Divergent Quiescent Transition Systems (extended version)*

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Abstract. Quiescence is a fundamental concept in modelling system behaviour, as it explicitly represents the fact that no output is produced in certain states. The notion of quiescence is also essential to model-based testing: if a particular implementation under test does not provide any output, then the test evaluation algorithm must decide whether or not to allow this behaviour. To explicitly model quiescence in all its glory, we introduce Divergent Quiescent Transition Systems (DQTSs).

DQTSs model quiescence using explicit δ-labelled transitions, analogous to Suspension Automata (SAs) in the well-known `ioco` framework. Whereas SAs have only been defined implicitly, DQTSs for the first time provide a fully-formalised framework for quiescence. Also, while SAs are restricted to convergent systems (i.e., without τ-cycles), we show how quiescence can be treated naturally using a notion of fairness, allowing systems exhibiting divergence to be modelled as well. We study compositionality under the familiar automata-theoretical operations of determinisation, parallel composition and action hiding. We provide a non-trivial algorithm for detecting divergent states, and discuss its complexity. Finally, we show how to use DQTSs in the context of model-based testing, for the first time presenting a full-fledged theory that allows `ioco` to be applied to divergent systems.

1 Introduction

Quiescence is a fundamental concept in modelling system behaviour. It explicitly represents the fact that in certain states no output is provided. The absence of outputs is often essential: an ATM, for instance, should deliver money only once per transaction. This means that its state just after payment should be quiescent: it should not produce any output until further input is given. On the other hand, the state before payment should clearly not be quiescent. Hence, quiescence may or may not be considered erroneous behaviour. Consequently, the notion of quiescence is essential in model-based testing, where it is detected by means of a timeout. If a particular implementation under test does not provide

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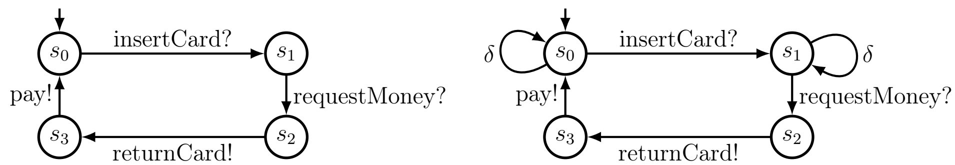


(a) A very basic ATM model.
(b) An SA for the ATM model.

Fig. 1: Deriving a suspension automaton.[1]

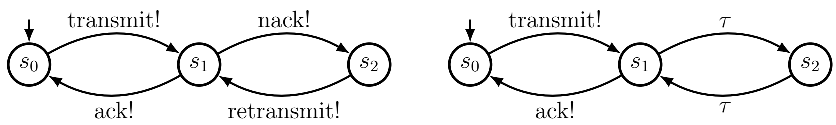
any output, then the test evaluation algorithm must decide whether to produce a pass verdict (allowing quiescence at this point) or a fail verdict (prohibiting quiescence at this point).

Origins. The notion of quiescence was first introduced by Vaandrager [1] to obtain a natural extension of blocking states: if a system is input-enabled (i.e., always ready to receive inputs), then no states are blocking, since each state has outgoing input transitions. Quiescence models the fact that a state would be blocking when considering only the internal and output actions. In the context of model-based testing, Tretmans introduced *repetitive quiescence* [2, 3]. This notion emerged from the need to continue testing, even in a quiescent state: in the ATM example above, we may need to test further behaviour arising from the (quiescent) state s_0. To accommodate this, Tretmans introduced the *Suspension Automaton* (SA) as an auxiliary concept [4]. An SA is obtained from an Input-Output Transition System (IOTS) by first adding a self-loop labelled by the quiescence label δ to each quiescent state and subsequently determinising the model. For instance, the ATM automaton in Fig. 1a has quiescent states s_0 and s_1; the corresponding SA is depicted in Fig. 1b.

Limitations of current treatments. While previous work [1–4] convincingly argued the need for quiescence, no comprehensive theory of quiescence existed thus far. A severe restriction is that SAs cannot cope with divergence (cycles consisting of internal actions only), since this may introduce newly quiescent states. The TGV framework [5] handles divergence by adding δ-labelled self-loops to such states. However, this treatment is in our opinion not satisfactory: quiescence due to divergence, expressing that no output will ever be produced, can in [5] be followed by an output action, which is counterintuitive. The current paper shows that an appropriate theory for quiescence that can cope with divergence is far from trivial.

Divergence does often occur in practice, e.g., due to action hiding. Therefore, current model-based testing approaches are not able to adequately handle such systems; in this paper, we fill this gap.

[1] Since we require systems to be input-enabled, these models are technically not correct. However, this could easily be fixed by adding self-loops to all states for each missing input. We chose to omit these for clarity of presentation.



(a) Original specification. (b) Divergence after hiding.

Fig. 2: A simple network protocol.

Example 1.1. Consider the simplified network protocol shown in Figure 2a. It is obtained as the parallel composition of a sending node (transmitting a message) and a receiving node (sending positive and negative acknowledgements). If only the initial transmission and success of this transmission are considered observable behaviour, the other actions (needed for parallel composition, but irrelevant in the final system) can be hidden, and the system shown in Figure 2b appears. Here, divergence may occur in states s_1 and s_2 (for instance, when retransmission was implemented erroneously and never succeeds). So, observation of quiescence is possible from these states, but simply adding δ-loops does not work anymore. After all, quiescence indicates the *indefinite* absence of outputs, and adding δ-loops to these states would allow outputs to occur after the δ-transitions. Hence, more sophisticated constructs are needed.

In addition to the divergence issue, quiescence was never treated as a first-class citizen: SAs cannot be built from scratch, and, even though important conformance relations such as `ioco` are defined in terms of them, SAs have been defined as an auxiliary construct and have never been studied extensively in isolation. In particular, their closure properties under standard operations like parallel composition and action hiding have not been investigated much.

Our approach. This paper remediates the shortcomings of previous work by introducing *Divergent Quiescent Transition Systems* (DQTSs). DQTSs represent quiescence explicitly using special δ-transitions. We stipulate four well-formedness rules that formalise when δ-transitions may occur. For instance, no δ-transition may be followed by an output transition, since this would contradict the meaning of quiescence. Key in our work is the treatment of divergence: a divergent path leads to the observation of quiescence if and only if it is fair, i.e., models a reasonable execution. We use the notion of fairness from Input-Output Automata (IOAs) [6], based on task partitions.

We show that well-formed DQTSs are closed under parallel composition, determinisation and action hiding. In this way, they constitute a compositional theory for quiescence. Additionally, we formally explain how to obtain a DQTS from an existing IOA by a process called *deltafication*, and show that deltafication is commutative with parallel composition and action hiding. The addition of divergence (and correspondingly fairness) brought about a more involved process of deltafication and action hiding (which may introduce divergence), requiring a novel algorithm for detecting divergent states. We provide this algorithm, which allows us to check well-formedness on a given DQTS as well. Finally, we redefine the `ioco` conformance relation based on DQTSs, allowing it to be applied in the presence of divergence and hence demonstrating the most important practical benefit of our model for testing: a more general class of systems can be handled.

A preliminary version of this work, already providing a fully formalised framework for dealing with quiescence as a first-class citizen, but not yet supporting divergence, appeared as [7].

Overview of the paper. Sec. 2 introduces the DQTS model, and Sec. 3 presents our well-formedness rules. Sec. 4 then provides operations and properties for DQTSs. In Sec. 5 we describe an algorithm to determine divergent states, and Sec. 6 discusses how to apply DQTSs in the `ioco` framework. Finally, conclusions and future work are presented in Sec. 7. Proofs for all our results can be found in Appendix A.

2 Divergent Quiescent Transition Systems

Preliminaries Given a set L, we use L^* to denote the set of all *finite sequences* $\sigma = a_1\, a_2\, \dots\, a_n$ over L. We write $|\sigma| = n$ for the *length* of σ, and ϵ for the *empty sequence*. We let L^ω denote the set of all *infinite sequences* over L, and use $L^\infty = L^* \cup L^\omega$. Given two sequences $\rho \in L^*$ and $\upsilon \in L^\infty$, we denote the *concatenation* of ρ and υ by $\rho\,\upsilon$. The *projection of an element* $a \in L$ *on* $L' \subseteq L$, denoted $a \restriction L'$, is a if $a \in L'$ and ϵ otherwise. The projection of a sequence $\sigma = a\,\sigma'$ is defined inductively by $(a\,\sigma') \restriction L' = (a \restriction L') \cdot (\sigma' \restriction L')$, and the projection of a set of sequences Z is defined as the sets of projections.

We use $\wp(L)$ to denote the *power set* of L. A set $P \subseteq \wp(L)$ such that $\varnothing \notin P$ is a *partition* of L if $\bigcup P = L$ and $p \neq q$ implies $p \cap q = \varnothing$ for all $p, q \in P$. Finally, we use the notation $\exists^\infty$ for 'there exist infinitely many'.

2.1 Basic Model and Definitions

Divergent Quiescent Transition Systems (DQTSs) are labelled transition systems that model quiescence, i.e., the absence of outputs or internal transitions, via a special δ-action. They are based on the well-known Input-Output Automata [8, 6]; in particular, their task partitions allow one to define fair paths.

Definition 2.1 (Divergent Quiescent Transition System). *A* Divergent Quiescent Transition System *(DQTS) is a tuple* $\mathcal{A} = \langle\, S, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow \rangle$, *where* S *is a set of states;* $S^0 \subseteq S$ *is a non-empty set of initial states;* L^{I}, L^{O} *and* L^{H} *are disjoint sets of input, output and internal labels, respectively;* P *is a partition of* $L^{\mathrm{O}} \cup L^{\mathrm{H}}$*; and* $\rightarrow\ \subseteq S \times L \cup \{\, \delta \,\} \times S$ *is the transition relation, where* $L = L^{\mathrm{I}} \cup L^{\mathrm{O}} \cup L^{\mathrm{H}}$*. We assume* $\delta \notin L$*.*

Given a DQTS $\mathcal{A}$*, we denote its components by* $S_{\mathcal{A}}, S^0_{\mathcal{A}}, L^{\mathrm{I}}_{\mathcal{A}}, L^{\mathrm{O}}_{\mathcal{A}}, L^{\mathrm{H}}_{\mathcal{A}}, P_{\mathcal{A}}, \rightarrow_{\mathcal{A}}$*. We omit the subscript when it is clear from the context.*

Example 2.1. The SA in Fig. 1b is a DQTS. □

Restrictions. We impose two important restrictions on DQTSs. (1) We require each DQTS $\mathcal{A}$ to be *input-enabled*, i.e., always ready to accept any input. Thus, we require that for each $s \in S$ and $a \in L^{\mathrm{I}}$, there exists an $s' \in S$ such that $(s, a, s') \in \rightarrow$. (2) We require each DQTS to be well-formed. Well-formedness requires technical preparation and is defined in Sec. 3.

Semantically, DQTSs assume progress. That is, DQTSs are not allowed to remain idle forever when output or internal actions are enabled. Without this assumption, each state would be potentially quiescent.

Actions. We use the terms label and action interchangeably. We often suffix a question mark (?) to input labels and an exclamation mark (!) to output labels. These are, however, not part of the label. A label without a suffix denotes an internal label. Output and internal actions are called *locally controlled*, because their occurrence is under the control of the DQTS. Thus, $L^{\mathrm{LC}} = L^{\mathrm{O}} \cup L^{\mathrm{H}}$ denotes the set of all locally controlled actions. The special label δ is used to denote the occurrence of quiescence (see Def. 2.10). The task partition P partitions the locally controlled actions into blocks, allowing one to reason about fairness: an execution is fair if every task partition that is enabled infinitely often, is also given control infinitely often (see Sec. 2.2).

We use the standard notations for transitions.

Definition 2.2 (Transitional notations). *Let $\mathcal{A}$ be a DQTS with $s, s' \in S$, $a, a_i \in L$, $b, b_i \in L^{\mathrm{I}} \cup L^{\mathrm{O}}$, and $\sigma \in (L^{\mathrm{I}} \cup L^{\mathrm{O}})^+$, then:*

$$
\begin{array}{lll}
s \xrightarrow{a} s' & =_{\mathrm{def}} & (s, a, s') \in \rightarrow \\
s \xrightarrow{a} & =_{\mathrm{def}} & \exists\, s'' \in S \,.\, s \xrightarrow{a} s'' \\
s \not\xrightarrow{a} & =_{\mathrm{def}} & \nexists\, s'' \in S \,.\, s \xrightarrow{a} s'' \\
s \xrightarrow{a_1 \cdot \ldots \cdot a_n} s' & =_{\mathrm{def}} & \exists\, s_0, \ldots, s_n \in S \,.\, s = s_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} s_n = s' \\
s \xRightarrow{\epsilon} s' & =_{\mathrm{def}} & s = s' \text{ or } \exists\, a_1, \ldots, a_n \in L^{\mathrm{H}} \,.\, s \xrightarrow{a_1 \cdot \ldots \cdot a_n} s' \\
s \xRightarrow{b} s' & =_{\mathrm{def}} & \exists\, s_0, s_1 \in S \,.\, s \xRightarrow{\epsilon} s_0 \xrightarrow{b} s_1 \xRightarrow{\epsilon} s' \\
s \xRightarrow{b_1 \cdot \ldots \cdot b_n} s' & =_{\mathrm{def}} & \exists\, s_0, \ldots, s_n \in S \,.\, s = s_0 \xRightarrow{b_1} \cdots \xRightarrow{b_n} s_n = s' \\
s \xRightarrow{\sigma} & =_{\mathrm{def}} & \exists\, s'' \in S \,.\, s \xRightarrow{\sigma} s''
\end{array}
$$

If $s \xrightarrow{a}$, we say that a is enabled *in s. We use $L(s)$ to denote the set of all actions $a \in L$ that are enabled in state $s \in S$, i.e., $L(s) = \{\, a \in L \mid s \xrightarrow{a} \,\}$. The notions are lifted to infinite traces in the obvious way.*

We use the following language notations for DQTSs and their behaviour.

Definition 2.3 (Language notations). *Let $\mathcal{A}$ be a DQTS, then:*

- *A finite path in $\mathcal{A}$ is a sequence $\pi = s_0\, a_1\, s_1\, a_2\, s_2\, \ldots\, s_n$ such that $s_{i-1} \xrightarrow{a_i} s_i$ for all $1 \leq i \leq n$. Infinite paths are defined analogously. The set of all paths in $\mathcal{A}$ is denoted $paths(\mathcal{A})$.*
- *Given any path, we write $first(\pi) = s_0$. Also, we denote by $states(\pi)$ the set of states that occur on π, and by ω-$states(\pi)$ the set of states that occur infinitely often. That is, $\omega\text{-}states(\pi) = \{\, s \in states(\pi) \mid \exists^{\infty} j \,.\, s_j = s \,\}$.*

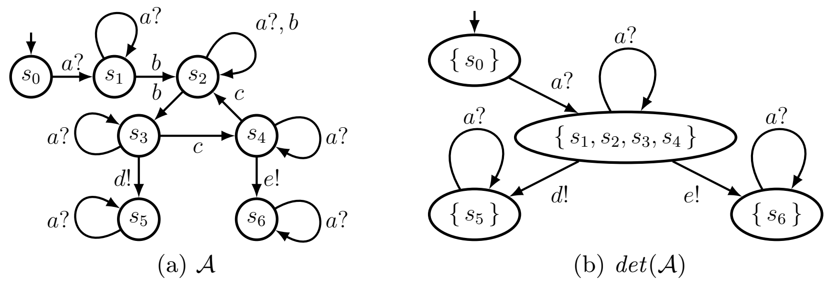


Fig. 3: Visual representations of the DQTSs $\mathcal{A}$ and $det(\mathcal{A})$.

- *We define* $trace(\pi) = \pi \upharpoonright (L^{\mathrm{I}} \cup L^{\mathrm{O}})$, *and say that* $trace(\pi)$ *is the* trace *of* π. *For every* $s \in S$, $traces(s)$ *is the set of all traces corresponding to paths that start in* s, *i.e.,* $traces(s) = \{ trace(\pi) \mid \pi \in paths(\mathcal{A}) \ \wedge\ first(\pi) = s \}$. *We define* $traces(\mathcal{A}) = \bigcup_{s \in S^0} traces(s)$, *and say that two DQTSs* $\mathcal{B}$ *and* $\mathcal{C}$ *are* trace-equivalent, *denoted* $\mathcal{B} \approx_{\mathrm{tr}} \mathcal{C}$, *if* $traces(\mathcal{B}) = traces(\mathcal{C})$.
- *For a finite trace* σ *and state* $s \in S$, $reach(s, \sigma)$ *denotes the set of states in* $\mathcal{A}$ *that can be reached from* s *via* σ, *i.e.,* $reach(s, \sigma) = \{ s' \in S \mid s \overset{\sigma}{\Longrightarrow} s' \}$. *For a set of states* $S' \subseteq S$, *we define* $reach(S', \sigma) = \bigcup_{s \in S'} reach(s, \sigma)$.

When needed, we add subscripts to indicate the DQTS these notions refer to.

Definition 2.4 (Determinism). *A DQTS* $\mathcal{A}$ *is* deterministic *if* $s \xrightarrow{a} s'$ *and* $s \xrightarrow{a} s''$ *imply* $a \notin L^{\mathrm{H}}$ *and* $s' = s''$, *for all* $s, s', s'' \in S$ *and* $a \in L$. *Otherwise,* $\mathcal{A}$ *is* nondeterministic.

Each DQTS has a trace-equivalent deterministic DQTS [9, 10]. Determinisation is carried out using the well-known subset construction procedure. This construction yields a system in which every state has a unique target per action, and internal transitions are not present anymore.

Definition 2.5 (Determinisation). *The* determinisation *of a DQTS* $\mathcal{A} = \langle S, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow \rangle$ *is the DQTS* $det(\mathcal{A}) = \langle T, \{ S^0 \}, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_{\mathrm{D}} \rangle$, *with* $T = \wp(S) \setminus \varnothing$ *and* $\rightarrow_{\mathrm{D}} = \{ (U, a, V) \in T \times L \times T \mid V = reach_{\mathcal{A}}(U, a) \ \wedge\ V \neq \varnothing \}$.

Example 2.2. The DQTS $\mathcal{A}$ in Fig. 3a is nondeterministic; its determinisation $det(\mathcal{A})$ is shown in Fig. 3b. □

2.2 Fairness and Divergence

The notion of fairness also plays a crucial role in DQTSs. The reason for this is that parallel composition may yield unreasonable divergences. For instance, if the DQTS in Fig. 4 is the composition of a system consisting solely of an internal a-loop and a system outputting a b precisely once, the progress assumption on the

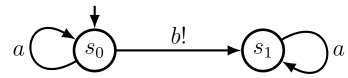


Fig. 4: Visual representation of a DQTS $\mathcal{B}$.

second component tells us that at some point we should observe this b-output. Therefore, we want to prohibit the divergent path $\pi = s_0\, a\, s_0\, a\, s_0\, \ldots$.

The following definition stems from [8, 6, 11], and states that if a subcomponent of the system infinitely often wants to execute some of its actions, it will indeed infinitely often execute some. Note that finite paths are fair by default.

Definition 2.6 (Fair path). *Let $\mathcal{A}$ be a DQTS and $\pi = s_0\, a_1\, s_1\, a_2\, s_2\, \ldots$ a path of $\mathcal{A}$. Then, π is* fair *if, for every $A \in P$ such that $\exists^\infty j\ .\ L(s_j) \cap A \neq \varnothing$, we have $\exists^\infty j\ .\ a_j \in A$. The set of all fair paths of a DQTS $\mathcal{A}$ is denoted $\mathit{fpaths}(\mathcal{A})$, and the set of corresponding traces is denoted $\mathit{ftraces}(\mathcal{A})$.*

Unfair paths are considered not to occur, so from now on we only consider $\mathit{fpaths}(\mathcal{A})$ and $\mathit{ftraces}(\mathcal{A})$ for the behaviour of $\mathcal{A}$.

Example 2.3. Consider again the DQTS $\mathcal{B}$ in Fig. 4. The infinite path $\pi = s_0\, a\, s_0\, a\, s_0\, \ldots$ would not be fair if $P_\mathcal{B} = \{\{a\}, \{b\}\}$, as the b-output is ignored forever. It would however be fair if $P_\mathcal{B} = \{\{a, b\}\}$. □

We can now formally define divergence: fair infinite internal behaviour.

Definition 2.7 (Divergent path). *Let $\mathcal{A}$ be a DQTS and $\pi \in \mathit{fpaths}(\mathcal{A})$ a fair infinite path. The path π is* divergent *if it contains only transitions labelled with internal actions, i.e., $a_i \in L^{\mathrm{H}}_{\mathcal{A}}$ for every action a_i on π. The set of all divergent paths of $\mathcal{A}$ is denoted $\mathit{dpaths}(\mathcal{A})$.*

Example 2.4. Consider the DQTS $\mathcal{A}$ in Fig. 3a with $L^{\mathrm{H}}_{\mathcal{A}} = \{b, c\}$. The infinite paths $s_2\, b\, s_2\, b\, s_2\, \ldots$ and $s_2\, b\, s_3\, c\, s_4\, c\, s_2\, b\, s_3\, \ldots$ are both divergent. Note that divergent traces are not preserved by determinisation. □

In contrast to SAs, we do allow divergent paths to occur in DQTSs. However, we assume that each divergent path in a DQTS only contains a finite number of states. This restriction serves to ensure that the deltafication of a DQTS, discussed in Sec. 4.1, always results in a correct DQTS. Since DQTSs typically contain a finite number of states, and even in infinite systems divergence often results from internal loops, this restriction is not a severe one.

Definition 2.8 (State-finite path). *Let $\mathcal{A}$ be a DQTS and let $\pi \in \mathit{fpaths}(\mathcal{A})$ be an infinite path. If $|\mathit{states}(\pi)| < \infty$, then π is* state-finite.

When the system is on a state-finite divergent path, it continuously loops through a finite number of states on this path. We call these states divergent.

