Spectral properties of birth-death polynomials

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Abstract. We consider sequences of polynomials that are defined by a three-terms recurrence relation and orthogonal with respect to a positive measure on the nonnegative axis. By a famous result of Karlin and McGregor such sequences are instrumental in the analysis of birth-death processes. Inspired by problems and results in this stochastic setting we present necessary and sufficient conditions in terms of the parameters in the recurrence relation for the smallest or second smallest point in the support of the orthogonalizing measure to be larger than zero, and for the support to be discrete with no finite limit point.

Keywords: birth-death process, orthogonal polynomials, orthogonalizing measure, spectrum, Stieltjes moment problem

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1 Introduction

We are concerned with a sequence of polynomials $\{P_n\}$ defined by the three-terms recurrence relation

$$\begin{aligned} &P_{n+1}(x) = (x - \lambda_n - \mu_n)P_n(x) - \lambda_{n-1}\mu_n P_{n-1}(x), \quad n > 0, \\ &P_1(x) = x - \lambda_0 - \mu_0, \quad P_0(x) = 1, \end{aligned} \tag{1}$$

where $\lambda_n > 0$ for $n \geq 0$, $\mu_n > 0$ for $n \geq 1$ and $\mu_0 \geq 0$. Since polynomial sequences of this type play an important role in the analysis of *birth-death processes* – continuous-time Markov chains on an ordered set with transitions only to neighbouring states – we will refer to $\{P_n\}$ as the sequence of *birth-death polynomials* associated with the *birth rates* λ_n and *death rates* μ_n. For more information on the relation between a sequence of birth-death polynomials and the corresponding birth-death process we refer to the seminal papers of Karlin and McGregor [18] and [19].

By Favard's theorem (see, for example, Chihara [8]) there exists a probability measure (a Borel measure of total mass 1) on $\mathbb{R}$ with respect to which the polynomials P_n are orthogonal. In the terminology of the theory of moments the *Hamburger moment problem* associated with the polynomials P_n is solvable. Actually, as shown by Karlin and McGregor [18] and Chihara [6] (see also [8, Theorem I.9.1 and Corollary]), even the *Stieltjes moment problem* associated with $\{P_n\}$ is solvable, which means that there exists an orthogonalizing measure ψ for $\{P_n\}$ with support on the nonnegative axis, that is,

$$\int_{[0,\infty)} P_n(x)P_m(x)\psi(dx) = k_n\delta_{nm}, \quad n,\ m \geq 0, \tag{2}$$

with $k_n > 0$. The Stieltjes moment problem associated with $\{P_n\}$ is said to be *determined* if ψ is uniquely determined by (2), and *indeterminate* otherwise. In the latter case there is, by [7, Theorem 5], a unique orthogonalizing measure for which the infimum of its support is maximal. We will refer to this measure as the *natural* measure for $\{P_n\}$. In what follows ψ will always refer to the natural measure for $\{P_n\}$ if the Stieltjes moment problem associated with $\{P_n\}$ is indeterminate.

Of particular interest to us are the quantities ξ_i, recurrently defined by

$$\xi_1 := \inf \operatorname{supp}(\psi), \tag{3}$$

and

$$\xi_{i+1} := \inf\{\operatorname{supp}(\psi) \cap (\xi_i, \infty)\}, \quad i \geq 1, \tag{4}$$

where $\operatorname{supp}(\psi)$ denotes the support of the measure ψ, also referred to as the *spectrum* of ψ (or of the polynomials P_n). In addition we let

$$\sigma := \lim_{i\to\infty} \xi_i, \tag{5}$$

the first limit point of $\operatorname{supp}(\psi)$ if it exists, and infinity otherwise. It is clear from the definition of ξ_i that, for all $i \geq 1$,

$$\xi_{i+1} \geq \xi_i \geq 0,$$

and

$$\xi_i = \xi_{i+1} \iff \xi_i = \sigma.$$

In the analysis of a birth-death process on a countable state space – a birth-death process on the nonnegative integers with birth rate λ_n and death rate μ_n in state n, say – the question of whether the time-dependent transition probabilities of the process converge to their limiting values exponentially fast as time goes to infinity has attracted considerable attention. This question may be translated into the setting of the polynomials P_n of (1) by asking whether $\xi_1 > 0$, and if not, whether $\xi_2 > 0$, since the exponential rate of convergence (or *decay parameter*) α of the birth-death process satisfies

$$\alpha = \begin{cases} \xi_1 & \text{if} \quad \xi_1 > 0 \\ \xi_2 & \text{if} \quad \xi_2 > \xi_1 = 0 \\ 0 & \text{if} \quad \xi_2 = \xi_1 = 0 \end{cases}$$

