# Queneau Numbers — Recent Results and a Bibliography

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**Abstract** — Publications dealing with Queneau numbers can be divided roughly into two categories. In the first one the relation to poetry and, more generally, to literature is the central issue. The second one consists of papers that investigate the mathematical properties of the Queneau numbers and of integer sequences that are closely related. Our comments in this survey are restricted to this second category.

**Keywords**: integer sequence, Queneau-Daniel permutation, Queneau number, distribution of prime numbers, Artin's conjecture (on primitive roots).

## 1 Introduction

The origin of the Queneau numbers stems from an old verse form: the sextine (or *sestina* in Italian), introduced by the 12th century troubadour Arnaut Daniel in his lyric poem "Lo ferm voler". For the complete text of this poem together with a translation, more medieval context and the history of Queneau numbers up to the 21st century, we refer to the splendid survey of M.P. Saclolo [20] and the references mentioned there.

Apart from the last three lines, a sextine constists of six stanzas each of which contains six lines: the last words of these lines in each stanza are permuted by the cycle (1 2 4 5 3 6) to obtain the last words in the next stanza. This permutation has been generalized to arbitrary $n$, resulting in the so-called *Queneau-Daniel permutation* $Q_n$; cf. Section 2 for a definition. The numbers $n$ for which $Q_n$ has order $n$ —and of which 6 is the prototypical example— are called the *Queneau numbers*.

A gentle introduction to the Queneau numbers is M.P. Saclolo's recent paper [20]; we assume the reader to be familiar with this excellent overview. The remaining sections of the present paper, preceding the bibliography in Section 6, focus on some recent mathematical results that were established after the publication of [20].

In collecting publications, that deal with Queneau numbers, one meets a few difficulties. First of all, some publications —particularly the older ones— are hard to access: notably this applies to [6] which provides the first complete characterization of the Queneau numbers. Secondly, the list of publications in Section 6 allows for a kind of dichotomy.

On the one hand, there are the publications on the poetry that serves as the starting point for the definition of the Queneau-Daniel permutation and of the Queneau numbers. In this category the emphasis is on the literary aspects, as well as on the possible generalizations and variations of the sextine, whereas mathematical considerations are rather secondary. Most papers in this category —like [2, 3, 4, 7, 8, 9, 11, 12, 14, 15]— are from members of the French OULIPO movement ("Ouvroir de Littérature Potentielle") of which Raymond Queneau was one of the founders.

On the other hand, some old contributions [5, 6] and many recent papers [13, 18, 21, 22] dealing with Queneau numbers are purely mathematical: they provide (partial or complete) characterizations of the Queneau numbers or they investigate the properties of (subsets of) the set of all Queneau numbers without hardly any reference to their poetical origins.

As to be expected the present paper fits into this second category. So in Section 2 we recall the basic definitions in order to fix the terminology and our notation. In Section 3 we comment on Queneau's conjecture, to which we may refer nowadays as Queneau's Lemma. Section 4 presents a catalogue of characterization results for the Queneau numbers (including some recent ones) in chronological order, and so it provides a rather complete[1] historic overview of characterization results for the Queneau numbers. Section 5 is devoted to density properties of the set of prime numbers associated with the Queneau numbers (the so-called *Queneau prime numbers*) and some of its subsets. As mentioned before, Section 6 consists of the actual bibliography.

The two results in Section 3 are given together with their proofs from [6]. The reason to do so is twofold: the accessibility of [6] is poor and the proof techniques used to obtain these results are applicable in other situations as well. The characterizations of the Queneau numbers in Section 4 are quoted without their proofs because the papers, from which they were taken, are easy to retrieve; see also [20] and [19] for some examples of proofs.

## 2 Definitions

First, we recall some definitions and terminology from [5], i.e., from the first major contribution to the mathematical properties of the Queneau numbers by M. Bringer.