Definition 2.9 (Divergent state). *Let $\mathcal{A}$ be a DQTS. A state $s \in S$ is* divergent*, denoted $d(s)$, if there is a (state-finite and fair) divergent path on which s occurs infinitely often, i.e., if there is a path $\pi \in dpaths(\mathcal{A})$ such that $s \in \omega\text{-}states(\pi)$. The set of all divergent states of $\mathcal{A}$ is denoted $d(\mathcal{A})$.*

Example 2.5. Consider the DQTS $\mathcal{A}$ in Fig. 3a. The path $\pi_1 = s_1\, b\, s_2\, b\, s_2 \ldots$ is state-finite, fair and divergent. Since s_2 occurs infinitely often on π_1, it is divergent; s_1, on the other hand, is not. Whether s_3 is divergent depends on the task partition P. If P contains an element A such that $\{ c, d, e \} \subseteq A$, then $\pi_2 = s_3\, c\, s_4\, c\, s_2\, b\, s_3 \ldots$ is fair and s_3 is divergent; otherwise, it is not. □

2.3 Quiescence

Definition 2.10 (Quiescent state). *Let $\mathcal{A}$ be a DQTS. A state $s \in S$ is* quiescent*, denoted $q(s)$, if it has no locally-controlled actions enabled. That is, $q(s)$ if $s \not\xrightarrow{a}$ for all $a \in L^{\mathrm{LC}}$. The set of all quiescent states of $\mathcal{A}$ is denoted $q(\mathcal{A})$.*

Example 2.6. States s_0, s_5 and s_6 of the DQTS $\mathcal{A}$ in Fig. 3a are quiescent. □

Divergent paths in DQTSs may yield observations of quiescence in states that are not necessarily quiescent. Consider the DQTS $\mathcal{B}$ in Fig. 4. State s_0 is not quiescent, since it enables output b. Nevertheless, this output is never observed on the divergent path $\pi = s_0\, a\, s_0\, a \ldots$. Hence, quiescence might be observed in a non-quiescent state (here, if π is fair). After observing quiescence due to a divergent path, the system will reside in one of the divergent states on that path.

3 Well-formed DQTSs

To be meaningful, DQTSs have to adhere to four well-formedness rules that formalize the semantics of quiescence. As indicated before, we assume all DQTSs to do so.

Definition 3.1 (Well-formedness). *A DQTS $\mathcal{A}$ is* well-formed *if it satisfies the following rules for all $s, s', s'' \in S$ and $a \in L^{\mathrm{I}}$:*

Rule R1 (Quiescence should be observable)**:** if $q(s)$ or $d(s)$, then $s \xrightarrow{\delta}$.
This rule requires that each quiescent or divergent state has an outgoing δ-transition, since in these states quiescence may be observed.

Rule R2 (Quiescent state after quiescence observation)**:** if $s \xrightarrow{\delta} s'$, then $q(s')$.
Since there is no notion of timing in DQTSs, there is no particular observation duration associated with quiescence. Hence, the execution of a δ-transition represents that the system has not produced any outputs indefinitely; therefore, enabling any outputs after a δ-transition would clearly be erroneous.

Note that, even though the δ-transition may be due to divergence, it would not suffice to require $q(s') \vee d(s')$. After all, $d(s')$ does not exclude output actions from s', and these should not be enabled directly after a δ-transitions.

Rule R3 (No new behaviour after quiescence observation)**:** if $s \xrightarrow{\delta} s'$, then $traces(s') \subseteq traces(s)$.

There is no notion of timing in DQTSs. Hence, behaviour that is possible after an observation of quiescence, must also be possible beforehand. Still, the observation of quiescence may indicate the outcome of an earlier nondeterministic choice, thereby reducing possible behaviour. Hence, the potential inequality.

Rule R4 (Continued quiescence preserves behaviour)**:** if $s \xrightarrow{\delta} s'$ and $s' \xrightarrow{\delta} s''$, then $traces(s'') = traces(s')$.

Since quiescence represents the fact that no outputs are observed, and there is no notion of timing in the DQTS model, there can be no difference between observing quiescence once or multiple times in succession.

In [12], four similar, but more complex, rules for *valid* SAs are discussed. However, these did not account for divergence.

Note that, by definition of divergent states, rule R1 does not require δ-transitions from states that have outgoing divergent paths on which they occur only finitely often. This simplifies the deltafication procedure, as will be made clear in Example 4.1. Also note that a path of a DQTS may contain multiple successive δ-transitions. This corresponds to the practical testing scenario of observing a time-out rather than an output more than once in a row [2, 3].

Since SAs are derived from IOTSs, and we assume that these IOTSs correctly capture system behaviour, we find that SAs are 'well-formed' in the sense that their observable behaviour (including quiescence) corresponds to that of realistic specifications. Since we also desire this property to hold for well-formed DQTSs, the above rules have been carefully crafted in such a way that well-formed DQTSs and SAs are equivalent in terms of expressible observable behaviour. The following theorem characterises this core motivation behind our design decisions: it shows that every trace in a DQTS can be obtained by starting with a traditional IOTS and adding δ-loops as for SAs, and vice versa. Hence, except for divergences, their expressivity coincides.

Theorem 3.1. *For every SA $\mathcal{S}$ there exists a well-formed DQTS $\mathcal{D}$ such that $\mathcal{S} \approx_{\mathrm{tr}} \mathcal{D}$, and vice versa.*

Verifying rule R1 requires identifying divergent states; in Sec. 5 we provide an algorithm to do so. Rule R2 can be checked trivially, while R3 and R4 in practice could be checked heuristically. For R3, verify whether $s \xrightarrow{\delta} s'$ and $s' \xrightarrow{a?} s''$ imply $s \xrightarrow{a?} s''$, and for R4, verify whether $s \xrightarrow{\delta} s'$ and $s' \xrightarrow{\delta} s''$ imply that $s' = s''$.

4 Operations and Properties

4.1 Deltafication: from IOA to DQTS

Usually, specifications are modelled as IOAs (or IOTSs, which can easily be converted to IOAs by taking $L^{\mathrm{H}} = \{\, \tau \,\}$ and $P = \{\, L^{\mathrm{LC}} \,\}$). During testing, however, we typically observe the outputs of the system generated in response to inputs from the tester; thus, it is useful to be able to refer to the absence of outputs explicitly. Hence, we need a way to convert an IOA to a well-formed DQTS that captures all possible observations of it, including quiescence. This conversion is called *deltafication*. It uses the notions of quiescence, divergence and state-finiteness, which were defined for DQTSs, but can just as well be used for IOAs (interpreting them as non-well-formed DQTSs without any δ-transitions). As for DQTSs, we require all IOAs to be input-enabled.

To satisfy rule R1, every state in which quiescence may be observed must have an outgoing δ-transition. When constructing SAs, δ-labelled self-loops are added to all quiescent states. This would not work for divergent states, however, since divergent states have outgoing internal transitions and possibly even output transitions (as in Fig. 4). So, a δ-labelled self-loop would contradict rule R2.

Our solution is to introduce a new state qos_s for every divergent state s, which acts as its *quiescence observation state*. When quiescence is observed in s, a δ-transition will lead to qos_s. To preserve the original behaviour, all inputs that are enabled in s must still be enabled in qos_s, and must lead to the same states that the original input transitions led to. All these considerations together lead to the following definition for the deltafication procedure for IOAs.

Definition 4.1 (Deltafication). *Let* $\mathcal{A} = \langle S_{\mathcal{A}}, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_{\mathcal{A}} \rangle$ *be an IOA with* $\delta \notin L$*. The* deltafication *of* $\mathcal{A}$ *is* $\delta(\mathcal{A}) = \langle S_\delta, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_\delta \rangle$*. We define* $S_\delta = S_{\mathcal{A}} \cup \{ qos_s \mid s \in d(\mathcal{A}) \}$*, i.e.,* S_δ *contains a new state* $qos_s \notin S_{\mathcal{A}}$ *for every divergent state* $s \in S_{\mathcal{A}}$ *of* $\mathcal{A}$*. The transition relation* $\rightarrow_\delta$ *is as follows:*

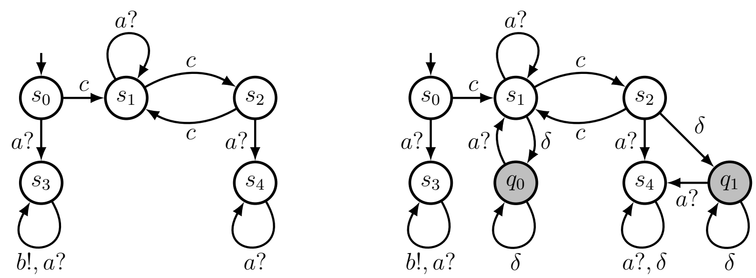
$$\begin{aligned} \rightarrow_\delta = \rightarrow_{\mathcal{A}} \; & \cup \; \{ (s, \delta, s) \quad\quad\; \mid s \in q(\mathcal{A}) \} \\ & \cup \; \{ (s, \delta, qos_s) \;\; \mid s \in d(\mathcal{A}) \} \cup \{ (qos_s, \delta, qos_s) \mid s \in d(\mathcal{A}) \} \\ & \cup \; \{ (qos_s, a?, s') \mid s \in d(\mathcal{A}) \;\wedge\; a? \in L^{\mathrm{I}} \;\wedge\; s \xrightarrow{a?}_{\mathcal{A}} s' \} \end{aligned}$$

Thus, the deltafication of an IOA adds δ-labelled self-loops to all quiescent states. Furthermore, a new quiescence observation state qos_s is introduced for every divergent state $s \in S$, alongside the required inputs and δ-transitions.

Note that computing $q(\mathcal{A})$ is trivial: simply identify all states without outgoing output or internal transition. Determining $d(\mathcal{A})$ is more complex; an algorithm to do so is provided in Sec. 5.

Example 4.1. See Fig. 5 for IOA $\mathcal{A}$ and its deltafication, given $P_{\mathcal{A}} = \{ \{ b, c \} \}$. Hence, s_1 and s_2 are divergent, and q_0 and q_1 quiescence observation states. Note that s_0 has an outgoing divergent path, while in accordance to rule R1 it is not given an outgoing δ-transition. The reason is that, when observing quiescence, our progress assumption prescribes that the system can only reside in s_1 or s_2. Hence, quiescence cannot be observed from s_0, and therefore also the a-transition to s_3 should not be possible anymore after observation of quiescence. This is now taken care of by not having a direct δ-transition from s_0. Because of this, no trace first having δ and then having the $b!$ output is present. □

As expected, deltafication indeed yields a well-formed DQTS.



(a) $\mathcal{A}$ (b) $\delta(\mathcal{A})$

Fig. 5: An IOA $\mathcal{A}$ and its deltafication $\delta(\mathcal{A})$. Newly introduced states are grey.

Theorem 4.1. *Given an IOA $\mathcal{A}$ with $\delta \notin L$ such that all divergent paths in $\mathcal{A}$ are state-finite, $\delta(\mathcal{A})$ is a well-formed DQTS.*

4.2 Operations on DQTSs

We introduce several standard operations on well-formed DQTSs. First, we define the well-known parallel composition operator. As usual, it requires every locally controlled action to be under the control of at most one component [6].

Definition 4.2 (Compatibility). *Two DQTSs $\mathcal{A}$ and $\mathcal{B}$ are* compatible *if $L^{\mathrm{O}}_{\mathcal{A}} \cap L^{\mathrm{O}}_{\mathcal{B}} = \varnothing$, $L^{\mathrm{H}}_{\mathcal{A}} \cap L_{\mathcal{B}} = \varnothing$, and $L^{\mathrm{H}}_{\mathcal{B}} \cap L_{\mathcal{A}} = \varnothing$.*

Definition 4.3 (Parallel composition). *Given two well-formed compatible DQTSs $\mathcal{A}$ and $\mathcal{B}$, the* parallel composition *of $\mathcal{A}$ and $\mathcal{B}$ is the DQTS $\mathcal{A} \parallel \mathcal{B}$, with $S_{\mathcal{A}\parallel\mathcal{B}} = S_{\mathcal{A}} \times S_{\mathcal{B}}$, $S^0_{\mathcal{A}\parallel\mathcal{B}} = S^0_{\mathcal{A}} \times S^0_{\mathcal{B}}$, $L^{\mathrm{I}}_{\mathcal{A}\parallel\mathcal{B}} = (L^{\mathrm{I}}_{\mathcal{A}} \cup L^{\mathrm{I}}_{\mathcal{B}}) \setminus (L^{\mathrm{O}}_{\mathcal{A}} \cup L^{\mathrm{O}}_{\mathcal{B}})$, $L^{\mathrm{O}}_{\mathcal{A}\parallel\mathcal{B}} = L^{\mathrm{O}}_{\mathcal{A}} \cup L^{\mathrm{O}}_{\mathcal{B}}$, $L^{\mathrm{H}}_{\mathcal{A}\parallel\mathcal{B}} = L^{\mathrm{H}}_{\mathcal{A}} \cup L^{\mathrm{H}}_{\mathcal{B}}$, $P_{\mathcal{A}\parallel\mathcal{B}} = P_{\mathcal{A}} \cup P_{\mathcal{B}}$, and*

$$
\begin{aligned}
\rightarrow_{\mathcal{A}\parallel\mathcal{B}} = \; & \{ ((s,t), a, (s',t')) \in S_{\mathcal{A}\parallel\mathcal{B}} \times ((L_{\mathcal{A}} \cap L_{\mathcal{B}}) \cup \{ \delta \}) \times S_{\mathcal{A}\parallel\mathcal{B}} \mid \\
& \quad s \xrightarrow{a}_{\mathcal{A}} s' \;\wedge\; t \xrightarrow{a}_{\mathcal{B}} t' \} \\
\cup \; & \{ ((s,t), a, (s',t)) \in S_{\mathcal{A}\parallel\mathcal{B}} \times (L_{\mathcal{A}} \setminus L_{\mathcal{B}}) \times S_{\mathcal{A}\parallel\mathcal{B}} \mid s \xrightarrow{a}_{\mathcal{A}} s' \} \\
\cup \; & \{ ((s,t), a, (s,t')) \in S_{\mathcal{A}\parallel\mathcal{B}} \times (L_{\mathcal{B}} \setminus L_{\mathcal{A}}) \times S_{\mathcal{A}\parallel\mathcal{B}} \mid t \xrightarrow{a}_{\mathcal{B}} t' \}
\end{aligned}
$$

We have $L_{\mathcal{A}\parallel\mathcal{B}} = L^{\mathrm{I}}_{\mathcal{A}\parallel\mathcal{B}} \cup L^{\mathrm{O}}_{\mathcal{A}\parallel\mathcal{B}} \cup L^{\mathrm{H}}_{\mathcal{A}\parallel\mathcal{B}} = L_{\mathcal{A}} \cup L_{\mathcal{B}}$.

Note that we require DQTSs to synchronise on δ-transitions, as a parallel composition of two DQTSs can only be quiescent when both components are.

It is often useful to hide certain output actions of a given well-formed DQTS, treating them as internal actions. For example, actions used for synchronisation are often not needed anymore in the parallel composition. Action hiding is slightly more complicated for DQTSs than for IOAs, as transforming output actions to internal actions can lead to newly divergent states. Still, whereas in SAs this was forbidden, in DQTSs it is allowed. Consequently, after hiding, new quiescence observation states may have to be added for newly divergent states.

Definition 4.4 (Action hiding). *Let $\mathcal{A} = \langle S_{\mathcal{A}}, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_{\mathcal{A}} \rangle$ be a well-formed DQTS and $H \subseteq L^{\mathrm{O}}$ a set of outputs, then* hiding H in $\mathcal{A}$ *yields the DQTS $\mathcal{A} \setminus H = \langle S_H, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}_H, L^{\mathrm{H}}_H, P, \rightarrow_H \rangle$, with $L^{\mathrm{O}}_H = L^{\mathrm{O}} \setminus H$, $L^{\mathrm{H}}_H = L^{\mathrm{H}} \cup H$, and $S_H = S_{\mathcal{A}} \cup \{ qos_s \mid s \in d(\mathcal{A} \setminus H) \setminus d(\mathcal{A}) \}$. Finally, $\rightarrow_H$ is defined by*

$$\begin{aligned}
\rightarrow_H = \rightarrow_{\mathcal{A}} & \cup \{ (s, \delta, qos_s) && \mid s \in d(\mathcal{A} \setminus H) \setminus d(\mathcal{A}) \} \\
& \cup \{ (qos_s, \delta, qos_s) && \mid s \in d(\mathcal{A} \setminus H) \setminus d(\mathcal{A}) \} \\
& \cup \{ (qos_s, a?, s') && \mid s \in d(\mathcal{A} \setminus H) \setminus d(\mathcal{A}) \wedge a? \in L^{\mathrm{I}} \wedge s \xrightarrow{a}_{\mathcal{A}} s' \}
\end{aligned}$$

So, similar to deltafication, quiescence observation states are added for all newly divergent states, along with the required input transitions to preserve behaviour.

4.3 Properties of DQTSs

We present several important results regarding DQTSs. First, it turns out that well-formed DQTSs are closed under all operations defined thus far.

Theorem 4.2. *Well-formed DQTSs are closed under the operations of determinisation, parallel composition, and action hiding, i.e., given two well-formed and compatible DQTSs $\mathcal{A}$ and $\mathcal{B}$, and a set of labels $H \subseteq L^{\mathrm{O}}_{\mathcal{A}}$, we find that $det(\mathcal{A})$, $\mathcal{A} \setminus H$, and $\mathcal{A} \parallel \mathcal{B}$ are also well-formed DQTSs.*

Next, we investigate the commutativity of function composition of deltafication with the operations. We consider the function compositions of two operations to be commutative if the end results of applying both operations in either order are trace equivalent. After all, trace-equivalent DQTSs behave in the same way. (Note that this is not the case for IOAs or IOTSs, as trace-equivalent variants of such systems might have different quiescence behaviour.) We show that parallel composition and action hiding can safely be swapped with deltafication, but note that deltafication has to precede determinisation to get sensible results. This is immediate, since determinisation does not preserve quiescence.

Proposition 4.1. *Deltafication and determinisation do not commute, i.e., given an IOA $\mathcal{A}$ such that $\delta \notin L$, not necessarily $det(\delta(\mathcal{A})) \approx_{\mathrm{tr}} \delta(det(\mathcal{A}))$.*

Consequently, when transforming a nondeterministic IOA $\mathcal{A}$ to a deterministic, well-formed DQTS, one should first derive $\delta(\mathcal{A})$ and afterwards determinise.

Deltafication does commute with action hiding and parallel composition. In the following theorem we use $\setminus_{\mathrm{I}}$ to denote basic action hiding for IOAs, and $\setminus_{\mathrm{D}}$ to denote action hiding for DQTSs (conform Def. 4.4).

Theorem 4.3. *Deltafication and action hiding commute: given an IOA $\mathcal{A}$ such that $\delta \notin L$ and a set of labels $H \subseteq L^{\mathrm{O}}_{\mathcal{A}}$, we have $\delta(\mathcal{A} \setminus_{\mathrm{I}} H) \approx_{\mathrm{tr}} \delta(\mathcal{A}) \setminus_{\mathrm{D}} H$.*

Theorem 4.4. *Deltafication and parallel composition commute: given two compatible IOAs $\mathcal{A}$, $\mathcal{B}$, such that $\delta \notin L_{\mathcal{A}} \cup L_{\mathcal{B}}$, we have $\delta(\mathcal{A} \parallel \mathcal{B}) \approx_{\mathrm{tr}} \delta(\mathcal{A}) \parallel \delta(\mathcal{B})$.*

The above results allow great modelling flexibility. After all, hiding and parallel composition are often already applied to the IOAs that describe a specification. We now showed that after deltafication these then yield the same well-formed DQTSs as in the case these operations are applied after deltafication.

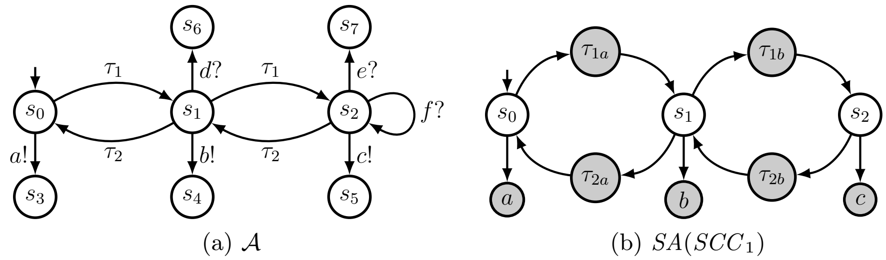


Fig. 6: An IOA $\mathcal{A}$ and the Streett automaton $SA(SCC_1)$.

5 Algorithm for detecting divergent states

We present an algorithm to detect divergent states in an IOA or DQTS. This is vital for verifying conformance to well-formedness rule R1, and for deltafication, since additional states have to be added for all divergent states in the original IOA. Recall from Def. 2.9 that a state s is divergent if there exists a fair divergent path on which s occurs infinitely often. Consequently, we need to find a fair cycle that starts at s and consists of only internal transitions. The presence of 'internal' cycles can be determined using Tarjan's well-known and efficient strongly connected components (SCCs) algorithm [13].

One way to efficiently verify fairness is to utilise Streett automata [14], which form a variation on Büchi automata [15]. The acceptance condition for a Streett automaton depends on pairs of sets of states (E_i, F_i) (called Streett pairs), that together form the acceptance component Ω. An ω-word is accepted with $\Omega = \{ (E_1, F_1), \ldots, (E_k, F_k)) \}$, if there exists a corresponding run that, for each j, only visits a state from F_j infinitely often if it visits a state from E_j infinitely often. This acceptance condition corresponds nicely with our notion of fairness.

Given an internal cycle $\pi = s_0\, a_1\, s_1\, a_2 \ldots a_n\, s_0$ with $a_i \in L^{\mathrm{H}}$, let $L(\pi) = \{ a_1, a_2, \ldots, a_n \}$ be the set of actions executed on the path π, and $L^{\mathrm{LC}}(s_i)$ be the set of locally controlled actions enabled at a state $s_i \in \mathit{states}(\pi)$. Because we require every divergent path to be state-finite (see Def. 2.8), these sets can always be calculated. If the cycle π is to be fair, then for every component $A_i \in P$ such that $A_i \cap L^{\mathrm{LC}}(s_i) \neq \varnothing$ for some $s_i \in \mathit{states}(\pi)$, there must be an action $a_i \in A_i$ such that $a_i \in L(\pi)$. By introducing additional states that, when visited, represent the fact that a particular locally controlled action is executed, we translate this fairness condition to a nonemptiness check on a Streett automaton.