(see, for example, [14]). Note that

$$\alpha > 0 \iff 0 < \sigma \leq \infty, \tag{6}$$

so the above question may be rephrased by asking whether $0 < \sigma \leq \infty$. Recent results, in particular in the Chinese literature, have culminated in a complete solution of the problem in the stochastic setting by revealing simple and easily verifiable conditions for exponential convergence in terms of the birth and death rates. The purpose of this paper is to present these results in an orthogonal-polynomial context, and to provide new proofs for some of the results by using tools from the orthogonal-polynomial toolbox. Our methods enable us also to establish a simple, necessary and sufficient condition for $\sigma = \infty$, that is, for the spectrum of the orthogonalizing measure to be discrete with no finite limit point, thus extending another recent result.

Before stating the results in Section 3 and discussing proofs in Section 4 we present a number of preliminary results in Section 2. Additional information on related literature and some concluding remarks will be given in Section 5.

2 Preliminaries

It will be convenient to define the constants π_n by

$$\pi_0 := 1 \quad \text{and} \quad \pi_n := \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}, \quad n > 0. \tag{7}$$

and to use the shorthand notation

$$K_n := \sum_{i=0}^{n} \pi_i, \quad n \geq 0, \quad K_\infty := \sum_{i=0}^{\infty} \pi_i, \tag{8}$$

and

$$L_n := \sum_{i=0}^{n} (\lambda_i \pi_i)^{-1}, \quad n \geq 0, \quad L_\infty := \sum_{i=0}^{\infty} (\lambda_i \pi_i)^{-1}. \tag{9}$$

With the convention that the measure ψ in (2) is interpreted as the natural measure if the Stieltjes moment problem associated with $\{P_n\}$ is indeterminate, the quantities ξ_i and σ of (3)–(5) may be defined alternatively in terms of the (simple and positive) zeros of the polynomials $P_n(x)$ (see [8, Section II.4]). Namely, with $x_{n1} < x_{n2} < \dots < x_{nn}$ denoting the n zeros of $P_n(x)$, we have the classic separation result

$$0 < x_{n+1,i} < x_{ni} < x_{n+1,i+1}, \quad i = 1, 2, \dots, n, \ n \geq 1,$$

so that the limits as $n \to \infty$ of x_{ni} exist, while

$$\lim_{n\to\infty} x_{ni} = \xi_i, \quad i = 1, 2, \ldots.$$

If the Stieltjes moment problem associated with $\{P_n\}$ is indeterminate then, by [7, Theorems 4 and 5], we have $\xi_{i+1} > \xi_i > 0$ for all $i \geq 1$ and $\sigma = \lim_{i\to\infty} \xi_i = \infty$, so that the spectrum of the (natural) measure ψ actually coincides with the set $\{\xi_1, \xi_2, \ldots\}$. So in this setting the questions of whether $\xi_1 > 0$ and the spectrum is discrete with no finite limit point can be answered in the affirmative. It is therefore no restriction to assume in what follows that

$$K_\infty + L_\infty = \infty, \tag{10}$$

which, by [9, Theorem 2], is necessary – and, if $\mu_0 = 0$, also sufficient – for the Stieltjes moment problem associated with $\{P_n\}$ to be determined.

