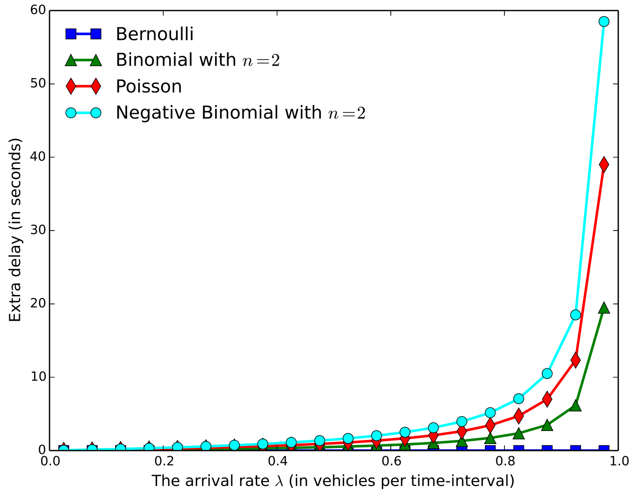


Figure 5: The expected extra delay due to the one-vehicle assumption as compared to the FCTL assumption.

Next we consider the distribution of $X_{dif}(z)$, i.e., the distribution of the queue length in a bottleneck. It is plotted in Figure 6 for rate $\lambda = 0.3, 0.5, 0.7, 0.9$ vehicles per time-interval. As we see, the queue with higher variance of the arrival process has a thicker tail than the queue with lower variance. The difference between distributions increases with the arrival rate.

As we derived, the absolute difference between FCTL and one-vehicle models is quite small. However, it is also interesting to know the relative difference. For the same settings as in Figure 3 we plot the relative difference of the expected queue length in Figure 7. For smaller green time the delay of FCTL model is already very high and the expected difference $X'_{dif}(1)$ is small. Thus, the relative difference is very small. For larger green time the delay is smaller and the arrival rate is bigger, so the relative difference is bigger. As we see for $g = 40$, this relative difference reaches 10% for negative binomial arrivals with $n = 2$.

Disruption of the traffic

In this subsection, we consider two types of a traffic disruption. The first one is the pedestrian/cyclist disruption and the second one is the ship/train disruption. Suppose that cyclists need 5 time-intervals, i.e., 10 seconds, to cross the road. There are two ways to provide the required green time. We can either shorten the green time of one lane, or, alternatively, extend the total cycle time and, hence, add extra red time to all lanes. Let p be the probability of cyclists arrival during the cycle. In Figure 8, we plot the overflow queue as a function of the rate for different p and g and fixed $c = 60$. Each graph we plot only up to load $x = 0.975$. The arrival process is assumed to be Poisson. As we see, the first way to deal with cyclists is highly disadvantageous for the lane. It significantly decreases the capacity of the lane and increases the overflow queue. The effect

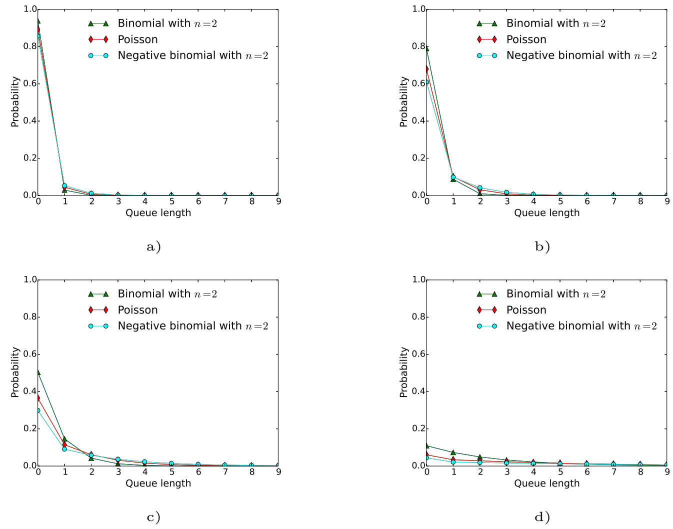


Figure 6: Distribution of $X_{dif}(z)$ for a) $\lambda = 0.3$, b) $\lambda = 0.5$, c) $\lambda = 0.7$ and d) $\lambda = 0.9$.