To clarify this construction, assume we wish to obtain the deltafication of the IOA $\mathcal{A}$ shown in Fig. 6a given $P = \{ A_1, A_2, A_3 \}$, where $A_1 = \{ a, \tau_1 \}$, $A_2 = \{ b, \tau_2 \}$, and $A_3 = \{ c \}$. First, we calculate the SCCs of $\mathcal{A}$, while only considering the internal transitions; in this case, there is only one: $SCC_1 = \{ s_0, s_1, s_2 \}$. To illustrate the conditions for an internal cycle to be fair, consider $\pi = s_0\, \tau_1\, s_1\, \tau_2\, s_0$. Since $L^{\mathrm{LC}}(s_0) \cap A_1 = \{ a, \tau_1 \}$ and $L^{\mathrm{LC}}(s_1) \cap A_2 = \{ b, \tau_2 \}$, it follows that for π

to be fair, there must be actions $a_i \in A_1$ and $a_j \in A_2$ such that $a_i \in L(\pi)$ and $a_j \in L(\pi)$. This indeed is the case for π, i.e., it is fair.

However, we do not know a priori that the fair path π exists. To find it, consider Fig. 6b. There, we introduced intermediate 'transition' states (marked grey) for every locally controlled transition in and leading out of SCC_1. For state s_0 to be visited infinitely often, it follows from $L^{\mathrm{LC}}(s_0) \cap A_1 \neq \varnothing$ and $L^{\mathrm{LC}}(s_0) \cap A_2 = L^{\mathrm{LC}}(s_0) \cap A_3 = \varnothing$ that there must be actions $a_i \in A_1$ that are executed infinitely often as well. Hence, one of the states a, τ_{1a}, τ_{1b} of $SA(SCC_1)$ must be visited infinitely often if s_0 is. For s_1, in addition, actions from A_2 must occur infinitely often. Finally, for s_2 similar reasoning applies. All this yields $\Omega = \{ (E_1, F_1), (E_2, F_2), (E_3, F_3), (E_4, F_4), (E_5, F_5) \}$, where $(E_1, F_1) = (\{ a, \tau_{1a}, \tau_{1b} \}, \{ s_0 \})$, $(E_2, F_2) = (\{ a, \tau_{1a}, \tau_{1b} \}, \{ s_1 \})$, $(E_3, F_3) = (\{ b, \tau_{2a}, \tau_{2a} \}, \{ s_1 \})$, $(E_4, F_4) = (\{ b, \tau_{2a}, \tau_{2a} \}, \{ s_2 \})$ and $(E_5, F_5) = (\{ c \}, \{ s_2 \})$. As mentioned earlier, an accepting run in $SA(SCC_1)$ must satisfy all Streett pairs in Ω. Consequently, if such an accepting run exists, then it immediately follows that a fair internal cycle exists in $\mathcal{A}$. Such a nonemptiness check can be carried out efficiently using an optimised algorithm by Henzinger and Telle [16].

However, a fair internal cycle only gives us a subset of all divergent states. To find all of them, we need to verify for every state if a fair internal cycle exists that contains that particular state. Therefore, if we wish to check if, e.g., state s_0 is divergent, we need to extend acceptance component Ω with an additional Streett pair to obtain $\Omega_{s_0} = \Omega \cup \{ (\{ s_0 \}, SCC_1) \}$. This way, we ensure that internal cycles in SCC_1 are only considered fair if they also contain state s_0. Hence, $SA(SCC_1)$ has an accepting run with acceptance component Ω_{s_0} if and only if s_0 is divergent. In a similar way, we can construct $\Omega_{s_1} = \Omega \cup (\{ s_1 \}, SCC_1)$ and $\Omega_{s_2} = \Omega \cup (\{ s_2 \}, SCC_1)$ to check whether s_1 and s_2 are divergent, respectively.

Based on the above, we give an algorithm (Fig. 7) to determine divergent states. For clarity, we range over all states s and check nonemptiness using their acceptance condition Ω_s. A trivial improvement would be to, when a fair cycle is found, mark all its states as divergent and refrain from checking Ω_{s_i} for them.

Complexity. We discuss the worst-case time complexity of this algorithm given a DQTS with n states, m transitions and k partitions.

First note that the size of the acceptance condition of the Streett automaton for an SCC of n' states and m' transitions is worst-case in $O(n'k + n'm'))$. After all, each of the n' states yields at most k Streett pairs (yielding the term $n'k$). Moreover, all Streett pairs corresponding to a state, together contain at most all states that represent transitions, of which there are m' (yielding the term $n'm'$).

The time complexity of `construct_streett_automaton(C)` is bounded by the size of the acceptance condition, and hence is in $O(n'(k + m'))$ (with n' and m' taken from C). As the function is called once for each SCC of the system, the total contribution of this function to the full algorithm is in $O(n(k + m))$. Additionally, Tarjan is called once, adding $O(n + m)$. Finally, in the worst-case scenario, the Henzinger/Telle algorithm, which is in

$$O(m \min\{\sqrt{m \log n}, k, n\} + n(k + m) \min\{\log n, k\})$$

```
algorithm determine_divergent_states is
  input: IOA 𝒜 = ⟨ S, S⁰, Lᴵ, Lᴼ, Lᴴ, → ⟩
  output: d(𝒜): a set containing all divergent states of 𝒜

  d(𝒜) := ∅

  // Use a modified version of Tarjan's algorithm to determine SCCs(𝒜)
  SCCs(𝒜) := the set of all SCCs of 𝒜 that are connected with internal transitions

  for each C ∈ SCCs(𝒜)
    // Build the Streett automaton SA(C) corresponding to SCC C
    ⟨ S_SA, →_SA, Ω ⟩ := construct_streett_automaton(C)

    for each state s in C
      // Add an additional Streett pair to ensure s is on any accepting cycle
      Ω_s := Ω ∪ ({ s }, S_C)

      // Use the algorithm by Henziger and Telle to check the emptiness of SA(C)
      if SA(C) has an accepting run with acceptance component Ω_s
        d(𝒜) := d(𝒜) ∪ { s }
    end for
  end for

  // Auxiliary function to construct the Streett automaton SA(C), alongside acceptance
  // component Ω, for the given SCC C
  function construct_streett_automaton(C)
    input: SCC C = ⟨ S_SCC, Lᴵ, Lᴼ, Lᴴ, P, →_SCC ⟩
    output: a Streett automaton SA(C) = ⟨ S_SA, →_SA, Ω ⟩

    S_SA := S_SCC
    →_SA := Ω := ts_map := ∅

    // First construct the Streett automaton
    for each (s, a, t) ∈ →_SCC such that s ∈ S_SCC and a ∈ L^LC
      // We need to insert a transition state for the transition (s, a, t)
      let ts_(s,a,t) ∉ S_SA be a new state
      S_SA := S_SA ∪ { ts_(s,a,t) }

      if t ∈ S_SCC then →_SA := →_SA ∪ { (s, a, ts_(s,a,t)), (ts_(s,a,t), a, t) }
                   else →_SA := →_SA ∪ { (s, a, ts_(s,a,t)) }

      let A ∈ P be the component such that a ∈ A
      ts_map(A) := ts_map(A) ∪ { ts_(s,a,t) }
    end for

    // Now construct the acceptance component Ω
    for each s ∈ S_SCC
      // Add a new Streett pair for every component whose actions are enabled in s
      for each A ∈ P such that s -a->_SCC for some a ∈ A
        Ω := Ω ∪ { (ts_map(A), { s }) }
      end for
    end for

    return ⟨ S_SA, →_SA, Ω ⟩
  end function
```

Fig. 7: Algorithm for detecting divergent states.

as shown in [16], is called once for each state. Together, this yields

$$O(n(k+m)+(n+m)+n(m\min\{\sqrt{m\log n},k,n\}+n(k+m)\min\{\log n,k\}))$$

Under the reasonable assumption that k is bounded, and after simplification, we find that the worst-case time complexity is in $O(n^2m)$.

6 DQTSs in a testing context

Our main motivation for introducing and studying the DQTS model was to enable a clean theoretical framework for model-based testing. Earlier, the TGV framework [5] already defined `ioco` also in the presence of divergence. Although this was an important first step, it is not completely satisfactory in the sense that quiescence observations may be followed by output actions; this is counterintuitive to our notion of quiescence. Now, we illustrate how DQTSs can be incorporated in the `ioco` testing theory without having this problem.

The core of the `ioco` framework is its *conformance relation*, relating specifications to implementations if and only if the latter is 'correct' with respect to the former. For `ioco`, this means that the implementation never provides an unexpected output (including quiescence) when it is only fed inputs that are allowed by the specification. Traditionally, this was formalised based on the SAs corresponding to the implementation and the specification. Now, we can apply well-formed DQTSs, as they already model the expected absence of outputs by explicit δ-transitions. In addition, since DQTSs support divergence, using them as opposed to SAs also allows ioco to be applied in the presence of divergence.

Definition 6.1 (`ioco`). *Let $\mathcal{A}_{impl}, \mathcal{A}_{spec}$ be well-formed DQTSs over the same alphabet. Then, $\mathcal{A}_{impl} \sqsubseteq_{\texttt{ioco}} \mathcal{A}_{spec}$ if and only if*

$$\forall\, \sigma \in \mathit{traces}(\mathcal{A}_{spec}) \,.\, \mathit{out}_{\mathcal{A}_{impl}}(\sigma) \subseteq \mathit{out}_{\mathcal{A}_{spec}}(\sigma),$$

where $\mathit{out}_{\mathcal{A}}(\sigma) = \{a \in L^{\mathrm{O}} \cup \{\delta\} \mid \sigma a \in \mathit{traces}(\mathcal{A})\}$.

Since all DQTSs are required to be input-enabled, it is easy to see that `ioco`-conformance precisely corresponds to traditional trace inclusion over well-formed DQTSs.

This improved notion of `ioco`-correspondence can be used as before [4, 17], at each point in time during testing choosing to either try to provide an input, observe the behaviour of the system or stop testing. As long as the trace obtained this way (including the δ actions, which can now be the result of either quiescence or divergence) is also a trace of the specification, the implementation is correct.

Note that the implementation and specification do not necessarily need to have the same task partition. After all, these are only needed to establish fair paths and hence divergences. This is used during deltafication, to determine which states are divergent. Although this influences `ioco` conformance (since it induces δ transitions), the conformance relation itself is not concerned with the task partitions anymore.

7 Conclusions and Future Work

In this paper, we introduced Divergent Quiescent Transition Systems (DQTSs) and investigated their properties. Also, we showed how to detect divergent states in order to construct the deltafication of an IOA, and discussed its complexity. Like SAs, DQTSs can be used to describe all possible observations of a system, including the observation of quiescence, i.e., the absence of outputs. Hence, DQTSs are especially useful to model specifications of reactive systems in the context of model-based testing. DQTSs for the first time allow the modelling of systems that exhibit divergence and explicit quiescence.

There are two advantages of using DQTSs rather than SAs for model-based testing. First, DQTSs allow more systems to be modelled naturally, as convergence is not required. Second, DQTSs are stand-alone entities whose properties have been investigated thoroughly. Hence, DQTSs are a formal and comprehensive theory to model and analyse quiescence, even in the presence of divergence.

We have shown that DQTSs are equally potent as SAs in terms of expressible observable behaviour, and that DQTSs can be used as a drop-in replacement for SAs in the `ioco` framework. Furthermore, we have proven that well-formed DQTSs exhibit desirable compositional properties. Consequently, composite systems can be represented as the parallel composition of smaller subcomponents.

Future Work. The action hiding operation for the DQTS model is quite complex, as outlined in Def. 4.4. To improve this, it might be useful to investigate a different strategy to mark quiescent and divergent states, e.g., using state labels. Also, `ioco`-based model-based testing tools like TorX internally still use the SA model to represent the specification of the system under test, and an SA-like model to represent the actual test cases. Hence, such tools should be adapted to utilise the improved `ioco` framework based on DQTSs. Work is currently already underway to adapt the TorX tool. Finally, it would be interesting to see if our notions could be phrased in a coalgebraic setting.

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A Proofs

Lemma A.1. *For every SA $\mathcal{S}$ there exists a well-formed DQTS $\mathcal{D}$ such that $\mathcal{S} \approx_{\text{tr}} \mathcal{D}$*

Proof. Let $\mathcal{A} = \langle S, S^0, L^{\text{I}}, L^{\text{O}}, \rightarrow_{\mathcal{A}} \rangle$ be an IOTS, and $\mathcal{S}$ the corresponding SA. Hence, as defined in [2,3], $\mathcal{S}$ is the determinisation of the IOTS $\mathcal{A}' = \langle S, S^0, L^{\text{I}}, L^{\text{O}}, \rightarrow'_{\mathcal{A}} \rangle$, where $\rightarrow'_{\mathcal{A}}$ is defined as follows:

$$\rightarrow'_{\mathcal{A}} = \rightarrow_{\mathcal{A}} \cup \{ (s, \delta, s) \in S \times \{ \delta \} \times S \mid q(s) \text{ holds in } \mathcal{A} \}$$

Let $\mathcal{B}$ be the simplest IOA that is isomorphic, and therefore trace-equivalent, to the IOTS $\mathcal{A}$, i.e., $\mathcal{B} = \langle S, S^0, L^{\text{I}}, L^{\text{O}}, L^{\text{H}}, P, \rightarrow_{\mathcal{A}} \rangle$, where $L^{\text{H}} = \{ \tau \}$ and $P = \{ L^{\text{O}} \cup L^{\text{H}} \}$. Since the IOTS $\mathcal{A}$ must be strictly convergent because it otherwise cannot be converted to a SA [2,3], it follows that $\mathcal{B}$ also cannot contain divergent paths, and therefore no divergent states.

Now, observe that $\mathcal{A}'$ was obtained from $\mathcal{A}$ by adding δ-labelled self-loops to all quiescent states. Applying the deltafication procedure for DQTSs (Def. 4.1) to $\mathcal{B}$ will result in exactly the same δ-transitions being added to the same states in $\mathcal{B}$, as a state in $\mathcal{B}$ is only quiescent when its isomorphic state in $\mathcal{A}$ is, and vice versa. Furthermore, $\mathcal{B}$ does not contain any divergent states. Hence, $\mathcal{A}'$ and $\delta(\mathcal{B})$ are isomorphic, and consequently they are trace-equivalent. Furthermore, by Thm. 4.1, $\delta(\mathcal{B})$ is a well-formed DQTS. Since $\mathcal{S}$ is obtained by determinising $\mathcal{A}'$, we find that $\mathcal{S}$ is also trace-equivalent to $\delta(\mathcal{B})$. □

Lemma A.2. *For every well-formed DQTS $\mathcal{D}$ there exists an SA $\mathcal{S}$ such that $\mathcal{D} \approx_{\text{tr}} \mathcal{S}$*

Proof. Let $\mathcal{D} = \langle S, S^0, L^{\text{I}}, L^{\text{O}}, L^{\text{H}}, P, \rightarrow_{\mathcal{D}} \rangle$ be a well-formed DQTS. Without loss of generality, we assume the following two properties of $\mathcal{D}$:

1. $\mathcal{D}$ does not contain any path of the form $s \xrightarrow{\delta}_{\mathcal{D}} t \xrightarrow{\delta}_{\mathcal{D}} u$ with $t, u \in S$ and $t \neq u$. This can be assumed, since rule R4 prescribes that in such a case the traces of t and u should coincide. Therefore, they can be merged to remove the unwanted path fragment, without changing the traces of $\mathcal{D}$.
2. $\mathcal{D}$ is deterministic. This can be assumed, since determinisation preserves traces [10].

Note that the first assumption implies that there are no cycles in $\mathcal{D}$ consisting solely of δ-transitions, except for self-loops.

Since SAs cannot be built from scratch, but only arise implicitly by adding δ-transitions to IOTSs, as discussed above, we construct an IOTS $\mathcal{A}$ such that the SA $\mathcal{S}$ obtained from $\mathcal{A}$ is trace-equivalent to the DQTS $\mathcal{D}$. Now, let $\mathcal{A} = \langle S, S^0, L^{\text{I}}, L^{\text{O}}, \rightarrow_{\mathcal{A}} \rangle$ be an IOTS, where $\rightarrow_{\mathcal{A}}$ is defined as follows:

$$\begin{aligned} \rightarrow_{\mathcal{A}} \; = \; & \{ (s, a, t) \in \rightarrow_{\mathcal{D}} \mid a \neq \delta \} \\ \cup \; & \{ (s, \tau, t) \in S \times \{ \tau \} \times S \mid (s, \delta, t) \in \rightarrow_{\mathcal{D}} \wedge s \neq t \} \end{aligned}$$

Note that, by assumption (1), indeed $\rightarrow_{\mathcal{D}} \subseteq S \times (L^{\mathrm{I}} \cup L^{\mathrm{O}} \cup \{ \tau \}) \times S$, and hence we have defined a proper IOTS. As earlier, the corresponding SA $\mathcal{S}$ is the determinisation of the IOTS $\mathcal{A}' = \langle S, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, \rightarrow'_{\mathcal{A}} \rangle$, where $\rightarrow'_{\mathcal{A}}$ is defined by

$$\rightarrow'_{\mathcal{A}} = \rightarrow_{\mathcal{A}} \cup \{ (s, \delta, s) \in S \times \{ \delta \} \times S \mid q(s) \text{ holds in } \mathcal{A} \}$$

Since, as mentioned before, determinisation preserves traces, we will only show that $\mathcal{A}'$ is trace-equivalent to $\mathcal{D}$. It then follows immediately that the SA S is also trace-equivalent to $\mathcal{D}$. Hence, we need to show that $\mathit{traces}(\mathcal{D}) = \mathit{traces}(\mathcal{A}')$, i.e., that both $\mathit{traces}(\mathcal{D}) \subseteq \mathit{traces}(\mathcal{A}')$ and $\mathit{traces}(\mathcal{A}') \subseteq \mathit{traces}(\mathcal{D})$. We will first prove the former, then the latter.

1. First, we prove that $\mathit{traces}(\mathcal{D}) \subseteq \mathit{traces}(\mathcal{A}')$. Let $\sigma \in \mathit{traces}(\mathcal{D})$. We must prove that also $\sigma \in \mathit{traces}(\mathcal{A}')$. If $\sigma \in \mathit{traces}(\mathcal{D})$, there exists a path $\pi = s_0\, a_1\, s_1\, a_2\, s_2\, \ldots\, a_n\, s_n$ in $\mathcal{D}$ such that $\mathit{trace}(\pi) = \sigma$, $s_i \in S$, $a_i \in L \cup \{ \delta \}$, and $s_0 \in S^0$. By backwards induction on the length of π, we show for every suffix $\pi' = s_k\, a_{k+1}\, s_{k+1}\, \ldots\, a_n\, s_n$ of π that $\mathit{trace}(\pi') \in \mathit{traces}_{\mathcal{A}'}(s_k)$. This then implies that for $\sigma = \mathit{trace}(\pi)$ we have $\sigma \in \mathit{traces}_{\mathcal{A}'}(s_0)$, and since $\mathit{traces}_{\mathcal{A}'}(s_0) = \mathit{traces}(\mathcal{A}')$, we have then proven that $\sigma \in \mathit{traces}(\mathcal{A}')$.

   Base case. For $k = n$, we have $\pi' = s_n$ and hence $\mathit{trace}(\pi') = \epsilon$. In this case, we obviously have $\mathit{trace}(\pi') \in \mathit{traces}_{\mathcal{A}'}(s_n)$.

   Inductive case. Assume $\mathit{trace}(\pi'') \in \mathit{traces}_{\mathcal{A}'}(s_{k+1})$ for the path $\pi'' = s_{k+1}\, a_{k+2}\, s_{k+2}\, \ldots\, a_n\, s_n$. We now must show that $\mathit{trace}(\pi') \in \mathit{traces}_{\mathcal{A}'}(s_k)$ for $\pi' = s_k\, a_{k+1}\, s_{k+1}\, a_{k+2}\, s_{k+2}\, \ldots\, a_n\, s_n$. Note that $\mathit{trace}(\pi') = a_{k+1} \cdot \mathit{trace}(\pi'')$, since there are no internal transitions in $\mathcal{D}$, which follows from the second assumption made above on the structure of $\mathcal{D}$. We make a case distinction based on whether (a) $a_{k+1} \neq \delta$, (b) $a_{k+1} = \delta$ and $s_k = s_{k+1}$, and (c) $a_{k+1} = \delta$ and $s_k \neq s_{k+1}$.

   (a) If $a_{k+1} \neq \delta$, then by definition of $\mathcal{A}$ and $\mathcal{A}'$ we have $s_k \xrightarrow{a_{k+1}}_{\mathcal{A}'} s_{k+1}$ in $\mathcal{A}'$. Hence, since $\pi'' \in \mathit{traces}_{\mathcal{A}'}(s_{k+1})$, it immediately follows that $\pi' \in \mathit{traces}_{\mathcal{A}'}(s_k)$.

   (b) If $a_{k+1} = \delta$ and $s_k = s_{k+1}$, then it follows from rule R2 that s_k is quiescent in $\mathcal{D}$. Furthermore, by the assumption that $\mathcal{D}$ is deterministic, there cannot exist any other outgoing δ-transitions from s_k in $\mathcal{D}$, and therefore no τ-transitions are added to s_k in the construction of $\mathcal{A}$. Consequently, s_k is also quiescent in $\mathcal{A}$, and hence we find that indeed $s_k \xrightarrow{\delta}_{\mathcal{A}'} s_{k+1}$ in $\mathcal{A}'$, by definition of $\mathcal{A}'$. Hence, since $\pi'' \in \mathit{traces}_{\mathcal{A}'}(s_{k+1})$, it immediately follows that $\pi' \in \mathit{traces}_{\mathcal{A}'}(s_k)$.

   (c) If $a_{k+1} = \delta$ and $s_k \neq s_{k+1}$, then due to rule R2 we find that s_{k+1} is quiescent, and it follows from rule R1 that s_{k+1} must have an outgoing δ-transition. By the assumption that no path fragment of the form $s \xrightarrow{\delta}_{\mathcal{D}} t \xrightarrow{\delta}_{\mathcal{D}} u$ with $t, u \in S$ and $t \neq u$ is present in $\mathcal{D}$, this implies that $s_{k+1} \xrightarrow{\delta}_{\mathcal{D}} s_{k+1}$. It then follows by definition of $\mathcal{A}'$ that no τ-transition is added to s_{k+1} in the construction of $\mathcal{A}$, and therefore s_{k+1} is also quiescent in $\mathcal{A}$.

