Stochastic Scheduling on Unrelated Machines

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Abstract

Two important characteristics encountered in many real-world scheduling problems are heterogeneous machines/processors and a certain degree of uncertainty about the actual sizes of jobs. The first characteristic entails machine dependent processing times of jobs and is captured by the classical unrelated machine scheduling model.The second characteristic is adequately addressed by stochastic processing times of jobs as they are studied in classical stochastic scheduling models. While there is an extensive but separate literature for the two scheduling models, we study for the first time a combined model that takes both characteristics into account simultaneously.

Here, the processing time of job j on machine i is governed by random variable P_{ij}, and its actual realization becomes known only upon job completion. With w_j being the given weight of job j, we study the classical objective to minimize the expected total weighted completion time $\mathbb{E}\left[\sum_j w_j C_j\right]$, where C_j is the completion time of job j. By means of a novel time-indexed linear programming relaxation, we compute in polynomial time a scheduling policy with performance guarantee $(3+\Delta)/2+\varepsilon$. Here, $\varepsilon > 0$ is arbitrarily small, and Δ is an upper bound on the squared coefficient of variation of the processing times. We show that the dependence of the performance guarantee on Δ is tight, as we obtain a $\Delta/2$ lower bound for the type of policies that we use. When jobs also have individual release dates r_{ij}, our bound is $(2+\Delta)+\varepsilon$. Via $\Delta = 0$, currently best known bounds for deterministic scheduling are contained as a special case.

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1 Introduction

Deterministic scheduling. The problem to minimize the total weighted completion time on unrelated parallel machines, denoted $\mathrm{R}\,|\,(r_{ij})\,|\,\sum w_jC_j$ in the three-field notation of Graham et al. [7], is one of the most important classical problems in the theory of deterministic scheduling. Each job j has a weight w_j, possibly an individual release date r_{ij} before which job j must not be scheduled on machine i, and the processing time of job j on machine i is p_{ij}. Each job has to be processed non preemptively on any one of the machines, and each machine can process at most one job at a time. The objective is to find a schedule minimizing the total weighted completion time $\sum_j w_jC_j$, where C_j denotes the completion time of job j in the schedule.

The special case with identical parallel machines is already known to be strongly NP-hard [12] but there do exist polynomial time approximation schemes [1, 30]. The general setting of unrelated parallel machines turns out to be significantly harder and there is a complexity gap compared to identical parallel machines: Hoogeveen, Schuurman and Woeginger [10] prove MaxSNP-hardness and hence there is no polynomial time approximation scheme. On the positive side, the currently best known approximation algorithms for unrelated parallel machines have performance guarantees 3/2 and 2, for the problem without and with release dates, respectively [3, 24, 26, 28]. Improving these bounds is considered to be among the most important open problems in scheduling [25] which is also an indication of the high significance of unrelated machine scheduling.

Stochastic scheduling. We consider for the first time the stochastic variant of unrelated machine scheduling. Here, the processing time of a job j on machine i is given by random variable P_{ij}. In stochastic scheduling, we are asked to compute a non-anticipatory scheduling policy. Roughly spoken, a scheduling policy makes scheduling decisions at certain decision times t, and these decisions are based on the observed past up to time t as well as the a priori knowledge of the input data of the problem. The policy, however, must not anticipate information about the future, such as the actual realizations of the processing times of jobs which have not yet been completed by time t.

We refer to Möhring, Radermacher and Weiss [16] for the formal definition of stochastic scheduling policies, and here confine ourselves with an intuitive description that puts stochastic scheduling in the framework of stochastic dynamic optimization: Actions of a scheduling policy at a time t consists of a set of jobs, possibly empty, to be started on a set of idle machines, together with a tentative next decision time $t^* > t$. The next action of the policy is due at t^*, or the time of the next job completion, or the time when the next job is released, whatever occurs first. Depending on the action of the policy, the next decision time as well as the state of the schedule at the next decision time is realized according to the probability distributions of the jobs' processing times. A non-anticipatory policy may learn over time, but it has only access to distributional information about remaining processing times of unfinished jobs, conditioned on the state of the schedule at time t.[1]

As all previous work in the area, we assume that the random variables P_{ij} are stochastically independent across jobs. For any given non-anticipatory scheduling policy, the possible outcome

[1] A concrete example may help: Imagine a job j which has processing time either small (ε) or large (M), both with probability 1/2. For a scheduling policy that starts this job at time t, it can make sense to define a tentative next decision time at $t^* = t + \varepsilon$, because then it learns with certainty what the actual processing time of job j is. Using such building blocks, one can even show that an optimal scheduling policy is generally not work conserving, i. e., machines are left deliberately idle [33].

of the objective function $\sum_j w_j C_j$ is a random variable, and our goal is to minimize its expected value, which by linearity of expectation equals $\sum_j w_j \mathbb{E}[C_j]$.

Related work. Generalizing a well known result of Smith [31] for deterministic single machine scheduling, Rothkopf [20] proved in 1966 that the WSEPT rule[2] minimizes the expected total weighted completion time on a single machine. Apart from Weiss' results on the asymptotic optimality of WSEPT in stochastic scheduling on identical parallel machines [34, 35], the first constant factor approximation algorithms for stochastic scheduling on identical parallel machines have been obtained in 1999 by Möhring, Schulz and Uetz [17]. Next to a linear programming (LP) based analysis of the WSEPT rule, they define list scheduling policies which are based on linear programming relaxations in completion time variables. The performance bounds are constant whenever the coefficients of variation of the jobs' processing times are bounded by a constant. As usual in stochastic scheduling, all bounds hold with respect to any non-anticipatory scheduling policy.