Let $\mathfrak{S}_n$ be the symmetric group on $n$ elements. The spiral permutation or *Queneau-Daniel* permutation $Q_n$ is defined by

$$Q_n(m) = m/2 \qquad \text{if } m \text{ is even, and}$$
$$Q_n(m) = n - (m-1)/2 \qquad \text{if } m \text{ is odd.}$$

The cyclic subgroup $\langle Q_n \rangle$ of $\mathfrak{S}_n$ generated by $Q_n$ is called the *Queneau-Daniel* group (associated with the permutation $Q_n$). If the Queneau-Daniel group $\langle Q_n \rangle$ is of order $n$,

[1] This overview is "complete" in the sense that —contrary to contributions like [9, 13, 15, 18, 19, 20]— it does refer to the (indeed less accessible) paper by C.W. Carroll & W.F. Orr [6] and its rôle in characterizing the Queneau numbers. Viz. [6] contains the first set of necessary and sufficient conditions for a number to be a Queneau number.

then the number $n$ is called *admissible* or a *Queneau number*. Equivalently, the number $n$ is a Queneau number, if the permutation $Q_n$ consists of a single cycle of length $n$. We will denote the set of Queneau numbers by $P(Q)$. The first few elements of $P(Q)$, viewed as integer sequence, are:

2, 3, 5, 6, 9, 11, 14, 18, 23, 26, 29, 30, 33, 35, 39, 41, 50, 51, 53, 65, 69, 74, 81, 83, 86, 89, 90, 95, 98, 99, 105, 113, 119, 131, 134, 135, 146, 155, 158, 173, 174, 179, 183, 186, 189, 191, 194, 209, 210, 221, 230, 231, 233, 239, 243, 245, 251, 254, 261, 270, ....

For more Queneau numbers we refer to the sequence A054639 in [33].

The inverse of the Queneau-Daniel permutation $Q_n$ equals the *twist* permutation $T_n$ which plays an important rôle in some parts of theoretical computer science, viz. in automata theory [27, 28, 29] and in combinatorics on words [21]. From [21] we quote the following notation and terminology. The twist permutation $T_n$ is defined by

$$\begin{array}{ll} T_n(m) = 2m & \text{if } 2m \leq n, \\ T_n(m) = 2(n-m)+1 & \text{otherwise,} \end{array}$$

or, rather

$$\begin{array}{ll} T_n(m) \equiv +2m \pmod{2n+1} & \text{if } 2m \leq n, \\ T_n(m) \equiv -2m \pmod{2n+1} & \text{otherwise.} \end{array}$$

A number $n$ is called a *twist prime* or a *$T$-prime* if $\langle T_n \rangle$ —i.e., the cyclic subgroup of $\mathfrak{S}_n$ generated by $T_n$— has order $n$ or, equivalently, if the permutation $T_n$ consists of a single cycle of length $n$. The set of $T$-primes will be denoted by $P(T)$.

It is easy to see that for each $n$, we have $Q_n^{-1} = T_n$; consequently, $n$ is a Queneau number if and only if $n$ is a $T$-prime, i.e., $P(Q) = P(T)$. This equivalence has another advantage: sometimes it is more convenient to use $Q_n^{-1}$ (i.e., $T_n$) in proofs instead of $Q_n$ itself or, the other way around (using $Q_n$ rather than $T_n$). For an example we refer to the proof of Lemma 3.1 below.

In view of [21], we may call a Queneau number a *$Q$-prime* as well, but we will not do so, as this would mean the introduction of a fourth name for a single object: admissible number, Queneau number, $T$-prime, and $Q$-prime. We restrict ourselves to both "Queneau number" and "$T$-prime" which allows us to refer implicitly to the underlying family of permutations: $\{Q_n\}_{n\geq 1}$ and $\{T_n\}_{n\geq 1}$, respectively.

Permutations, like $T_n$, are often related to shuffling a deck of cards in a non-standard way. For $T_n$ we cut a deck of cards into two equal parts and before the perfect interleaving process we put the top half upside down. Another example is the so-called Monge's shuffle [11, 18] which is related to the Archimedes' permutation $A_0$ of [21]; cf. [18] for an overview of several shuffles and their permutations. The "ordinary way" of shuffling a deck of cards by perfect interleaving is a starting point in modeling parallel processes in computer science; cf., e.g., [26] for an automaton-based approach. More on ways of shuffling and their permutations can be found in [14, 18, 21].