Hence, we have $s_{k+1} \xrightarrow{\delta}_{\mathcal{A}'} s_{k+1}$. Also, since $s_k \xrightarrow{\delta}_{\mathcal{D}} s_{k+1}$, we can conclude by the definitions of $\mathcal{A}$ and $\mathcal{A}'$ that $s_k \xrightarrow{\tau}_{\mathcal{A}'} s_{k+1}$. Consequently, in $\mathcal{A}'$ there exists a path $s_k \xrightarrow{\tau}_{\mathcal{A}'} s_{k+1} \xrightarrow{\delta}_{\mathcal{A}'} s_{k+1}$ and therefore a trace δ from s_k to s_{k+1} . Thus, since $\pi'' \in \mathit{traces}_{\mathcal{A}'}(s_{k+1})$, it immediately follows that $\pi' \in \mathit{traces}_{\mathcal{A}'}(s_k)$.s

2. Next, we prove that $\mathit{traces}(\mathcal{A}') \subseteq \mathit{traces}(\mathcal{D})$. Let $\sigma \in \mathit{traces}(\mathcal{A}')$. We must prove that also $\sigma \in \mathit{traces}(\mathcal{D})$. If $\sigma \in \mathit{traces}(\mathcal{A}')$, there exists a path $\pi = s_0\, a_1\, s_1\, a_2\, s_2\, \ldots\, a_n\, s_n$ in $\mathcal{A}'$ such that $\mathit{trace}(\pi) = \sigma$, $s_i \in S$, $a_i \in L \cup \{\tau, \delta\}$, and $s_0 \in S^0$. By backwards induction on the length of π, we show for every suffix $\pi' = s_k\, a_{k+1}\, s_{k+1}\, \ldots\, a_n\, s_n$ of π that $\mathit{trace}(\pi') \in \mathit{traces}_{\mathcal{D}}(s_k)$. This then implies that for $\sigma = \mathit{trace}(\pi)$ we have $\sigma \in \mathit{traces}_{\mathcal{D}}(s_0)$, and since $\mathit{traces}_{\mathcal{D}}(s_0) = \mathit{traces}(\mathcal{D})$, we have then proven that $\sigma \in \mathit{traces}(\mathcal{D})$.

   Base case. For $k = n$, we $\pi' = s_n$ and hence $\mathit{trace}(\pi') = \epsilon$. In this case, we obviously have $\mathit{trace}(\pi') \in \mathit{traces}_{\mathcal{D}}(s_n)$.

   Inductive case. Assume $\mathit{trace}(\pi'') \in \mathit{traces}_{\mathcal{D}}(s_{k+1})$ for the path $\pi'' = s_{k+1}\, a_{k+2}\, s_{k+2}\, \ldots\, a_n\, s_n$. We now must show that $\mathit{trace}(\pi') \in \mathit{traces}_{\mathcal{D}}(s_k)$ for $\pi' = s_k\, a_{k+1}\, s_{k+1}\, a_{k+2}\, s_{k+2}\, \ldots\, a_n\, s_n$. Note that $\pi' = a_{k+1} \cdot \pi''$ if $a_{k+1} \neq \tau$ and $\pi' = \pi''$ if $a_{k+1} = \tau$. We make a case distinction based on whether (a) $a_{k+1} \neq \delta$ and $a_{k+1} \neq \tau$, (b) $a_{k+1} = \delta$, (c) $a_{k+1} = \tau$ and $s_k \xrightarrow{\tau}_{\mathcal{D}} s_{k+1}$, and (d) $a_{k+1} = \tau$ and $s_k \xrightarrow{\delta}_{\mathcal{D}} s_{k+1}$.

   (a) If $a_{k+1} \neq \delta$ and $a_{k+1} \neq \tau$, then we can conclude from the definitions of $\mathcal{A}$ and $\mathcal{A}'$ that $s_k \xrightarrow{a_{k+1}}_{\mathcal{D}} s_{k+1}$. Hence, since $\pi'' \in \mathit{traces}_{\mathcal{D}}(s_{k+1})$, it immediately follows that $\pi' \in \mathit{traces}_{\mathcal{D}}(s_k)$.
   (b) If $a_{k+1} = \delta$, then it follows from the definitions of $\mathcal{A}$ and $\mathcal{A}'$ that it must have been added during the construction of $\mathcal{A}'$ (and hence it follows that $s_{k+1} = s_k$), since s_k was quiescent in $\mathcal{A}$. Therefore, s_k is also quiescent in $\mathcal{D}$ (since $\mathcal{D}$ cannot have more output transitions or internal transitions than $\mathcal{A}$), and consequently $s_k \xrightarrow{\delta}_{\mathcal{D}} s_k$ by rule R1. Thus, since $\pi'' \in \mathit{traces}_{\mathcal{D}}(s_{k+1})$ and $s_{k+1} = s_k$, it immediately follows that $\pi' \in \mathit{traces}_{\mathcal{D}}(s_k)$.
   (c) and (d). If $a_{k+1} = \tau$, then $\pi' = \pi''$. If this transition was added due to the presence of the transition $s_k \xrightarrow{\tau}_{\mathcal{D}} s_{k+1}$, then, since $\pi'' \in \mathit{traces}_{\mathcal{D}}(s_{k+1})$, it immediately follows that $\pi' \in \mathit{traces}_{\mathcal{D}}(s_k)$.
   Otherwise, if this transition was added due to the transition $s_k \xrightarrow{\delta}_{\mathcal{D}} s_{k+1}$, then from rule R3 it follows that $\mathit{traces}_{\mathcal{D}}(s_{k+1}) \subseteq \mathit{traces}_{\mathcal{D}}(s_k)$. Thus, since $\pi'' \in \mathit{traces}_{\mathcal{D}}(s_{k+1})$, this implies that $\pi' \in \mathit{traces}_{\mathcal{D}}(s_k)$. □

Theorem 3.1. *For every SA $\mathcal{S}$ there exists a well-formed DQTS $\mathcal{D}$ such that $\mathcal{S} \approx_{\mathrm{tr}} \mathcal{D}$, and vice versa.*

Proof. Follows directly from Lemma A.1 and Lemma A.2. □

Theorem 4.1. *Given an IOA $\mathcal{A}$ with $\delta \notin L$ such that all divergent paths in $\mathcal{A}$ are state-finite, $\delta(\mathcal{A})$ is a well-formed DQTS.*

Proof. Let $\mathcal{A} = \langle S, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_{\mathcal{A}} \rangle$ be an IOA with $\delta \notin L$ such that all divergent paths in $\mathcal{A}$ are state-finite, and let $\delta(\mathcal{A}) = \langle S_\delta, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_\delta \rangle$ be its deltafication, as defined in Def. 4.1. To show that $\delta(\mathcal{A})$ is a well-formed DQTS, we need to prove that $\delta(\mathcal{A})$ satisfies each of the rules R1, R2, R3 and R4. In the following, we use $\mathit{traces}_\delta(s)$ to denote the set of all traces of $\delta(\mathcal{A})$ starting in the state $s \in S_\delta$.

1. To prove that $\delta(\mathcal{A})$ satisfies rule R1, we must show that for all states $s \in S_\delta$:

   $$\text{if } q(s) \text{ or } d(s), \text{ then } s \xrightarrow{\delta}_\delta$$

   Since $s \in S_\delta$ and $q(s)$ or $d(s)$ holds in $\delta(\mathcal{A})$, it follows from Def. 4.1 that the following cases are possible: (a) $s \in S$ and $q(s)$ holds in $\delta(\mathcal{A})$; (b) $s \in S$ and $d(s)$ in $\delta(\mathcal{A})$; and (c) $s \in S_\delta \setminus S$ (and $q(s)$ holds in $\delta(\mathcal{A})$). Clearly, it is not possible that $s \in S_\delta \setminus S$ and $d(s)$ holds in $\delta(\mathcal{A})$.

   (a) Assume $s \in S$ and $q(s)$ holds in $\delta(\mathcal{A})$. Since deltafication does not hide or remove any existing output or internal transitions, $q(s)$ then also holds in $\mathcal{A}$. By Def. 4.1, we have $(s, \delta, s) \in \rightarrow_\delta$ after deltafication and therefore $s \xrightarrow{\delta}_\delta$.

   (b) Assume $s \in S$ and $d(s)$ holds in $\delta(\mathcal{A})$. In other words, s occurs infinitely often on a divergent path π in $\delta(\mathcal{A})$. Since deltafication does not hide any existing output transitions, nor creates any new internal transitions, the divergent path π must also be present in $\mathcal{A}$. Consequently, $d(s)$ also holds in $\mathcal{A}$. By Def. 4.1, we have $(s, \delta, \mathit{qos}_s) \in \rightarrow_\delta$ after deltafication, where qos_s is a new quiescence observation state for s. Thus, $s \xrightarrow{\delta}_\delta$.

   (c) Assume $s \in S_\delta \setminus S$. Hence, s is a newly added quiescence observation state for some divergent state, and by Def. 4.1 we have both $q(s)$ and $s \xrightarrow{\delta}_\delta s$.

2. To prove that $\delta(\mathcal{A})$ satisfies rule R2, we must show that for all states $s, s' \in S_\delta$:

   $$\text{if } s \xrightarrow{\delta}_\delta s', \text{ then } q(s')$$

   Since $s, s' \in S_\delta$ and $s \xrightarrow{\delta}_\delta s'$, it follows from Def. 4.1 that the following cases are possible: (a) $s, s' \in S$; (b) $s \in S$ and $s' \in S_\delta \setminus S$; and (c) $s, s' \in S_\delta \setminus S$. Clearly, it is not possible that $s \in S_\delta \setminus S$, $s' \in S$, and $s \xrightarrow{\delta}_\delta s'$.

   (a) Assume $s, s' \in S$ and $s \xrightarrow{\delta}_\delta s'$. By Def. 4.1, we have $s = s'$, and s (and therefore also s') is quiescent.

   (b) Assume $s \in S$, $s' \in S_\delta \setminus S$, and $s \xrightarrow{\delta}_\delta s'$. From Def. 4.1, it follows that s' is the quiescence observation state for the divergent state s, and s' is quiescent.

   (c) Assume $s, s' \in S_\delta \setminus S$ and $s \xrightarrow{\delta}_\delta s'$. From Def. 4.1, it follows that s' is a quiescence observation state, $s = s'$, and s' is quiescent.

3. To prove that $\delta(\mathcal{A})$ satisfies rule R3, we must show that for all states $s, s' \in S_\delta$:

$$\text{if } s \xrightarrow{\delta}_\delta s', \text{ then } \mathit{traces}_\delta(s') \subseteq \mathit{traces}_\delta(s)$$

Since $s, s' \in S_\delta$ and $s \xrightarrow{\delta}_\delta s'$, it follows from Def. 4.1 that the following cases are possible: (a) $s, s' \in S$; (b) $s \in S$ and $s' \in S_\delta \setminus S$; and (c) $s, s' \in S_\delta \setminus S$. Clearly, it is not possible that $s \in S_\delta \setminus S$, $s' \in S$, and $s \xrightarrow{\delta}_\delta s'$.

(a) Assume $s, s' \in S$ and $s \xrightarrow{\delta}_\delta s'$. By Def. 4.1, we have $s = s'$, and therefore $\mathit{traces}_\delta(s') \subseteq \mathit{traces}_\delta(s)$.

(b) Assume $s \in S$, $s' \in S_\delta \setminus S$ and $s \xrightarrow{\delta}_\delta s'$. From Def. 4.1, it follows that s' is a quiescence observation state for the divergent state s. Let $\sigma \in \mathit{traces}_\delta(s')$. We have to show that also $\sigma \in \mathit{traces}_\delta(s)$. There are two cases to consider: either $|\sigma| = 0$ or $|\sigma| \geq 1$. If $|\sigma| = 0$, then $\sigma = \epsilon$, and by definition $\sigma \in \mathit{traces}_\delta(s)$. If $|\sigma| \geq 1$, then, by Def. 4.1, $\sigma = a \cdot \sigma'$, where either $a = \delta$, or $a \in L^{\mathrm{I}}(s)$. In the first case we have $s' \xrightarrow{\delta}_\delta s'$ and $s' \xRightarrow{\sigma'}_\delta$. Since also $s \xrightarrow{\delta}_\delta s'$, it directly follows that $\sigma \in \mathit{traces}_\delta(s)$. In the second case we have $s' \xrightarrow{a}_\delta s''$ and $s'' \xRightarrow{\sigma'}_\delta$ for some $s'' \in S$. By Def. 4.1, we then must have $s \xrightarrow{a}_{\mathcal{A}} s''$, and therefore also $s \xrightarrow{a}_\delta s''$. Hence, since we have $s'' \xRightarrow{\sigma'}_\delta$, we find $\sigma \in \mathit{traces}_\delta(s)$.

(c) Assume $s, s' \in S_\delta \setminus S$ and $s \xrightarrow{\delta}_\delta s'$. From Def. 4.1, it follows that s is a quiescence observation state and $s = s'$. Thus, $\mathit{traces}_\delta(s') \subseteq \mathit{traces}_\delta(s)$.

4. To prove that $\delta(\mathcal{A})$ satisfies rule R4, we must show that for all states $s, s', s'' \in S_\delta$:

$$\text{if } s \xrightarrow{\delta}_\delta s' \text{ and } s' \xrightarrow{\delta}_\delta s'', \text{ then } \mathit{traces}_\delta(s'') = \mathit{traces}_\delta(s')$$

Since $s, s', s'' \in S_\delta$, $s \xrightarrow{\delta}_\delta s'$ and $s' \xrightarrow{\delta}_\delta s''$, it follows from Def. 4.1 that the following cases are possible: (a) $s, s', s'' \in S$; (b) $s \in S$ and $s', s'' \in S_\delta \setminus S$; and (c) $s, s', s'' \in S_\delta \setminus S$. All other permutations are not possible.

(a) Assume $s, s', s'' \in S$, $s \xrightarrow{\delta}_\delta s'$ and $s' \xrightarrow{\delta}_\delta s''$. By Def. 4.1, we have $s = s' = s''$, and therefore $\mathit{traces}_\delta(s') = \mathit{traces}_\delta(s'')$.

(b) Assume $s \in S$, $s', s'' \in S_\delta \setminus S$, $s \xrightarrow{\delta}_\delta s'$ and $s' \xrightarrow{\delta}_\delta s''$. From Def. 4.1, it follows that s' is the quiescence observation state for the divergent state s, and $s' = s''$. Clearly then, $\mathit{traces}_\delta(s'') = \mathit{traces}_\delta(s')$.

(c) Assume $s, s', s'' \in S_\delta \setminus S$, $s \xrightarrow{\delta}_\delta s'$ and $s' \xrightarrow{\delta}_\delta s''$. From Def. 4.1, it follows that s is a quiescence observation state and $s = s' = s''$. Thus, $\mathit{traces}_\delta(s'') = \mathit{traces}_\delta(s')$. □

Lemma A.3. *Well-formed DQTSs are closed under determinisation, i.e., given a well-formed DQTS $\mathcal{A}$, $\mathit{det}(\mathcal{A})$ is also a well-formed DQTS.*

Proof. Let $\mathcal{A} = \langle S, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_{\mathcal{A}} \rangle$ be a well-formed DQTS and let $\mathit{det}(\mathcal{A}) = \langle S_{\mathrm{D}}, S^0_{\mathrm{D}}, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_{\mathrm{D}} \rangle$ be its determinisation, as defined in Def. 2.5. To prove that well-formed DQTSs are closed under determinisation we must show that $\mathit{det}(\mathcal{A})$ is a well-formed DQTS, i.e., that it satisfies each of the rules R1, R2, R3 and R4. In the following, we use $\mathit{traces}_{\mathrm{D}}(U)$ to denote the set of all traces of $\mathit{det}(\mathcal{A})$ starting in the state $U \in S_{\mathrm{D}}$.

1. To prove that $det(\mathcal{A})$ satisfies rule R1, we must show that for all states $U \in S_{\mathrm{D}}$:

$$\text{if } q(U) \text{ or } d(U) \text{, then } U \xrightarrow{\delta}_{\mathrm{D}}$$

By Def. 2.5, there are no more internal transitions present after determinisation. Hence, there can be no $U \in S_{\mathrm{D}}$ such that $d(U)$ holds in $det(\mathcal{A})$. Instead, assume $q(U)$ holds in $det(\mathcal{A})$ for an $U \in S_{\mathrm{D}}$. This implies that all states $s \in U$ are quiescent in $\mathcal{A}$. From rule R1 it follows that for every state $s \in U$ there exists another state $s' \in S$ such that $s \xrightarrow{\delta}_{\mathcal{A}} s'$. Therefore $reach_{\mathcal{A}}(U, \delta) \neq \varnothing$. By Def. 2.5, we then have $(U, \delta, reach_{\mathcal{A}}(U, \delta)) \in \rightarrow_{\mathrm{D}}$. Consequently, $U \xrightarrow{\delta}_{\mathrm{D}}$.
2. To prove that $det(\mathcal{A})$ satisfies rule R2, we must show that for all states $U, V \in S_{\mathrm{D}}$:

$$\text{if } U \xrightarrow{\delta}_{\mathrm{D}} V \text{, then } q(V)$$

Consider any transition $U \xrightarrow{\delta}_{\mathrm{D}} V$ with $U, V \in S_{\mathrm{D}}$. If $U \xrightarrow{\delta}_{\mathrm{D}} V$, then, by Def. 2.5, $V = reach_{\mathcal{A}}(U, \delta)$ and $V \neq \varnothing$. Hence, for every state $s' \in V$ there exists a state $s \in U$ such that $s \xrightarrow{\delta}_{\mathcal{A}} s'$. Using rule R2 we can then conclude that every $s' \in V$ is quiescent in $\mathcal{A}$, thus $q(V)$ holds in $det(\mathcal{A})$.
3. To prove that $det(\mathcal{A})$ satisfies rule R3, we must show that for all states $U, V \in S_{\mathrm{D}}$:

$$\text{if } U \xrightarrow{\delta}_{\mathrm{D}} V \text{, then } traces_{\mathrm{D}}(V) \subseteq traces_{\mathrm{D}}(U)$$

Consider any transition $U \xrightarrow{\delta}_{\mathrm{D}} V$ with $U, V \in S_{\mathrm{D}}$. Assume $\sigma \in traces_{\mathrm{D}}(V)$. We must show that also $\sigma \in traces_{\mathrm{D}}(U)$. If $\sigma \in traces_{\mathrm{D}}(V)$, then there clearly must exist a state $s' \in V$ such that $s' \xRightarrow{\sigma}_{\mathcal{A}}$. Since $U \xrightarrow{\delta}_{\mathrm{D}} V$, it follows from Def. 2.5 that $V = reach_{\mathcal{A}}(U, \delta)$ and $V \neq \varnothing$. Hence, there must exist a state $s \in U$ such that $s \xrightarrow{\delta}_{\mathcal{A}} s'$. Using rule R3 we can then conclude that $traces_{\mathcal{A}}(s') \subseteq traces_{\mathcal{A}}(s)$, and therefore $s \xRightarrow{\sigma}_{\mathcal{A}}$. Since $s \in U$, it follows that $\sigma \in traces_{\mathrm{D}}(U)$.
4. To prove that $det(\mathcal{A})$ satisfies rule R4, we must show that for all states $U, V, W \in S_{\mathrm{D}}$:

$$\text{if } U \xrightarrow{\delta}_{\mathrm{D}} V \text{ and } V \xrightarrow{\delta}_{\mathrm{D}} W \text{, then } traces_{\mathrm{D}}(W) = traces_{\mathrm{D}}(V)$$

Consider any pair of transitions $U \xrightarrow{\delta}_{\mathrm{D}} V$ and $V \xrightarrow{\delta}_{\mathrm{D}} W$, with $U, V, W \in S_{\mathrm{D}}$. To prove that $traces_{\mathrm{D}}(W) = traces_{\mathrm{D}}(V)$, we must show that both $traces_{\mathrm{D}}(W) \subseteq traces_{\mathrm{D}}(V)$ and $traces(V) \subseteq traces_{\mathrm{D}}(W)$. The former follows directly from rule R3, so all that's left to prove is that $traces_{\mathrm{D}}(V) \subseteq traces_{\mathrm{D}}(W)$.
Assume $\sigma \in traces_{\mathrm{D}}(V)$. We must show that also $\sigma \in traces_{\mathrm{D}}(W)$. If $\sigma \in traces_{\mathrm{D}}(V)$, then there clearly must exist a state $s' \in V$ such that $s' \xRightarrow{\sigma}_{\mathcal{A}}$. Since $U \xrightarrow{\delta}_{\mathrm{D}} V$, it follows that there exists a state $s \in U$ such that $s \xrightarrow{\delta}_{\mathcal{A}} s'$. Furthermore, it follows from rule R2 that V is quiescent, and therefore all states in V are quiescent, including s'. Since $V \xrightarrow{\delta}_{\mathrm{D}} W$, we have $W = reach(V, \delta)$ and $W \neq \varnothing$. We can then conclude, using rule R1, that there must exist a state $s'' \in W$ such that $s' \xrightarrow{\delta}_{\mathcal{A}} s''$. Thus, we have $s \xrightarrow{\delta}_{\mathcal{A}} s' \xrightarrow{\delta}_{\mathcal{A}} s''$.

From rule R4 it then follows that $traces(s'') = traces(s')$ and consequently $s'' \xRightarrow{\sigma}_{\mathcal{A}}$. Since $s'' \in W$, it follows that $\sigma \in traces_{\mathrm{D}}(W)$. □

Lemma A.4. *Well-formed DQTSs are closed under parallel composition, i.e., given two compatible well-formed DQTSs $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \parallel \mathcal{B}$ is also a well-formed DQTS.*

Proof. Given two well-formed DQTSs $\mathcal{A} = \langle S_{\mathcal{A}}, S^0_{\mathcal{A}}, L^{\mathrm{I}}_{\mathcal{A}}, L^{\mathrm{O}}_{\mathcal{A}}, L^{\mathrm{H}}_{\mathcal{A}}, P_{\mathcal{A}}, \rightarrow_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle S_{\mathcal{B}}, S^0_{\mathcal{B}}, L^{\mathrm{I}}_{\mathcal{B}}, L^{\mathrm{O}}_{\mathcal{B}}, L^{\mathrm{H}}_{\mathcal{B}}, P_{\mathcal{B}}, \rightarrow_{\mathcal{B}} \rangle$ that are compatible, let the DQTS $\mathcal{A} \parallel \mathcal{B} = \langle S_{\mathcal{A}\parallel\mathcal{B}}, S^0_{\mathcal{A}\parallel\mathcal{B}}, L^{\mathrm{I}}_{\mathcal{A}\parallel\mathcal{B}}, L^{\mathrm{O}}_{\mathcal{A}\parallel\mathcal{B}}, L^{\mathrm{H}}_{\mathcal{A}\parallel\mathcal{B}}, P_{\mathcal{A}\parallel\mathcal{B}}, \rightarrow_{\mathcal{A}\parallel\mathcal{B}} \rangle$ be their parallel composition, as defined in Def. 4.3. To prove that well-formed DQTSs are closed under parallel composition we need to show that $\mathcal{A} \parallel \mathcal{B}$ is a well-formed DQTS, i.e., we need to prove that $\mathcal{A} \parallel \mathcal{B}$ satisfies each of the rules R1, R2, R3 and R4.