By using an idea from Chekuri et al. [2], that approach was extended to stochastic scheduling problems with precedence constrains by Skutella and Uetz [29]. Subsequently, Megow, Uetz and Vredeveld [14] combined the stochastic scheduling model with online scheduling, and derived combinatorial, constant competitive algorithms for that model. All these results, including [14], are based on essentially one and the same linear programming relaxation, namely that of [17]. With respect to the underlying relaxation, Schulz [23] goes one step further, and uses the mean busy time relaxation that was previously used also by Correa and Wagner [4], yet its validity in stochastic scheduling still relies on the validity of the completion time relaxation of [17]. Nevertheless, in comparison to [14], the clever use of an optimal solution to an equivalent time-indexed LP relaxation for deterministic scheduling yields improved and simpler results.

Two other, important research directions are related to our work in that they derive performance bounds for stochastic scheduling problems and minsum objective, yet for different models and independent of the techniques of [17] as well as ours. One is the approximation algorithms for preemptive stochastic scheduling by Megow and Vredeveld [15]. They use a single machine relaxation that is optimally solved by a Gittins index policy, and thereby achieve a competitive ratio of 2 for preemptive online stochastic scheduling on parallel identical machines. The other is work by Scharbrodt et al. [21] and Souza and Steger [32], who analyze the expected competitive ratio rather than the expected performance of a policy. In that model, one analyzes the expected ratio of the performance $v(\Pi)$ of a non-anticipatory policy Π over the value of an offline optimum solution $v(\text{Offline-Opt})$. In other words, [21, 32] analyze the ratio $\mathbb{E}[v(\Pi)/v(\text{Offline-Opt})]$, while we follow [14, 17, 23, 29] and focus on the ratio $\mathbb{E}[v(\Pi)]/\mathbb{E}[v(\Pi^{\text{Opt}})]$ instead. While there are very good reasons for the expected competitive ratio, discussed e.g. in [21], we believe that both models have their justification. One argument in favor of the analysis used here is that it is based on a "fair" adversary, in that the adversary is restricted to be non-anticipatory, too[3]. This is in contrast to the offline optimum adversary used in [21, 32], and more generally, in competitive analysis. In fact, the strength of the offline optimum adversary is one of the main critiques of competitive analysis in general, because it often renders competitive analysis too pessimistic[4].

[2]Weighted shortest expected processing time first: schedule jobs in order of non-increasing ratios $w_j/\mathbb{E}[P_j]$.

[3]The restriction of the adversary's power was called *comparative analysis* by Koutsoupias and Papadimitriou [11].

[4]It is therefore noticeable that [21, 32] achieve constant bounds on the expected competitive ratio. For example for exponentially distributed processing times, the expected competitive ratio of the WSEPT rule on identical, parallel machines is no more than $3 - 1/m$ [32].

stochastic scheduling model	worst case performance guarantee arbitrary P_{ij}	$\mathbb{CV}[P_{ij}] \leq 1$	reference
$\mathrm{P} \,|\, | \,\mathbb{E}\big[\sum w_j C_j\big]$	$1 + \frac{(m-1)(\Delta+1)}{2m}$	$2 - 1/m$	[17, 14]
$\mathrm{P} \,|\, r_j \,|\, \mathbb{E}\big[\sum w_j C_j\big]$	$2 + \Delta$	3	[23]
$\mathrm{R} \,|\, | \,\mathbb{E}\big[\sum w_j C_j\big]$	$1 + \frac{\Delta+1}{2} + \varepsilon$	$2 + \varepsilon$	this paper
$\mathrm{R} \,|\, r_{ij} \,|\, \mathbb{E}\big[\sum w_j C_j\big]$	$2 + \Delta + \varepsilon$	$3 + \varepsilon$	this paper

Table 1: Performance bounds for nonpreemptive stochastic machine scheduling problems. Parameter $\varepsilon > 0$ can be chosen arbitrarily small. Parameter Δ upper bounds the squared coefficient of variation $\mathbb{CV}^2[P_{ij}] = \mathbb{V}\mathrm{ar}[P_{ij}]/\mathbb{E}^2[P_{ij}]$ for all P_{ij}. The third column shows the results for $\mathbb{CV}[P_{ij}] \leq 1$; e. g., uniform, exponential, or Erlang distributions. As usual in stochastic scheduling, these bounds hold with respect to the expected performance of any non-anticipatory scheduling policy.

Note that all mentioned results are restricted to identical parallel machines. Table 1 gives a brief overview of currently best known performance bounds in nonpreemptive stochastic scheduling with minsum objective, next to the results obtained in this paper for unrelated parallel machines.

With respect to algorithmic ideas and techniques, the evolution of stochastic scheduling has largely benefited in the past from progress being made for the corresponding deterministic scheduling problems. For example, all LP-based approximation results for stochastic scheduling on identical parallel machines outlined above build upon a class of linear programming relaxations in completion time variables that dates back to Wolsey [36] and Queyranne [19] (for single machine scheduling), and was later generalized to identical parallel machines by Schulz [22] and Hall et al. [9] who also presented LP-based approximation algorithms for deterministic scheduling problems.

Our contribution. We obtain the first approximation algorithms for stochastic scheduling on unrelated machines. Despite the fact that the unrelated machine scheduling model is significantly richer than identical machine scheduling, our bounds essentially match all previous performance bounds that have been obtained for the corresponding stochastic scheduling problems on identical parallel machines; see Table 1. We also give a tight lower bound, showing that the dependence of the performance bound on the squared coefficient of variation Δ is unavoidable for the class of policies that we use. For the first time we completely depart from the LP relaxation of Möhring et al. [17], and show how to put a novel, time-indexed linear programming relaxation to work in stochastic machine scheduling. We are optimistic that this novel approach will inspire further research and prove useful for other stochastic optimization problems in scheduling and related areas.

