Approximate Order-up-to Policies for Inventory Systems with Binomial Yield

Wanrong Ju, Adriana F. Gabor

Department of Econometrics, Erasmus University Rotterdam
Burgmeester Oudlaan 50, 3062 PA Rotterdam, the Netherlands

Jan-Kees C.W. van Ommeren

Faculty of Mathematical Sciences, University of Twente
P.O.Box 217, 7500 AE Enschede, the Netherlands

Abstract

This paper studies an inventory policy for a retailer who orders his products from a supplier whose deliveries only partially satisfy the quality requirements. We model this situation by an infinite-horizon periodic-review model with binomial random yield and positive lead time. We propose an order-up-to policy based on approximating the inventory model with unreliable supplier by a model with a reliable supplier and suitably modified demand distribution. The performance of the order-up-to policy is verified by comparing it with both the optimal policy and the safety stock policy proposed in [11]. Further, we extend our approximation to a dual sourcing model with one slow, unreliable and one fast, fully reliable supplier. Compared to the dual-index order-up-to policy for the model with full information on the yield, the proposed approximation gives promising results.

Keywords: Inventory management, Yield uncertainty, Order-up-to policy

1. Introduction

Rising with the prevalence of outsourcing activities, supply risk has recently attracted great attention from the OR research community. One important type of risk in outsourcing processes is the uncertainty in the

Email addresses: ju@ese.eur.nl (Wanrong Ju), gabor@ese.eur.nl (Adriana F. Gabor), Jan-Kees C.W.vanOmmeren@math.utwente.nl (J.C.W. van Ommeren)

order quantities that turn out to be usable at the buyer companies. This uncertainty is often referred to as yield uncertainty in the literature. There are many reasons that may lead to yield uncertainty. When goods are transported from a global supplier or transported goods are delicate parts, yield uncertainty is often related to damage during transportation due to humidity, collision and other reasons. Part of the goods received may also fail the quality inspection of the buyers. For example, in the semiconductor industry, the yield rate may drop below 50% due to strict requirements on quality [6]. Yield uncertainty is also encountered in industries where production is influenced by exogenous factors, like weather and diseases. [12] reports that in agriculture, yield rate can be as low as 30%.

Yield uncertainty significantly increases the difficulty of inventory management. Numerous papers have studied optimal or heuristic policies for inventory systems with uncertain yield. However, few have taken into account the effect of lead time. Lead time refers to the time span between the moment an order is placed by the buyer and the moment when ordered goods are delivered. It consists of the order processing time, production time and transportation time and is sometimes as long as several months. In practice, lead times can hardly be neglected especially in the case of global sourcing. In this paper, we study the inventory system of a retailer with positive lead time and yield uncertainty. The retailer has a global supplier whose deliveries only partially satisfy the quality requirements. We study the case in which failure of different units in an order is uncorrelated and each unit has the same probability of failing. This is often the situation if the uncertain yield is caused by damage during transportation or failure at quality inspection. The retailer checks his inventory level periodically and decides the quantity to order based on his inventory control policy. Unsatisfied demand is fully backlogged. The number of usable units in an order becomes known only when the order physically arrives at the retailer. The total inventory costs of the retailer consist of the holding cost, penalty (backlogging) cost and ordering cost. Inventory holding costs are incurred for the items in inventory at the end of a period. On the other hand, penalty costs are incurred when there is not enough inventory to satisfy customer demand. For this model, we propose a simple order-up-to policy (the OPMD heuristic) based on the optimal policy in an approximate model with a modified demand distribution and a reliable supplier. We then consider the case where the risk posed by the unreliable supplier is mitigated by ordering a part of the units from a more expensive and reliable supplier. To the best of our knowledge, this model has not been previously discussed in the OR literature. For this model, we propose a dual-index order-up-to policy (the DOPMD heuristic) based on an approximate model with two reliable suppliers and modified demand distribution.

The remainder of the paper is organized as follows. In Section 2, we give a brief review of the related literature. In Section 3, we formulate the single sourcing model with positive lead time and yield uncertainty. Subsequently, we propose a simple order-up-to heuristic and derive the optimal order-up-to level based on a reduction to a model with full returns. An extension of our heuristic to a dual sourcing model with general lead times and yield uncertainty is presented in Section 4. In Section 5, we present numerical results on the performance of the proposed heuristics. For the single sourcing model, we compare our heuristic with the optimal policy and a recently proposed heuristic [11]. In the case of dual sourcing, we compare the proposed heuristic and the dual-index order-up-to policy for the model with perfect yield information (perfect information DOP).

2. Literature Review

Yield uncertainty has drawn extensive attention in inventory management research in the past several decades. There are three types of random yield that have been considered in the literature: binomial yield [11], stochastically proportional yield [7, 1, 4, 10, 13, 8, 11] and interrupted geometric yield [11]. Binomial yield is used when failures of different units in a batch are uncorrelated and occur with the same probability. Stochastically proportional yield is on the other hand, used to characterize the situation in which a random process affects whole batches and the proportion of usable units in an order is a random variable. Models using interrupted geometric yield assume that good items are generated independently with a fixed probability until a failure occurs and thereafter all items are defective.

Most papers consider the effect of random yield under the assumption of zero lead time. [7] is among the first to study the structure of optimal policies in single-sourcing periodic review systems with random yield. They show that, despite the existence of a reorder point, the optimal order quantity is not linear in the inventory position. [4] and [10] revisit this problem and prove that the infinite-horizon periodic-review model can be reduced to a newsvendor problem. However, the distribution of the key variable in the newsvendor problem depends on the order quantity in each period. They therefore propose several myopic heuristics. [13] finds upper and lower bounds for the optimal reorder point and order quantity in an infinite-horizon model and gives valuable insights into the structure of the optimal policies.

Among the well performing heuristics proposed for the inventory optimization problem with one unreliable supplier, many fall into the class of 'linear inflation rules'. 'Linear inflation rules' restrict the order quantity to a linear function of inventory position with two parameters, called the 'order-up-to level' and the 'inflation factor'. Some of the myopic heuristics proposed by [4] fall into this class. [8] finds the optimal policy within this class and proves that the average total cost is convex in the order-up-to level for any given inflation factor. [11] is one of the few that considers the effect of positive lead time. The authors capture the two sources of uncertainty, i.e. yield and demand uncertainty, by the safety stock variable. Under the assumption that safety stock follows a normal distribution, they find the optimal safety stock levels for three different types of random yield. [9] proposes two approaches to derive the optimal and near-optimal values for the order-up-to level for a given inflation factor. The first approach models the on-hand inventory by a Markov chain and is exact for zero lead time. For general lead time, the approximate approach is analyzed by assuming a standard or gamma distribution of the on-hand inventory.