1. To prove that $\mathcal{A} \parallel \mathcal{B}$ satisfies rule R1, we must show that for every state $(s,t) \in S_{\mathcal{A}\parallel\mathcal{B}}$:

   $$\text{if } q((s,t)) \text{ or } d((s,t)), \text{ then } (s,t) \xrightarrow{\delta}_{\mathcal{A}\parallel\mathcal{B}}$$

   Let $(s,t) \in S_{\mathcal{A}\parallel\mathcal{B}}$. We will look at the cases for $q((s,t))$ and $d((s,t))$ separately.

   First, assume $q((s,t))$ holds in $\mathcal{A} \parallel \mathcal{B}$. In this case, there is no $a \in L^{\mathrm{O}}_{\mathcal{A}\parallel\mathcal{B}} \cup L^{\mathrm{H}}_{\mathcal{A}\parallel\mathcal{B}}$ such that $(s,t) \xrightarrow{a}_{\mathcal{A}\parallel\mathcal{B}}$. Since both $\mathcal{A}$ and $\mathcal{B}$ are input-enabled, it follows from Def. 4.3 that there is no $a \in L^{\mathrm{O}}_{\mathcal{A}} \cup L^{\mathrm{H}}_{\mathcal{A}}$ such that $s \xrightarrow{a}_{\mathcal{A}}$ and no $a \in L^{\mathrm{O}}_{\mathcal{B}} \cup L^{\mathrm{H}}_{\mathcal{B}}$ such that $t \xrightarrow{a}_{\mathcal{B}}$. Hence, both s and t are quiescent, and by rule R1 we have $s \xrightarrow{\delta}_{\mathcal{A}}$ and $t \xrightarrow{\delta}_{\mathcal{B}}$. From Def. 4.3 it then follows that $(s,t) \xrightarrow{\delta}_{\mathcal{A}\parallel\mathcal{B}}$.

   Now, assume $d((s,t))$ holds in $\mathcal{A} \parallel \mathcal{B}$, i.e., there exists a divergent path $\pi \in dpaths(\mathcal{A} \parallel \mathcal{B})$ such that $(s,t) \in \omega\text{-}states(\pi)$, i.e., the state (s,t) appears infinitely often on an infinite fair path π that is also divergent. By Def. 4.3, each step of path π is a transition by either $\mathcal{A}$ or $\mathcal{B}$, since the sets of internal transitions of $\mathcal{A}$ and $\mathcal{B}$ are disjoint, and they cannot synchronise on them. We can therefore distinguish three cases: (a) $\mathcal{A}$ and $\mathcal{B}$ both carry out an infinite number of internal transitions in the path π; (b) $\mathcal{A}$ carries out a finite number of internal transitions, and $\mathcal{B}$ an infinite number; and (c), $\mathcal{B}$ carries out a finite number of internal transitions, and $\mathcal{A}$ an infinite number. For each case, we will show that both $s \xrightarrow{\delta}_{\mathcal{A}}$ and $t \xrightarrow{\delta}_{\mathcal{B}}$, and therefore, by Def. 4.3, also $(s,t) \xrightarrow{\delta}_{\mathcal{A}\parallel\mathcal{B}}$.

   (a) Assume both $\mathcal{A}$ and $\mathcal{B}$ carry out an infinite number of internal transitions in the path π. Now assume that $\mathcal{A}$ carries out all the even transitions (i.e., the second, fourth, etc.) and $\mathcal{B}$ all the odd transitions (i.e., the first, third, etc.) in path π. However, the following proof can also be adapted for any other path π. Hence, path π is defined as follows:

   $$\begin{aligned}\pi \; = \; & (s_0,t_0) \xrightarrow{b_1}_{\mathcal{A}\parallel\mathcal{B}} (s_0,t_1) \xrightarrow{a_1}_{\mathcal{A}\parallel\mathcal{B}} (s_1,t_1) \xrightarrow{b_2}_{\mathcal{A}\parallel\mathcal{B}} \\ & (s_1,t_2) \xrightarrow{a_2}_{\mathcal{A}\parallel\mathcal{B}} (s_2,t_2) \ldots\end{aligned}$$

where $s_i \in S_{\mathcal{A}}$, $t_i \in S_{\mathcal{B}}$, $a_i \in L^{\mathrm{H}}_{\mathcal{A}}$ and $b_i \in L^{\mathrm{H}}_{\mathcal{B}}$. Since $(s,t) \in \omega\text{-}states(\pi)$, it follows that $\exists^{\infty} i, j$ such that $(s_i, t_j) = (s, t)$. Furthermore, by Def. 4.3, the construction of path π implies the existence of two infinite paths $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ in respectively $\mathcal{A}$ and $\mathcal{B}$, such that:

$$\begin{aligned} \pi_{\mathcal{A}} &= s_0 \xrightarrow{a_1}_{\mathcal{A}} s_1 \xrightarrow{a_2}_{\mathcal{A}} s_2 \xrightarrow{a_3}_{\mathcal{A}} \cdots \\ \pi_{\mathcal{B}} &= t_0 \xrightarrow{b_1}_{\mathcal{B}} t_1 \xrightarrow{b_2}_{\mathcal{B}} t_2 \xrightarrow{b_3}_{\mathcal{B}} \cdots \end{aligned}$$

Clearly, both paths $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are divergent, since $a_i \in L^{\mathrm{H}}_{\mathcal{A}}$ and $b_i \in L^{\mathrm{H}}_{\mathcal{B}}$. Since the path π is fair with respect to the task partition $P_{\mathcal{A}\|\mathcal{B}}$, it follows immediately that both paths $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are fair with respect to the task partitions $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$, respectively. To see this, recall that we have $L^{\mathrm{H}}_{\mathcal{A}} \cap L^{\mathrm{H}}_{\mathcal{B}} = \varnothing$, $L^{\mathrm{O}}_{\mathcal{A}} \cap L^{\mathrm{O}}_{\mathcal{B}} = \varnothing$ and both $\mathcal{A}$ and $\mathcal{B}$ are input-enabled. Furthermore, by Def. 4.3, any locally controlled actions that are enabled in all states $s_i \in S_{\mathcal{A}}$ and $t_j \in S_{\mathcal{B}}$ will also be enabled in $(s_i, j_i) \in S_{\mathcal{A}\|\mathcal{B}}$. Hence, since $P_{\mathcal{A}\|\mathcal{B}} = P_{\mathcal{A}} \cup P_{\mathcal{B}}$, it follows that if either $\pi_{\mathcal{A}}$ or $\pi_{\mathcal{B}}$ was not fair, then π could not be fair either. Consequently, $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are both divergent paths.

As mentioned before, we have that $\exists^{\infty} i, j$ such that (s_i, t_j) is a state on the path π and $(s_i, t_j) = (s, t)$. From this, it immediately follows that $\exists^{\infty} i$ such that s_i is a state on the path $\pi_{\mathcal{A}}$ and $s_i = s$, and $\exists^{\infty} j$ such that t_j is a state on the path $\pi_{\mathcal{B}}$ and $t_j = t$. Thus, $s \in \omega\text{-}states(\pi_{\mathcal{A}})$ and $t \in \omega\text{-}states(\pi_{\mathcal{B}})$. Since $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are divergent, it then follows that $d(s)$ holds in $\mathcal{A}$ and $d(t)$ in $\mathcal{B}$. By rule R1 we then must have $s \xrightarrow{\delta}_{\mathcal{A}}$ and $t \xrightarrow{\delta}_{\mathcal{B}}$.

(b) Assume $\mathcal{A}$ carries out a finite number of internal transitions in path π, and $\mathcal{B}$ an infinite number. Since π is infinite and the number of internal transitions of $\mathcal{A}$ is finite, this means that π can always be split into a finite path π' and an infinite path π'' such that all internal transitions carried out by $\mathcal{A}$ in π are on path π', and none are on path π''. Thus, the infinite path π'' only contains internal transitions of $\mathcal{B}$. Note that π' may consist of just a single state, in case $\mathcal{A}$ does not contribute to the path π at all. For example, assume path π is defined as follows:

$$\pi = u_0 \xrightarrow{a_1}_{\mathcal{A}\|\mathcal{B}} u_1 \xrightarrow{b_1}_{\mathcal{A}\|\mathcal{B}} u_2 \xrightarrow{a_2}_{\mathcal{A}\|\mathcal{B}} u_3 \xrightarrow{b_2}_{\mathcal{A}\|\mathcal{B}} u_4 \xrightarrow{b_3}_{\mathcal{A}\|\mathcal{B}} u_5 \xrightarrow{b_4}_{\mathcal{A}\|\mathcal{B}} \cdots$$

where $u_i \in S_{\mathcal{A}\|\mathcal{B}}$, $a_i \in L^{\mathrm{H}}_{\mathcal{A}}$ and $b_i \in L^{\mathrm{H}}_{\mathcal{B}}$. Hence, only internal transitions of $\mathcal{B}$ are executed after state u_3. Clearly then, a possible assignment for π' and π'' is the following:

$$\begin{aligned} \pi' &= u_0 \xrightarrow{a_1}_{\mathcal{A}\|\mathcal{B}} u_1 \xrightarrow{b_1}_{\mathcal{A}\|\mathcal{B}} u_2 \xrightarrow{a_2}_{\mathcal{A}\|\mathcal{B}} u_3 \\ \pi'' &= u_3 \xrightarrow{b_2}_{\mathcal{A}\|\mathcal{B}} u_4 \xrightarrow{b_3}_{\mathcal{A}\|\mathcal{B}} u_5 \xrightarrow{b_4}_{\mathcal{A}\|\mathcal{B}} \cdots \end{aligned}$$

Since $\mathcal{A}$ and $\mathcal{B}$ cannot synchronise on internal transitions, it follows that path π'' is defined as follows:

$$\pi'' = (s_0, t_0) \xrightarrow{b_1}_{\mathcal{A}\|\mathcal{B}} (s_0, t_1) \xrightarrow{b_2}_{\mathcal{A}\|\mathcal{B}} (s_0, t_2) \xrightarrow{b_3}_{\mathcal{A}\|\mathcal{B}} (s_0, t_3) \xrightarrow{b_4}_{\mathcal{A}\|\mathcal{B}} \cdots$$

where $s_0 \in S_{\mathcal{A}}$, $t_i \in S_{\mathcal{B}}$, and $b_i \in L^{\mathrm{H}}_{\mathcal{B}}$. Since path π is divergent, path π'' is also divergent. Furthermore, if $(s,t) \in \omega\text{-}states(\pi)$, then also $(s,t) \in \omega\text{-}states(\pi'')$. We must show that $s \xrightarrow{\delta}_{\mathcal{A}}$ and $t \xrightarrow{\delta}_{\mathcal{B}}$. We will do this by proving that $q(s)$ holds in $\mathcal{A}$ and $d(t)$ holds in $\mathcal{B}$. The desired result then follows directly from rule R1.

First, we will prove that $q(s)$ holds in $\mathcal{A}$. Since $(s,t) \in \omega\text{-}states(\pi'')$, it follows from the above definition of π'' that $s = s_0$. Let $L^{\mathrm{L}}_{\mathcal{A}}(s_0)$ denote the set of all locally controlled actions of $\mathcal{A}$ that are enabled in the state s_0. To prove that $q(s)$ holds in $\mathcal{A}$, we must show that $L^{\mathrm{L}}_{\mathcal{A}}(s_0) = \varnothing$. We do this by assuming the opposite, i.e., $L^{\mathrm{L}}_{\mathcal{A}}(s_0) \neq \varnothing$, and show that this leads to a contradiction.

From the definition of π'' and Def. 4.3 it follows that $L^{\mathrm{L}}_{\mathcal{A}}(s_0) \subseteq L(u)$ for all states $u \in S_{\mathcal{A}\|\mathcal{B}}$ on the path π''. If $L^{\mathrm{L}}_{\mathcal{A}}(s_0) \neq \varnothing$, then, by Def. 2.6, there is an $A \in P_{\mathcal{A}}$ such that $A \cap L^{\mathrm{L}}_{\mathcal{A}}(s_0) \neq \varnothing$. Consequently, since path π'' is fair, it must be the case that $\exists^{\infty} j$ such that a_j is an action executed on the path π'' and $a_j \in A$. However, only internal transitions from $\mathcal{B}$ are executed on path π'' and by Def. 4.2 we have $L^{\mathrm{H}}_{\mathcal{B}} \cap L_{\mathcal{A}} = \varnothing$.

Now, all that's left to prove is that $d(t)$ holds in $\mathcal{B}$. Since $s = s_0$, π'' is defined as follows:

$$\pi'' = (s,t_0) \xrightarrow{b_1}_{\mathcal{A}\|\mathcal{B}} (s,t_1) \xrightarrow{b_2}_{\mathcal{A}\|\mathcal{B}} (s,t_2) \xrightarrow{b_3}_{\mathcal{A}\|\mathcal{B}} (s,t_3) \xrightarrow{b_4}_{\mathcal{A}\|\mathcal{B}} \cdots$$

where $t_i \in S_{\mathcal{B}}$, and $b_i \in L^{\mathrm{H}}_{\mathcal{B}}$. Hence, by Def. 4.3, we have the following infinite path $\pi_{\mathcal{B}}$ in $\mathcal{B}$:

$$\pi_{\mathcal{B}} = t_0 \xrightarrow{b_1}_{\mathcal{B}} t_1 \xrightarrow{b_2}_{\mathcal{B}} t_2 \xrightarrow{b_3}_{\mathcal{B}} t_3 \xrightarrow{b_4}_{\mathcal{B}} \cdots$$

Clearly, path $\pi_{\mathcal{B}}$ is divergent, since $b_i \in L^{\mathrm{H}}_{\mathcal{B}}$. Since the path π'' is fair with respect to the task partition $P_{\mathcal{A}\|\mathcal{B}}$, it follows immediately that $\pi_{\mathcal{B}}$ is also fair with respect to the task partitioning $P_{\mathcal{B}}$. To see this, recall that we have $L^{\mathrm{H}}_{\mathcal{A}} \cap L^{\mathrm{H}}_{\mathcal{B}} = \varnothing$, $L^{\mathrm{O}}_{\mathcal{A}} \cap L^{\mathrm{O}}_{\mathcal{B}} = \varnothing$ and both $\mathcal{A}$ and $\mathcal{B}$ are input-enabled. Furthermore, by Def. 4.3, any locally controlled actions that are enabled in all states $t_j \in S_{\mathcal{B}}$ will also be enabled in $(s,t_j) \in S_{\mathcal{A}\|\mathcal{B}}$. Hence, since $P_{\mathcal{B}} \subseteq P_{\mathcal{A}\|\mathcal{B}}$, it follows that if $\pi_{\mathcal{B}}$ was not fair, then π could not be fair either. Consequently, $\pi_{\mathcal{B}}$ is a divergent path.

Furthermore, as we observed earlier, we have $(s,t) \in \omega\text{-}states(\pi'')$. From this, and the definition of $\pi_{\mathcal{B}}$, it follows that $\exists^{\infty} j$ such that t_j is a state on the path $\pi_{\mathcal{B}}$ and $t_j = t$. Hence, $t \in \omega\text{-}states(\pi_{\mathcal{B}})$. Since $\pi_{\mathcal{B}}$ is also divergent, it then follows that $d(t)$ holds in $\mathcal{B}$.

(c) Assume $\mathcal{B}$ carries out a finite number of internal transitions in path π, and $\mathcal{A}$ an infinite number. The proof for this case is then symmetric to the proof for the previous case.

2. To prove that $\mathcal{A} \parallel \mathcal{B}$ satisfies rule R2, we must show that for all pairs of states $(s,t),(s',t') \in S_{\mathcal{A}\|\mathcal{B}}$:

$$\text{if } (s,t) \xrightarrow{\delta}_{\mathcal{A}\|\mathcal{B}} (s',t'), \text{ then } q((s',t'))$$

Consider any transition $(s,t) \xrightarrow{\delta}_{\mathcal{A}\|\mathcal{B}} (s',t')$ with $(s,t),(s',t') \in S_{\mathcal{A}\|\mathcal{B}}$. From Definition 4.3 it then follows that $s \xrightarrow{\delta}_{\mathcal{A}} s'$ and $t \xrightarrow{\delta}_{\mathcal{B}} t'$. By rule R2, both s' and t' are quiescent. Thus, by Definition 4.3, $q((s',t'))$ holds in $\mathcal{A} \parallel \mathcal{B}$.

3. To prove that $\mathcal{A} \parallel \mathcal{B}$ satisfies rule R3, we must show that for all pairs of states $(s,t),(s',t') \in S_{\mathcal{A}\|\mathcal{B}}$:

$$\text{if } (s,t) \xrightarrow{\delta}_{\mathcal{A}\|\mathcal{B}} (s',t'), \text{ then } \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s',t')) \subseteq \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s,t))$$

Consider any transition $(s,t) \xrightarrow{\delta}_{\mathcal{A}\|\mathcal{B}} (s',t')$ with $(s,t),(s',t') \in S_{\mathcal{A}\|\mathcal{B}}$. Assume $\sigma \in \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s',t'))$. We have to show that also $\sigma \in \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s,t))$. Since $(s,t) \xrightarrow{\delta}_{\mathcal{A}\|\mathcal{B}} (s',t')$, it follows from Def. 4.3 that $s \xrightarrow{\delta}_{\mathcal{A}} s'$ and $t \xrightarrow{\delta}_{\mathcal{B}} t'$. By rule R3, we then have $\mathit{traces}_{\mathcal{A}}(s') \subseteq \mathit{traces}_{\mathcal{A}}(s)$ and $\mathit{traces}_{\mathcal{B}}(t') \subseteq \mathit{traces}_{\mathcal{B}}(t)$.
Additionally, note that $\sigma \in \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s',t'))$ implies that there is a path

$$\pi = (s'_0,t'_0) \xrightarrow{a_1}_{\mathcal{A}\|\mathcal{B}} (s'_1,t'_1) \xrightarrow{a_2}_{\mathcal{A}\|\mathcal{B}} \cdots \xrightarrow{a_{n-1}}_{\mathcal{A}\|\mathcal{B}} (s'_{n-1},t'_{n-1}) \xrightarrow{a_n}_{\mathcal{A}\|\mathcal{B}} (s'_n,t'_n)$$

for some $n \geq |\sigma|$, where $(s'_0,t'_0) = (s',t')$ and $\mathit{trace}(\pi) = \sigma$. Note that some of the actions a_i can be equal to τ, and that not all states s_i and t_i have to be distinct.
We prove by induction on the length of the path π that (1) $s' \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}} s'_n$ and $t' \xRightarrow{\rho_{\mathcal{B}}}_{\mathcal{B}} t'_n$, where $\rho_{\mathcal{A}} = \sigma \restriction (L_{\mathcal{A}} \cup \{\,\delta\,\})$ and $\rho_{\mathcal{B}} = \sigma \restriction (L_{\mathcal{B}} \cup \{\,\delta\,\})$, that (2) $s \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}}$ and $t \xRightarrow{\rho_{\mathcal{B}}}_{\mathcal{B}}$, and that (3) $(s,t) \xRightarrow{\sigma}_{\mathcal{A}\|\mathcal{B}} (s_m,t_m)$ for every pair $(s_m,t_m) \in \mathit{reach}_{\mathcal{A}}(s,\rho_{\mathcal{A}}) \times \mathit{reach}_{\mathcal{B}}(t,\rho_{\mathcal{B}})$. Note that the last part implies that $\sigma \in \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s,t))$, which is what we needed to show (the first two parts are needed for the induction).

Base case. Let $|\pi| = 0$, i.e., π is the empty path and $(s'_n,t'_n) = (s',t')$. This implies that $\sigma = \rho_{\mathcal{A}} = \rho_{\mathcal{B}} = \epsilon$, and hence $s' \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}} s'_n$ and $t' \xRightarrow{\rho_{\mathcal{B}}}_{\mathcal{B}} t'_n$. Also, $s \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}}$ and $t \xRightarrow{\rho_{\mathcal{B}}}_{\mathcal{B}}$ since $\epsilon \in \mathit{traces}_{\mathcal{A}}(s)$ and $\epsilon \in \mathit{traces}_{\mathcal{B}}(t)$. To see why $(s,t) \xRightarrow{\sigma}_{\mathcal{A}\|\mathcal{B}} (s_m,t_m)$ for every $(s_m,t_m) \in \mathit{reach}_{\mathcal{A}}(s,\rho_{\mathcal{A}}) \times \mathit{reach}_{\mathcal{B}}(t,\rho_{\mathcal{B}})$, note that since $\sigma = \rho_{\mathcal{A}} = \rho_{\mathcal{B}} = \epsilon$, $\mathit{reach}_{\mathcal{A}}(t,\rho_{\mathcal{A}})$ and $\mathit{reach}_{\mathcal{B}}(t,\rho_{\mathcal{B}})$ contain precisely all states that can be reached from s and t, respectively, by only taking τ-transitions. By Def. 4.3, these τ-transitions (if any) can also be executed in all possible interleavings starting from (s,t), since $\mathcal{A}$ and $\mathcal{B}$ do not synchronise on τ-transitions.