## 3 Queneau's Conjecture/Lemma

R. Queneau tried to characterize the natural numbers that are not Queneau numbers: he conjectured that numbers $n$ satisfying the relation $n = 2xy + x + y$ for some $x$, $y \geq 1$, are not Queneau numbers [2, 3]. This conjecture has been proven in [5] by M. Bringer and independently by C.W. Carroll and W.F. Orr [6]. So we now may refer to this result as "Queneau's Lemma".

**Lemma 3.1 / Queneau's Lemma.** [5, 6] *If there exist integers $x$ and $y$ with $x$, $y \geq 1$ such that $n = 2xy + x + y$, then $n$ is not a Queneau number.*

*Proof.* We use $T_n$ instead of $Q_n$; so we will show that $n$ is not a $T$-prime.

Suppose there exist integers $x$, $y \geq 1$ such that $n = 2xy+x+y$; then we have $2x+1 < n$. Next we consider the multiples of $2x + 1$ that are less than or equal to $n$ and their images under the permutation $T_n$. For multiples $m(2x+1)$ with $1 \leq m(2x+1) < \lceil (n+1)/2 \rceil$ and with $\lceil (n + 1)/2 \rceil \leq m(2x + 1) \leq n$, we have respectively,

$$T_n(m(2x + 1)) = 2m(2x + 1),$$

and

$$\begin{aligned} T_n(m(2x + 1)) &= 2(n - m(2x + 1)) + 1 = 2(2xy + x + y - 2mx - m) + 1 \\ &= 4xy + 2y - 4mx - 2m + 2x + 1 = (2x + 1)(2y - 2m + 1). \end{aligned}$$

So every multiple of $2x + 1$ is mapped by $T_n$ on another multiple of $2x + 1$. For $n$ to be $T$-prime, $T_n$ must consists of a single cycle of length $n$, which implies that all $l$ with $1 \leq l \leq n$ must be divisible by $2x + 1$. But this is impossible since $2x + 1 > 1$ for $x \geq 1$. Consequently, $n$ is not $T$-prime and therefore not a Queneau number. □

Slight variations of this proof are applicable in similar situations. For example, in [16, 21] we showed that the condition in Queneau's Lemma also implies that $n$ is neither a $J_2$-prime nor a $\overline{J_2}$-prime. The $J_2$-primes are related to the so-called Josephus problem (i.e., a generalization of "eeny, meeny, miny, moe"); see [32] or §3.3 in [23] for an introduction. The $\overline{J_2}$-primes stem from the dual of the Josephus problem; cf. [16, 21].

Queneau's Lemma implies the following result which is crucial in all characterizations of the Queneau numbers (Section 4). This important property has been established by M. Bringer [5] and independently in [6] by C.W. Carroll and W.F. Orr. It will also play a principal part in Section 5.

**Proposition 3.2.** [5, 6] *If $n$ is a Queneau number, then $2n + 1$ is a prime number.*

*Proof.* Assume to the contrary that $2n+1$ is not prime. Since $2n+1$ is an odd integer, it must be the product of two odd integers strictly greater than 1: $(2x+1)(2y+1) = 2n+1$ with $x$, $y \geq 1$. This yields $4xy+2x+2y+1 = 2n+1$, or $2xy+x+y = n$. From Queneau's Lemma (Lemma 3.1) it then follows that $n$ is not a Queneau number. □

Statements similar to Proposition 3.2 hold for the $J_2$-primes and the $\overline{J_2}$-primes as well [16, 21]. See also the remark after Theorem 4.5 in Section 4.

## 4 Characterizations of Queneau Numbers

As usual $\mathbb{Z}$ is the set of all integers and for a prime number $p$, $\mathbb{Z}_p$ denotes the finite field of integers modulo $p$ and $\mathbb{Z}_p^\star$ denotes the cyclic multiplicative group of $\mathbb{Z}_p$. Remember that $\mathbb{Z}_p^\star$ has order $p-1$. Let $G_p$ be the set of all elements in $\mathbb{Z}_p^\star$ of order $p-1$, i.e., the set of all possible generators of $\mathbb{Z}_p^\star$. Note that in number theory the elements of $G_p$ are called *primitive roots* modulo $p$ [24, 35].