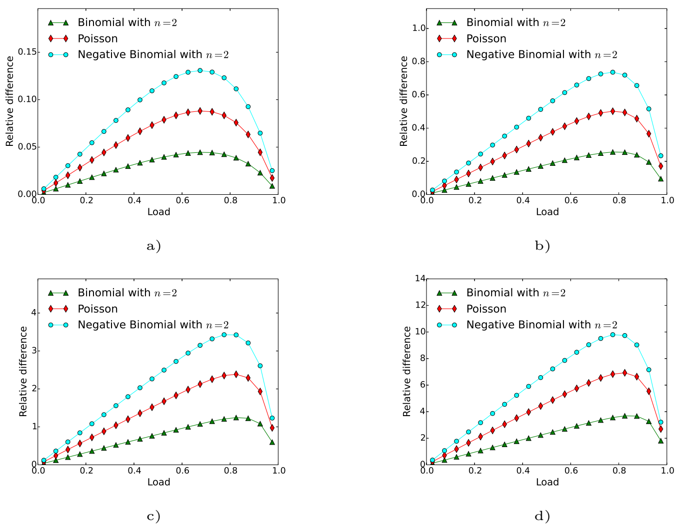


Figure 7: The relative difference (in %) of the expected delay in the FCTL and one-vehicle models as a function of load for binomial, Poisson and negative binomial arrivals for $c = 60$, $n = 2$ and a) $g = 5$ b) $g = 15$ c) $g = 30$ d) $g = 40$.

is stronger for smaller green time. The conclusion is that it is better, if possible, to increase the cycle time.

As we also see from the Figure, the system with uncertainty, i.e., $0 < p < 1$, has a larger overflow queue than the system without uncertainty for the same load. Consider two systems: one has no uncertainty, and another has uncertainty. Suppose that both the average green time and average cycle time are the same for these systems. Then the second system will have a bigger overflow queue, as shown in Figure 9.

Let us now consider the interruption by trains/ships. We assume that with probability p all green time during the cycle is effectively red. If the probability is $p = 0.0(3)$, $c = 60$, then on average there is one lost green time during an hour. In Figure 10, we plot the overflow queue as a function of the arrival rate. We see that the impact on busy lanes (with bigger green time) is larger than for lanes with short green time. Also even for small rate on both lanes the overflow queue is non-zero, as it is in the case of cyclists interruption.

Uncertain departure times

Consider the FCTL model with uncertain departure times that was presented in subsection 5.2. In Figure 11, we plot the expected overflow queue as a function of the arrival rate for different probability of departure. We suppose that, on average, a vehicle departs in 2 seconds. The length of the time-interval is set to be equal to $\tau = 2p$ seconds. The arrival process is assumed to be Poisson. We fixed the cycle time (in seconds) and consider two different green times.

The uncertainty in departure times does not influence the capacity of the system but only increases the the overflow queue and, consequently, the delay. The increase in the overflow queue seems to be the same. In fact, it is a bit bigger (about 0.1 difference) for the lane with smaller green time. However, this lane has a bigger overflow queue, and the relative difference for it is smaller.

Green time allocation problem

In this subsection, we consider the green time allocation problem. In [11], Webster proposed to provide each lane with a part of the total green time proportional to the arrival rate, i.e., such that the load is the same for each lane. On the one hand, as we see in Figure 3, the lane with the smallest green time, i.e., with the smallest arrival rate, in this case faces the greatest delay. On the other hand, this delay is experienced by a small part of the vehicles. Let us consider an example. Suppose we have three lanes with rates' ratio $\lambda_1 : \lambda_2 : \lambda_3 = 5 : 15 : 30 = 1 : 3 : 6$. Suppose also that we assign to them, in total, 50 green time-intervals out of 60 time-intervals in a cycle. We consider the following ways to assign green time:

- green time is proportional to the arrival rate,
- green time is allocated by minimizing the expected total queue length,
- green time is allocated by minimizing the maximum expected delay per lane.

Due to the computational efficiency of our method we minimize queue length or delay by using a simple exhaustive search.