Inductive case. Let π' be the path from (s'_0,t'_0) to (s'_{n-1},t'_{n-1}), and let $\sigma' = \mathit{trace}(\pi')$. Assume that (1) $s' \xRightarrow{\rho'_{\mathcal{A}}}_{\mathcal{A}} s'_{n-1}$ and $t' \xRightarrow{\rho'_{\mathcal{B}}}_{\mathcal{B}} t'_{n-1}$, where $\rho'_{\mathcal{A}} = \sigma' \restriction (L_{\mathcal{A}} \cup \{\,\delta\,\})$ and $\rho'_{\mathcal{B}} = \sigma' \restriction (L_{\mathcal{B}} \cup \{\,\delta\,\})$, that (2) $s \xRightarrow{\rho'_{\mathcal{A}}}_{\mathcal{A}}$ and $t \xRightarrow{\rho'_{\mathcal{B}}}_{\mathcal{B}}$, and that (3) $(s,t) \xRightarrow{\sigma'}_{\mathcal{A}\|\mathcal{B}} (s_m,t_m)$ for every pair $(s_m,t_m) \in \mathit{reach}_{\mathcal{A}}(s,\rho'_{\mathcal{A}}) \times \mathit{reach}_{\mathcal{B}}(t,\rho'_{\mathcal{B}})$. Let $\sigma = \sigma'\, a = \mathit{trace}(\pi)$. Since $\sigma \in \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s',t'))$, we have $a \in L_{\mathcal{A}\|\mathcal{B}} \cup \{\,\epsilon,\delta\,\}$. We look at the cases $a = \epsilon$, $a \in L_{\mathcal{A}} \setminus L_{\mathcal{B}}$, $a \in L_{\mathcal{B}} \setminus L_{\mathcal{A}}$, and $a \in (L_{\mathcal{A}} \cap L_{\mathcal{B}}) \cup \{\,\delta\,\}$ separately.

– If $a = \epsilon$, then apparently $a_n = \tau$ and $\sigma = \sigma'\epsilon = \sigma'$. By Def. 4.3, this implies that either $s'_{n-1} = s'_n$ and $t'_{n-1} \xrightarrow{\tau}_{\mathcal{B}} t'_n$, or $t'_{n-1} = t'_n$ and $s'_{n-1} \xrightarrow{\tau}_{\mathcal{A}} s'_n$. Both cases imply that $s' \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}} s'_n$ and $t' \xRightarrow{\rho_{\mathcal{B}}}_{\mathcal{B}} t'_n$, since $\rho_i = \rho'_i \cdot (a \upharpoonright (L_{\mathcal{A}} \cup \{\delta\})) = \rho'_i \cdot (\epsilon \upharpoonright (L_{\mathcal{A}} \cup \{\delta\})) = \rho'_i$ for $i \in \{\mathcal{A}, \mathcal{B}\}$ and we assumed $s' \xRightarrow{\rho'_{\mathcal{A}}}_{\mathcal{A}} s'_{n-1}$ and $t' \xRightarrow{\rho'_{\mathcal{B}}}_{\mathcal{B}} t'_{n-1}$. Also, since $\rho'_i = \rho_i$ and $\sigma' = \sigma$, by the induction hypothesis we have $s \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}}$, $t \xRightarrow{\rho_{\mathcal{B}}}_{\mathcal{B}}$, and $(s,t) \xRightarrow{\sigma}_{\mathcal{A}\|\mathcal{B}} (s_m, t_m)$ for every $(s_m, t_m) \in \mathit{reach}_{\mathcal{A}}(s, \rho_{\mathcal{A}}) \times \mathit{reach}_{\mathcal{B}}(t, \rho_{\mathcal{B}})$.

– If $a \in L_{\mathcal{A}} \setminus L_{\mathcal{B}}$, then $a_n = a$ and $(s'_{n-1}, t'_{n-1}) \xrightarrow{a_n}_{\mathcal{A}\|\mathcal{B}} (s'_n, t'_n)$ implies, by Def. 4.3, that $s'_{n-1} \xrightarrow{a}_{\mathcal{A}} s'_n$ and $t'_{n-1} = t'_n$. Since $s' \xRightarrow{\rho'_{\mathcal{A}}}_{\mathcal{A}} s'_{n-1}$ and $\rho_{\mathcal{A}} = \rho'_{\mathcal{A}} \cdot a$, this implies that $s' \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}} s'_n$, and since $t' \xRightarrow{\rho'_{\mathcal{B}}}_{\mathcal{B}} t'_{n-1}$ and $\rho'_{\mathcal{B}} = \rho_{\mathcal{B}}$, we have $t' \xRightarrow{\rho_{\mathcal{B}}}_{\mathcal{B}} t'_n$. Since $\mathit{traces}_{\mathcal{A}}(s') \subseteq \mathit{traces}_{\mathcal{A}}(s)$ and $\mathit{traces}_{\mathcal{B}}(t') \subseteq \mathit{traces}_{\mathcal{B}}(t)$, also $s \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}}$ and $t \xRightarrow{\rho_{\mathcal{B}}}_{\mathcal{B}}$. Clearly, $\mathit{reach}_{\mathcal{B}}(t, \rho_{\mathcal{B}}) = \mathit{reach}_{\mathcal{B}}(t, \rho'_{\mathcal{B}})$, since $\rho_{\mathcal{B}} = \rho'_{\mathcal{B}}$. Furthermore, for every state $v \in \mathit{reach}_{\mathcal{A}}(s, \rho_{\mathcal{A}})$ there exists a state $u \in \mathit{reach}_{\mathcal{A}}(s, \rho'_{\mathcal{A}})$ such that $u \xRightarrow{a}_{\mathcal{A}} v$. Hence, since $(s,t) \xRightarrow{\sigma'}_{\mathcal{A}\|\mathcal{B}} (s_m, t_m)$ for every pair $(s_m, t_m) \in \mathit{reach}_{\mathcal{A}}(s, \rho'_{\mathcal{A}}) \times \mathit{reach}_{\mathcal{B}}(t, \rho'_{\mathcal{B}})$, by Def. 4.3 also $(s,t) \xRightarrow{\sigma}_{\mathcal{A}\|\mathcal{B}} (s_n, t_n)$ for every pair $(s_n, t_n) \in \mathit{reach}_{\mathcal{A}}(s, \rho_{\mathcal{A}}) \times \mathit{reach}_{\mathcal{B}}(t, \rho_{\mathcal{B}})$.

– If $a \in L_{\mathcal{B}} \setminus L_{\mathcal{A}}$, the proof is symmetrical to the previous case.

– If $a \in L_{\mathcal{A}} \cap L_{\mathcal{B}}$ or $a = \delta$, then $a_n = a$ and $(s'_{n-1}, t'_{n-1}) \xrightarrow{a_n}_{\mathcal{A}\|\mathcal{B}} (s'_n, t'_n)$ implies, by Def. 4.3, that $s'_{n-1} \xrightarrow{a}_{\mathcal{A}} s'_n$ and $t'_{n-1} \xrightarrow{a}_{\mathcal{B}} t'_n$. Since $s' \xRightarrow{\rho'_{\mathcal{A}}}_{\mathcal{A}} s'_{n-1}$ and $\rho_{\mathcal{A}} = \rho'_{\mathcal{A}} \cdot a$, this implies that $s' \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}} s'_n$; $t' \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{B}} t'_n$ follows symmetrically. Since $\mathit{traces}_{\mathcal{A}}(s') \subseteq \mathit{traces}_{\mathcal{A}}(s)$ and $\mathit{traces}_{\mathcal{B}}(t') \subseteq \mathit{traces}_{\mathcal{B}}(t)$, also $s \xRightarrow{\rho_{\mathcal{A}}}_{\mathcal{A}}$ and $t \xRightarrow{\rho_{\mathcal{B}}}_{\mathcal{B}}$. Furthermore, for every state $v \in \mathit{reach}_{\mathcal{A}}(s, \rho_{\mathcal{A}})$ there exists a state $u \in \mathit{reach}_{\mathcal{A}}(s, \rho'_{\mathcal{A}})$ such that $u \xRightarrow{a}_{\mathcal{A}} v$; for $\mathit{reach}_{\mathcal{B}}(t, \rho_{\mathcal{B}})$ the same property (but with $\mathit{reach}_{\mathcal{B}}(t, \rho'_{\mathcal{B}})$ rather than $\mathit{reach}_{\mathcal{A}}(s, \rho'_{\mathcal{A}})$) holds. Hence, since $(s,t) \xRightarrow{\sigma'}_{\mathcal{A}\|\mathcal{B}} (s_m, t_m)$ for every pair $(s_m, t_m) \in \mathit{reach}_{\mathcal{A}}(s, \rho'_{\mathcal{A}}) \times \mathit{reach}_{\mathcal{B}}(t, \rho'_{\mathcal{B}})$, by Def. 4.3 also $(s,t) \xRightarrow{\sigma}_{\mathcal{A}\|\mathcal{B}} (s_n, t_n)$ for every pair $(s_n, t_n) \in \mathit{reach}_{\mathcal{A}}(s, \rho_{\mathcal{A}}) \times \mathit{reach}_{\mathcal{B}}(t, \rho_{\mathcal{B}})$.

4. To prove that $\mathcal{A} \parallel \mathcal{B}$ satisfies rule R4, we must show that for all pairs of states $(s,t), (s',t'), (s'',t'') \in S_{\mathcal{A}\|\mathcal{B}}$:

$$\text{if } (s,t) \xrightarrow{\delta}_{\mathcal{A}\|\mathcal{B}} (s',t') \text{ and } (s',t') \xrightarrow{\delta}_{\mathcal{A}\|\mathcal{B}} (s'',t''), \\ \text{then } \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s',t')) = \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s'',t''))$$

Consider any pair of transitions $(s,t) \xrightarrow{\delta}_{\mathcal{A}\|\mathcal{B}} (s',t')$ and $(s',t') \xrightarrow{\delta}_{\mathcal{A}\|\mathcal{B}} (s'',t'')$ with $(s,t), (s',t'), (s'',t'') \in S_{\mathcal{A}\|\mathcal{B}}$. From Def. 4.3 it follows that $s \xrightarrow{\delta}_{\mathcal{A}} s'$, $s' \xrightarrow{\delta}_{\mathcal{A}} s''$, $t \xrightarrow{\delta}_{\mathcal{B}} t'$ and $t \xrightarrow{\delta}_{\mathcal{B}} t''$. By rule R4, we then have $\mathit{traces}_{\mathcal{A}}(s') = \mathit{traces}_{\mathcal{A}}(s'')$ and $\mathit{traces}_{\mathcal{B}}(t') = \mathit{traces}_{\mathcal{B}}(t'')$. To prove that $\mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s',t')) = \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s'',t''))$, we must prove that both $\mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s',t')) \subseteq \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s'',t''))$ and $\mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s'',t'')) \subseteq \mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s',t'))$. The latter follows directly from rule R3, so all that's left to show is $\mathit{traces}_{\mathcal{A}\|\mathcal{B}}((s',t')) \subseteq$

$traces_{\mathcal{A}\|\mathcal{B}}((s'', t''))$. The proof for this is similar to the proof for rule R3, but using the fact that $traces_{\mathcal{A}}(s') = traces_{\mathcal{A}}(s'')$ and $traces_{\mathcal{B}}(t') = traces_{\mathcal{B}}(t'')$, instead of $traces_{\mathcal{A}}(s') \subseteq traces_{\mathcal{A}}(s)$ and $traces_{\mathcal{B}}(t') \subseteq traces_{\mathcal{B}}(t)$. □

Lemma A.5. *Well-formed DQTSs are closed under action hiding, i.e., given a well-formed DQTS $\mathcal{A}$ and a set of labels $H \subseteq L^{\mathrm{O}}_{\mathcal{A}}$, $\mathcal{A} \setminus H$ is also a well-formed DQTS.*

Proof. Let $\mathcal{A} = \langle S, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_{\mathcal{A}} \rangle$ be a well-formed DQTS and let $H \subseteq L^{\mathrm{O}}$ be a set of outputs. We then have $\mathcal{A} \setminus H = \langle S_H, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}_H, L^{\mathrm{H}}_H, P, \rightarrow_H \rangle$, as defined in Def. 4.4. To prove that well-formed DQTSs are closed under action hiding we must show that $\mathcal{A} \setminus H$ is a well-formed DQTS, i.e., that it satisfies each of the rules R1, R2, R3 and R4. In the following, we use $traces_H(s)$ to denote the set of all traces of $\mathcal{A} \setminus H$ starting in the state $s \in S$.

1. To prove that $\mathcal{A} \setminus H$ satisfies rule R1, we must show that for all states $s \in S_H$:

   $$\text{if } q(s) \text{ or } d(s), \text{ then } s \xrightarrow{\delta}_H$$

   Since $s \in S_H$ and $q(s)$ or $d(s)$ holds, it follows from Def. 4.4 that only the following cases are possible: (a) $s \in S$ and $q(s)$ holds in $\mathcal{A} \setminus H$; (b) $s \in S$ and $d(s)$ holds in $\mathcal{A} \setminus H$; and (c) $s \in S_H \setminus S$ (and $q(s)$ holds in $\mathcal{A} \setminus H$).

   (a) Assume $s \in S$ and $q(s)$ holds in $\mathcal{A} \setminus H$. Since hiding of actions effectively relabels output-transitions to internal transitions, it follows that $q(s)$ must also hold in $\mathcal{A}$. By rule R1, we then have $s \xrightarrow{\delta}_{\mathcal{A}}$. Since hiding does not affect existing δ-transitions, we then also have $s \xrightarrow{\delta}_H$.
   (b) Assume $s \in S$ and $d(s)$ holds in $\mathcal{A} \setminus H$. We can distinguish two cases: either $d(s)$ also holds in $\mathcal{A}$, or it does not. In the first case, we have, by rule R1, $s \xrightarrow{\delta}_{\mathcal{A}}$. Since hiding does not affect existing δ-transitions, we then also have $s \xrightarrow{\delta}_H$. If $d(s)$ does not hold in $\mathcal{A}$, then s has become newly divergent in $\mathcal{A} \setminus H$. By Def. 4.4, we then have $s \xrightarrow{\delta}_H$.
   (c) Assume $s \in S_H \setminus S$. Hence, s is a newly added quiescence observation state for some newly divergent state, and by Def. 4.4 we have $s \xrightarrow{\delta}_H s$.

2. To prove that $\mathcal{A} \setminus H$ satisfies rule R2, we must show that for all states $s, s' \in S_H$:

   $$\text{if } s \xrightarrow{\delta}_H s', \text{ then } q(s')$$

   Since $s, s' \in S_H$ and $s \xrightarrow{\delta}_H s'$, it follows from Def. 4.4 that only the following cases are possible: (a) $s, s' \in S$; (b) $s \in S$ and $s' \in S_H \setminus S$; and (c) $s, s' \in S_H \setminus S$.

   (a) Assume $s, s' \in S$ and $s \xrightarrow{\delta}_H s'$. Since hiding of actions does not result in the addition of new δ-transitions between states that already existed before the hiding operation took place, it follows that we also have $s \xrightarrow{\delta}_{\mathcal{A}} s'$. Rule R2 then implies that $q(s')$ holds in $\mathcal{A}$, and therefore, by Def. 4.4, hiding will not introduce any new outgoing transitions for this state. Consequently, $q(s')$ also holds in $\mathcal{A} \setminus H$.

(b) Assume $s \in S$, $s' \in S_H \setminus S$ and $s \xrightarrow{\delta}_H s'$. From Def. 4.4, it follows that s' is a newly created quiescence observation state for the newly divergent state s, and s' is quiescent.

(c) Assume $s, s' \in S_H \setminus S$ and $s \xrightarrow{\delta}_H s'$. From Def. 4.4, it follows that s' is a newly created quiescence observation state, $s = s'$, and s' is quiescent.

3. To prove that $\mathcal{A} \setminus H$ satisfies rule R3, we must show that for all states $s, s' \in S_H$:

$$\text{if } s \xrightarrow{\delta}_H s', \text{ then } \mathit{traces}_H(s') \subseteq \mathit{traces}_H(s)$$

Since $s, s' \in S_H$ and $s \xrightarrow{\delta}_H s'$, it follows from Def. 4.4 that only the following cases are possible: (a) $s, s' \in S$; (b) $s \in S$ and $s' \in S_H \setminus S$; and (c) $s, s' \in S_H \setminus S$.

(a) Assume $s, s' \in S$ and $s \xrightarrow{\delta}_H s'$. Since hiding of actions does not result in the addition of new δ-transitions between states that already existed before the hiding operation took place, it follows that we also have $s \xrightarrow{\delta}_{\mathcal{A}} s'$. Rule R3 then implies that $\mathit{traces}_{\mathcal{A}}(s') \subseteq \mathit{traces}_{\mathcal{A}}(s)$. From Rule R2 we can also conclude that $q(s')$ holds in $\mathcal{A}$, and therefore, by Def. 4.4, hiding will not introduce any new outgoing transitions for state s'. Consequently, it follows that $\mathit{traces}_H(s') = \mathit{traces}_{\mathcal{A}}(s')$. Furthermore, by Def. 4.4, we have $\mathit{traces}_{\mathcal{A}}(s) \subseteq \mathit{traces}_H(s)$, since new traces may be added by the hiding operation (when s is newly divergent), but existing traces are preserved. From this, it directly follows that $\mathit{traces}_H(s') \subseteq \mathit{traces}_H(s)$.

(b) Assume $s \in S$, $s' \in S_H \setminus S$ and $s \xrightarrow{\delta}_H s'$. From Def. 4.4, it follows that s' is a newly added quiescence observation state for the newly divergent state s. Let $\sigma \in \mathit{traces}_H(s')$. We have to show that also $\sigma \in \mathit{traces}_H(s)$. There are two cases to consider: either $|\sigma| = 0$ or $|\sigma| \geq 1$. If $|\sigma| = 0$, then $\sigma = \epsilon$, and by definition $\sigma \in \mathit{traces}_H(s)$. If $|\sigma| \geq 1$, then, by Def. 4.4, $\sigma = a \cdot \sigma'$, where either $a = \delta$, or $a \in L^{\mathrm{I}}(s)$. In the first case we have $s' \xrightarrow{\delta}_H s'$ and $s' \xRightarrow{\sigma'}_H$. Since also $s \xrightarrow{\delta}_H s'$, it directly follows that $\sigma \in \mathit{traces}_H(s)$. In the second case we have $s' \xrightarrow{a}_H s''$ and $s'' \xRightarrow{\sigma'}_H$ for some $s'' \in S$. By Def. 4.4, we then must have $s \xrightarrow{a}_{\mathcal{A}} s''$, and therefore also $s \xrightarrow{a}_H s''$. Hence, since we have $s'' \xRightarrow{\sigma'}_H$, we find $\sigma \in \mathit{traces}_H(s)$.

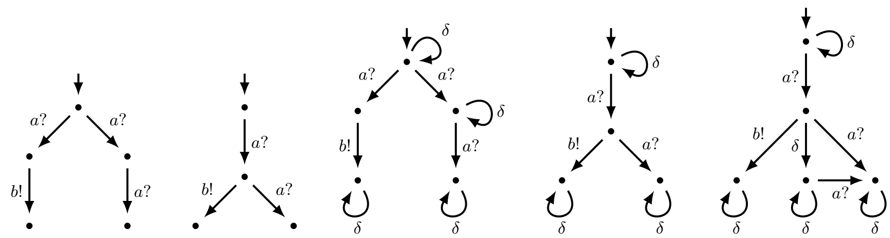
(c) Assume $s, s' \in S_H \setminus S$ and $s \xrightarrow{\delta}_H s'$. From Def. 4.4, it follows that s is a quiescence observation state and $s = s'$. Thus, $\mathit{traces}_H(s') \subseteq \mathit{traces}_H(s)$.

4. To prove that $\mathcal{A} \setminus H$ satisfies rule R4, we must show that for all states $s, s', s'' \in S_H$:

$$\text{if } s \xrightarrow{\delta}_H s' \text{ and } s' \xrightarrow{\delta}_H s'', \text{ then } \mathit{traces}_H(s') = \mathit{traces}_H(s'')$$

Since $s, s', s'' \in S_H$, $s \xrightarrow{\delta}_H s'$ and $s' \xrightarrow{\delta}_H s''$, it follows from Def. 4.4 that only the following cases are possible: (a) $s, s', s'' \in S$; (b) $s \in S$ and $s', s'' \in S_H \setminus S$; and (c) $s, s', s'' \in S_H \setminus S$.

(a) Assume $s, s', s'' \in S$, $s \xrightarrow{\delta}_H s'$ and $s' \xrightarrow{\delta}_H s''$. It then follows from Def. 4.4 that also $s \xrightarrow{\delta}_{\mathcal{A}} s'$ and $s' \xrightarrow{\delta}_{\mathcal{A}} s''$; and therefore, by rule R4,



(a) $\mathcal{A}$ (b) $det(\mathcal{A})$ (c) $\delta(\mathcal{A})$ (d) $\delta(det(\mathcal{A}))$ (e) $det(\delta(\mathcal{A}))$

Fig. 8: The determinisation and deltafication of the IOA $\mathcal{A}$ do not commute. Note that some a-labelled self-loops have been left out to reduce clutter.

$traces_{\mathcal{A}}(s') = traces_{\mathcal{A}}(s'')$. From Rule R2 we can also conclude that $q(s')$ and $q(s'')$ hold in $\mathcal{A}$, and therefore, by Def. 4.4, hiding will not introduce any new outgoing transitions for both states s' and s''. Consequently, it follows that $traces_H(s') = traces_{\mathcal{A}}(s')$ and $traces_H(s'') = traces_{\mathcal{A}}(s'')$. From this, it directly follows that $traces_H(s') = traces_H(s'')$.

(b) Assume $s \in S$, $s', s'' \in S_H \setminus S$, $s \xrightarrow{\delta}_H s'$ and $s' \xrightarrow{\delta}_H s''$. From Def. 4.4, it follows that s' is the newly added quiescence observation state for the newly divergent state s, and $s' = s''$. Clearly then, $traces_\delta(s'') = traces_\delta(s')$.

(c) Assume $s, s', s'' \in S_H \setminus S$, $s \xrightarrow{\delta}_H s'$ and $s' \xrightarrow{\delta}_H s''$. From Def. 4.4, it follows that s is a newly added quiescence observation state and $s = s' = s''$. Thus, $traces_\delta(s'') = traces_\delta(s')$. □

Theorem 4.2. *Well-formed DQTSs are closed under the operations of determinisation, parallel composition, and action hiding, i.e., given two well-formed and compatible DQTSs $\mathcal{A}$ and $\mathcal{B}$, and a set of labels $H \subseteq L^{\mathrm{O}}_{\mathcal{A}}$, we find that $det(\mathcal{A})$, $\mathcal{A} \setminus H$, and $\mathcal{A} \parallel \mathcal{B}$ are also well-formed DQTSs.*

Proof. Follows directly from Lemma A.3, Lemma A.4 and Lemma A.5.