Time-indexed linear programming relaxations have played a pivotal role in the development of constant factor approximation algorithms for deterministic scheduling on unrelated parallel machines [24]. In spite of that, it remained unclear and a major open problem how to come up with a meaningful time-indexed LP relaxation for stochastic scheduling problems [13]. Here the main difficulty is that, in contrast to deterministic schedules that can be fully described by time-indexed 0-1-variables, scheduling *policies* feature a considerably richer structure including complex dependencies between the execution of different jobs which cannot be easily described by time-indexed

variables.

In Section 3 we show how to overcome this difficulty and present the first time-indexed LP relaxation for stochastic scheduling on unrelated parallel machines. Here, the value of the time-indexed variable x_{ijt} represents the probability of job j being started on machine i at time t.[5] While writing down the machine capacity constraints[6] is rather easy for deterministic scheduling in this formulation, the situation is somewhat more complicated in the stochastic setting and we require a fair amount of information about the exact probability distributions of random variables P_{ij}.

Notice that, due to the stochastic nature of processing times, even a schedule produced by an optimal policy can be arbitrarily long such that infinitely many variables x_{ijt} may take positive values. Nonetheless, in Appendix A we show how to overcome this difficulty and obtain an (almost) optimal LP solution efficiently, i. e., in polynomial time.

In Section 4 we discuss how to turn a feasible solution to the time-indexed LP relaxation into a simple scheduling policy. Our approach is inspired by the randomized rounding algorithm for deterministic scheduling on unrelated parallel machines in [24]. Each job j is randomly assigned to a machine i with probability $\sum_t x_{ijt}$; then, on each machine i, the WSEPT policy is used to schedule the jobs assigned to i. The analysis, however, is based on a somewhat more elaborate, random sequencing of jobs which is determined by a two-stage random process.

Since each job is immediately and irrevocably assigned to a machine, our scheduling policies fall into the special class of *fixed assignment policies*. Notice that these policies ignore the additional information that evolves over time in the form of the actual realizations of processing times. Not surprisingly, this ignorance comes at a price. In Section 5 we prove a lower bound of $\Delta/2$ on the performance guarantee of *any* fixed assignment policy. This negative result nicely complements our positive results; see Table 1.

In order to keep the presentation as simple as possible, we ignore release dates and restrict to the problem $\mathrm{R}\,|\,|\,\mathbb{E}\big[\sum w_j C_j\big]$ throughout most of the paper. Only in Section 6 we show how release dates can be taken care of in our approach.

Parallel to Stochastic Knapsack. There is an interesting parallel of the present work on stochastic scheduling with that on stochastic knapsack problems[7]. The first study of approximation algorithms for stochastic knapsack problems is due to Dean, Goemans and Vondrák [5], presenting constant factor approximation algorithms along with an analysis of the adaptivity gap[8]. Their results are based on a linear programming relaxation that is essentially the deterministic knapsack LP where item sizes and weights are replaced by expected values. In that sense, methodology-wise their linear program parallels that of [17] in stochastic scheduling on parallel machines. Recently, Gupta et al. [8] were able to obtain constant factor approximation algorithms for a much broader class of stochastic knapsack problems (and other problems, too). Key to these results is a more sophisticated, time-indexed linear programming relaxation, based on the same type of variables as we use here. It is interesting to note that in their paper as well as in ours, moving from "natural yet simple" LP relaxations to richer time-indexed LP relaxations is key to more general results.

[5] Even for simple scheduling policies like the WSEPT rule, determining this probability is highly non-trivial.

[6] The machine capacity constraints say that each machine can process at most one job at a time.

[7] Note that a stochastic knapsack problem can be reinterpreted as a single machine stochastic scheduling problem where all jobs have due date 1, and with weighted earliness objective.

[8] In stochastic scheduling, this would correspond to the gap between the best static list scheduling policy and an optimal (adaptive) scheduling policy.

2 Notation and preliminaries

We are given a set of jobs J of cardinality n with job weights $w_j \in \mathbb{Z}_{>0}$, $j \in J$, and a set of unrelated parallel machines M of cardinality m. Moreover, for every job $j \in J$ and every machine $i \in M$, we are given a random variable P_{ij}. Each job j needs to be executed on any one of the machines $i \in M$, and each machine can process at most one job at a time. If job j is processed on machine i, its processing time is P_{ij}. However, the actual realization of the processing time is only known upon j's completion and we are thus looking for a non-anticipatory scheduling policy which minimizes the expected total weighted completion time $\mathbb{E}\left[\sum_j w_j C_j\right]$, where C_j denotes the completion time of job j.

Later, in Section 6, we consider a slightly more general model where each job $j \in J$ also comes with a machine dependent release date $r_{ij} \in \mathbb{Z}_{\geq 0}$ before which job j must not be scheduled on machine i. One can think of applications where some job $j \in J$ might not be processed on a certain machine $i \in M$, i. e., $\mathbb{E}[P_{ij}] = \infty$. For the sake of simplicity of presentation, we assume in this paper that $\mathbb{E}[P_{ij}]$ is finite for all $i \in M$ and $j \in J$. But all presented results also hold for the more general case where $\mathbb{E}[P_{ij}] = \infty$ for certain pairs i, j.

Throughout this paper we assume that the random variables P_{ij}, $i \in M$, $j \in J$, take positive integral values only. The following lemma states that this assumption costs at most a factor $1 + \varepsilon$ in the objective function value.