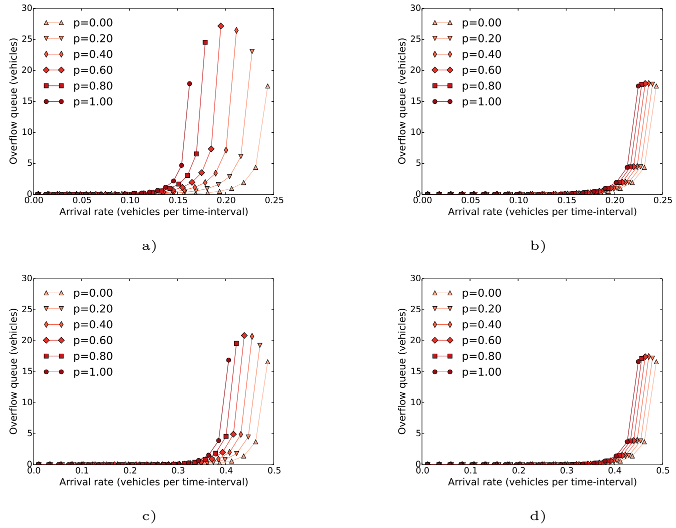


Figure 8: The overflow queue in case of cyclists as function of arrival rate. The cycle time is $c = 60$. The green time in figures a) and b) is $g = 15$ and on c) and d) $g = 30$. In figures a) and c) the green time for cyclists (5 time-intervals) is given from the green time of the lane. In figures b) and d) the extra time is added to the cycle.

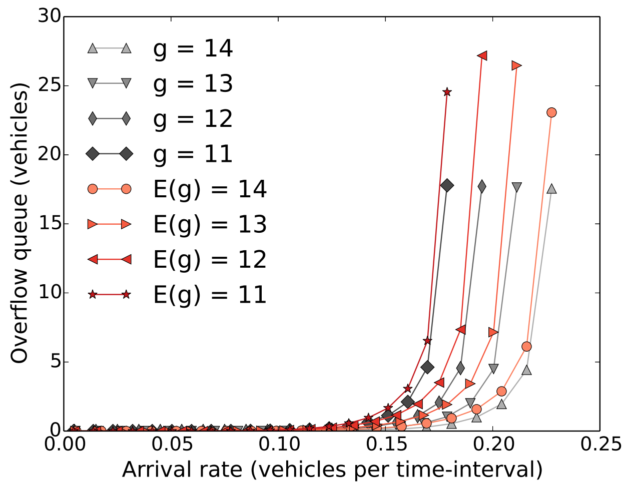
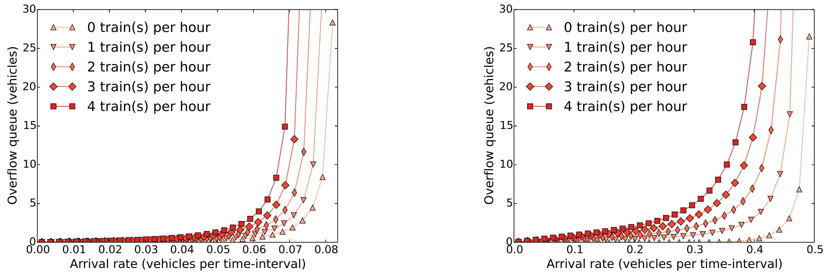
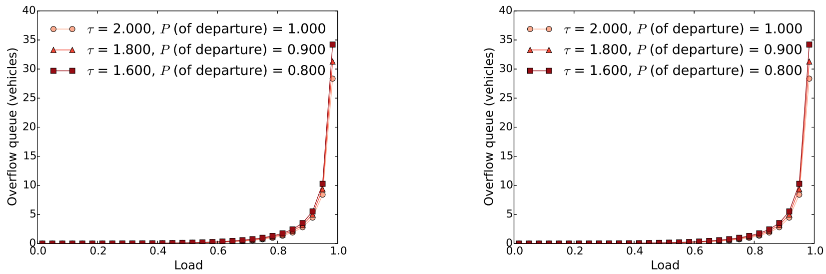


Figure 9: First four graphs represent overflow queue with fixed green time $g = 14, 13, 12, 11$. Last four graphs represent the overflow queue in case of cyclists. The cycle time is $c = 60$, $g = 15$. With probabilities $p = 0.2, 0.4, 0.6, 0.8$ the cyclists arrive and the green time is smaller by 5 time-intervals. The corresponding expected green time is $\mathbb{E}(g) = 14, 13, 12, 11$.



a) b)

Figure 10: The overflow queue for FCTL model with train disruption. The cycle time $c = 60$, the green times are a) $g = 5$ and b) $g = 30$.



a) b)

Figure 11: The overflow queue for FCTL model with uncertain departure times. The cycle time is 120 seconds, the green times are a) 10 and b) 60 seconds.