Proposition 4.1. *Deltafication and determinisation do not commute, i.e., given an IOA $\mathcal{A}$ such that $\delta \notin L$, not necessarily $det(\delta(\mathcal{A})) \approx_{\mathrm{tr}} \delta(det(\mathcal{A}))$.*

Proof. Consider the IOTS $\mathcal{A}$, and its determinisation $det(\mathcal{A})$ and deltafication $\delta(\mathcal{A})$, shown in Fig. 8. Clearly, the deltafication of the determinisation of $\mathcal{A}$ (i.e., $\delta(det(\mathcal{A}))$), shown in Fig. 8d, results in an incorrect observation automaton, as it does not model the fact that in the nondeterministic DQTS $\delta(\mathcal{A})$ quiescence may be observed after an initial a input, as required by rule R1.

Contrary to the deltafication of the determinisation of $\mathcal{A}$, the determinisation of the deltafication of $\mathcal{A}$ (i.e., $det(\delta(\mathcal{A}))$), which is shown in Fig. 8e, does preserve the fact that quiescence may be observed after an initial a input. This should not come as a surprise, since for any IOA $\mathcal{A}$ the determinisation $det(\mathcal{A})$ is trace equivalent to the original automaton [10]. □

Theorem 4.3. *Deltafication and action hiding commute: given an IOA $\mathcal{A}$ such that $\delta \notin L$ and a set of labels $H \subseteq L^{\mathrm{O}}_{\mathcal{A}}$, we have $\delta(\mathcal{A} \setminus_{\mathrm{I}} H) \approx_{\mathrm{tr}} \delta(\mathcal{A}) \setminus_{\mathrm{D}} H$.*

Proof. Let $\mathcal{A} = \langle S, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow \rangle$ be an IOA such that $\delta \notin L$, and let $H \subseteq L^{\mathrm{O}}$. Furthermore, let $\mathcal{B} = \mathcal{A} \setminus_{\mathrm{I}} H = \langle S, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}_{\mathcal{B}}, L^{\mathrm{H}}_{\mathcal{B}}, P, \rightarrow_{\mathcal{B}} \rangle$, and let $\mathcal{C} = \delta(\mathcal{A}) = \langle S_{\mathcal{C}}, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}, L^{\mathrm{H}}, P, \rightarrow_{\mathcal{C}} \rangle$, as defined in Def. 4.1. Finally, let $\mathcal{D} = \delta(\mathcal{A} \setminus_{\mathrm{I}} H) = \langle S_{\mathcal{D}}, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}_{\mathcal{D}}, L^{\mathrm{H}}_{\mathcal{D}}, P, \rightarrow_{\mathcal{D}} \rangle$, as defined in Def. 4.1, and let $\mathcal{E} = \delta(\mathcal{A}) \setminus_{\mathrm{D}} H = \langle S_{\mathcal{E}}, S^0, L^{\mathrm{I}}, L^{\mathrm{O}}_{\mathcal{E}}, L^{\mathrm{H}}_{\mathcal{E}}, P, \rightarrow_{\mathcal{E}} \rangle$, as defined in Def. 4.4. Note that $L^{\mathrm{O}}_{\mathcal{B}} = L^{\mathrm{O}}_{\mathcal{D}} = L^{\mathrm{O}}_{\mathcal{E}}$ and $L^{\mathrm{H}}_{\mathcal{B}} = L^{\mathrm{H}}_{\mathcal{D}} = L^{\mathrm{H}}_{\mathcal{E}}$, since the same set of outputs H is being hidden.

To prove that $\delta(\mathcal{A} \setminus_{\mathrm{I}} H) \approx_{\mathrm{tr}} \delta(\mathcal{A}) \setminus_{\mathrm{D}} H$, we show that $traces(\delta(\mathcal{A} \setminus_{\mathrm{I}} H)) = traces(\delta(\mathcal{A}) \setminus_{\mathrm{D}} H)$. Hence, we need to show that $traces(\delta(\mathcal{A} \setminus_{\mathrm{I}} H)) \subseteq traces(\delta(\mathcal{A}) \setminus_{\mathrm{D}} H)$ and $traces(\delta(\mathcal{A} \setminus_{\mathrm{I}} H)) \subseteq traces(\delta(\mathcal{A}) \setminus_{\mathrm{D}} H)$, i.e., that $traces(\mathcal{D}) \subseteq traces(\mathcal{E})$ and $traces(\mathcal{E}) \subseteq traces(\mathcal{D})$. We will only prove the former; the proof for the latter is largely symmetrical and therefore omitted.

Let $\sigma \in traces(\mathcal{D})$; we must show that also $\sigma \in traces(\mathcal{E})$. Assume $\sigma = a_1\, a_2\, \ldots\, a_n$ with $a_i \in L_{\mathcal{D}}$. Since $\mathcal{D} = \delta(\mathcal{A} \setminus_{\mathrm{I}} H)$, $\mathcal{D}$ was obtained from the IOA $\mathcal{B}$ by applying deltafication. Consequently, the trace σ can either contain δ-transitions that were newly added by the deltafication procedure, or it contains no δ-transitions at all. We will look at both cases separately.

1. Assume the trace σ does not contain any δ-transitions. In this case, we obviously have $\sigma \in traces(\mathcal{B})$. Since $\mathcal{B} = \mathcal{A} \setminus_{\mathrm{I}} H$, it follows that there exists a trace $\rho \in traces(\mathcal{A})$ such that $\rho \restriction (L_{\mathcal{A}} \setminus H) = \sigma$. Hence, $\rho = B_1\, a_1\, C_1\, B_2\, a_2\, C_2\, \ldots\, B_n\, a_n\, C_n$, with $B_i, C_i \in H^*$. Because $\mathcal{C} = \delta(\mathcal{A})$, and deltafication does not remove existing transitions, it then immediately follows that also $\rho \in traces(\mathcal{C})$. Consequently, there exists a path $\pi = s_0 \xRightarrow{B_1}_{\mathcal{C}} t_0 \xrightarrow{a_1}_{\mathcal{C}} u_0 \xRightarrow{C_1}_{\mathcal{C}} s_1 \xRightarrow{B_2}_{\mathcal{C}} t_1 \xrightarrow{a_2}_{\mathcal{C}} u_1 \xRightarrow{C_2}_{\mathcal{C}} \cdots \xRightarrow{B_n}_{\mathcal{C}} t_{n-1} \xrightarrow{a_n}_{\mathcal{C}} u_{n-1} \xRightarrow{C_n}_{\mathcal{C}} s_n$ in $\mathcal{C}$ with $s_0 \in S^0$, $s_i, t_i, u_i \in S_{\mathcal{C}}$, and $B_i, C_i \in H^*$. From Def. 4.4, it then follows that there must be a path $\pi' = s_0 \xRightarrow{a_1}_{\mathcal{E}} s_1 \xRightarrow{a_2}_{\mathcal{E}} \cdots \xRightarrow{a_n}_{\mathcal{E}} s_n$ in $\mathcal{E}$, since $\mathcal{E} = \mathcal{C} \setminus_{\mathrm{D}} H$. Thus, since $trace(\pi') = a_1\, a_2\, \ldots\, a_n = \sigma$, we find $\sigma \in traces(\mathcal{E})$.
2. Now, we look at the case that the deltafication of $\mathcal{B}$ did introduce new δ-transitions to the trace σ. Assume, without loss of generality, that a_j with $1 \leq j \leq n$ is the only such δ-transition in the trace σ, i.e., $\sigma = a_1\, \ldots\, a_{j-1}\, \delta\, a_{j+1}\, \ldots\, a_n$. Note that by rule R2, a_{j+1} cannot be an output. Let $\sigma' = a_1\, \ldots\, a_{j-1}$ and $\sigma'' = a_{j+1}\, \ldots\, a_n$; thus, $\sigma = \sigma'\, \delta\, \sigma''$. Since $\sigma \in traces(\mathcal{D})$, it follows there exist states $s \in S^0$ and $s', s'', s''' \in S_{\mathcal{D}}$ such that $s \xRightarrow{\sigma'}_{\mathcal{D}} s'$, $s' \xrightarrow{\delta}_{\mathcal{D}} s''$, and $s'' \xRightarrow{\sigma''}_{\mathcal{D}} s'''$. Hence, the new δ-transition has been created between states s' and s''. Since $\mathcal{D}$ is the deltafication of $\mathcal{B}$, from Def. 4.1 we can conclude that in this case either $q(s)$ holds in $\mathcal{B}$ and $s' = s''$,

or $d(s)$ holds in $\mathcal{B}$ and s'' is the quiescence observation state of s'. In both cases, we find that since $s'' \xRightarrow{\sigma''}_{\mathcal{D}} s'''$, then also $s' \xRightarrow{\sigma''}_{\mathcal{D}} s'''$. Hence, since $s \xRightarrow{\sigma'}_{\mathcal{D}} s'$ and $s' \xRightarrow{\sigma''}_{\mathcal{D}} s'''$, and neither σ' nor σ'' contains δ-transitions, we also have $s \xRightarrow{\sigma'}_{\mathcal{B}} s'$ and $s' \xRightarrow{\sigma''}_{\mathcal{B}} s'''$.

Since $s \xRightarrow{\sigma'}_{\mathcal{B}} s'$, $s' \xRightarrow{\sigma''}_{\mathcal{B}} s'''$ and $\mathcal{B} = \mathcal{A} \setminus_{\mathrm{I}} H$, it follows that there exist traces $\rho', \rho'' \in \mathit{traces}(\mathcal{A})$ such that $\rho' \restriction (L_{\mathcal{A}} \setminus H) = \sigma'$, $\rho'' \restriction (L_{\mathcal{A}} \setminus H) = \sigma''$, $s \xRightarrow{\rho'}_{\mathcal{A}} s'$ and $s' \xRightarrow{\rho''}_{\mathcal{A}} s'''$. Hence, $\rho' = B_1\, a_1\, C_1 \, \ldots \, B_{j-1}\, a_{j-1}\, C_{j-1}$ and $\rho'' = B_{j+1}\, a_{j+1}\, C_{j+1} \, \ldots \, B_n\, a_n\, C_n$, with $B_i, C_i \in H^*$. Note that, as mentioned above, a_{j+1} cannot be an output. Since deltafication does not remove any existing transitions, and $\mathcal{C} = \delta(\mathcal{A})$, we also have $s \xRightarrow{\rho'}_{\mathcal{C}} s'$ and $s' \xRightarrow{\rho''}_{\mathcal{C}} s'''$.

We now have to consider two different cases, as mentioned above: either (a) $q(s')$ holds in $\mathcal{B}$ and $s' = s''$, or (b) $d(s')$ holds in $\mathcal{B}$ and s'' is the quiescence observation state of s' in $\mathcal{D}$.

(a) Assume $q(s')$ holds in $\mathcal{B}$ and $s' = s''$. In this case, it follows that $q(s')$ must also hold in $\mathcal{A}$. During deltafication, a δ-labelled self-loop is then added to the state s' in $\mathcal{C}$, and we have $s' \xrightarrow{\delta}_{\mathcal{C}} s'$. Putting this all together yields the path $\pi = s \xRightarrow{\rho'}_{\mathcal{C}} s' \xrightarrow{\delta}_{\mathcal{C}} s' \xRightarrow{\rho''}_{\mathcal{C}} s'''$ in $\mathcal{C}$. Hence, $\pi = s \xRightarrow{B_1}_{\mathcal{C}} t_0 \xrightarrow{a_1}_{\mathcal{C}} u_0 \xRightarrow{C_1}_{\mathcal{C}} \ldots \xRightarrow{B_{j-1}}_{\mathcal{C}} t_{j-2} \xrightarrow{a_{j-1}}_{\mathcal{C}} u_{j-2} \xRightarrow{C_{j-1}}_{\mathcal{C}} s' \xrightarrow{\delta}_{\mathcal{C}} s' \xRightarrow{B_{j+1}}_{\mathcal{C}} t_j \xrightarrow{a_{j+1}}_{\mathcal{C}} u_j \xRightarrow{C_{j+1}}_{\mathcal{C}} \ldots \xRightarrow{B_n}_{\mathcal{C}} t_{n-1} \xrightarrow{a_n}_{\mathcal{C}} u_{n-1} \xRightarrow{C_n}_{\mathcal{C}} s'''$ with $t_i, u_i \in S_{\mathcal{C}}$, and $B_i, C_i \in H^*$. From Def. 4.4, it then follows that there must be a path $\pi' = s \xRightarrow{\sigma'}_{\mathcal{E}} s' \xRightarrow{\delta}_{\mathcal{C}} s' \xRightarrow{\sigma''}_{\mathcal{C}} s'''$ in $\mathcal{E}$, since $\mathcal{E} = \mathcal{C} \setminus_{\mathrm{D}} H$, $\sigma' = a_1 \, \ldots \, a_{j-1}$, and $\sigma'' = a_{j+1} \, \ldots \, a_n$. Thus, since $\mathit{trace}(\pi') = \sigma' \, \delta \, \sigma'' = \sigma$, we have $\sigma \in \mathit{traces}(\mathcal{E})$.

(b) Assume $d(s')$ holds in $\mathcal{B}$ and s'' is the quiescence observation state of s' in $\mathcal{D}$. Since $\mathcal{B} = \mathcal{A} \setminus_{\mathrm{I}} H$, there are two possibilities: either $d(s')$ also holds in $\mathcal{A}$, or not. We will look at these cases separately.
First, assume $d(s')$ also holds in $\mathcal{A}$. Since $\mathcal{C}$ is the deltafication of $\mathcal{A}$, it follows from Def. 4.1 that a quiescence observation state $\mathit{qos}_{s'}$ is added for the state s' in $\mathcal{C}$, and we have $s' \xrightarrow{\delta}_{\mathcal{C}} \mathit{qos}_{s'}$. Furthermore, for every $a \in L^{\mathrm{I}}$ and $t \in S_{\mathcal{A}}$ such that $s' \xrightarrow{a}_{\mathcal{A}} t$, we have $\mathit{qos}_{s'} \xrightarrow{a}_{\mathcal{C}} t$. Since the first label in the trace ρ" cannot be an output, as mentioned above, it follows from the fact that $s' \xRightarrow{\rho''}_{\mathcal{C}} s'''$, that also $\mathit{qos}_{s'} \xRightarrow{\rho''}_{\mathcal{C}} s'''$. Consequently, we find that the path $\pi = s \xRightarrow{\rho'}_{\mathcal{C}} s' \xrightarrow{\delta}_{\mathcal{C}} \mathit{qos}_{s'} \xRightarrow{\rho''}_{\mathcal{C}} s'''$ exists in $\mathcal{C}$. Hence, $\pi = s \xRightarrow{B_1}_{\mathcal{C}} t_0 \xrightarrow{a_1}_{\mathcal{C}} u_0 \xRightarrow{C_1}_{\mathcal{C}} \ldots \xRightarrow{B_{j-1}}_{\mathcal{C}} t_{j-2} \xrightarrow{a_{j-1}}_{\mathcal{C}} u_{j-2} \xRightarrow{C_{j-1}}_{\mathcal{C}} s' \xrightarrow{\delta}_{\mathcal{C}} \mathit{qos}_{s'} \xRightarrow{B_{j+1}}_{\mathcal{C}} t_j \xrightarrow{a_{j+1}}_{\mathcal{C}} u_j \xRightarrow{C_{j+1}}_{\mathcal{C}} \ldots \xRightarrow{B_n}_{\mathcal{C}} t_{n-1} \xrightarrow{a_n}_{\mathcal{C}} u_{n-1} \xRightarrow{C_n}_{\mathcal{C}} s'''$ with $t_i, u_i \in S_{\mathcal{C}}$, and $B_i, C_i \in H^*$. From Def. 4.4, it then follows that there must be a path $\pi' = s \xRightarrow{\sigma'}_{\mathcal{E}} s' \xRightarrow{\delta}_{\mathcal{E}} \mathit{qos}_{s'} \xRightarrow{\sigma''}_{\mathcal{E}} s'''$ in $\mathcal{E}$, since $\mathcal{E} = \mathcal{C} \setminus_{\mathrm{D}} H$. Thus, since $\mathit{trace}(\pi') = \sigma' \, \delta \, \sigma'' = \sigma$, we have $\sigma \in \mathit{traces}(\mathcal{E})$.
Now, assume that $d(s')$ does not hold in $\mathcal{A}$. In this case, the hiding of the output set H has made the state s' newly divergent in $\mathcal{B}$. Hence, by

Def. 2.9, there must exist a fair infinite path $\pi = t_0\, b_1\, t_1\, b_2\, \dots$ in $\mathcal{A}$ with $t_i \in S_{\mathcal{A}}$, $b_i \in L_{\mathcal{A}}$, such that $b_i \in (L^{\mathrm{H}}_{\mathcal{A}} \cup H)$ for all $1 \leq i \leq n$, and $s' \in \omega\text{-}states(\pi)$. Note that for at least one b_i we must have $b_i \in H$, otherwise s' would also be divergent in $\mathcal{A}$. Clearly, π is also a fair infinite path of $\mathcal{C}$, since during deltafication the task partition P remains unchanged and no new output transitions or internal transitions are created. Subsequently hiding the output set H makes π a divergent path, since all actions on path π are either internal actions, or actions from the set H. Hence, since $s' \in \omega\text{-}states(\pi)$, $d(s')$ holds in $\mathcal{E} = \mathcal{C} \setminus_{\mathrm{D}} H$, and is therefore newly divergent. Consequently, by Def. 4.1, a new quiescence observation state $qos_{s'}$ is created by the hiding operation for the state s', and we have $s' \xrightarrow{\delta}_{\mathcal{E}} qos_{s'}$.

Because $s \xRightarrow{\rho'}_{\mathcal{C}} s'$ and $s' \xRightarrow{\rho''}_{\mathcal{C}} s'''$, we have $s \xRightarrow{\sigma'}_{\mathcal{E}} s'$ and $s' \xRightarrow{\sigma''}_{\mathcal{E}} s'''$, since $\rho' \upharpoonright (L_{\mathcal{A}} \setminus H) = \sigma'$, $\rho'' \upharpoonright (L_{\mathcal{A}} \setminus H) = \sigma''$. Like in the previous case, it follows from the facts that $s' \xRightarrow{\sigma''}_{\mathcal{E}} s'''$, $qos_{s'}$ is the quiescence observation state of s', and σ'' does not start with an output, that also $qos_{s'} \xRightarrow{\sigma''}_{\mathcal{E}} s'''$. Hence, $\pi' = s \xRightarrow{\sigma'}_{\mathcal{E}} s' \xrightarrow{\delta}_{\mathcal{E}} qos_{s'} \xRightarrow{\sigma''}_{\mathcal{E}} s'''$ is a path in $\mathcal{E}$. As $trace(\pi') = \sigma'\, \delta\, \sigma'' = \sigma$, we have $\sigma \in traces(\mathcal{E})$. □

Theorem 4.4. *Deltafication and parallel composition commute: given two compatible IOAs $\mathcal{A}$, $\mathcal{B}$, such that $\delta \notin L_{\mathcal{A}} \cup L_{\mathcal{B}}$, we have $\delta(\mathcal{A} \parallel \mathcal{B}) \approx_{\mathrm{tr}} \delta(\mathcal{A}) \parallel \delta(\mathcal{B})$.*

Proof. Let $\mathcal{A} = \langle S_{\mathcal{A}}, S^0_{\mathcal{A}}, L^{\mathrm{I}}_{\mathcal{A}}, L^{\mathrm{O}}_{\mathcal{A}}, L^{\mathrm{H}}_{\mathcal{A}}, P_{\mathcal{A}}, \rightarrow_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle S_{\mathcal{B}}, S^0_{\mathcal{B}}, L^{\mathrm{I}}_{\mathcal{B}}, L^{\mathrm{O}}_{\mathcal{B}}, L^{\mathrm{H}}_{\mathcal{B}}, P_{\mathcal{B}}, \rightarrow_{\mathcal{B}} \rangle$ be compatible IOAs with $\delta \notin L_{\mathcal{A}} \cup L_{\mathcal{B}}$. Let $\delta(\mathcal{A} \parallel \mathcal{B}) = \langle S_{\mathcal{C}}, S^0_{\mathcal{C}}, L^{\mathrm{I}}_{\mathcal{C}}, L^{\mathrm{O}}_{\mathcal{C}}, L^{\mathrm{H}}_{\mathcal{C}}, P_{\mathcal{C}}, \rightarrow_{\mathcal{C}} \rangle$ and $\delta(\mathcal{A}) \parallel \delta(\mathcal{B}) = \langle S_{\mathcal{D}}, S^0_{\mathcal{D}}, L^{\mathrm{I}}_{\mathcal{D}}, L^{\mathrm{O}}_{\mathcal{D}}, L^{\mathrm{H}}_{\mathcal{D}}, P_{\mathcal{D}}, \rightarrow_{\mathcal{D}} \rangle$, as defined by Def. 4.1 and Def. 4.3. We have $S^0_{\mathcal{C}} = S^0_{\mathcal{D}} = S^0_{\mathcal{A}} \times S^0_{\mathcal{B}}$, and $L_{\mathcal{C}} = L_{\mathcal{D}} = L_{\mathcal{A}} \cup L_{\mathcal{B}}$. To prove that $\delta(\mathcal{A} \parallel \mathcal{B}) \approx_{\mathrm{tr}} \delta(\mathcal{A}) \parallel \delta(\mathcal{B})$, we will prove a stronger property: we will show that they are isomorphic. Clearly, two automata that are isomorphic are also trace equivalent. Hence, we will show that there exists a bijection $h\colon S_{\mathcal{C}} \rightarrow S_{\mathcal{D}}$ such that the following holds:

1. for every state $(s_0, t_0) \in S^0_{\mathcal{C}}$ there exists a state $(u_0, v_0) \in S^0_{\mathcal{D}}$ such that $h((s_0, t_0)) = (u_0, v_0)$, and vice versa;
2. $(s, t) \xrightarrow{a}_{\mathcal{C}} (s', t')$ if and only if $h((s,t)) \xrightarrow{a}_{\mathcal{D}} h((s', t'))$, for all $(s,t), (s', t') \in S_{\mathcal{C}}$ and $a \in L_{\mathcal{C}} \cup \{\, \delta \,\}$.