Dual sourcing is often used for balancing cost and service level. [17] proves that for periodic review models and difference in leadtime between the two suppliers equal to one, the optimal policy is a dual-index order-up-to policy (DOP). However, when the difference between lead times is larger than one, the optimal policy is hard to derive. Therefore several heuristics have been proposed in the literature. [16] shows that DOP performs well in dual sourcing models with general lead times and proves that for any given difference between the order-up-to levels, the optimal expedited order-up-to level can be found by solving a specific newsvendor problem. However, for finding the distribution of the demand in the newsvendor problem, they rely on simulation. [3] proposes an approximation of this distribution, which is exact when the difference between the order-up-to levels is one or approaches infinity. [14] generalizes DOP and proposes three new policies for the same model, namely the vector base-stock policy, the weighted DOP and the demand allocation policy. The first two policies use an order-up-to rule for the expedited supplier and the state information for deciding the regular order quantities. The last policy uses an order-up-to rule for the regular supplier and allocates demand between the two suppliers based on myopic costs. The authors show numerically that the three policies outperform on average the optimal DOP in either cost saving or computational time. Besides DOP, other types of heuristics have also been proposed. [15] considers an order-up-to policy which places regular orders periodically to restore the inventory position to the target level and emergency orders only when the likelihood of a stockout is very high. [2] studies a continuous review inventory model with two suppliers and proposes a tailored base-surge policy for this model. The cheap, offshore supplier is considered as the 'base' from which the buyer replenishes at a constant rate while the responsive, nearshore supplier acts as the 'surge' from which the buyer replenishes only when on-hand inventory is below a certain level. They present bounds on the optimal cost and an asymptotically optimal policy for a high volume system. A simple 'square-root' formula is presented which gives valuable insight into how to allocate orders between the two sources.

Statement of contribution: The contributions of this paper to the literature can be summarized as follows. First, we develop a simple order-up-to heuristic (the OPMD heuristic) for a single sourcing model with positive lead time and binomial yield. The proposed order-up-to level is found based on an approximating inventory model with modified demand distribution and reliable supplier. We show that our heuristic performs well by comparing it with the optimal policy and the heuristic proposed in [11]. Second, we consider the model in which an expedited, reliable supplier is used for mitigating the risk posed by the unreliable supplier. To the best of our knowledge, this model has not been previously studied by the OR community. To solve it, we propose a dual-index order-up-to policy (the DOPMD heuristic), based on an approximate model with two reliable suppliers and modified demand distribution. When compared to the optimal dual-index-order-up-to policy for the model with perfect yield information, our heuristic gives promising results.

3. The Single Sourcing Inventory Model with Unreliable Supplier

We consider an infinite-horizon periodic-review model with an unreliable supplier. For each order X placed with the supplier, only a binomial random portion $B(X, p)$ is returned, where $0 < p < 1$ is the long-run average fraction of orders being returned. We assume that p is known in advance. Demand in different periods, denoted as D_n, $n = 1, 2, \cdots$, is assumed to be independent and identically distributed, with $E(D) < \infty$. Revealed demand is fulfilled from on hand inventory I and unsatisfied demand is fully backlogged. Ordered items are delivered after a positive lead time l. The exact number of units returned remains unknown until delivery. The retailer pays a variable ordering cost c for each ordered unit. We assume zero fixed ordering cost. Backlogged demand is charged a penalty cost b per unit per period while inventory carried at the end of a period is charged a holding cost h per unit per period.

The sequence of events in each period is as follows. First, on-hand inventory is observed. Second, an order is placed according to the inventory control policy that is applied. Third, a binomial random portion of the order placed l periods in the past arrives. Fourth, demand of this period is revealed and fulfilled or backlogged.

We are interested in finding an efficient inventory control policy that minimizes the long-run average total cost given by $\lim_{N\mapsto\infty} \frac{\sum_{n=1}^{N} TC_n}{N}$, with

$$TC_n = cX_n + hI_n^+ + bI_n^-,$$

where X_n and I_n are the order placed and the on hand inventory in period n respectively, $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$.

Notations used in this paper are summarized in Table 1.

Table 1: Notations and Descriptions

Notations	Descriptions	Notations	Descriptions
n	Period index	c	Per unit ordering cost
I_n	On-hand inventory in period n	h	Inventory holding cost per unit per period
IP_n	Inventory position in period n	b	Penalty cost per unit per period
X_n	Order placed in period n	l	lead time
D_n	Demand in period n	p	Success rate of the Binomial yield distribution
f_U	Probability density function of random variable U	F_U	Cumulative distribution function of random variable U

An order-up-to policy with modified demand (OPMD)

The optimal policy for the single-sourcing model with yield uncertainty can in principle be found by using Markov decision process. Due to state space explosion of the underlying Markov chain, this approach is computationally intractable for large lead times. We therefore propose an order-up-to heuristic with optimal order-up-to levels decided based on an approximate inventory model with full returns.

Without loss of generality, we assume that the system starts with zero item in transit, in other words, $X_0 = 0$.

To motivate our approximation, consider the single-sourcing inventory model described above with the order X_{n+1} in period $n+1$ defined by

$$X_{n+1} = B(X_{n-l}, 1-p) + D_n. \tag{1}$$

Lemma 1 The sequence of orders X_n, $n = 1, 2, 3, ...$ has a limiting distribution.

Proof By using iteratively (1), we obtain

$$\begin{aligned} X_{n+1} &= D_n + B(D_{n-l-1}, 1-p) + B(D_{n-2l-2}, (1-p)^2) + \cdots \\ &= \sum_{k=0}^{\lfloor \frac{n}{l+1} \rfloor} R_{n,k}, \end{aligned}$$

with $R_{n,k} = B(D_{n-k(l+1)}, (1-p)^k)$. Note that since demand in different periods is i.i.d., the distribution of $R_{n,k}$ does not depend on n. For simplicity, we will hereafter omit the index n and refer to $R_{n,k}$ as R_k. We will show that $S_m = \sum_{k=0}^{m} R_k$ converges almost surely, which implies that X_n converges almost surely.