Let the total load be equal to $x = 0.9$. The results for Bernoulli and Poisson cases are given in Table 3. In the table, the total delay is a sum of average delays per lane. Even though it has no physical meaning, it measures the change in the expected delays.

Note that for both types of arrivals all three ways of the green time allocation give similar results. However, the difference in the delay between different ways is significant, especially for the first lane. So, as we see, the proportional green time is the most beneficial for the busiest lane but very unfair for the lane with the smallest rate. Using either minimal total delay or minimal delay per lane policy improves significantly (2 times) the situation for the lane with the smallest rate but increases the delay for the busiest lane.

For smaller load these three ways of the green time allocation work completely different. For the same settings $c = 60$, total red time equal to 10 and arrivals' rates proportional as $5 : 15 : 30$, we consider our three ways of allocation for different total loads. The result can be found in the figures 12 and 13. In the first one, the delay per lane in each type of allocation is plotted as a function of load. The second shows the allocated green time. The arrival process is assumed to be Poisson.

As we see, the minimum delay per lane policy suggests almost equal green time allocation for low total load, while the minimum total queue length policy suggests to give the largest part (more than proportional 30 time-intervals) of

Table 3: The comparison of the expected delay and queue length for different green time allocation policies. The delay is given in seconds and the green time in time-intervals. A time-interval is set to be equal to 2 seconds.

	Bernoulli arrivals				Poisson arrivals			
	Lane 1	Lane 2	Lane 3	Total	Lane 1	Lane 2	Lane 3	Total
Arrival rate λ	0.075	0.225	0.450	0.750	0.075	0.225	0.450	0.750
	Proportional green time							
Green time	5	15	30		5	15	30	
Delay	139.626	61.731	31.752	233.109	147.906	68.992	37.909	254.807
Queue length	5.236	6.945	7.144	19.325	5.546	7.762	8.529	21.838
	Minimal total queue length							
Green time	6	15	29		6	15	29	
Delay	68.881	61.731	38.096	168.708	71.097	68.992	48.670	188.759
Queue length	2.583	6.945	8.572	18.099	2.666	7.762	10.951	21.378
	Minimal delay per lane							
Green time	7	15	28		6	15	29	
Delay	56.267	61.731	55.355	173.354	71.097	68.992	48.670	188.759
Queue length	2.110	6.945	12.455	21.510	2.666	7.762	10.951	21.378

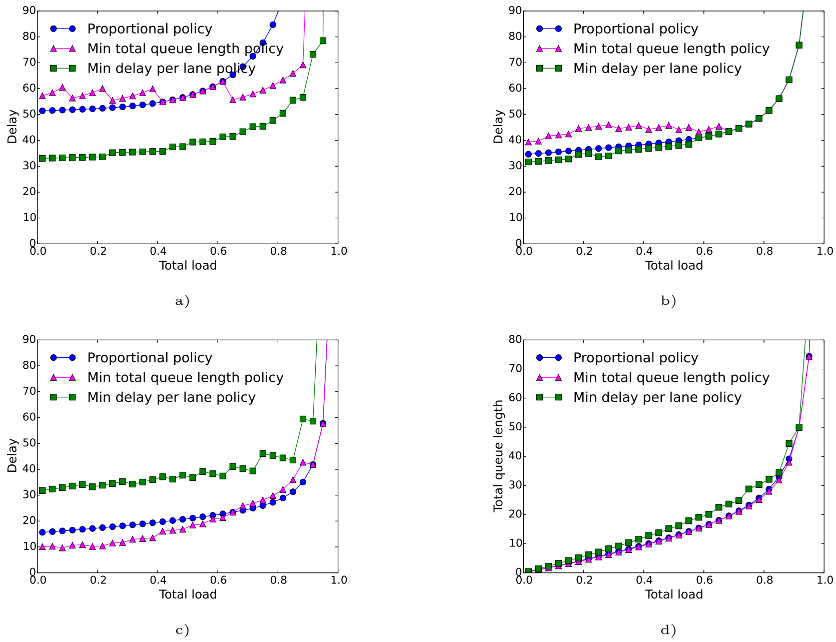


Figure 12: The delay per lane and the total queue length as function of total load for different ways of green time allocation. Figures a), b) and c) represent delay on lanes with low, medium and high arrivals' rates. Figure d) is the graph of the total queue length.

green time to the third lane with the highest arrival rate. The difference in delay for different policies is clearer for the lowest and highest arrival rate lanes. We also see that the total queue length for proportional policy and minimum total length are almost the same. However, we already saw from the table how significant the difference in the delay is (especially for the first lane).