First, we define the function h. By Def. 4.1, the deltafication procedure creates new quiescence observation states for divergent states. As a consequence, we have $S_{\mathcal{C}} \supseteq S_{\mathcal{A}} \times S_{\mathcal{B}}$ and $S_{\mathcal{D}} \supseteq S_{\mathcal{A}} \times S_{\mathcal{B}}$, but it is not necessarily the case that $S_{\mathcal{C}} = S_{\mathcal{D}}$ due to the presence of the quiescence observation states. Therefore, we define the function h as follows:

$$
\begin{aligned}
h \;=\; & \{\, ((s,t),(s,t)) && \mid (s,t) \in S_{\mathcal{A}} \times S_{\mathcal{B}} \,\} \\
\cup\; & \{\, (qos_{(s,t)}, (qos_s, qos_t)) && \mid qos_{(s,t)} \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}}) \wedge \\
& && \;\; s \in d(\mathcal{A}) \;\wedge\; t \in d(\mathcal{B}) \,\} \\
\cup\; & \{\, (qos_{(s,t)}, (qos_s, t)) && \mid qos_{(s,t)} \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}}) \wedge \\
& && \;\; s \in d(\mathcal{A}) \;\wedge\; t \in q(\mathcal{B}) \,\} \\
\cup\; & \{\, (qos_{(s,t)}, (s, qos_t)) && \mid qos_{(s,t)} \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}}) \wedge \\
& && \;\; s \in q(\mathcal{A}) \;\wedge\; t \in d(\mathcal{B}) \,\}
\end{aligned}
$$

Hence, the function h maps all states in $S_{\mathcal{A}} \times S_{\mathcal{B}}$ to themselves, as these states exist in both $S_{\mathcal{C}}$ and $S_{\mathcal{D}}$. All states that are in $S_{\mathcal{C}}$ but not in $S_{\mathcal{A}} \times S_{\mathcal{B}}$ are newly created quiescence observation states for divergent states in $S_{\mathcal{A}} \times S_{\mathcal{B}}$. As we have seen in the proof for Lemma A.4, when $d((s,t))$ holds for some state $(s,t) \in \mathcal{A} \parallel \mathcal{B}$, there are three possibilities for the component states $s \in S_{\mathcal{A}}$ and $t \in S_{\mathcal{B}}$: $d(s)$ and $d(t)$ hold in $\mathcal{A}$ and $\mathcal{B}$, respectively; $d(s)$ and $q(t)$ hold in $\mathcal{A}$ and $\mathcal{B}$, respectively; or $q(s)$ and $d(t)$ hold in $\mathcal{A}$ and $\mathcal{B}$, respectively. In the first case, we can simply map $qos_{(s,t)}$ to the composite state (qos_s, qos_t) in $S_{\mathcal{D}}$, as the deltafications of $\mathcal{A}$ and $\mathcal{B}$ will have created the quiescence observation states qos_s and qos_t for the divergent states s and t. In the second case, however, t is quiescent rather than divergent in $\mathcal{B}$. Hence, a quiescence observation state will be created for the divergent state s, but not for t, since t acts as its own quiescence observation state. Consequently, we map $qos_{(s,t)}$ to the composite state (qos_s, t) in this case. The same principle applies for the third case.

We have to prove that h is indeed a bijection, i.e., that is it both injective and surjective. First, we show that h is injective. Consider two states $(s,t),(u,v) \in S_{\mathcal{C}}$ such that $(s,t) \neq (u,v)$. Clearly, if $(s,t),(u,v) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$, then $h((s,t)) = (s,t) \neq (u,v) = h((u,v))$. If $(s,t) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$ and $(u,v) \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$, then (u,v) is a quiescence observation state, and is therefore mapped by h to a state $(x,y) \in S_{\mathcal{D}}$, where either x or y, or both, are quiescence observation states. Since $(s,t) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$, it directly follows that $h((s,t)) = (s,t) \neq (x,y) = h((u,v))$. A similar argument shows that if $(u,v) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$ and $(s,t) \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$, then also $h((s,t)) \neq h((u,v))$. Now, assume $(s,t),(u,v) \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$. In this case, both (s,t) and (u,v) are quiescence observation states, for some states (s',t') and (u',v') in $S_{\mathcal{A}} \times S_{\mathcal{B}}$. Consequently, (s,t) is mapped to either $(qos_{s'}, qos_{t'})$, $(qos_{s'}, t')$, or $(s', qos_{t'})$. Similarly, (u,v) is mapped to either $(qos_{u'}, qos_{v'})$, $(qos_{u'}, v')$, or $(u', qos_{v'})$. Since $qos_{s'} \neq qos_{u'}$ if $s' \neq u'$, and $qos_{t'} \neq qos_{v'}$ if $t' \neq v'$, it immediately follows that $h((s,t)) \neq h((u,v))$.

Next, we show that h is also surjective. Let (u,v) be some state in $S_{\mathcal{D}}$. We have to show that there exists a state $(s,t) \in S_{\mathcal{C}}$ such that $h((s,t)) = (u,v)$. If $(u,v) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$, then we can take $(s,t) = (u,v)$, since $h((u,v)) = (u,v)$ and $(u,v) \in S_{\mathcal{C}}$. Assume $(u,v) \in S_{\mathcal{D}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$. Hence, (u,v) is either equal to $(qos_{u'}, qos_{v'})$, $(qos_{u'}, v')$, or $(u', qos_{v'})$, for states $u' \in S_{\mathcal{A}}$, $v' \in S_{\mathcal{B}}$. For all these cases, we have $h((qos_{u',v'})) = (u,v)$.

Now that we have a bijection h that maps all elements from $S_{\mathcal{C}}$ to elements of $S_{\mathcal{D}}$, we need to prove that this bijection satisfies the two conditions outlined above. Since $S^0_{\mathcal{C}} = S^0_{\mathcal{D}}$ and $S^0_{\mathcal{C}} \subseteq S_{\mathcal{A}} \times S_{\mathcal{B}}$, clearly for all $s_0 \in S^0_{\mathcal{C}}$ there exists a $t_0 \in S^0_{\mathcal{D}}$ such that $h(s_0) = t_0$, namely $t_0 = s_0$; and symmetrically for all $t_0 \in S^0_{\mathcal{D}}$. To prove that $(s,t) \xrightarrow{a}_{\mathcal{C}} (s',t')$ if and only if $h((s,t)) \xrightarrow{a}_{\mathcal{D}} h((s',t'))$, we must show that if $(s,t) \xrightarrow{a}_{\mathcal{C}} (s',t')$, then $h((s,t)) \xrightarrow{a}_{\mathcal{D}} h((s',t'))$, and if $h((s,t)) \xrightarrow{a}_{\mathcal{D}} h((s',t'))$, then $(s,t) \xrightarrow{a}_{\mathcal{C}} (s',t')$. We will only prove the former case, the proof for the latter case is largely symmetrical. We look at the cases (1) $a \in L^{\mathrm{H}}_{\mathcal{C}}$; (2) $a = \delta$; (3) $a \in L^{\mathrm{I}}_{\mathcal{C}}$; and (4) $a \in L^{\mathrm{O}}_{\mathcal{C}}$, separately.

1. Assume $a \in L^{\mathrm{H}}_{\mathcal{C}}$, i.e., $(s,t) \xrightarrow{a}_{\mathcal{C}} (s',t')$ for some $a \in L^{\mathrm{H}}_{\mathcal{C}}$. In this case, we have $(s,t),(s',t') \in S_{\mathcal{A}} \times S_{\mathcal{B}}$, since, by Def. 4.1, quiescence observation states cannot have incoming or outgoing internal transitions. Consequently, we must show that also $(s,t) \xrightarrow{a}_{\mathcal{D}} (s',t')$, since $h((s,t)) = (s,t)$ and $h((s',t')) = (s',t')$. As deltafication does not affect nor introduce internal transitions, $(s,t) \xrightarrow{a}_{\mathcal{C}} (s',t')$ implies, by Def. 4.3, either $s \xrightarrow{a}_{\mathcal{A}} s'$ and $t = t'$, or $t \xrightarrow{a}_{\mathcal{B}} t'$ and $s = s'$. In both cases, these transitions will still exist after the deltafication of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, it follows directly from Def. 4.3 that also $(s,t) \xrightarrow{a}_{\mathcal{D}} (s',t')$.
2. Assume $a = \delta$, i.e., $(s,t) \xrightarrow{\delta}_{\mathcal{C}} (s',t')$. From Def. 4.1 we can conclude that there are three possible cases: (a) $(s,t),(s',t') \in S_{\mathcal{A}} \times S_{\mathcal{B}}$; (b) $(s,t) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$ and $(s',t') \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$; or (c), $(s,t),(s',t') \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$. We will look at these cases separately.
   (a) Assume $(s,t),(s',t') \in S_{\mathcal{A}} \times S_{\mathcal{B}}$. By Def. 4.1, the state (s,t) is quiescent in $\mathcal{A} \parallel \mathcal{B}$ and we have $(s,t) = (s',t')$. Furthermore, we have $h((s,t)) = (s,t)$, and therefore also $h((s',t')) = (s',t')$. Since $\mathcal{A}$ and $\mathcal{B}$ are input-enabled, we can conclude from Def. 4.3 that both s and t must also be quiescent in $\mathcal{A}$ and $\mathcal{B}$, respectively. Hence, after deltafication of $\mathcal{A}$ and $\mathcal{B}$, both s and t will have δ-labelled self-loops. Consequently, by Def. 4.3, $(s,t) \xrightarrow{\delta}_{\mathcal{D}} (s',t')$.
   (b) Assume $(s,t) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$ and $(s',t') \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$. In this case, by Def. 4.1, the state (s',t') is the quiescence observation state for the state (s,t), and the state (s,t) is divergent in $\mathcal{A} \parallel \mathcal{B}$. Furthermore, we have $h((s,t)) = (s,t)$. The state that (s',t') is mapped to by h depends on whether the states s and t are quiescent or divergent. As discussed above, there are three cases to consider: (i) $d(s)$ holds in $\mathcal{A}$ and $d(t)$ holds in $\mathcal{B}$; (ii) $d(s)$ holds in $\mathcal{A}$ and $q(t)$ holds in $\mathcal{B}$; and (iii), $q(s)$ holds in $\mathcal{A}$ and $d(t)$ holds in $\mathcal{B}$. We will look at each of those cases in turn.
      i. Assume $d(s)$ holds in $\mathcal{A}$ and $d(t)$ holds in $\mathcal{B}$. In this case, we have $h(s',t') = (qos_s, qos_t)$. We must show that $(s,t) \xrightarrow{\delta}_{\mathcal{D}} (qos_s, qos_t)$. By Def. 4.1, we have $s \xrightarrow{\delta}_{\delta(\mathcal{A})} qos_s$ and $t \xrightarrow{\delta}_{\delta(\mathcal{B})} qos_t$. It then follows directly from Def. 4.3 that $(s,t) \xrightarrow{\delta}_{\mathcal{D}} (qos_s, qos_t)$.
      ii. Assume $d(s)$ holds in $\mathcal{A}$ and $q(t)$ holds in $\mathcal{B}$. In this case, we have $h(s',t') = (qos_s, t)$. We must show that $(s,t) \xrightarrow{\delta}_{\mathcal{D}} (qos_s, t)$. By

Def. 4.1, we have $s \xrightarrow{\delta}_{\delta(\mathcal{A})} qos_s$ and $t \xrightarrow{\delta}_{\delta(\mathcal{B})} t$. It then follows directly from Def. 4.3 that $(s,t) \xrightarrow{\delta}_{\mathcal{D}} (qos_s, t)$.

iii. Assume $q(s)$ holds in $\mathcal{A}$ and $d(t)$ holds in $\mathcal{B}$. The proof for this case is symmetrical to the proof for the previous case.

(c) Finally, assume $(s,t),(s',t') \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$. In this case, by Def. 4.1, we have $(s,t) = (s',t')$, and the state (s,t) is the quiescence observation state for some divergent state (s'',t'') in $\mathcal{A} \parallel \mathcal{B}$. The state that (s,t) is mapped to by h depends on whether the states s'' and t'' are quiescent or divergent. Thus, as above, there are three cases to consider: (i) $d(s'')$ holds in $\mathcal{A}$ and $d(t'')$ holds in $\mathcal{B}$; (ii) $d(s'')$ holds in $\mathcal{A}$ and $q(t'')$ holds in $\mathcal{B}$; and (iii) $q(s'')$ holds in $\mathcal{A}$ and $d(t'')$ holds in $\mathcal{B}$. We will look at each of those cases in turn.

i. Assume $d(s'')$ holds in $\mathcal{A}$ and $d(t'')$ holds in $\mathcal{B}$. In this case, we have $h(s,t) = (qos_{s''}, qos_{t''})$. We must show that $(qos_{s''}, qos_{t''}) \xrightarrow{\delta}_{\mathcal{D}} (qos_{s''}, qos_{t''})$. By Def. 4.1, we have $qos_{s''} \xrightarrow{\delta}_{\delta(\mathcal{A})} qos_{s''}$ and $qos_{t''} \xrightarrow{\delta}_{\delta(\mathcal{B})} qos_{t''}$. It then follows directly from Def. 4.3 that $(qos_{s''}, qos_{t''}) \xrightarrow{\delta}_{\mathcal{D}} (qos_{s''}, qos_{t''})$.

ii. Assume $d(s'')$ holds in $\mathcal{A}$ and $q(t'')$ holds in $\mathcal{B}$. In this case, we have $h(s,t) = (qos_{s''}, t'')$. We must show that $(qos_{s''}, t'') \xrightarrow{\delta}_{\mathcal{D}} (qos_{s''}, t'')$. By Def. 4.1, we have $qos_{s''} \xrightarrow{\delta}_{\delta(\mathcal{A})} qos_{s''}$ and $t'' \xrightarrow{\delta}_{\delta(\mathcal{B})} t''$. It then follows directly from Def. 4.3 that $(qos_{s''}, t'') \xrightarrow{\delta}_{\mathcal{D}} (qos_{s''}, t'')$.

iii. Assume $q(s)$ holds in $\mathcal{A}$ and $d(t)$ holds in $\mathcal{B}$. The proof for this case is symmetrical to the proof for the previous case.

3. Assume $a \in L^{\mathrm{I}}_{\mathcal{C}}$, i.e., $(s,t) \xrightarrow{a}_{\mathcal{C}} (s',t')$ for some $a \in L^{\mathrm{I}}_{\mathcal{C}}$. From Def. 4.1 we can conclude that there are two possible cases: either $(s,t),(s',t') \in S_{\mathcal{A}} \times S_{\mathcal{B}}$, or $(s',t') \in S_{\mathcal{A}} \times S_{\mathcal{B}}$ and $(s,t) \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$. We will look at these cases separately.

Assume $(s,t),(s',t') \in S_{\mathcal{A}} \times S_{\mathcal{B}}$. In this case, we have $h((s,t)) = (s,t)$ and $h((s',t')) = (s',t')$. Consequently, we must show that $(s,t) \xrightarrow{a}_{\mathcal{D}} (s',t')$. As deltafication does not affect nor introduce input-labelled transitions, it follows from Def. 4.3 that there are three possibilities:

(a) $s \xrightarrow{a}_{\mathcal{A}} s'$ and $t \xrightarrow{a}_{\mathcal{B}} t'$.
(b) $s \xrightarrow{a}_{\mathcal{A}} s'$, $t = t'$ and $a \notin L_{\mathcal{B}}$.
(c) $t \xrightarrow{a}_{\mathcal{B}} t'$, $s = s'$ and $a \notin L_{\mathcal{A}}$.

In all cases, these transitions will still exist after the deltafication of $\mathcal{A}$ and $\mathcal{B}$. Neither will $L_{\mathcal{A}}$ nor $L_{\mathcal{B}}$ change. Thus, it follows directly from Definition 4.3 that also $(s,t) \xrightarrow{a}_{\mathcal{D}} (s',t')$.

Now, assume $(s',t') \in S_{\mathcal{A}} \times S_{\mathcal{B}}$ and $(s,t) \in S_{\mathcal{C}} \setminus (S_{\mathcal{A}} \times S_{\mathcal{B}})$. In this case, we have $h((s',t')) = (s',t')$. By Def. 4.1, the state (s,t) is the quiescence observation state of some divergent state (s'',t''), i.e., $(s,t) = qos_{(s'',t'')}$. We then also have $(s'',t'') \xrightarrow{a}_{\mathcal{C}} (s',t')$. The state that (s,t) is mapped to by h depends on whether the states s'' and t'' are quiescent or divergent. Again, there are three cases to consider: (a) $d(s'')$ holds in $\mathcal{A}$ and $d(t'')$ holds in $\mathcal{B}$; (b) $d(s'')$ holds in $\mathcal{A}$ and $q(t'')$ holds in $\mathcal{B}$; and (c), $q(s'')$ holds in $\mathcal{A}$ and $d(t'')$ holds in $\mathcal{B}$. We will look at each of those cases in turn.

(a) Assume $d(s'')$ holds in $\mathcal{A}$ and $d(t'')$ holds in $\mathcal{B}$. In this case, we have $h(s,t) = (qos_{s''}, qos_{t''})$. We must show that $(qos_{s''}, qos_{t''}) \xrightarrow{a}_{\mathcal{D}} (s',t')$. Since $(s'',t'') \xrightarrow{a}_{\mathcal{C}} (s',t')$, it follows from Def. 4.3 that there are three possibilities:

i. $s'' \xrightarrow{a}_{\mathcal{A}} s'$ and $t'' \xrightarrow{a}_{\mathcal{B}} t'$. By Def. 4.1, we then have $qos_{s''} \xrightarrow{a}_{\delta(\mathcal{A})} s'$ and $qos_{t''} \xrightarrow{a}_{\delta(\mathcal{B})} t'$. It then follows directly from Def. 4.3 that $(qos_{s''}, qos_{t''}) \xrightarrow{a}_{\mathcal{D}} (s',t')$.

ii. $s'' \xrightarrow{a}_{\mathcal{A}} s'$, $t'' = t'$ and $a \notin L_{\mathcal{B}}$. By Def. 4.1, we then have $qos_{s''} \xrightarrow{a}_{\delta(\mathcal{A})} s'$. Since $a \notin L_{\mathcal{B}}$, it follows from Def. 4.3 that $(qos_{s''}, qos_{t''}) \xrightarrow{a}_{\mathcal{D}} (s',t')$.

iii. $t'' \xrightarrow{a}_{\mathcal{B}} t'$, $s'' = s'$ and $a \notin L_{\mathcal{A}}$. The proof for this case is symmetrical to the proof for the previous case.

(b) Assume $d(s'')$ holds in $\mathcal{A}$ and $q(t'')$ holds in $\mathcal{B}$. In this case, we have $h(s,t) = (qos_{s''}, t'')$. We must show that $(qos_{s''}, t'') \xrightarrow{a}_{\mathcal{D}} (s',t')$. Since $(s'',t'') \xrightarrow{a}_{\mathcal{C}} (s',t')$, it follows from Def. 4.3 that there are three possibilities:

i. $s'' \xrightarrow{a}_{\mathcal{A}} s'$ and $t'' \xrightarrow{a}_{\mathcal{B}} t'$. By Def. 4.1, we then have $qos_{s''} \xrightarrow{a}_{\delta(\mathcal{A})} s'$ and $t'' \xrightarrow{a}_{\delta(\mathcal{B})} t'$. It then follows directly from Def. 4.3 that $(qos_{s''}, t'') \xrightarrow{a}_{\mathcal{D}} (s',t')$.

ii. $s'' \xrightarrow{a}_{\mathcal{A}} s'$, $t'' = t'$ and $a \notin L_{\mathcal{B}}$. By Def. 4.1, we then have $qos_{s''} \xrightarrow{a}_{\delta(\mathcal{A})} s'$. Since $a \notin L_{\mathcal{B}}$, it follows from Def. 4.3 that $(qos_{s''}, t'') \xrightarrow{a}_{\mathcal{D}} (s',t')$.

iii. $t'' \xrightarrow{a}_{\mathcal{B}} t'$, $s'' = s'$ and $a \notin L_{\mathcal{A}}$. Since $a \notin L_{\mathcal{A}}$, it follows from Def. 4.3 that $(qos_{s''}, t'') \xrightarrow{a}_{\mathcal{D}} (s',t')$.

(c) Assume $q(s'')$ holds in $\mathcal{A}$ and $d(t'')$ holds in $\mathcal{B}$. The proof for this case is symmetrical to the proof for the previous case.

4. Finally, assume $a \in L^{\mathrm{O}}_{\mathcal{C}}$, i.e., $(s,t) \xrightarrow{a}_{\mathcal{C}} (s',t')$ for some $a \in L^{\mathrm{O}}_{\mathcal{C}}$. Similar to the case for $a \in L^{\mathrm{H}}_{\mathcal{C}}$, we have $(s,t), (s',t') \in S_{\mathcal{A}} \times S_{\mathcal{B}}$, since, by Def. 4.1, quiescence observation states cannot have incoming or outgoing output transitions. As a result, we must show that also $(s,t) \xrightarrow{a}_{\mathcal{D}} (s',t')$, since $h((s,t)) = (s,t)$ and $h((s',t')) = (s',t')$. As deltafication does not affect nor introduce output-labelled transitions, it follows from Def. 4.3 that there are four possibilities:

(a) $s \xrightarrow{a}_{\mathcal{A}} s'$, $t \xrightarrow{a}_{\mathcal{B}} t'$ and $a \in L^{\mathrm{O}}_{\mathcal{A}}$, $a \in L^{\mathrm{I}}_{\mathcal{B}}$.
(b) $s \xrightarrow{a}_{\mathcal{A}} s'$, $t \xrightarrow{a}_{\mathcal{B}} t'$ and $a \in L^{\mathrm{I}}_{\mathcal{A}}$, $a \in L^{\mathrm{O}}_{\mathcal{B}}$.
(c) $s \xrightarrow{a}_{\mathcal{A}} s'$, $t = t'$ and $a \notin L_{\mathcal{B}}$.
(d) $t \xrightarrow{a}_{\mathcal{B}} t'$, $s = s'$ and $a \notin L_{\mathcal{A}}$.

In all four cases, these transitions will still exist after the deltafication of $\mathcal{A}$ and $\mathcal{B}$. Neither will $L_{\mathcal{A}}$ or $L_{\mathcal{B}}$ change. Thus, it follows directly from Def. 4.3 that also $(s,t) \xrightarrow{a}_{\mathcal{D}} (s',t')$. □