The probability generating function $\hat{R}_k$ of R_k is given by $\hat{R}_k(z) = \hat{D}(q_k z + (1-q_k))$, where $q_k = (1-p)^k$ and $\hat{D}$ is the probability generating function of D. Since

$$\begin{aligned} P(R_{n+1} \geq \frac{1}{n^2}) &= 1 - P(R_{n+1} = 0) \\ &= 1 - \hat{D}(1 - (1-p)^{n+1}) \\ &= (1-p)^{n+1} E(D) + o((1-p)^{n+1}) \end{aligned}$$

$E(D) < \infty$ and $0 < p < 1$, based on Borel Cantelli lemma (Proposition 2.8, [5]), we can conclude that S_n converges almost surely. □

Let F_∞ be the limiting distribution of X_n. Consider a sequence of independent variables Y_n, $n = 1, 2, ...$, distributed according to F_∞. We approximate the model with uncertain yield with a model with full returns and demand in period n given by

$$D'_n = B(Y_n, 1-p) + D_n.$$

We call D'_n the virtual demand in the model with full returns. Observe that although the variables $B(X_n, 1-p) + D_n$, $n = 1, 2, ...$ are dependent, by our choice of Y_n, the variables D'_n, $n = 1, 2, ...$ are independent.

Remark In the model with full returns, the next recursion holds

$$I_{n+1} = I_n + D'_{n-l-1} - D'_n,$$

whereas in the model with binomial return, we have

$$\begin{aligned} I_{n+1} &= I_n + B(X_{n-l}, p) - D_n \\ &= I_n + X_{n-l} - [B(X_{n-l}, 1-p) + D_n]. \end{aligned}$$

When $n \mapsto \infty$, X_{n-l} has the same limiting distribution as D'_{n-l-1} and $B(X_{n-l}, 1-p) + D_n$ the same limiting distribution as D'_n.

It is well known that in the classic model with full returns, the order-up-to policy is optimal and that in each period, the order placed is equal to the demand in the previous period. Therefore the next equation holds

$$I_n = z - (D'_{n-l-1} + D'_{n-l} + \cdots + D'_{n-1})$$

where z is the order-up-to level. So the optimal order-up-to-level in the approximate system can be found by solving a newsvendor problem, i.e.

$$z^* = F^{-1}_{D'(l+1)}(\frac{b}{b+h}),$$

where $F_{D'(l+1)}$ is the cumulative distribution function of $\sum_{k=0}^{l} D'_{n-k}$ for all n.

The performance of the proposed heuristic (OPMD) in the original problem will be tested in Section 5 by comparing it with the optimal policy derived by dynamic programming and the safety stock policy proposed by [11].

4. The Dual-Sourcing Inventory Model with Unreliable Supplier

In this section, we consider the inventory system of a retailer who sources from two suppliers, a regular (r) and an expedited (e) supplier. The lead time l_r of the regular supplier is longer than the lead time l_e of the expedited supplier, while the the ordering cost c_r of the regular supplier is lower than the cost c_e of the expedited one. Moreover, the regular supplier has binomial random yield, which means that, out of an order X_n^r placed with him in period n, only a random portion $B(X_n^r, p)$ turns out to be usable when the order is delivered in period $n+l_r$. On the other hand, if an order X_n^e is placed with the expedited supplier in period n, the whole order will be delivered in period $n + l_e$. To the best of our knowledge, this model seems not to have been studied before in the literature.

For the case with two reliable suppliers, [16] shows that the performance of a dual-index order-up-to policy (DOP) is close to that of the optimal policy. Therefore, in this section, we focus on finding the optimal DOP for the model with two suppliers one of which is unreliable.

A DOP is characterised by two order-up-to levels: one for the expedited supplier, z_e, and one for the regular supplier, z_r. In each period $n \geq l_r$, there are l_r regular and l_e expedited orders in pipeline, denoted by $< X^r_{n-l_r}, ..., X^r_{n-1} >$, and $< X^e_{n-l_e}, ..., X^e_{n-1} >$, respectively. The expedited inventory position in period n, IP^e_n, is comprised of on-hand inventory and all the orders due to arrive in the next l_e periods, while the regular inventory position IP^r_n is comprised of on-hand inventory and all the orders that will arrive in the next l_r periods. More precisely,

$$IP^e_n = I_n + (X^e_{n-l_e} + ... + X^e_{n-1}) + (X^r_{n-l_r} + ... + X^r_{n-l-1})$$
$$IP^r_n = I_n + (X^e_{n-l_e} + ... + X^e_{n-1}) + (X^r_{n-l_r} + ... + X^r_{n-1})$$

where $l = l_r - l_e$.

In each period n, the following sequence of events takes place. First, an expedited order X^e_n is placed, to restore the inventory position IP^e_n to the value z_e. Observe that when the size of X^e_n is decided, X^r_{n-l} enters the information horizon. Thus, one first checks if there is a surplus, i.e., whether $IP^e_n + X^r_{n-l} > z_e$. If this is the case, no expedited order is placed. Otherwise, an expedited order equal to the deficit $X^e_n = z_e - (IP^e_n + X^r_{n-l})$ is placed. Then the expedited order X^e_n is added to the inventory position of the regular supplier, IP^r_n and a regular order $X^r_n = z_r - (IP^r_n + X^e_n)$ is placed. Finally, the orders due to arrive in this period, $X^r_{n-l_r}$ and $X^e_{n-l_e}$ arrive. Note that since the regular supplier is unreliable, only $B(X^r_n, p)$ units are usable. Finally, demand D_n is revealed and satisfied from the on-hand inventory if available. Unsatisfied demand is back-ordered. The inventory level is then updated and holding or penalty costs are incurred.

In the literature, the quantity $O_n = (IP^e_n + X^r_{n-l} - z_e)^+$ is known as *the overshoot*. The overshoot and the inventory positions of the regular and expedited supplier satisfy the following equations

$$IP^e_n + X^r_{n-l} + X^e_n = z_e + O_n \tag{2}$$
$$IP^r_n + X^e_n + X^r_n = z_r. \tag{3}$$

Substracting (2) from (3), we obtain

$$\sum_{k=0}^{l-1} X_{n-k}^r = z_r - z_e - O_n$$

and

$$\sum_{k=0}^{l-1} E(X_{n-k}^r) = z_r - z_e - E(O_n).$$

The optimal DOP can be found by formulating the problem as a Markov decision process. However, since a state contains all the pipeline information, for large l_r, the optimization problem becomes intractable. In the next section, we propose a dual-index order-up-to heuristic that can be used for large values of l_r.