The first major attempt to describe the Queneau numbers mathematically is [5] by M. Bringer: it results in a partial characterization of $P(Q)$. The main results from [5] can be summarized as follows.

**Theorem 4.1.** [5] *Let $n$ be a natural number.*

(1) *If $n$ is a Queneau number, then $2n+1$ is a prime number.*

(2) *If $2n+1$ is a prime number and $+2$ belongs to $G_{2n+1}$, then $n$ is a Queneau number.*

(3) *If both $n$ and $2n+1$ are prime numbers, then $n$ is a Queneau number.*

(4) *If $n$ is of the form $n = 2p$ where $p$ and $4p+1$ are prime numbers ($p \geq 3$), then $n$ is a Queneau number.*

(5) *Numbers of the form $2^k$ ($k \geq 2$), $2^k - 1$ ($k \geq 3$), and $4k$ ($k \geq 1$) are not Queneau numbers.* □

Notice that the first statement of Theorem 4.1(5) follows from the third one.

A complete characterization of the Queneau numbers has been obtained by C.W. Carroll and W.F. Orr in [6], which does not refer to [5]. We slightly reformulate the main result from [6] which results in following characterization of the Queneau numbers.[2]

**Theorem 4.2.** [6] *A natural number $n$ is a Queneau number if and only if*

(1) *$2n+1$ is a prime number, and*

(2) *at least one of $-2$ and $+2$ belongs to $G_{2n+1}$ i.e., $\{-2, +2\} \cap G_{2n+1} \neq \varnothing$.* □

Although Theorem 4.2 provides necessary and sufficient conditions for a number to be a Queneau number there is ample space for refinement.

A first attempt by J. Roubaud in [9] ended up in a failure: it resulted in a partial characterization only together with the erroneous conclusion that 141 is a Queneau number. The fact that $141 \notin P(Q)$ has been observed in [10].

J. Roubaud's approach has been completed successfully by J.-G. Dumas in [13], which —as J. Roubaud in [9]— refers to [5] but not to [6]. From [13] we quote the following two complete, refined characterizations (Theorem 2 and Corollary 1 in [13], respectively). These results together with their proofs are also quoted in [20] and in [19].

**Theorem 4.3.** [13] *If $n$ is a natural number and $p = 2n+1$, then $n$ is a Queneau number if and only if $p$ is a prime number and either 2 is of order $2n$ in $\mathbb{Z}/p\mathbb{Z}$, or $n$ is odd and 2 is of order $n$ in $\mathbb{Z}/p\mathbb{Z}$.* □

[2]In [6] condition (2) reads: "either +2 or −2 belongs to $G_{2n+1}$". If "either ⋯ or ⋯" stands for the *exclusive* or, then this version of the result is definitely incorrect; cf. Theorem 4.5 or Theorem 4.6.

**Theorem 4.4.** [13] *If $n$ is a natural number and $p = 2n+1$, then $n$ is a Queneau number if and only if $p$ is a prime number and either 2 is of order $2n$ in $\mathbb{Z}/p\mathbb{Z}$ and $n \equiv 1$ or 2 (mod 4), or 2 is of order $n$ in $\mathbb{Z}/p\mathbb{Z}$ and $n \equiv 3$ (mod 4).* □

In [16] and [21], which contain references to [5, 6, 13] and to [5, 6, 9, 13] respectively, we established necessary and sufficient conditions on a number to be a $T$-prime. Rephrased in terms of Queneau numbers instead of $T$-primes, it yields another refined characterization of $P(Q)$ that reads as follows.

**Theorem 4.5.** [16, 21] *A natural number $n$ is a Queneau number if and only if $2n+1$ is a prime number and exactly one of the following three conditions holds:*

(1) $n \equiv 1 \pmod 4$ *and* $+2 \in G_{2n+1}$ *and* $-2 \notin G_{2n+1}$.