Lemma 1. *For any fixed $\varepsilon > 0$, while only loosing a factor $1+\varepsilon$ in the objective function value, an arbitrary instance can be modified such that the random variables P_{ij}, $i \in M$, $j \in J$, take positive integral values only.*

Proof. If $\mathbb{E}[P_{ij}] = 0$ and $r_{ij} = 0$ for some pair i, j, then we can ignore job j since it can be scheduled at no further cost on machine i at time 0. We can thus assume from now on that $\mathbb{E}[P_{ij}] > 0$ or $r_{ij} > 0$ for all pairs i, j. By scaling processing times and release dates appropriately, we can make sure that $\mathbb{E}[P_{ij}] \geq \frac{n}{\varepsilon}$ or $r_{ij} \geq \frac{n}{\varepsilon}$ for each pair i, j. As a result of this scaling step we know that, for any scheduling policy, $\mathbb{E}[C_j] \geq n/\varepsilon$ for each job $j \in J$. Rounding up all processing times to the nearest positive integer therefore increases the (expected) completion time of any job j by at most $n \leq \varepsilon\, \mathbb{E}[C_j]$. The overall increase in the objective function is thus bounded by a factor $1+\varepsilon$. □

Given that all processing times are integral, we can obviously assume with no further loss of generality that jobs can only be started at integral points in time $t \in \mathbb{Z}_{\geq 0}$.

In order to write down an LP relaxation in time-indexed variables, we require a fair amount of information about the exact probability distributions of random variables P_{ij}. More precisely, besides the expectations $\mathbb{E}[P_{ij}]$, we also need the values

$$q_{ijr} := \mathbb{P}\mathrm{r}[P_{ij} \geq r + 1] \qquad \text{for } i \in M,\ j \in J, \text{ and } r \in \mathbb{Z}_{\geq 0}.$$

This, of course, raises questions about the input size of the problem. Here we make the following assumption. In the input we are given, for each job $j \in J$ and each machine $i \in M$, the expected processing time $\mathbb{E}[P_{ij}]$. Moreover, we have access to an oracle which, for any triple i, j, r returns q_{ijr}.

We emphasize that, in order for our approach to work, it suffices to get these values within some finite precision at the expense of an additional factor $1 + \varepsilon$ in the performance guarantee of our algorithms. More precisely, it suffices to get the values q_{ijr} rounded to multiples of ε/n, which, in particular, can be encoded polynomially in the input size. Notice that such an oracle can be

simulated by a polynomial-time Monte Carlo algorithm that can sample from the distribution of the random variables P_{ij}. Having said that, in order to keep the presentation simple we neglect these aspects throughout the paper and assume that we have access to the exact values q_{ijr}.

In the analysis of our algorithm we need the following standard property of the moments of random variable P_{ij}.

Lemma 2. *Let $j \in J$ and $i \in M$. Then,*

$$\sum_{r \in \mathbb{Z}_{\geq 0}} q_{ijr} = \mathbb{E}[P_{ij}] \qquad \text{and} \qquad \sum_{r \in \mathbb{Z}_{\geq 0}} (r + \tfrac{1}{2})\, q_{ijr} = \frac{1 + \mathbb{CV}[P_{ij}]^2}{2}\, \mathbb{E}[P_{ij}]^2 \ ,$$

where $\mathbb{CV}[P_{ij}]^2 := \big(\mathbb{E}[P_{ij}^2] - E[P_{ij}]^2\big)/\mathbb{E}[P_{ij}]^2$ is the squared coefficient of variation of P_{ij}.

The proof of the lemma is based on standard results for the nth moment of a random variable, see, e. g. [6, V.6, Lemma 1]. For completeness, we give the simple proof for the discrete case here.

Proof. First,

$$\sum_{r \geq 0} q_{ijr} = \sum_{r \geq 0} \sum_{q \geq r} \mathbb{P}\mathrm{r}[P_{ij} = q + 1] = \sum_{r \geq 0} (r+1)\, \mathbb{P}\mathrm{r}[P_{ij} = r + 1] = \mathbb{E}[P_{ij}] \ .$$

For the second claim,

$$\sum_{r \geq 0} (r + \tfrac{1}{2})\, q_{ijr} = \sum_{r \geq 0} \sum_{q \geq r} (r + \tfrac{1}{2})\, \mathbb{P}\mathrm{r}[P_{ij} = q + 1] = \sum_{r \geq 0} \tfrac{1}{2}\, (r+1)^2\, \mathbb{P}\mathrm{r}[P_{ij} = r + 1] = \tfrac{1}{2}\, \mathbb{E}[P_{ij}^2] \ .$$

The claim now follows by definition of the coefficient of variation. □

3 Time-indexed LP relaxation

In the following we derive an LP relaxation of the stochastic scheduling problem under consideration. For a given non-anticipatory scheduling policy Π, let x_{ijt} be the probability that Π starts job $j \in J$ on machine $i \in M$ at time $t \in \mathbb{Z}_{\geq 0}$. Notice that this random decision may depend on the actual processing times of other jobs started by Π before time t. On the other hand, due to the non-anticipatory nature of policy Π, the random variable P_{ij} is independent of Π's random decision to start job j on machine i at time t.

As the x_{ijt}'s are going to be the variables of our LP relaxation, we derive crucial properties that are going to be the constraints of the LP relaxation. If job $j \in J$ is started on machine $i \in M$ at time $t \in \mathbb{Z}_{\geq 0}$, due to the non-anticipative nature of policy Π, j's expected completion time is $t + \mathbb{E}[P_{ij}]$. Thus, by linearity of expectation, the expected completion time of j is

$$\mathbb{E}[C_j] = \sum_{i \in M} \sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt} \big(t + \mathbb{E}[P_{ij}]\big) \ .$$

A more careful look at j's behavior reveals the following property. Conditioning on j being started on machine i at time t, the probability that j is still occupying machine i within the later time interval $[s, s+1]$, $s \in \mathbb{Z}_{\geq t}$, is equal to $q_{ij\,s-t}$ by definition. Unconditioning yields

$$\mathbb{P}\mathrm{r}\big[i \text{ processes } j \text{ in } [s, s+1]\big] = \sum_{t=0}^{s} x_{ijt}\, q_{ij\,s-t} \ . \tag{1}$$