Under these circumstances we know from [19] (or from classic results on the moment problem in [26]) that

$$\psi(\{0\}) = \begin{cases} \dfrac{1}{K_\infty} & \text{if } \mu_0 = 0 \text{ and } K_\infty < \infty \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

so that

$$\mu_0 > 0 \text{ or } (\mu_0 = 0 \text{ and } L_\infty < \infty) \implies \xi_1 > 0 \text{ or } \sigma = 0. \tag{12}$$

Actually, under the premise in (12) the measure ψ has a finite moment of order -1, since, by [19, (9.9) and (9.14)],

$$\int_0^\infty \frac{\psi(dx)}{x} = \frac{L_\infty}{1 + \mu_0 L_\infty}, \tag{13}$$

which, if $L_\infty = \infty$, should be interpreted as infinity if $\mu_0 = 0$ and as μ_0^{-1} if $\mu_0 > 0$.

3 Results

In what follows we maintain the assumption $K_\infty + L_\infty = \infty$. Our first proposition deals with a simple case.

Proposition 1 If $K_\infty = L_\infty = \infty$ then $\sigma = 0$.

Indeed, for $\mu_0 = 0$ this result follows immediately from (11) and (13), while it is known (see [14, p. 527]) that changing the value of μ_0 (or, for that matter, of any finite number of birth and death rates) does not affect the value of σ.

Our next result is a proposition on the basis of which all the remaining results of this section can be obtained using orthogonal-polynomial techniques.

Proposition 2 Let $K_\infty < \infty$ and $\mu_0 > 0$. Then

$$\frac{1}{4R} \le \xi_1 \le \frac{1}{R}$$

if $R := \sup_n L_n(K_\infty - K_n) < \infty$, and $\xi_1 = 0$ otherwise.

This proposition was stated explicitly for the first time (in terms of the decay parameter of an absorbing birth-death process) by Sirl et al. [27]. These authors do not provide a proof, but note that the techniques employed by Chen to analyse *ergodic* birth-death processes – which in our setting correspond to the case $K_\infty < \infty$ and $\mu_0 = 0$ – are applicable to absorbing birth-death processes as well (see in particular [2, Theorem 3.5]). Mu-Fa Chen himself stated the result of Proposition 2 explicitly in [4, Theorem 4.2]. Chen's technique involves *Dirichlet forms*, but recently Proposition 2 was proven in [17] using orthogonal-polynomial and eigenvalue techniques. A sketch of the argument employed in [17], emphasizing and elucidating the orthogonal-polynomial aspects, will be given in Section 4.

We next list a number of results as corollaries of the Propositions 1 and 2.

Corollary 1 (i) If $K_\infty < \infty$ and $\mu_0 > 0$, then

$$\xi_1 > 0 \iff 0 < \sigma \le \infty \iff \sup_n L_n(K_\infty - K_n) < \infty.$$

(ii) If $K_\infty < \infty$ and $\mu_0 = 0$, then $\xi_1 = 0$ and

$$\xi_2 > 0 \iff 0 < \sigma \le \infty \iff \sup_n L_n(K_\infty - K_n) < \infty.$$

(iii) If $L_\infty < \infty$, then

$$\xi_1 > 0 \iff 0 < \sigma \le \infty \iff \sup_n K_n(L_\infty - L_n) < \infty.$$

Corollary 2 If $\sigma = \infty$ then $K_\infty < \infty$ or $L_\infty < \infty$. Moreover,

(i) if $K_\infty < \infty$, then

$$\sigma = \infty \iff \lim_{n\to\infty} L_n(K_\infty - K_n) = 0;$$

(ii) if $L_\infty < \infty$, then

$$\sigma = \infty \iff \lim_{n\to\infty} K_n(L_\infty - L_n) = 0.$$

Corollary 1 (i) is [27, Corollary 1]. Corollary 1 (ii) (in the setting of birth-death processes) is the oldest result and was first presented by Mu-Fa Chen in [2]. Together with many related and more refined results, the statements (i) and (iii) of Corollary 1 appear in the survey paper [4]. Corollary 2 (i) for the case $\mu_0 = 0$ was presented by Mao in [22], but announced already as a result of Mao's in [3]. In its generality Corollary 2 is new.