(2) $n \equiv 2 \pmod 4$ *and both* $-2 \in G_{2n+1}$ *and* $+2 \in G_{2n+1}$.

(3) $n \equiv 3 \pmod 4$ *and* $-2 \in G_{2n+1}$ *and* $+2 \notin G_{2n+1}$. □

Dropping condition (3) in Theorem 4.5 yields a characterization of the set $P(J_2)$ of $J_2$-primes. Similarly, deleting condition (1) in Theorem 4.5 results in a characterization of the set $P(\overline{J_2})$ of $\overline{J_2}$-primes [16, 21]. The subsets of $P(Q)$ characterized by conditions (1), (2) and (3) solely, are denoted by $P(A_1^+)$, $P(A_0)$ and $P(A_1^-)$, respectively; they are related to permutations defined by Archimedes' spirals [16, 21, 18]. So $P(A_0)$ is the set of even Queneau numbers, and $P(A_1) = P(A_1^+) \cup P(A_1^-)$ is the set of odd Queneau numbers. Consequently, we obtain for the set $P(Q)$ of Queneau numbers, that

$$P(Q) = P(J_2) \cup P(\overline{J_2}) = P(A_0) \cup P(A_1) = P(A_0) \cup P(A_1^+) \cup P(A_1^-),$$

where $P(A_0)$, $P(A_1^+)$ and $P(A_1^-)$ are mutually disjoint, whereas

$P(J_2) \cap P(\overline{J_2}) = P(A_0)$,

$P(J_2) = P(A_0) \cup P(A_1^+)$, and

$P(\overline{J_2}) = P(A_0) \cup P(A_1^-)$.

Theorems 4.3, 4.4 and 4.5 inspired J.-G. Dumas to establish in [18] the following necessary and sufficient conditions for a number to belong to $P(Q)$.

**Theorem 4.6.** [18] *A natural number $n$ is a Queneau number if and only if $2n+1$ is a prime number and exactly one of the following three conditions holds:*

(1) $n \equiv 1 \pmod 4$ *and* $+2 \in G_{2n+1}$ *and* $-2$ *is of order $n$ in* $\mathbb{Z}/(2n+1)\mathbb{Z}$.

(2) $n \equiv 2 \pmod 4$ *and both* $-2 \in G_{2n+1}$ *and* $+2 \in G_{2n+1}$.

(3) $n \equiv 3 \pmod 4$ *and* $-2 \in G_{2n+1}$ *and* $+2$ *is of order $n$ in* $\mathbb{Z}/(2n+1)\mathbb{Z}$. □

It is not difficult to show that Theorems 4.5 and 4.6 are logically equivalent: given Theorem 4.4, Theorem 4.5 implies Theorem 4.6, and vice versa.

## 5 Queneau Prime Numbers and Their Density

In this section we address the problem whether there are infinitely many Queneau numbers.

Simply counting Queneau numbers as one counts, for instance, prime numbers as in §2.6 of [35], results in Table 1 where we also include the counts for the $A_0$-, $A_1$-, $A_1^+$-, $A_1^-$-,