As machine i can process at most one job at a time, also the *expected* number of jobs being processed by i in $[s, s+1]$ is bounded by 1. That is, by linearity of expectation,

$$\sum_{j \in J} \sum_{t=0}^{s} x_{ijt}\, q_{ij\, s-t} \leq 1 \ .$$

Finally, since policy Π has to process all jobs, we get for every job j

$$\sum_{i \in M} \sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt} = 1 \ .$$

Thus, the probabilities x_{ijt} corresponding to policy Π form a feasible solution to the following LP relaxation, and the value of this LP solution x is equal to the expected value of the schedule produced by policy Π:

$$\begin{aligned}
\min \quad & \sum_{j \in J} w_j\, C_j^{\mathrm{LP}} & & \\
\text{s.t.} \quad & \sum_{i \in M} \sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt} = 1 & \text{for all } j \in J, & \quad (2) \\
& \sum_{j \in J} \sum_{t=0}^{s} x_{ijt}\, q_{ij\, s-t} \leq 1 & \text{for all } i \in M,\ s \in \mathbb{Z}_{\geq 0}, & \quad (3) \\
& C_j^{\mathrm{LP}} = \sum_{i \in M} \sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt}\, (t + \mathbb{E}[P_{ij}]) & \text{for all } j \in J, & \quad (4) \\
& x_{ijt} \geq 0 & \text{for all } j \in J,\ i \in M,\ t \in \mathbb{Z}_{\geq 0}. &
\end{aligned}$$

Notice that the LP variables C_j^{LP} are uniquely determined by the x-variables and could as well be omitted by replacing them in the objective function with the right hand side of (4).

Also notice that this linear program suffers from infinitely many variables and constraints. In Appendix A, we argue that this is only a minor problem that can be dealt with at the expense of an additional factor $1 + \varepsilon$ in the performance guarantee of our algorithms.

4 Turning an LP solution into a scheduling policy

For a feasible LP solution x, let $X_{ij} := \sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt}$ for $i \in M$, $j \in J$. LP constraints (2) imply that $\sum_{i \in M} X_{ij} = 1$ for every job $j \in J$.

Given the values X_{ij} corresponding to a feasible LP solution x, our scheduling policy $\textsc{Assign}(X)$ assigns each job $j \in J$ independently at random to one machine $i \in M$ with probability X_{ij}. Then, on each machine $i \in M$, it sequences jobs assigned to i according to the WSEPT rule.

Theorem 1. *The expected value of the schedule constructed by policy* $\textsc{Assign}(X)$ *is at most* $\frac{3}{2} + \frac{\Delta}{2}$ *times the value of the underlying LP solution* x*.*

Notice that Theorem 1 and Theorem 5 (see Appendix) together imply the existence of a polynomial-time algorithm that, for any given instance of our stochastic scheduling problem and for any $\varepsilon > 0$, finds a scheduling policy with performance guarantee $\frac{3}{2} + \frac{\Delta}{2} + \varepsilon$. Remember that Δ upper

bounds the squared coefficient of variation $\mathbb{CV}[P_{ij}]^2$ for all P_{ij}. It is not difficult to see that, instead of the random assignment of jobs to machines, we can use a deterministic assignment obtained via the method of conditional probabilities and still get the same performance guarantee.

The proof of Theorem 1 is based on a refined, somewhat more complicated policy, that takes the entire LP solution x into account and yields a worse schedule in expectation. It is therefore sufficient to prove the bound stated in Theorem 1 for this alternative policy which we refer to as Assign&Sequence(x).

Assign&Sequence(x)

1. For every job $j \in J$, choose a pair (i,t) independently at random with probability x_{ijt} and some $r \in \mathbb{Z}_{\geq 0}$ independently at random with probability $q_{ijr}/\mathbb{E}[P_{ij}]$; assign job j to machine i and set its *tentative start time* s to $s := t + r$ (we write "$j \to i, s$" for short).
2. On each machine $i \in M$, sequence all jobs assigned to i in order of increasing tentative start times; ties are broken randomly.

Notice that, as in the simpler policy Assign(X), job j is assigned to machine i with probability $\sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt} = X_{ij}$. Since Assign($X$) sequences the jobs on every machine in an optimal way, it is superior to policy Assign&Sequence(x).

By construction of policy Assign&Sequence(x), the probability of assigning job $j \in J$ to machine $i \in M$ and setting its tentative start time to $s \in \mathbb{Z}_{\geq 0}$ is

$$\Pr[j \to i, s] = \sum_{t=0}^{s} x_{ijt} \, \frac{q_{ij\,s-t}}{\mathbb{E}[P_{ij}]} \ . \tag{5}$$

We prove the following job-by-job performance guarantee for Assign&Sequence(x).

Theorem 2. *For every job $j \in J$, the expected value of j's completion time in the schedule constructed by policy* Assign&Sequence(x) *is at most $(\frac{3}{2} + \frac{\Delta_j}{2})\, C_j^{\mathrm{LP}}$ where $\Delta_j := \max_{i \in M} \mathbb{CV}[P_{ij}]^2$.*

By linearity of expectation, Theorem 2 immediately implies Theorem 1. In the proof of Theorem 2 we make use of the following lemma.