4 Proofs

Obviously, Corollary 1 (i) follows immediately from (12) and Proposition 2, and the first statement of Corollary 2 from Proposition 1. The proofs of the remaining statements in the Corollaries 1 and 2 will be given in three steps. In the first step, elaborated in Subsection 4.1, we will show that by employing the duality concept for birth-death processes introduced by Karlin and McGregor [18, 19] one can show that the results of both corollaries for the case $L_\infty < \infty$ are implied by the results for the case $K_\infty < \infty$.

In the second step, elaborated in Subsection 4.2, we will show that by using properties of *co-recursive* polynomials the statements of the corollaries for the case $K_\infty < \infty$ and $\mu_0 = 0$ are implied by the results for the case $K_\infty < \infty$ and $\mu_0 > 0$.

In Subsection 4.3 we will apply results on *associated* polynomials to obtain the statement of Corollary 2 for the case $K_\infty < \infty$ and $\mu_0 > 0$ from Corollary 1 (i). As announced, we conclude in Subsection 4.4 with a sketch of the proof of Proposition 2 presented in [17], and some elucidative remarks.

4.1 Dual polynomials

Our point of departure in this subsection is a sequence of birth-death polynomials $\{P_n\}$ satisfying the recurrence relation (1) with $\mu_0 > 0$. Following Karlin and McGregor [18, 19], we define the *dual polynomials* P_n^d by a recurrence relation similar to (1) but with parameters λ_n^d and μ_n^d given by $\mu_0^d = 0$ and

$$\lambda_n^d := \mu_n, \quad \mu_{n+1}^d := \lambda_n, \quad n \geq 0.$$

Accordingly, we define $\pi_0^d = 1$ and, for $n \geq 1$,

$$\pi_n^d = \frac{\lambda_0^d \lambda_1^d \dots \lambda_{n-1}^d}{\mu_1^d \mu_2^d \dots \mu_n^d} = \frac{\mu_0 \mu_1 \dots \mu_{n-1}}{\lambda_0 \lambda_1 \dots \lambda_{n-1}},$$

and note that

$$\pi_{n+1}^d = \mu_0 (\lambda_n \pi_n)^{-1} \quad \text{and} \quad (\lambda_n^d \pi_n^d)^{-1} = \mu_0^{-1} \pi_n. \tag{14}$$

So the assumption (10) is equivalent to

$$\sum_{n=0}^{\infty} \left(\pi_n^d + (\lambda_n^d \pi_n^d)^{-1} \right) = \infty.$$

The polynomials P_n and P_n^d are easily seen to be related by

$$P_{n+1}^d(x) = P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0. \tag{15}$$

In the terminology of Chihara [8, Section I.7-9] the polynomials P_n are the *kernel polynomials* (with κ-parameter 0) corresponding to the polynomials P_n^d. As a consequence, there is a unique (natural) measure ψ^d on the nonnegative real axis with respect to which the polynomials P_n^d are orthogonal. By [18, Lemma 3] we actually have

$$\mu_0 \psi([0, x]) = x \psi^d([0, x]), \quad x \geq 0.$$

With ξ_i^d and σ^d denoting the quantities defined by (3)–(5) if we replace ψ by ψ^d, we thus have, for $i \geq 1$,

$$\xi_i = \begin{cases} \xi_{i+1}^d & \text{if } \ \xi_1^d = 0 \ \text{ and } \ \sigma^d > 0 \\ \xi_i^d & \text{otherwise,} \end{cases} \tag{16}$$

and

$$\sigma^d = \sigma. \tag{17}$$

With (14) and (16) it is now easy to see that statement (iii) of Corollary 1 is implied by statement (ii) if $\mu_0 > 0$, and by statement (i) if $\mu_0 = 0$. Also, statement (ii) of Corollary 2 follows from statement (i), as a consequence of (17).