A dual-index order-up-to policy with modified demand (DOPMD)

As in the single sourcing case, we propose to approximate the dual sourcing model with uncertain yield with a model with full returns, but with modified demand distribution.

Note that in the dual sourcing model with uncertain returns, the following recursion holds

$$\begin{aligned} I_{n+1} &= I_n + X_{n-l_e}^e + B(X_{n-l_r}^r, p) - D_n \\ &= I_n + X_{n-l_e}^e + X_{n-l_r}^r - (D_n + B(X_{n-l_r}^r, 1-p)). \end{aligned}$$

If the variables $D_n + B(X_{n-l_r}^r, 1-p)$ were independent and their distribution easy to calculate, we could reduce the model with uncertain returns to a model with full returns and demand defined as $D'_n = D_n + B(X_{n-l_r}^r, 1-p)$. However, a regular order depends on the orders placed in the previous l_r periods, making thus the distribution of X_n^r difficult to find. Therefore we propose to use the following approximation.

Let Y_n be a random variable distributed according to F_∞, the limiting distribution of the orders in a system where the only supplier is the regular supplier. Observe that in the dual sourcing model, X_n^r is usually smaller than Y_n, since part of the orders are delivered by the expedited supplier. We assume that $X_n^r = B(Y_n, \alpha)$, with $\alpha \in [0,1]$. Thus, each unit that would be ordered from the regular supplier if he was the only supplier is now ordered with probability $1-\alpha$ from the expedited supplier. To find an appropriate α, recall that $\sum_{k=0}^{l-1} E(X_{n-k}^r) = z_r - z_e - E(O_n)$. Since $E(O_n) \geq 0$ and $E(X_n^r) = \alpha E(Y_n)$, it holds that $\alpha l E(Y_n) \leq z_r - z_e$. We therefore propose to choose $\alpha = \min\{\frac{\Delta}{lE(Y_n)}, 1\}$, where $\Delta = z_r - z_e$. Since the cumulative distribution function of Y_n is $F_\infty(\cdot)$, $E(Y_n) = \frac{E(D_n)}{p}$ and $\alpha = \min\{\frac{\Delta p}{lE(D_n)}, 1\}$.

We are now able to describe the approximate dual sourcing model with full returns. In the approximate model, both retailers are assumed reliable. Their costs and lead times are as in the initial model. We define the demand in period n as

$$D'_n = D_n + B(Y_n, \alpha(1-p)). \tag{4}$$

where $\alpha = \min\{\frac{\Delta p}{lE(D_n)}, 1\}$. Since the variables Y_n are independent and identically distributed, so are the variables D'_n, $n = 1, 2, \cdots$.

It has been proven that for any fixed Δ, the optimal expedited order-up-to level in the dual sourcing model with full returns can be found by solving a newsvendor problem [16], i.e.

$$z_e^* = F^{-1}_{D'(l_e+1)-O}\left(\frac{b}{b+h}\right),$$

where $F_{D'(l+1)-O}$ is the cumulative distribution function of $\sum_{k=0}^{l} D'_{n-k} - O_{n-l}$ for all n. As in [16], for each Δ, we derive the distribution of O_n by simulation and then determine the optimal expedited order-up-to level and the optimal total cost. Subsequently, we use one-dimensional search to find the optimal value for Δ. Note that, in order to reduce computation times, the distribution of O_n could also be approximated as described in [3]. This is, however, not the focus of this paper.

In Section 5, we testify the performance of the DOPMD heuristic by comparing it with DOP under perfect information about the returned order quantities.

5. Numerical Results

In this section, we present numerical results on the performance of the proposed heuristics for the single and dual sourcing models.

5.1. Performance of the OPMD heuristic for the single sourcing model

To study the influence of the parameters on the performance of the OPMD heuristic, we construct 74 different scenarios. We start with a base case in which the parameters take the values $h = 5$, $c = 150$, $l = 2$, $p = 0.8$, $b = 495$ and $D \sim U\{0, 1, \cdots, 4\}$[1]. Subsequently, we vary the values of

[1] $U\{0, 1, \cdots, n\}$ denotes the discrete uniform distribution on $\{0, 1, \cdots, n\}$

one or two parameters and keep the others as in the base case. The optimal order-up-to level for the OPMD heuristic is found by solving the newsvendor problem in the approximate model with full returns. The average total cost for given optimal order-up-to level is calculated as the long run average cost of the underlying Markov chain. For small instances, we compare the OPMD heuristic with both the optimal policy and the safety stock policy proposed in **(author?)** [11]. The optimal policy is derived by using dynamic programming. For large instances, we only compare the OPMD heuristic with the safety stock policy.

In Section 5.1.1 to 5.1.4, we study the impact of lead time, yield rate, penalty cost and demand distribution on the performance of t respectively. To keep the dynamic program tractable, we focus on discrete demand distributions with bounded support. As $b, h > 0$, we restrict the backlogs and on-hand inventory to $[0, \lceil \frac{(l+1)D_{max}}{p} \rceil]$, where D_{max} denotes the maximum demand and $\lceil x \rceil$ denotes the minimum integer that is larger than or equal to x. Notice that the probability of backlog being larger than $\lceil \frac{(l+1)D_{max}}{p} \rceil$ is smaller than $(Pr(D = D_{max}))^{(l+1)}$ and that of on-hand inventory being larger than $\lceil \frac{(l+1)D_{max}}{p} \rceil$ is smaller than $(Pr(D = 0))^{(l+1)}$. The order quantity is restricted to $[0, \lceil \frac{2D_{max}}{p} \rceil]$. Note that since every ordered unit is returned with probability p, the expected number of units needed to be ordered to get one unit returned is $\frac{1}{p}$. Hence the probability of order quantity exceeding $\lceil \frac{2D_{max}}{p} \rceil$ is very small. Moreover, in all our numerical experiments, the order quantities in the optimal policy did not exceed $\lceil \frac{2D_{max}}{p} \rceil$.