Lemma 3. *Let $j \in J$, $i \in M$, and $s \in \mathbb{Z}_{\geq 0}$. If $j \to i, s$, then the expected total processing time of jobs that policy* Assign&Sequence(x) *schedules on machine i before job j is at most $s + \frac{1}{2}$.*

Proof. We first bound the expected total processing time of jobs $k \neq j$ with $k \to i, s'$ for some fixed $s' \in \mathbb{Z}_{\geq 0}$:

$$\sum_{k \neq j} \mathbb{E}[P_{ik}] \Pr[k \to i, s'] \overset{(5)}{=} \sum_{k \neq j} \sum_{t'=0}^{s'} x_{ikt'} \, q_{ik\,s'-t'} \leq 1 \qquad \text{by (3)}.$$

Thus, the expected[9] total processing times of jobs processed before job j on machine i is at most

$$\sum_{k \neq j} \mathbb{E}[P_{ik}] \left(\sum_{s'=0}^{s-1} \Pr[k \to i, s'] + \tfrac{1}{2} \Pr[k \to i, s] \right) \leq s + \tfrac{1}{2} \ .$$

This concludes the proof. □

[9] Notice that the expectation is taken with respect to both the random decisions of our policy Assign&Sequence(x) as well as the random processing times of jobs $k \neq j$.

Proof of Theorem 2. By Lemma 3 we get

$$\mathbb{E}[C_j \mid j \to i, s] \leq s + \tfrac{1}{2} + \mathbb{E}[P_{ij}] \tag{6}$$

for every job $j \in J$, machine $i \in M$, and tentative start time $s \in \mathbb{Z}_{\geq 0}$. Unconditioning the expectation yields

$$\mathbb{E}[C_j] = \sum_{i \in M} \sum_{s \in \mathbb{Z}_{\geq 0}} \mathbb{E}\big[C_j \mid j \to i, s\big] \, \Pr[j \to i, s] \ .$$

Applying inequality (6) and equation (5) we get

$$\mathbb{E}[C_j] \leq \sum_{i=1}^{m} \sum_{s \in \mathbb{Z}_{\geq 0}} \big(s + \tfrac{1}{2} + \mathbb{E}[P_{ij}]\big) \sum_{t=0}^{s} x_{ijt} \, \frac{q_{ij\,s-t}}{\mathbb{E}[P_{ij}]} \ .$$

Exchanging the order of summation of s and t, and setting $r := s - t$ yields

$$\begin{aligned}
\mathbb{E}[C_j] &\leq \sum_{i=1}^{m} \sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt} \left(t + \mathbb{E}[P_{ij}] + \sum_{r \in \mathbb{Z}_{\geq 0}} (r + \tfrac{1}{2}) \, \frac{q_{ijr}}{\mathbb{E}[P_{ij}]} \right) \\
&= \sum_{i=1}^{m} \sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt} \left(t + \left(\frac{3}{2} + \frac{\mathbb{CV}[P_{ij}]^2}{2} \right) \mathbb{E}[P_{ij}] \right) \\
&\leq \left(\frac{3}{2} + \frac{\Delta_j}{2} \right) C_j^{\mathrm{LP}}
\end{aligned}$$

by Lemma 2 and (4). This concludes the proof. □

5 Lower bound on the performance of fixed assignment policies

In this section we show that the dependence of our performance guarantee on the squared coefficients of variation $\Delta = \max_{ij} \mathbb{CV}[P_{ij}]^2$ has the right order of magnitude for all algorithms that use fixed job assignments to machines, i. e., for all policies where the assignment of all jobs to machines is fixed right in the beginning and not changed after learning about the processing time realizations of already completed jobs. This even holds for the case of identical parallel machines where, for each job $j \in J$, there is one random variable P_j such that $P_{ij} = P_j$ for all machines $i \in M$.

Theorem 3. *Even for the special case of identical parallel machines, the performance ratio of any fixed-assignment policy is at least* $\frac{(1-\delta)\Delta}{2}$ *for any* $\delta > 0$.

This theorem shows that our approximation result cannot be significantly improved without considering adaptive policies.

Proof. Let $\varepsilon > 0$ be a fixed small constant. Consider an instance consisting of m identical parallel machines and m^2 jobs with unit weights $w_j = 1$ and stochastic processing times P_j with

$$P_j = \begin{cases} 1 & \text{with probability } \frac{1-\varepsilon}{m}, \\ 0 & \text{with probability } 1 - \frac{1-\varepsilon}{m}. \end{cases}$$

The expected processing time of each job is $\mathbb{E}[P_j] = \mathbb{P}\mathrm{r}[P_j = 1] = \frac{1-\varepsilon}{m}$ and the total expected processing time of all jobs is

$$\mathbb{E}\left[\sum_{j\in J} P_j\right] = (1-\varepsilon)\, m \ .$$

The squared coefficient of variation of random variable P_j is

$$\Delta = \mathbb{CV}[P_j]^2 = \frac{m}{1-\varepsilon} - 1 \ .$$

The probability that our instance has at least $\frac{1}{1-\varepsilon}\,\mathbb{E}\left[\sum_{j\in J} P_j\right] = m$ jobs with processing times equal to one is upper bounded by $e^{-\frac{1}{4}m\,\varepsilon^2/(1-\varepsilon)}$ by the Chernoff bounds for the sum of independent Boolean random variables (see, e. g., [18, Theorems 4.1 and 4.3]). Note also that, under any realization of processing times, the value of the schedule computed by the optimal policy is upper bounded by m^3.

In order to derive an upper bound on the optimal policy, consider the following adaptive (but non-anticipatory) policy. Initially all machines are available. Start one job on each available machine. If some of the jobs are immediately finished (i. e., they have processing time 0), then start one job per available machine again. Once we have a job that has not been finished immediately, we declare the machine where this job is processed *un-available* and continue to assign jobs to available machines. The expected value of the schedule produced by the optimal policy can thus be bounded from above by

$$\begin{aligned}
&\mathbb{P}\mathrm{r}\left[\sum_{j\in J} P_j < m\right]\mathbb{E}\left[\sum_{j\in J} P_j \Big| \sum_{j\in J} P_j < m\right] + \mathbb{P}\mathrm{r}\left[\sum_{j\in J} P_j \geq m\right] m^3 \\
&\quad\leq \mathbb{E}\left[\sum_{j\in J} P_j\right] + \mathbb{P}\mathrm{r}\left[\sum_{j\in J} P_j \geq m\right] m^3 \\
&\quad\leq (1-\varepsilon)\, m + e^{-\frac{1}{4}m\,\varepsilon^2/(1-\varepsilon)}\, m^3 < m \ ,
\end{aligned}$$

where the last inequality holds for large enough m.