For lanes 1 and 2 (low and medium rate) the minimum delay per lane policy is the most beneficial, while for the third lane it is the most disadvantageous. The minimum total length policy is conversely the most beneficial for the busiest lane and the most disadvantageous for the first two lanes. However, for higher load it gives smaller delay than the proportional policy.

Conclusions

We presented in this paper a new method to calculate the expectation and distribution of the queue length for a number of discrete-time queueing systems. We applied this method to the FCTL model and several of its extensions.

First, we studied the impact of arrival variability. The numerical results show that higher variance of the arrival process results in higher average delay, but the difference is noticeable only for a load higher than 0.8.

We compared the FCTL and one-vehicle assumptions. The absolute difference is quite small for many relevant applications, however the relative difference can be quite big. Additionally, we proved the decomposition rule for the one-vehicle model.

Comparing different disruptions of the traffic we found several interesting results. For example, the green time for cyclists or pedestrians is better to add to the cycle length than to give from the green time of one of the lines. Also the disruptions caused by trains and ships have a serious impact on the expected overflow queue and, consequently, the average delay. This kind of disruption causes the increase in the overflow queue even for a low load, while the disruption by cyclists and the uncertainty in departure time result in a overflow queue that is close to zero for low load. The uncertainty in departure times has the smallest impact on the overflow queue among considered types of disruption. It does not influence the stability of the system and the change in the overflow queue is relatively small.

Finally, we compared different green time allocation policies and concluded that the proportional policy may be extremely disadvantageous for the lanes with small arrival rate. The minimal total queue length policy may give too much priority to the busiest lanes, while the minimal delay per lane policy favours the lanes with lowest arrival lane. The choice of policy heavily depends on the goals of control.

The future work will be on extending this method to a wider range of models. For example, for the actuated control of an intersection.

Appendix. The proof of Theorem 1

Recall that we need to prove that $zB(w) \neq wB(z)$ for $z \neq w$, $z, w \in \bar{D}_1$. Consider Taylor expansion $\sum_{j=0}^{\infty} b_j z^j$ of $B(z)$ around zero. Let $\tilde{B}(z) = \sum_{j=2}^{\infty} b_j z^j$. Then, it is equivalent to prove that $z(b_0 + \tilde{B}(w)) \neq w(b_0 + \tilde{B}(z))$ for $z \neq w$,

$z, w \in \bar{D}_1$. Now, we can reformulate the theorem in the following way: there is not more than one root of the equation $z = a(b_0 + \tilde{B}(z))$ in $\bar{D}_1$ for each $a \in \mathbb{C}$. Number a here represents the value $\frac{w}{b_0+\tilde{B}(w)}$ for some $w \in \bar{D}_1$. To make such a reformulation we need to show that $b_0 + \tilde{B}(w) \neq 0$ for all $w \in \bar{D}_1$.

First we show it for $w = 1$. Consider the derivative of $\tilde{B}(z)$ at 1. On one hand, according to the stability assumption, we get that $\tilde{B}'(1) = B'(1) - b_1 < 1 - b_1 = \tilde{B}(1) + b_0$. On the other hand, $b_k \geqslant 0$ for $k \in \mathbb{N}$, and, thus,

$$\tilde{B}'(1) = \sum_{j=2}^{\infty} j b_j \geqslant \sum_{j=2}^{\infty} 2 b_j = 2\tilde{B}(1).$$

Finally, we get that $2\tilde{B}(1) \leqslant \tilde{B}'(1) < \tilde{B}(1) + b_0$. Therefore, $b_0 > \tilde{B}(1)$.