5.1.1. Impact of Yield Rate

Next we examine the performance of the OPMD heuristic under different yield rates. We vary $p \in \{0.4, 0.6, 0.8, 1\}$ and $D \sim U\{0, 1, \cdots, n\}$, $n = 2, 4$ and compare the performance of the OPMD heuristic, the optimal policy and the safety stock policy. The results are shown in Table 2. The average relative difference between the OPMD heuristic and the optimal policy is 0.97% and the maximum difference is 2.35%. As shown in column 4 of Table 2, the performance of the OPMD heuristic improves when the yield rate increases. This is due to the fact that the OPMD heuristic assumes independent virtual demands, which holds if orders from different periods are independent. When the yield rate is high, the unreturned order quantities are relatively small, which leads to less correlation among orders.

On the other hand, the performance of the safety stock policy improves when the yield rate decreases, which can be seen in column 5 of Table 2. The average and maximum difference between the safety stock and the

Table 2: Impact of yield rate
(h=5, l=2, b=495, c=150)

		Optimal policy	OPMD heuristic	Safety stock policy
p	Demand dist.	Average total cost	% above optimal	% above optimal
0.4	U{0,1,2}	400.08	**2.35**	**0.31**
0.6	U{0,1,2}	273.01	**1.53**	**0.76**
0.8	U{0,1,2}	208.11	0.71	1.06
1	U{0,1,2}]	165.00	0.00	3.86
0.4	U{0,1,2,3,4}	789.94	**1.68**	**0.52**
0.6	U{0,1,2,3,4}	537.07	**1.11**	**0.85**
0.8	U{0,1,2,3,4}	408.87	0.40	1.50
1	U{0,1,2,3,4}	329.00	0.00	2.00

optimal policy are 1.36% and 3.86% respectively. As the results in Table 2 indicate, when the yield rate is relatively high, our heuristic performs better than the safety stock policy. The reverse seems to hold for low yield rates. The same patterns hold for larger instances shown in Table 3, where $D \sim U\{0, 1, ..., 8\}$, $l \in \{2, 4, 8, 20\}$ and $p \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Note that for these instances, since the state space of the dynamic program grows too large, we only compare the OPMD heuristic with the safety stock policy.

5.1.2. Impact of Lead Time

To study the impact of lead time on the performance of the OPMD heuristic, we first compare it with the optimal and the safety stock policy in small instances. For this, we modify the base case by first taking $D \sim U\{0, 1, 2\}$ and $l \in \{1, 2, 4, 6, 7\}$ and then $D \sim U\{0, 1, 2, 3, 4\}$ and $l \in \{1, 2, 4, 6\}$. For larger lead times, due to state space explosion, it is computationally intensive to find the optimal policy by dynamic programming. Therefore we only compare the OPMD heuristic with the safety stock policy for these instances. The results are summarized in Table 3 and Table 4 .

The average and maximum difference (over all 9 scenarios in Table 4) between the OPMD heuristic and the optimal policy is 0.49% and 0.89% respectively. We observe that the OPMD heuristic deviates slightly less from the optimal policy when lead time increases. To explain this, recall that the OPMD heuristic assumes independent virtual demands, hence, independent order quantities in the original model. Since an order depends only on the orders placed $k(l + 1)$ periods in the past, with $k \geq 1$, the larger the lead time, the less is the correlation among different orders. Moreover, we notice that the performance of the OPMD heuristic seems insensitive to changes in lead time. On the other hand, as column 5 in Table 4 shows, the safety

Table 3: Impact of yield rate and lead time
$(h = 5, b = 495, c = 150, D \sim U\{0, 1, \cdots, 8\})$

		OPMD heuristic	Safety stock policy
l	p	Average total cost	% above proposed heuristic
2	0.1	6171.94	**-1.46**
2	0.3	2101.52	**-1.06**
2	0.5	1278.58	**-0.09**
2	0.7	922.49	1.07
2	0.9	723.02	2.21
4	0.1	6216.61	**-1.85**
4	0.3	2129.63	**-1.46**
4	0.5	1301.46	**-0.46**
4	0.7	942.59	0.72
4	0.9	743.11	1.73
8	0.1	6273.50	**-2.25**
8	0.3	2174.27	**-2.14**
8	0.5	1336.58	**-0.97**
8	0.7	973.78	0.20
8	0.9	771.22	1.14
20	0.1	6405.41	**-3.23**
20	0.3	2263.68	**-3.22**
20	0.5	1409.05	**-1.84**
20	0.7	1036.35	**-0.38**
20	0.9	827.79	0.55

stock policy performs significantly better for larger lead times.

To examine the performance of the OPMD heuristic for larger lead times, we refer to the rows corresponding to $l \in \{8, 20\}$ in Table 3. As column 4 in Table 3 indicates, the safety stock policy outperforms our heuristic for large lead times and relatively low yield rates. The reverse seems to hold for large lead times and high yield rates ($p = 0.9$).

5.1.3. Impact of Penalty Cost

In order to study the influence of the penalty cost, we set $b \in \{5, 15, 95, 495\}$. Note that the penalty cost influences the optimal order-up-to level through the optimal fractile in the newsvendor problem in the model with full returns. For $h = 5$, the optimal fractile $\frac{b}{b+h} \in \{0.5, 0.75, 0.95, 0.99\}$. Moreover, we vary the value of the ordering cost in $c \in \{5, 10, 50, 150\}$. As can be seen in Table 5, in general the deviation of the OPMD heuristic from the optimal policy increases when the penalty cost (the optimal fractile) increases. However, when the penalty cost is much lower than the ordering cost, e.g.