| $n$ | | $\pi(A_0,n)$ | $\pi(A_1,n)$ | $\pi(A_1^+,n)$ | $\pi(A_1^-,n)$ | $\pi(J_2,n)$ | $\pi(\overline{J_2},n)$ | $\pi(Q,n)$ |
|---|---|---|---|---|---|---|---|---|
| | $N$ | $\pi(a_0,N)$ | $\pi(a_1,N)$ | $\pi(a_1^+,N)$ | $\pi(a_1^-,N)$ | $\pi(j_2,N)$ | $\pi(\overline{j_2},N)$ | $\pi(q,N)$ |
| $10^1$ | $2{\cdot}10^1{+}1$ | 2 | 3 | 2 | 1 | 4 | 3 | 5 |
| $10^2$ | $2{\cdot}10^2{+}1$ | 11 | 19 | 10 | 9 | 21 | 20 | 30 |
| $10^3$ | $2{\cdot}10^3{+}1$ | 61 | 116 | 55 | 61 | 116 | 122 | 177 |
| $10^4$ | $2{\cdot}10^4{+}1$ | 418 | 839 | 421 | 418 | 839 | 836 | 1257 |
| $10^5$ | $2{\cdot}10^5{+}1$ | 3378 | 6706 | 3328 | 3378 | 6706 | 6756 | 10084 |
| $10^6$ | $2{\cdot}10^6{+}1$ | 27882 | 55702 | 27861 | 27841 | 55743 | 55723 | 83584 |
| $10^7$ | $2{\cdot}10^7{+}1$ | 237676 | 475478 | 237656 | 237822 | 475332 | 475498 | 713154 |
| $10^8$ | $2{\cdot}10^8{+}1$ | 2071170 | 4143232 | 2072304 | 2070928 | 4143474 | 4142098 | 6214402 |

Table 1: Counting $A_0$-, $A_1$-, $A_1^+$-, $A_1^-$-, $J_2$-, $\overline{J_2}$-primes and Queneau numbers.

$J_2$- and the $\overline{J_2}$-primes. Let $\pi(Q,n)$ be the number of Queneau numbers less than or equal to $n$; $\pi(X,n)$ is defined analogously for $X \in \{A_0, A_1, A_1^+, A_1^-, J_2, \overline{J_2}\}$. In Table 1 one should ignore for the moment the second row and the second column (in which $N$ is involved).

Table 2 contains the distribution of the $A_0$-, $A_1$-, $A_1^+$-, $A_1^-$-, $J_2$-, $\overline{J_2}$-primes and the Queneau numbers. This table shows that the distribution of the Queneau numbers resembles a "Prime Number Theorem"-like behavior [22]. An identical observation applies to the subsets $P(A_0)$, $P(A_1)$, $P(A_1^+)$, $P(A_1^-)$, $P(J_2)$ and $P(\overline{J_2})$ of the set of Queneau numbers.

Let $\mathbb{P}$ the set of odd prime numbers and let $\pi(\mathbb{P},n)$ the number of odd prime numbers less than or equal to $n$. Remember that the Prime Number Theorem reads as:

**Prime Number Theorem.** *The function $\pi(\mathbb{P},n)$ is asymptotic to $n/\ln n$. That is $\lim_{n\to\infty} \pi(\mathbb{P},n)\ln n/n = 1$.* □

From Table 2 we observe that the distributions of $X$-primes —where $X$ equals $A_0$, $A_1$, $A_1^+$, $A_1^-$, $J_2$ or $\overline{J_2}$— and the Queneau numbers show limiting values

$$\Lambda(X) = \lim_{n\to\infty} \pi(X,n)\ln n/n$$

unequal to 1. Rather than inferring some rough estimates for $\Lambda(X)$ from Table 2, we will follow another approach.

**Definition 5.1.** For $X$ equal to $A_0$, $A_1$, $A_1^+$, $A_1^-$, $J_2$, $\overline{J_2}$ and $Q$, let $P(x)$ be the set of prime numbers, given by

$$P(x) = \{2n{+}1 \mid n \in P(X)\}.$$

The members of $P(x)$ are called the *x-primes* or the *x-prime numbers*. The elements of $P(q)$ are the prime numbers associated with the Queneau numbers; they are called the *Queneau prime numbers*. □