Now, we need to use the following small lemma:

Lemma 4. *Consider an analytic function $C(z) = \sum_{j=0}^{\infty} c_j z^j$ in $D_{r+\delta}$, where $r \in \mathbb{R}$, $\delta > 0$. Suppose that $c_j \geqslant 0$ for all $j \in \mathbb{N} \cup \{0\}$, then the absolute value of $C(z)$ reaches the maximum value in r, i.e., $|C(z)| \leqslant C(r)$ for each $z \in \bar{D}_r$.*

Proof. Since the function $C(z)$ is analytic in disk $D_{r+\delta}$, Taylor series converges absolutely. Therefore, $|C(z)| \leqslant \sum_{j=0}^{\infty} c_j |z|^j \leqslant \sum_{j=0}^{\infty} c_j r^j = C(r)$. □

As a corollary we get that $|\tilde{B}(z)| < b_0$ for each $z \in \bar{D}_1$, and, consequently, there are no solutions of the equation $b_0 + \tilde{B}(z) = 0$ in $\bar{D}_1$. Hence, it is sufficient to prove that equations

$$z = a(b_0 + \tilde{B}(z)) \tag{35}$$

has not more than one root in $\bar{D}_1$ for each $a \in \mathbb{C}$.

If $a = 0$, we have the simple equation $z = 0$ that has not more than one solution. If $a \neq 0$, we can uniquely represent it as $te^{i\varphi}$, where $0 \leqslant \varphi < 2\pi$, $t > 0$. We want to prove, for a fixed value of φ, that the amount of roots inside the unit disk does not increase when t increases. To do so, we consider our roots as functions of t and prove that the absolute value of the root increases as t increases. Suppose $z(t)$ is a root of the equation (35) inside the unit disk. Consider its derivative:

$$\frac{dz}{dt} = \frac{d}{dt}(te^{i\varphi}(b_0 + \tilde{B}(z))) = e^{i\varphi}(b_0 + \tilde{B}(z)) + te^{i\varphi}\tilde{B}'(z)\frac{dz}{dt}.$$

Rearranging and plugging $\frac{z}{t(b_0+\tilde{B}(z))}$ instead of $e^{i\varphi}$ give us

$$t\frac{dz}{dt} = \frac{z}{1 - \frac{z\tilde{B}'(z)}{b_0+\tilde{B}(z)}}.$$

Thus, the derivative of the $|z(t)|^2$ is equal to

$$\frac{d(z\bar{z})}{dt} = \frac{dz}{dt}\bar{z} + \frac{d\bar{z}}{dt}z = 2Re\left(\frac{dz}{dt}\bar{z}\right) = \frac{2|z|^2}{tRe\left(1 - \frac{z\tilde{B}'(z)}{b_0+\tilde{B}(z)}\right)}.$$

To prove that this derivative is positive we only need to prove the following lemma.

Lemma 5. *There is some $\delta > 0$ such that for each $z \in D_{1+\delta}$*

$$Re\left(1 - \frac{z\tilde{B}'(z)}{b_0 + \tilde{B}(z)}\right) > 0. \tag{36}$$

Proof. First of all, it is sufficient to prove that (36) holds for each $z \in \bar{D}_1$. Indeed, if for some point z inequality (36) holds, then for some neighbourhood of z it also holds. Since $\bar{D}_1$ is compact, if (36) holds in $\bar{D}_1$, it also holds for small neighbourhood $D_{1+\delta}$, $\delta > 0$.

Now note that inequality (36) is equivalent to the following inequality:

$$Re\left((b_0 + \tilde{B}(z))(b_0 + \tilde{B}(\bar{z})) - z\tilde{B}'(z)(b_0 + \tilde{B}(\bar{z}))\right) > 0.$$

It can be rewritten as $b_0^2 > Re\left(b_0(z\tilde{B}'(z) - \tilde{B}(z) - \tilde{B}(\bar{z})) + (z\tilde{B}'(z) - \tilde{B}(z))\tilde{B}(\bar{z})\right)$. Since $Re(\tilde{B}(\bar{z})) = Re(\tilde{B}(z))$, we finally get that (36) is equivalent to

$$b_0^2 > Re\left(b_0(z\tilde{B}'(z) - 2\tilde{B}(z)) + (z\tilde{B}'(z) - \tilde{B}(z))\tilde{B}(\bar{z})\right). \tag{37}$$

Note that $Re(z) \leqslant |z|$ for each $z \in \mathbb{C}$, and, thus,

$$Re\left(b_0(z\tilde{B}'(z) - 2\tilde{B}(z)) + (z\tilde{B}'(z) - \tilde{B}(z))\tilde{B}(\bar{z})\right) \leqslant b_0|z\tilde{B}'(z) - 2\tilde{B}(z)| + |z\tilde{B}'(z) - \tilde{B}(z)| \cdot |\tilde{B}(z)|.$$