Table 4: Impact of lead time
(h=5, p=0.8, b=495 and c=150)

		Optimal policy	OPMD heuristic	Safety stock policy
l	Demand dist.	Average total cost	% above optimal	% above optimal
1	U{0,1,2}	203.56	0.89	2.00
2	U{0,1,2}	208.11	0.71	1.06
4	U{0,1,2}	214.76	0.60	0.98
6	U{0,1,2}	220.04	**0.47**	**0.29**
7	U{0,1,2}	222.37	**0.44**	**0.32**
1	U{0,1,2,3,4}	401.62	0.36	1.82
2	U{0,1,2,3,4}	408.87	0.40	1.50
4	U{0,1,2,3,4}	419.92	0.28	1.26
6	U{0,1,2,3,4}	428.68	0.24	0.51

Table 5: Impact of penalty cost
(h=5, l=2, p=0.8 and $D \sim U\{0,1,2,3,4\}$)

		Optimal policy	OPMD heuristic	Safety stock policy
b	c	Average total cost	% above optimal	% above optimal
5	5	23.29	1.15	6.06
15	5	29.65	1.39	8.11
95	5	39.62	1.78	13.01
495	5	46.37	3.46	14.70
5	10	35.79	0.85	2.76
15	10	42.15	1.08	5.65
95	10	52.12	1.44	9.88
495	10	58.87	2.69	12.00
5	50	65.00	109.38	111.12
15	50	142.15	0.32	0.89
95	50	152.12	0.49	3.85
495	50	158.87	0.90	4.40
5	150	65.00	493.05	498.63
15	150	195.00	101.01	102.66
95	150	402.12	0.03	1.43
495	150	408.87	0.40	1.50

$b = 5, c = 50, 150$ and $b = 15, c = 150$, the OPMD heuristic leads to a large deviation from the optimal policy. This phenomenon can also be seen when the safety stock policy is applied. The reason is that the optimal policy is influenced by the ordering costs, while both the OPMD heuristic and the safety stock policy are not. When the ordering cost is much higher than the penalty cost, it is more cost efficient to backlog demand and incur penalty cost than to order. Neither of the heuristics takes this aspect into account. If we exclude the three exceptional cases, the average deviation of the OPMD heuristic from the optimal policy is 1.20% with the maximum being 3.46%, while the average deviation of the safety stock policy is 6.36% with the maximum being 14.70%. The OPMD heuristic outperforms the safety stock policy in all cases shown in Table 5.

5.1.4. Impact of Mean, Variance and Skewness of Demand Distribution

In this section, we examine the influence of the demand distribution on the performance of the OPMD heuristic, by varying its mean, variance and skewness. In order to study the impact of mean, we choose demand distributions with the same variance and skewness but different means. For $k, n \in \mathbf{Z}_+$ and $k \leq n$, let $U\{n-k, n, n+k\}$ denote the distribution given by $Pr(D = n-k) = Pr(D = n) = Pr(D = n+k) = 1/3$. The skewness of this distribution is equal to 0. When $k = 1$, the variance of the distribution is $\frac{2}{3}$ and when $k = 2$, the variance of the distribution equals $\frac{8}{3}$. Table 6 contains the detailed results for this demand distribution. the OPMD heuristic seems robust under changes in mean demand, with an average deviation from the optimal policy of 0.41% and a maximum deviation of 0.67%. Moreover, the performance of the OPMD heuristic slightly improves when the mean demand increases.

Table 6: Impact of Mean Demand
(h=5, b=495, c=150, l=2 and p=0.8)

Demand dist.	Mean	Variance	Optimal policy Average total cost	OPMD heuristic % above optimal	Safety stock policy % above optimal
U{0,1,2}	1	2/3	208.15	0.66	0.92
U{1,2,3}	2	2/3	400.77	0.49	1.79
U{2,3,4}	3	2/3	588.73	0.26	0.37
U{3,4,5}	4	2/3	778.60	0.23	0.27
U{0,2,4}	2	8/3	412.33	**0.59**	**0.17**
U{1,3,5}	3	8/3	601.52	**0.44**	**0.08**
U{2,4,6}	4	8/3	790.80	**0.34**	**0.05**
U{3,5,7}	5	8/3	979.96	**0.27**	**0.08**

Next we change the variance of the demand distribution while keeping the mean and the skewness constant. The results shown in Table 7 testify that the performance of the OPMD heuristic is also robust against demand variability. The average deviation from the optimal policy is 0.45% and the maximum deviation is 0.88%. In our experiments, the OPMD heuristic outperforms the safety stock policy for small demand variances ($var(D) \in$

$\{0, 2/3\}$), while the safety stock policy gives better results for larger demand variability ($var(D) \geq 1$).

Table 7: Impact of Variance of Demand
(h=5, b=495, c=150, l=2 and p=0.8)

			Optimal policy	OPMD	Safety stock policy
Demand dist.	Mean	Variance	Average total cost	% above optimal	% above optimal
U{1}	1	0	198.11	0.88	4.35
U{0,1,2}	1	2/3	208.15	0.66	0.92
U{0,2}	1	1	210.79	**0.75**	**0.21**
U{2}	2	0	391.01	0.49	0.85
U{1,2,3}	2	2/3	400.77	0.49	1.79
U{0,1,2,3,4}	2	2	408.87	**0.37**	**0.15**
U{0,2,4}	2	8/3	412.33	**0.44**	**0.08**
U{0,4}	2	4	417.00	**0.46**	**0.28**
U{3}	3	0	582.12	0.23	1.32
U{2,3,4}	3	2/3	588.73	0.26	0.37
U{1,2,3,4,5}	3	2	598.22	**0.29**	**0.15**
U{1,3,5}	3	8/3	601.52	**0.44**	**0.07**
U{1,5}	3	4	605.45	**0.65**	**0.27**
U{0,6}	3	9	621.57	**0.76**	**0.72**
U{4}	4	0	772.56	0.16	0.76
U{3,4,5}	4	2/3	778.60	0.23	0.27
U{2,3,4,5,6}	4	2	787.56	**0.28**	**0.14**
U{2,4,6}	4	8/3	790.80	**0.35**	**0.03**
U{2,6}	4	4	795.01	**0.44**	**0.13**

In the end, we examine the influence of the skewness of demand distribution by choosing $D \sim NB(r, q)$, where $NB(r, q)$ denotes the negative binomial distribution with r being the number of failures until the experiment stops and q being the probability of success for each trial. In our experiments, we vary $r \in \{1, 2, 4, 6\}$ and $q \in \{0.2, 0.4\}$. In order to acquire distributions with different skewness, we truncate $NB(r, 0.2)$ to take values in $[0, r]$ and $NB(r, 0.4)$ to take values in $[0, 4r/3]$. The skewness for the truncated distributions is shown in column 2 of Table 8. As can be seen in column 4, the performance of OPMD is robust against changes in the skewness of demand distribution. The average deviation in average total costs from the optimal policy is 0.33% while the maximum deviation is 0.70%. Compared with the safety stock policy, OPMD performs better when the skewness is negative and has a large absolute value. When skewness is positive and has a small absolute value, the safety stock policy outperforms OPMD.