4.2 Co-recursive polynomials

Our point of departure in this subsection is the sequence of birth-death polynomials $\{P_n\}$ satisfying the recurrence relation (1) with $\mu_0 = 0$. With $\{P_n\}$ we associate a sequence of birth-death polynomials $\{P_n^*\}$ with parameters λ_n^* and μ_n^* that are identical to those of $\{P_n\}$ except that $\mu_0^* = c > 0$. So the polynomials P_n^* satisfy $P_0^*(x) = 1$ and

$$P_{n+1}^*(x) = (x - \lambda_n - \mu_n)P_n^*(x) - \lambda_{n-1}\mu_n P_{n-1}^*(x), \quad n > 0,$$

but

$$P_1^*(x) = x - \lambda_0 - c = P_1(x) - c.$$

Evidently, there is unique (natural) orthogonalizing measure ψ^* for the polynomials P_n^* and we can define quantities ξ_i^* and σ^* in terms of ψ^* analogously to (3)–(5). Moreover ξ_i^* is the limit as $n \to \infty$ of x_{ni}^*, the ith smallest zero of the polynomial $P_n^*(x)$.

Given the polynomials P_n, the polynomials P_n^* are called *co-recursive* polynomials and have been studied for the first time by Chihara [5]. In particular, applying [5, Theorem 1] to the situation at hand, we have

$$x_{n,i} < x_{n,i}^* < x_{n,i+1} < x_{n,i+1}^* \quad i = 1, \ldots, n-1, \ n > 0.$$

Subsequently letting n tend to infinity we obtain

$$\xi_i \le \xi_i^* \le \xi_{i+1} \le \xi_{i+1}^* \quad i \ge 1, \tag{18}$$

and hence

$$\sigma^* = \sigma. \tag{19}$$

We have now gathered sufficient information to conclude that statement (i) of Corollary 1 implies statement (ii). Indeed, suppose the parameters in the recurrence relation for the polynomials P_n satisfy $K_\infty < \infty$ and $\mu_0 = 0$. Then, by applying Corollary 1 (i) to the polynomials P_n^* we conclude that $\xi_1^* > 0$ is equivalent to $\sigma^* > 0$, and to $\sup_n L_n(K_\infty - K_n) < \infty$. But $\xi_1^* > 0$ is equivalent to $\xi_2 > 0$ since $\xi_1 \le \xi_1^* \le \xi_2 \le \xi_2^*$, by (18), while we cannot have $\xi_1^* = 0$ if $\xi_2^* > 0$, by (12). Finally, $\sigma^* > 0$ is equivalent to $\sigma > 0$ by (19).

In view of (19) it also follows that to prove Corollary 2 (i) it suffices to establish the result for $\mu_0 > 0$.

4.3 Associated polynomials

Throughout this subsection we assume $K_\infty < \infty$. The *associated* (or *numerator*) polynomials $P_n^{(k)}$ of order $k \ge 0$ associated with the sequence $\{P_n\}$ of (1) are given by the recurrence relation

$$\begin{aligned} &P_{n+1}^{(k)}(x) = (x - \lambda_{n+k} - \mu_{n+k})P_n^{(k)}(x) - \lambda_{n+k-1}\mu_{n+k}P_{n-1}^{(k)}(x), \quad n > 0, \\ &P_1^{(k)}(x) = x - \lambda_k - \mu_k, \quad P_0^{(k)}(x) = 1. \end{aligned}$$

Defining $\xi_i^{(k)}$ and $\sigma^{(k)}$ as in (3)–(5) with ψ replaced by $\psi^{(k)}$ we have

$$\xi_1^{(k)} \le \xi_1^{(k+1)}, \quad k \ge 0, \quad \text{and} \quad \lim_{k\to\infty} \xi_1^{(k)} = \sigma. \tag{20}$$

from [8, Theorem III.4.2] and [13, Theorem 1], respectively. Moreover, defining $\pi_n^{(k)}$, $K_n^{(k)}$, $K_\infty^{(k)}$ and $L_n^{(k)}$ as in (7)–(9) with λ_n and μ_n replaced by λ_{n+k} and μ_{n+k}, respectively, it is readily seen that $\pi_i^{(k)} = \pi_{i+k}/\pi_k$, so that

$$K_n^{(k)} = \frac{1}{\pi_k}\left(K_{n+k} - K_{k-1}\right), \quad K_\infty^{(k)} = \frac{1}{\pi_k}\left(K_\infty - K_{k-1}\right) < \infty,$$