Consider now any fixed-assignment policy that assigns k_i jobs to be processed on machine $i \in M$ with $\sum_{i\in M} k_i = m^2$. Since all jobs have identical distributions and weights, the optimal single machine policy is to process jobs according to an arbitrary permutation. The expected value of such schedule on machine $i \in M$ is

$$\sum_{q=1}^{k_i} q\,\frac{1-\varepsilon}{m} = (1-\varepsilon)\,\frac{k_i\,(k_i+1)}{2m} \ .$$

Thus, due to convexity, the value of the fixed assignment policy is

$$(1-\varepsilon)\sum_{i\in M} \frac{k_i\,(k_i+1)}{2m} \geq (1-\varepsilon)\,\frac{m\,(m+1)}{2} \ .$$

Therefore, the worst-case ratio between an optimal fixed assignment policy and an optimal policy is at least $(1-\varepsilon)\,\frac{m+1}{2}$. Since we can choose $\varepsilon > 0$ to be arbitrarily small and m arbitrarily large, we derive that this ratio is at least $\frac{(1-\delta)\Delta}{2}$ for any $\delta > 0$ and large enough parameter m. □

6 Adding release dates

In this section we show how to adapt our analysis for a more general problem where each job $j \in J$ comes with a machine dependent deterministic release date $r_{ij} \in \mathbb{Z}_{\geq 0}$ before which job j must not be scheduled on machine i. To handle release dates we add one additional family of constraints to our time-indexed LP relaxation:

$$x_{ijt} = 0 \qquad \text{for all } i \in M,\, j \in J,\, t < r_{ij}.$$

These constraints are obviously fulfilled by the probabilities x_{ijt} corresponding to an arbitrary scheduling policy Π as no job may be started before it is released. We consider the same LP based policy Assign&Sequence(x) for this more general problem.

Theorem 4. *In the presence of release dates, for every job $j \in J$, the expected value of j's completion time in the schedule constructed by policy* Assign&Sequence(x) *is at most $(2 + \Delta_j)\, C_j^{\mathrm{LP}}$ where $\Delta_j := \max_{i \in M} \mathbb{CV}[P_{ij}]^2$.*

Notice that the following proof of Theorem 4 is almost identical to the proof of Theorem 2.

Proof. Note that the release dates of all jobs that have tentative start times less than s is at most s. Thus, by Lemma 3, we get

$$\mathbb{E}[C_j \mid j \to i, s] \leq s + s + \tfrac{1}{2} + \mathbb{E}[P_{ij}] \tag{7}$$

for every job $j \in J$, machine $i \in M$, and tentative start time $s \in \mathbb{Z}_{\geq 0}$. Unconditioning the expectation yields

$$\mathbb{E}[C_j] = \sum_{i \in M} \sum_{s \in \mathbb{Z}_{\geq 0}} \mathbb{E}\big[C_j \mid j \to i, s\big]\, \mathbb{P}\mathrm{r}[j \to i, s] \ .$$

Applying inequality (7) and equation (5) we get

$$\mathbb{E}[C_j] \leq \sum_{i=1}^{m} \sum_{s \in \mathbb{Z}_{\geq 0}} \big(2s + \tfrac{1}{2} + \mathbb{E}[P_{ij}]\big) \sum_{t=0}^{s} x_{ijt}\, \frac{q_{ij\,s-t}}{\mathbb{E}[P_{ij}]} \ .$$

Exchanging the order of summation of s and t, and setting $r := s - t$ yields

$$\begin{aligned}
\mathbb{E}[C_j] &\leq \sum_{i=1}^{m} \sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt} \left(2t + \mathbb{E}[P_{ij}] + 2 \sum_{r \in \mathbb{Z}_{\geq 0}} (r + \tfrac{1}{2})\, \frac{q_{ijr}}{\mathbb{E}[P_{ij}]}\right) \\
&= \sum_{i=1}^{m} \sum_{t \in \mathbb{Z}_{\geq 0}} x_{ijt} \Big(2t + \big(2 + \mathbb{CV}[P_{ij}]^2\big)\, \mathbb{E}[P_{ij}]\Big) \leq \big(2 + \Delta_j\big)\, C_j^{\mathrm{LP}}
\end{aligned}$$

by Lemma 2 and (4). This concludes the proof. □

Corollary 1. *In the presence of release dates, the expected value of the schedule constructed by policy* Assign&Sequence(x) *is at most $2 + \Delta$ times the value of the underlying LP solution x. Thus, for any given instance of our stochastic scheduling problem and for any $\varepsilon > 0$, a $(2 + \Delta + \varepsilon)$-approximate scheduling policy can be found in polynomial time.*

7 Concluding remarks

Comparison of LP relaxations. One of the main technical contributions of this paper is to introduce the important concept of time-indexed linear programming relaxations to the area of stochastic scheduling. All previous approximation results in this area are restricted to the identical parallel machine setting and essentially based on variants of an LP relaxation in completion time variables introduced in [17]. For the special case of identical parallel machines, the new time-indexed LP relaxation introduced in this paper is always at least as strong as all known variants of the LP relaxation in completion time variables. More precisely, for any feasible solution x to the LP relaxation introduced in Section 3, the values of C_j^{LP}, $j \in J$, fulfill all known constraints of the LP relaxation in completion time variables.