| $n$ | $\pi(X,n)\ln n/n$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $A_0$ | $A_1$ | $A_1^+$ | $A_1^-$ | $J_2$ | $\overline{J_2}$ | $Q$ |
| $10^1$ | 0.4605 | 0.4605 | 0.4605 | 0.2303 | 0.9210 | 0.6908 | 1.1513 |
| $10^2$ | 0.5066 | 0.8750 | 0.4605 | 0.4145 | 0.9671 | 0.9210 | 1.3816 |
| $10^3$ | 0.4214 | 0.8013 | 0.3799 | 0.4214 | 0.8013 | 0.8427 | 1.2227 |
| $10^4$ | 0.3850 | 0.7727 | 0.3878 | 0.3850 | 0.7727 | 0.7700 | 1.1577 |
| $10^5$ | 0.3889 | 0.7721 | 0.3832 | 0.3889 | 0.7721 | 0.7778 | 1.1610 |
| $10^6$ | 0.3852 | 0.7696 | 0.3849 | 0.3846 | 0.7701 | 0.7698 | 1.1548 |
| $10^7$ | 0.3831 | 0.7664 | 0.3831 | 0.3833 | 0.7661 | 0.7664 | 1.1495 |
| $10^8$ | 0.3815 | 0.7632 | 0.3817 | 0.3815 | 0.7633 | 0.7630 | 1.1447 |

Table 2: Distribution of $A_0$-, $A_1$-, $A_1^+$-, $A_1^-$-, $J_2$-, $\overline{J_2}$-primes and Queneau numbers.

The counting results of the $x$-primes and the Queneau prime numbers are in Table 1 where we now should delete the first row and first column (in which $n$ occurs).

Note that the Queneau prime numbers are *not* the prime numbers among the Queneau numbers, but they are the prime numbers associated with the Queneau numbers by means of Proposition 3.2 and Theorems 4.2–4.6.

The shift of attention from Queneau numbers to Queneau prime numbers allows us to apply some results from number theory. The first one is a famous conjecture ("Artin's Conjecture on Primitive Roots") on the infinity of certain subsets of $\mathbb{P}$ and their density in $\mathbb{P}$ [31]. Remember that for a prime number $p$, the elements of $G_p$ are also called "primitive roots modulo $p$". Let $\mathbb{S}(g)$ is the set of prime numbers $p$ such that $g$ is a primitive root modulo $p$, i.e., $g \in G_p$.

**Artin's Conjecture on Primitive Roots (ACPR).** *Let $g$ be an integer which is not a perfect square and not equal to $-1$, and let $g = g_0h^2$ with $g_0$ square-free. Then*

(1) *$\mathbb{S}(g)$ is infinite, and $\mathbb{S}(g)$ has a positive asymptotic density in $\mathbb{P}$.*

(2) *If in addition $g$ is not a perfect power and if $g_0$ is not congruent* 1 *modulo* 4*, this density is independent of $g$ and equals Artin's constant* **A**. □

Artin's constant **A** is defined as the infinite product

$$\mathbf{A} = \prod_{p \text{ is prime}} \left(1 - \frac{1}{p(p-1)}\right) = 0.3739558136192022880547280543464164151\cdots.$$

From Definition 5.1, ACPR and characterizations for $P(J_2)$ and $P(\overline{J_2})$ [16, 21], similar to Theorem 4.5, it is straightforward to obtain the following result.

| $N$ | $\pi(x, N)\ln N/N$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $a_0$ | $a_1$ | $a_1^+$ | $a_1^-$ | $j_2$ | $\overline{j_2}$ | $q$ |
| $2{\cdot}10^1{+}1$ | 0.2900 | 0.4349 | 0.2900 | 0.1450 | 0.5799 | 0.4349 | 0.7249 |
| $2{\cdot}10^2{+}1$ | 0.2902 | 0.5013 | 0.2638 | 0.2375 | 0.5541 | 0.5277 | 0.7915 |
| $2{\cdot}10^3{+}1$ | 0.2317 | 0.4407 | 0.2089 | 0.2317 | 0.4407 | 0.4635 | 0.6724 |
| $2{\cdot}10^4{+}1$ | 0.2070 | 0.4154 | 0.2085 | 0.2069 | 0.4154 | 0.4139 | 0.6224 |
| $2{\cdot}10^5{+}1$ | 0.2062 | 0.4093 | 0.2031 | 0.2062 | 0.4093 | 0.4123 | 0.6154 |
| $2{\cdot}10^6{+}1$ | 0.2023 | 0.4041 | 0.2021 | 0.2020 | 0.4044 | 0.4042 | 0.6063 |
| $2{\cdot}10^7{+}1$ | 0.1998 | 0.3997 | 0.1998 | 0.1999 | 0.3995 | 0.3997 | 0.5995 |
| $2{\cdot}10^8{+}1$ | 0.1979 | 0.3960 | 0.1980 | 0.1979 | 0.3960 | 0.3959 | 0.5939 |

Table 3: Distribution of $a_0$-, $a_1$-, $a_1^+$-, $a_1^-$-, $j_2$-, $\overline{j_2}$-primes and Queneau prime numbers.