and

$$L_n^{(k)} = \pi_k(L_{n+k} - L_{k-1}).$$

(These relations are valid for $k \geq 0$ if we let $K_{-1} = L_{-1} = 0$.) It follows that $R^{(k)} := \sup_n L_n^{(k)}(K_\infty^{(k)} - K_n^{(k)})$ satisfies

$$R^{(k)} = \sup_n \ (L_{n+k} - L_{k-1})(K_\infty - K_{n+k}).$$

Applying Proposition 2 to $\xi_1^{(k)}$ we find that

$$\frac{1}{4R^{(k)}} \leq \xi_1^{(k)} \leq \frac{1}{R^{(k)}}, \quad k \geq 0,$$

so by (20) we have $\sigma = \infty$ if and only if $\lim_{k\to\infty} R^{(k)} = \infty$, which is easily seen to be equivalent to statement (i) of Corollary 2.

4.4 Proposition 2: Sketch of proof and remarks

The zeros x_{ni} of the polynomials P_n of (1) may be interpreted as eigenvalues of a symmetric tridiagonal matrix (or *Jacobi matrix*). Indeed, let I denote the identity matrix and

$$J_n := \begin{pmatrix} \lambda_0+\mu_0 & -\sqrt{\lambda_0\mu_1} & 0 & \cdots & 0 & 0 \\ -\sqrt{\lambda_0\mu_1} & \lambda_1+\mu_1 & -\sqrt{\lambda_1\mu_2} & \cdots & 0 & 0 \\ 0 & -\sqrt{\lambda_1\mu_2} & \lambda_2+\mu_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1}+\mu_{n-1} & -\sqrt{\lambda_{n-1}\mu_n} \\ 0 & 0 & 0 & \cdots & -\sqrt{\lambda_{n-1}\mu_n} & \lambda_n+\mu_n \end{pmatrix}.$$

Then, expanding $\det(xI - J_n)$ by its last row and comparing the result with the recurrence relation (1), it follows that we can identify $\det(xI - J_n)$ with the polynomial $P_{n+1}(x)$. So a representation for $\xi_1 = \lim_{n\to\infty} x_{n1}$ may be obtained by letting n tend to infinity in a representation of the smallest eigenvalue of the Jacobi matrix J_n. The latter may be obtained by minimizing the *Rayleigh quotient*

$$R(J_n, \boldsymbol{x}) := \frac{\boldsymbol{x}^T J_n \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}$$

of J_n over all nonzero vectors $\boldsymbol{x}$ (see, for example, [23, Section 7.5]). Actually, precisely this approach was adopted in [16, Section 5] to get representations for x_{n1} and ξ_1. However, to prove Proposition 2 a subtler approach is needed. Namely, replacing J_n by

$$\tilde{J}_n := \begin{pmatrix} \lambda_0+\mu_0 & -\sqrt{\lambda_0\mu_1} & 0 & \cdots & 0 & 0 \\ -\sqrt{\lambda_0\mu_1} & \lambda_1+\mu_1 & -\sqrt{\lambda_1\mu_2} & \cdots & 0 & 0 \\ 0 & -\sqrt{\lambda_1\mu_2} & \lambda_2+\mu_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1}+\mu_{n-1} & -\sqrt{\lambda_{n-1}\mu_n} \\ 0 & 0 & 0 & \cdots & -\sqrt{\lambda_{n-1}\mu_n} & \mu_n \end{pmatrix},$$

the polynomials $\tilde{P}_{n+1}(x) := \det(xI - \tilde{J}_n)$ are readily seen to satisfy

$$\tilde{P}_{n+1}(x) = P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0,$$

and can therefore be identified as *quasi-orthogonal polynomials* (see [8, Section II.5]). As a consequence $\tilde{P}_n(x)$ has real and simple zeros $\tilde{x}_{n1} < \tilde{x}_{n2} < \cdots < \tilde{x}_{nn}$, which are separated by the zeros of $P_n(x)$. Moreover, it is not difficult to verify that $\tilde{x}_{n1} < x_{n1}$, so that $\tilde{\xi}_1 := \lim_{n\to\infty} \tilde{x}_{n1} \leq \xi_1$. But, seeing (15), the polynomials $\tilde{P}_n$ can also be identified with the *dual* polynomials P_n^d introduced in Section 4.1. So it follows with (12) and (16) that in the setting at hand we actually have $\tilde{\xi}_1 = \xi_1^d = \xi_1$. To get a representation for ξ_1 we may therefore start with the representation for $\tilde{x}_{n1}$ obtained by minimizing the Rayleigh quotient of $\tilde{J}_n$ and subsequently let n tend to infinity. Proceeding in this way leads to the representation