Table 8: Impact of Skewness of Demand
(h=5, b=495, c=150, l=2 and p=0.8)

		Optimal policy	OPMD	Safety stock policy
Demand dist.	skewness	Average total cost	% above optimal	% above optimal
NB(1,0.2)	-23.44	163.45	0.34	0.51
NB(2,0.2)	-34.91	364.90	0.39	0.81
NB(4,0.2)	-159.86	761.98	0.17	0.40
NB(6,0.2)	-934.14	1148.30	0.18	0.52
NB(1,0.4)	0.87	162.21	**0.70**	**0.86**
NB(2,0.4)	0.11	363.86	**0.32**	**0.26**
NB(4,0.4)	0.02	772.82	**0.29**	**0.08**
NB(6,0.4)	0.01	1177.3	**0.25**	**0.12**

5.2. Performance of the DOPMD heuristic for the dual sourcing model

In this section, we study the performance of the DOPMD heuristic by comparing it with the dual-index order-up to policy for the model with perfect information on yield in which we assume that the returned quantities are known immediately after the orders are placed. We call this policy the perfect information DOP. The reason for using it as a benchmark is twofold: first, deriving the optimal policy for the dual sourcing model with general lead times and random yield is computationally intensive even for small lead times and demand; second, DOP has been proven to have a near optimal performance in dual sourcing models with general lead times [16]. Note that in the model with perfect information, one can directly work with the yield quantities of the regular orders and find the optimal dual-index order-up to policy by applying the solution procedure proposed in [16]. Since in the model with an unreliable regular supplier, on average $\frac{1}{p}$ units need to be ordered to get one unit of yield, in the model with perfect information we take the regular ordering cost as $\frac{c_r}{p}$. For our heuristic, the order-up-to levels are found by applying the solution procedure proposed in [16] to a dual sourcing model with full returns and modified demand defined by equation (4). When the optimal z_e takes negative values, we set $z_e^* = 0$. The average total costs, corresponding to these order-up-to levels, are derived by simulation. We run the simulation till the 95% confidence intervals for the expected order quantities, the expected on-hand inventory and the expected backlog are smaller than 0.02. One could also derive the average costs from the underlying Markov process, however, since the state space includes information on both regular and expedited orders in transit, the dynamic program becomes computationally intractable. Since we rely on simulation, occasionally, the average total costs obtained by the heuristic are slightly smaller than those obtained by the perfect information DOP.

As in Section 5.1, we start with a base case and construct 35 scenarios by modifying one or two of its parameters. In the base case, we choose $l_e = 1$, $l_r = 2$, $c_r = 100$, $c_e = 150$, $h = 5$, $b = 495$, $p = 0.8$ and $D \sim Pois(2)$, where $Pois(\lambda)$ denotes the Poisson distribution with mean λ. We fix the values of h, c_e and l_e in all instances and study the impact of c_r, p, l_r, b and demand on the performance of the DOPMD heuristic respectively. The parameter values used in this section are summarised in Table 9.

Table 9: Parameter Values in the Dual Sourcing Model

Parameter	Values
f_D	Poisson(λ), $\lambda \in \{2, 4, 6, 8, 10\}$
l_r	2, 4, 6, 8, 10
c_r	10, 40, 70, 100, 130
b	5, 7.5, 9, 12, 15, 95, 495
p	0.6, 0.7, 0.8, 0.9, 1

5.2.1. Impact of Yield Rate

We begin by examining the impact of yield rate on the performance of the DOPMD heuristic by taking $p \in \{0.6, 0.7, 0.8, 0.9, 1\}$, $l_r \in \{2, 4\}$ and all the other parameters as in the base case. The results can be found in Table 10. As can be seen in column 4, the maximum deviation of the DOPMD heuristic from the perfect information DOP is 2.73%, while the average deviation is 1.11%. Moreover, when p increases from 0.6 to 1, the performance of the DOPMD heuristic first worsens and then improves. When $p = 0.6$, the expected ordering cost for each unit of yield from the regular supplier is $\frac{100}{0.6}$ (note that $c_r = 100$) which is higher than 150 (i.e. c_e). Therefore, both the DOPMD heuristic and the perfect information DOP derive that single sourcing from the expedited supplier is optimal in this situation and lead to the same order-up-to levels. For large values of p (i.e. $p \in \{0.9, 1\}$), the system we study approaches the dual sourcing system with full returns (see equation 4) for which the DOPMD heuristic is the same as the perfect information DOP.

The deviation of the average total costs obtained by the DOPMD heuristic from those of the perfect information DOP for $p \in \{0.7, 0.8\}$ can be explained as follows. Recall that in the DOPMD heuristic, the regular order is taken as $B(Y, \alpha)$, where Y is the order quantity in a system with one unreliable supplier and $\alpha = min\{\frac{\Delta p}{l\mathbb{E}(D)}, 1\}$. Hence, for large values of Δp, $\alpha = 1$, in which case the modified demand in period n is equal to

$D_n + B(Y_n, 1-p)$. For $p \in \{0.7, 0.8\}$, the modified demand is too large, as it assumes unreturned quantities equal to those in a model with one unreliable supplier (and no expedited supplier). On the other hand, for the perfect information DOP, the unreturned quantity is smaller, as it takes into account the orders placed with the expedited supplier.

Table 10: Impact of yield rate
(h=5,l_e=1,c_e=150,c_r=100,b=495,$D \sim Pois(2)$)

		Perfect info. DOP	DOPMD heuristic
l_r	p	Average total cost	% above perfect info. DOP
2	0.6	309.48	-0.01
2	0.7	301.19	2.73
2	0.8	268.14	1.92
2	0.9	241.42	1.12
2	1	219.92	0.09
4	0.6	309.48	0.00
4	0.7	303.70	1.83
4	0.8	273.40	2.36
4	0.9	248.17	1.07
4	1	227.56	0.00

5.2.2. Impact of Regular Lead Time

To analyze the influence of the regular lead time, we take $l_r \in \{2, 4, 6, 8, 10\}$, $b \in \{95, 495\}$ and the other parameters as in the base case. The results are reported in Table 11. As can be seen in column 4, changing l_r does not have a significant effect on the relative performance of the DOPMD heuristic compared with the perfect information DOP. The maximum deviation of the DOPMD heuristic is 2.64%, while the average deviation is 2.03%.