Since functions $z\tilde{B}'(z) - 2\tilde{B}(z)$, $z\tilde{B}'(z) - \tilde{B}(z)$ and $\tilde{B}(z)$ are analytical and have positive coefficients in their Taylor expansion at 0, we can use lemma 4. This gives us that

$$Re\left(b_0(z\tilde{B}'(z) - 2\tilde{B}(z)) + (z\tilde{B}'(z) - \tilde{B}(z))\tilde{B}(\bar{z})\right) \leqslant$$
$$\leqslant b_0(\tilde{B}'(1) - 2\tilde{B}(1)) + (\tilde{B}'(1) - \tilde{B}(1)) \cdot (\tilde{B}(1)) < b_0(b_0 - \tilde{B}(1)) + b_0\tilde{B}(1) = b_0^2.$$

Here we used that $\tilde{B}'(1) < \tilde{B}(1) + b_0$. Thus, we proved for each $z \in \bar{D}_1$ inequality (37), which is equivalent to (36). □

We proved that $\frac{d|z\bar{z}|}{dt} > 0$ for each $z = z(t) \in D_{1+\delta} \setminus \{0\}$. Since $b_0 + \tilde{B}(0) \neq 0$, we get that $z(t) \neq 0$. Therefore, any root of the equation (35) for $t > 0$ goes outside $\bar{D}_{1+\delta}$. Hence, the amount of roots inside D_1 decreases when t increases. But for small $t > 0$ we can use Rouche's theorem to show that there is only one root of the equation (35). Thus, we have proved Theorem 1.

References

[1] I. J. Adan, J. van Leeuwaarden, and E. M. Winands. On the application of rouché's theorem in queueing theory. *Operations Research Letters*, 34(3):355–360, 2006.

[2] N. T. Bailey. On queueing processes with bulk service. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 80–87, 1954.

[3] J. Darroch. On the traffic-light queue. *The Annals of Mathematical Statistics*, 35(1):380–388, 1964.

[4] A. J. Janssen and J. S. van Leeuwaarden. Analytic computation schemes for the discrete-time bulk service queue. *Queueing Systems*, 50(2-3):141–163, 2005.

[5] D. R. McNeil. A solution to the fixed-cycle traffic light problem for compound poisson arrivals. *Journal of Applied Probability*, 5(3):624–635, 1968.

[6] A. J. Miller. Settings for fixed-cycle traffic signals. *OR*, pages 373–386, 1963.

[7] G. F. Newell. Approximation methods for queues with application to the fixed-cycle traffic light. *Siam Review*, 7(2):223–240, 1965.

[8] M. S. van den Broek, J. van Leeuwaarden, I. J. Adan, and O. J. Boxma. Bounds and approximations for the fixed-cycle traffic-light queue. *Transportation Science*, 40(4):484–496, 2006.

[9] J. S. van Leeuwaarden. Delay analysis for the fixed-cycle traffic-light queue. *Transportation Science*, 40(2):189–199, 2006.

[10] Ė. B. Vinberg. *A course in algebra*. Number 56. American Mathematical Soc., 2003.

[11] F. V. Webster. Traffic signal settings. Technical report, 1958.

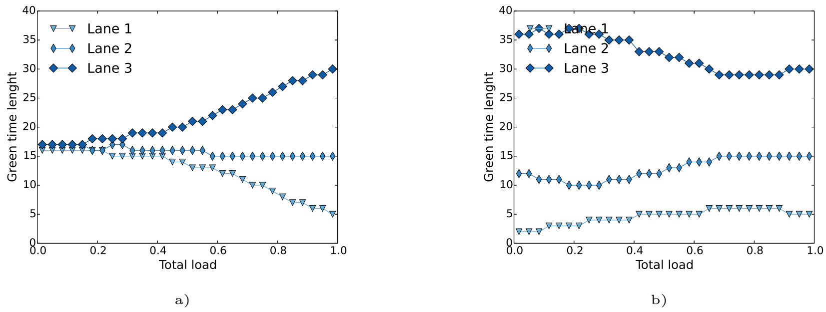


Figure 13: The green time allocated to each lane for a) minimum delay per lane policy and b) minimum total queue length. Lanes 1, 2, 3 correspond to the lanes with low, medium and high arrival rate.