$$\xi_1 = \inf_{\boldsymbol{x}} \left\{ \frac{\sum_{i=0}^{\infty} \mu_i \pi_i x_i^2}{\sum_{i=0}^{\infty} \pi_i \left(\sum_{j=0}^{i} x_j \right)^2} \right\}, \qquad (21)$$

where $\boldsymbol{x} = (x_0, x_1, \dots)$ is an infinite sequence of real numbers with finitely many nonzero elements. Proposition 2 emerges after applying the *weighted discrete Hardy's inequalities* given in [24]. For the details of the proof we refer to [17].

The results in [17] include representations in the spirit of (21) for the decay parameter of a birth-death process under all possible scenarios. The proofs of these results require a representation for the *second smallest* eigenvalue of a Jacobi matrix, which is obtained in [17] by applying the *Courant-Fischer theorem*, an extension of the method involving Rayleigh quotients used above to represent the smallest eigenvalue. Being content in this paper with criteria for positivity rather than representations, there is no need to appeal to the full Courant-Fischer theorem.

5 Related literature and concluding remarks

We have noted in the introduction that in the setting of birth-death processes it is of particular interest to be able to establish whether the transition probabilities converge to their limiting values exponentially fast. In view of (6) this question may be translated in the current setting by asking whether $0 < \sigma \leq \infty$, so Corollary 1 provides us with a simple means to check whether the decay parameter of a birth-death process is positive.

In the orthogonal-polynomial literature the question of whether the support of an orthogonalizing measure is discrete with no finite limit point has received some attention, notably in the work of Chihara (see [8, Chapter IV], [10], [11] and [12])). Chihara's point of departure usually is the three-terms recurrence relation

$$\begin{aligned} &P_{n+1}(x) = (x - c_n)P_n(x) - \rho_n P_{n-1}(x), \quad n > 0, \\ &P_1(x) = x - c_0, \quad P_0(x) = 1, \end{aligned} \tag{22}$$

where $\rho_n > 0$. Note that we regain the polynomials P_n of (1) if

$$c_n = \lambda_n + \mu_n, \quad \rho_{n+1} = \lambda_n \mu_{n+1}, \quad n \geq 0. \tag{23}$$

Interestingly, by [8, Corollary to Theorem I.9.1] the existence of positive numbers λ_n and μ_n (except $\mu_0 \geq 0$) satisfying (23) is not only sufficient, but also *necessary* for the Stieltjes moment problem associated with the polynomials $\{P_n\}$ of (22) to be solvable. Moreover, if such numbers exist one can always choose $\mu_0 = 0$. So in view of Corollary 2, and considering that the sequence $\{P_n\}$ is orthogonal with respect to a measure on $[a, \infty)$ if and only if the sequence $\{Q_n\}$, with $Q_n(x) := P_n(x-a)$, is orthogonal with respect to a measure on $[0,\infty)$, we can formulate the following result with regard to the polynomials (22).

Proposition 3 The polynomials $\{P_n\}$ of (22) are orthogonal with respect to a measure on the interval $[a,\infty)$ with discrete support and no finite limit point if and only if the numbers λ_n and μ_n recurrently defined by $\lambda_0 := c_0 - a$ and

$$\mu_n := \rho_{n+1}/\lambda_n,\ \lambda_n := c_n - a - \mu_n, \quad n = 1, 2, \ldots,$$

are all positive and – using the notation (7)–(9) – satisfy $L_n(K_\infty - K_n) \to 0$ or $K_n(L_\infty - L_n) \to 0$ as $n \to \infty$.