5.2.3. Impact of Penalty Cost

In this section, we study the influence of the penalty cost as well as the optimal fractile on the performance of the DOPMD heuristic. For this we take $b \in \{5, 7.5, 12, 15, 95, 495\}$, which results in an optimal fractile $\frac{b}{b+h} \in \{0.5, 0.6, 0.75, 0.95, 0.99\}$ for the newsvendor problem in the approximate model with full returns. Recall that the optimal fractile influences the expedited order-up-to level as $z_e^* = F^{-1}_{D'(l_e+1)-O}(\frac{b}{b+h})$. The results are shown in Table 12. As we can see, the performance of the DOPMD heuristic is robust under different values of b. Compared to the perfect information DOP, the proposed heuristic has an average deviation of 1.03% and a maximum deviation of 1.92%. In most cases, the relative performance of the DOPMD

Table 11: Impact of regular lead time
(h=5,l_e=1,c_e=150,c_r=100,p=0.8,$D \sim Pois(2)$)

		Perfect info. DOP	DOPMD heuristic
b	l_r	Average total cost	% above perfect info. DOP
495	2	268.14	1.92
495	4	273.39	2.36
495	6	275.51	2.64
495	8	276.91	2.47
495	10	277.83	2.39
95	2	262.77	1.50
95	4	268.04	1.73
95	6	270.57	1.60
95	8	271.91	1.91
95	10	272.83	1.82

heuristic worsens when penalty cost increases. A higher penalty cost results in a higher optimal fractile, which implies that the optimal z_e is equal to a larger percentile of the distribution of $D'^{(l_e+1)} - O$. This distribution is obtained by simulation as in [16]. Since it is more difficult to estimate the probabilities at the tails of the distribution function, the bad performance of the DOPMD heuristic could be caused by the difficulty in estimating the distribution of $D'^{(l_e+1)} - O$.

Table 12: Impact of penalty cost
(h=5,l_e=1,l_r=2,c_e=150,c_r=100,p=0.8,$D \sim Pois(2)$)

	Perfect info. DOP	DOPMD heuristic
b	Average total cost	% above perfect info. DOP
5	249.18	0.28
7.5	251.13	0.45
12	253.22	1.07
15	254.47	0.66
95	262.84	1.58
495	268.08	1.92

5.2.4. Impact of Regular Ordering Cost

To examine the impact of the regular ordering cost on the performance of the DOPMD heuristic, we vary in the base case $c_r \in \{10, 40, 70, 100, 130\}$ and $b \in \{95, 495\}$. The results can be found in Table 13. As can be seen in column 8, the average deviation of the DOPMD heuristic from the perfect information DOP is 3.59% while the maximum deviation is 11.36%.

Moreover, the relative performance of the DOPMD heuristic improves significantly as c_r increases. Based on columns 3, 4, 6 and 7, the order-up-to levels of both the DOPMD heuristic and the perfect information DOP are not influenced by the change in c_r when $c_r \in \{10, 40, 70, 100\}$. Therefore, in these cases, the absolute differences between the average total costs of the DOPMD heuristic and the perfect information DOP do not change, which leads to a decreasing relative difference. When $b = 495$, both the DOPMD heuristic and the perfect information DOP order from both suppliers in order to achive a high service level and avoid the large penalty cost. When $b = 95$, the DOPMD heuristic switches to single sourcing from the regular supplier while the perfect information DOP allocates more order quantities to the expedited supplier. When $c_r = 130$, the regular supplier becomes more expensive than the expedited one and both policies prefer single sourcing from the expedited supplier, in which case they derive the same policy. The large deviation, i.e. 11.36% and 8.36%, only occurs when $c_r \leq c_e/10$ which is less than one tenth of c_e. This situation rarely exists in practice.

Table 13: Impact of regular ordering cost
(h=5,l_e=1,l_r=2,c_e=150,p=0.8,$D \sim Pois(2)$)

		Perfect info. DOP			DOPMD heuristic		
b	cr	z_e	z_r	Average total cost	z_e	z_r	% above perfect info. DOP
495	10	3	11	51.39	3	13	11.36
495	40	3	11	129.32	3	13	4.59
495	70	3	11	195.89	3	13	2.93
495	100	3	11	268.14	8	13	1.92
495	130	8	8	309.48	8	8	0.00
95	10	5	9	46.11	0	11	8.36
95	40	5	9	118.36	0	11	3.31
95	70	5	9	190.60	0	11	1.97
95	100	5	9	262.84	0	11	1.44
95	130	7	7	306.52	7	7	-0.01

5.2.5. Impact of Demand Distribution

To examine the robustness of the DOPMD heuristic under different demand distribution, we change in the base case $D \sim Pois(\lambda)$, $\lambda \in \{2, 4, 6, 8, 10\}$. We focus on Poisson distribution because it is commonly used in the literature and is considered as a good approximation of the demand processes in practice. The results are shown in Table 14. As can be seen in column 3, the maximum and average deviation of the DOPMD heuristic from the perfect information DOP is 1.92% and 1.11% respectively. Moreover, the relative performance of the DOPMD heuristic seems to improve when λ increases.

Table 14: Impact of demand distribution
(h=5,l_e=1,l_r=2,c_e=150,c_r=100,p=0.8,b=495)

	Perfect info. DOP	DOPMD heuristic
λ	Average total cost	% above perfect info. DOP
2	268.08	1.92
4	540.24	1.20
6	805.20	0.98
8	1066.20	0.79
10	1325.00	0.65

6. Conclusions and Discussion

In this paper, we study both the single-sourcing and dual-sourcing inventory models with positive lead times and random yield. Yield uncertainty has rarely been considered in models with positive lead times and never in the dual-sourcing model with general lead times, which is the contribution of this paper. For both models, we propose simple order-up-to heuristics. The optimal order-up-to levels are derived based on approximate models with full returns and modified demand distributions. Numerical results show that the performance of the proposed heuristic in the single sourcing model is close to that of the optimal policy. Compared to the safety stock policy recently proposed by [11], our heuristic seems to perform better than the safety stock policy when yield rate is high or lead time is small. For the dual sourcing model, the numerical results indicate that the proposed heuristic gives, in most cases, results close to the DOP for the model with perfect yield information. Moreover, the performance is robust with respect to changes in the main parameters.

7. References

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