Execution of a policy. We have argued that the policy we propose can be computed in polynomial time, but so far did not discuss the computation time to actually execute the scheduling policy, or more generally, any stochastic scheduling policy. The major issue is how, and with which computational effort the scheduler learns about the next job completion when executing a set of jobs. Probabilistically, this event is described by the minimum of a set of random variables, of which we just sample while executing the policy. In general, and already if there is just one single job to be processed, there might of course be nonzero probability for a job to be exponentially longer than expected. But due to Markov's inequality, the probability for exceeding the expected processing time by an exponential factor is exponentially small, too. Therefore, with high probability the sampled processing times of jobs can be encoded polynomially in the input size of the problem. Apart from this minor issue inherent in all stochastic scheduling problems, we note that the policy $\textsc{Assign}(X)$ is in particular elementary [16], meaning that jobs are only started upon release times or completion times of other jobs. Hence, there is only a linear number of decision times.

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A Solving the LP relaxation efficiently

The time-indexed LP relaxation introduced in Section 3 suffers from its infinitely many variables and constraints. Notice, however, that sometimes infinitely many variables are needed in order to map an optimal scheduling policy Π to an LP solution as described in Section 3. Consider, for example, two jobs with exponentially distributed processing times that must be scheduled on a single machine. Notice that the start time of the second job cannot be bounded, i. e., for any $t \geq 0$, the second job is started with positive probability at time t or later.

Despite this peculiarity, we can show that an optimal solution to the LP relaxation has finite support. More precisely, we give a pseudo-polynomial upper bound on the largest time index t such that $x_{ijt} > 0$ in an optimal LP solution for any $j \in J$ and $i \in M$.

Any reasonable scheduling policy produces a schedule where the expected completion time of every job is at most

$$D := \max_{i \in M} \sum_{j \in J} \mathbb{E}[P_{ij}] \ .$$

In particular, the expected value of the schedule is at most $U := D \sum_{j \in J} w_j$ which is thus also an upper bound on the optimal LP solution. Moreover, let

$$R := 2n \max_{j \in J, i \in M} \mathbb{E}[P_{ij}] \ .$$

Lemma 4. *Let $T := 2U + R$. There is an optimal solution x to the LP relaxation such that $x_{ijt} = 0$ for $i \in M$, $j \in J$, and $t > T$.*

Proof. Consider an optimal LP solution x and an arbitrary machine $i \in M$. Since U is an upper bound on the value of x and since $w_j \geq 1$ for each job $j \in J$, it follows from Markov's inequality that

$$\sum_{j \in J} \sum_{t \geq 2U} x_{ijt} \leq \frac{1}{2} \ . \tag{8}$$

Using Markov's inequality once again, we derive that for each job $j \in J$ and $r \geq R$

$$q_{ijr} = \mathbb{P}\mathrm{r}[P_{ij} \geq r + 1] \leq \mathbb{P}\mathrm{r}\big[P_{ij} \geq 2n\, \mathbb{E}[P_{ij}]\big] \leq \frac{1}{2n} \ . \tag{9}$$

We now define a new LP solution x' by letting

$$x'_{ijt} := \begin{cases} x_{ijt} & \text{for } t < T, \\ \sum_{t' \geq T} x_{ijt'} & \text{for } t = T, \\ 0 & \text{for } t > T. \end{cases}$$

By definition, x' has the desired property stated in the lemma and its objective function value is bounded from above by the value of x. It remains to prove that x' is a feasible LP solution. It is clear that x' fulfills LP constraints (2) since x fulfills these constraints. It is, however, less obvious that x' also fulfills all LP constraints (3). For $s < T$

$$\sum_{j \in J} \sum_{t=0}^{s} x'_{ijt}\, q_{ij\, s-t} = \sum_{j \in J} \sum_{t=0}^{s} x_{ijt}\, q_{ij\, s-t} \leq 1$$

since x is a feasible LP solution and thus satisfies (3). Finally, for $s \geq T$,

$$\sum_{j \in J} \sum_{t=0}^{s} x'_{ijt}\, q_{ij\, s-t} = \sum_{j \in J} \sum_{t=0}^{2U-1} x_{ijt}\, q_{ij\, s-t} + \sum_{j \in J} \sum_{t=2U}^{s} x'_{ijt}\, q_{ij\, s-t}$$
$$\leq \frac{1}{2n} \sum_{j \in J} \sum_{t=0}^{2U-1} x_{ijt} + \sum_{j \in J} \sum_{t \geq 2U} x_{ijt} \leq \frac{1}{2} + \frac{1}{2} = 1$$

where the first inequality follows from (9) and the definition of x', and the last inequality follows from (2) and (8). This concludes the proof. □

It can be easily derived from the proof that the property claimed in Lemma 4 indeed holds for any optimal LP solution x.

It is important to notice that, for such a solution x, constraints (3) for $s > T$ are implied by the constraint for $s = T$ since q_{ijr} is non-increasing in r. We have thus reduced the problem of finding an optimal LP solution to solving a *truncated* time-indexed LP of pseudo-polynomial size. It is well known that such time-indexed LPs can be solved approximately in polynomial time at the expense of losing a factor $1+\varepsilon$ in the objective function. The underlying idea is to replace the discretization of time into unit size intervals by a slightly rougher discretization using intervals of geometrically increasing lengths. We refer to [27, Chapter 2.13] for a more thorough discussion of this point and summarize this section by stating the following theorem.

Theorem 5. *The infinite time-indexed LP relaxation introduced in Section 3 can be solved in pseudo-polynomial time in the input size. Moreover, a* $(1+\varepsilon)$*-approximate LP solution can be found in time polynomial in the input size and* $1/\varepsilon$*.*

We finally mention that the Theorem 5 can be easily generalized to the more general problem with release dates studied in Section 6.