The question of whether $\sigma = \infty$ in the specific setting of birth-death polynomials has been addressed by Chihara in [12], and earlier by Lederman and Reuter [20], Maki [21] and the present author [14]. By [8, Theorem IV.3.1] a necessary condition for $\sigma = \infty$ is $c_n \to \infty$, so an interesting case arises in the birth-death setting when

$$\lambda_n = an^\alpha + o(n^\alpha), \quad \mu_n = bn^\beta + o(n^\beta), \quad n \geq 0, \tag{24}$$

where $a,\ b,\ \alpha,\ \beta$ are nonnegative constants such that $\mu_0 \geq 0$ and $\lambda_n > 0$, $\mu_{n+1} > 0$ for $n \geq 0$. By employing a criterion involving *chain sequences* Chihara [12] proves that $\sigma = \infty$ if $\alpha \neq \beta$, or if $\alpha = \beta$ but $a \neq b$, a conclusion that may be reached also by applying Corollary 2. Chihara demonstrates in addition that both $\sigma = \infty$ and $\sigma < \infty$ may occur if $\alpha = \beta$, $a = b$ and $\alpha \leq 2$, thus refuting the conjecture in [25] that the spectrum in this case is continuous. Chihara suspects the claim in [25], that always $\sigma = \infty$ when $\alpha = \beta$, $a = b$ and $\alpha > 2$, to be true, but he can verify it only under additional assumptions on the rates. But actually, σ may be finite for all $\alpha > 0$, as the following example shows. Let

$$\lambda_0 = 1,\ \mu_0 = 0 \ \text{ and } \ \lambda_n = n^\alpha, \quad \mu_n = n^\alpha(1 + g_n), \quad n > 0,$$

where, for $k = 0, 1, \ldots,$

$$g_n = \begin{cases} \frac{1}{2k+1} & n = n_{2k} + 1, \ldots, n_{2k+1} \\ -\frac{1}{2k+2} & n = n_{2k+1} + 1, \ldots, n_{2k+2}, \end{cases}$$

with $n_0 = 0$ and $n_1 < n_2 < \dots$ successively chosen such that

$$G_{n_{2k+1}} > 1 \quad \text{and} \quad n_{2k+2}^{\alpha} G_{n_{2k+2}} < 1, \quad k = 0, 1, \dots,$$

where

$$G_n = \prod_{i=1}^{n} (1 + g_i), \quad n \geq 1.$$

Since

$$\pi_n = (n^{\alpha} G_n)^{-1}, \quad (\lambda_n \pi_n)^{-1} = G_n,$$

it follows that $K_\infty = L_\infty = \infty$. So by Proposition 1 we have $\sigma = 0$.

We conclude this section with the following observation. Letting

$$C := \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} K_n \quad \text{and} \quad D := \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} (K_\infty - K_n),$$

it is shown in [15, Theorem 2] that

$$C < \infty \text{ or } D < \infty \iff \sum_{i>1} \frac{1}{\xi_i} < \infty,$$

whence $\sigma = \infty$ if $C < \infty$ or $D < \infty$, a conclusion that may be drawn also from the main theorem in [1]. But, with $K_{-1} = L_{-1} = 0$, we actually have

$$\begin{aligned} C &= \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} K_n = \sum_{n=0}^{\infty} \pi_n (L_\infty - L_{n-1}) \\ &= \sum_{n=0}^{\infty} (K_n - K_{n-1})(L_\infty - L_{n-1}) \\ &= \sum_{n=0}^{\infty} \left\{ K_n (L_\infty - L_n) - K_{n-1}(L_\infty - L_{n-1}) + (\lambda_n \pi_n)^{-1} K_n \right\} \\ &= \lim_{n \to \infty} K_n (L_\infty - L_n) + C, \end{aligned}$$

so that

$$C < \infty \implies \lim_{n \to \infty} K_n (L_\infty - L_n) = 0.$$

Similarly,

$$D < \infty \implies \lim_{n \to \infty} L_n (K_\infty - K_n) = 0.$$

So the fact that $\sigma = \infty$ if $C < \infty$ or $D < \infty$ can be concluded from Corollary 2 as well. Note that our assumption (10) is equivalent to $C + D = \infty$.

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