**Theorem 5.2.** [22] *Under the assumption of ACPR, both $P(j_2)$ and $P(\overline{j_2})$ are infinite sets, and $\Lambda(j_2) = \Lambda(\overline{j_2}) = \mathbf{A}$.* □

Since $P(q) = P(j_2) \cup P(\overline{j_2})$ and $P(Q) = \{(n-1)/2 \mid n \in P(q)\}$, we obtain

**Corollary 5.3.** *Under the assumption of ACPR, the set $P(q)$ of Queneau prime numbers is infinite. Consequently, under the assumption of ACPR, the set $P(Q)$ of Queneau numbers is infinite as well.* □

In [25] it is proved that ACPR follows from the Generalized Riemann Hypothesis (GRH); so in Theorem 5.2 and Corollary 5.3 “ACPR” may be substituted by “GRH”.

Relying on the Generalized Riemann Hypothesis has the advantage that we can apply some other results from number theory. Many densities in $\mathbb{P}$ can be established from contributions like [34, 30] assuming GRH. Without digressing to all the details (for which we refer to [22]), we restrict ourselves to the following:

**Theorem 5.4.** [22] *Under the assumption of GRH, we have $\Lambda(a_0) = \Lambda(a_1^+) = \Lambda(a_1^-) = \mathbf{A}/2$, $\Lambda(a_1) = \mathbf{A}$, and $\Lambda(q) = 3\mathbf{A}/2$.* □

Recalling that $P(A_0)$ and $P(A_1)$ are the sets of even and odd Queneau numbers, respectively, results in

**Corollary 5.5.** [22] *Under the assumption of GRH, we have that*

(1) *the set $P(q)$ of Queneau prime numbers has density $3\mathbf{A}/2$ in $\mathbb{P}$,*

(2) *the set $P(a_0)$ of Queneau prime numbers associated with the even Queneau numbers has density $\mathbf{A}/2$ in $\mathbb{P}$, and*

(3) *the set $P(a_1)$ of Queneau prime numbers associated with the odd Queneau numbers has density $\mathbf{A}$ in $\mathbb{P}$.* □

For the Queneau numbers we may infer, under the assumption of GRH that

$\Lambda(Q) = 2\Lambda(q) = 3\mathbf{A} = 1.121867440857606864164184163039249245 3\cdots,$

but from a number-theoretical point of view this is less relevant than the fact that $\Lambda(q) = 3\mathbf{A}/2$, because $\Lambda(Q)$ cannot be interpreted as a density in $\mathbb{P}$. The same remark applies to the even Queneau numbers, for which we have —again assuming GRH—

$\Lambda(A_0) = 2\Lambda(a_0) = 2\mathbf{A}/2 = 0.373955813619202288054728054346416415 1\cdots,$

and to the odd Queneau numbers with

$\Lambda(A_1) = 2\Lambda(a_1) = 2\mathbf{A} = 0.747911627238404576109456108692832830 2\cdots,$

which are less important than the respective equalities $\Lambda(a_0) = \mathbf{A}/2$ and $\Lambda(a_1) = \mathbf{A}$ from Theorem 5.4; cf. also Table 2.

In the study of Queneau numbers $+2$ and $-2$ play a principal part as primitive roots modulo $p$ (elements of $G_p$), particularly in characterizing $P(Q)$ and $P(q)$ as well as in determining $\Lambda(q)$. It is an interesting research problem whether we could obtain similar results for other small primitive roots ($g = \pm 3, -4, \pm 5, \pm 6, \pm 7, \ldots$). The zigzag permutations $Z_g$ of [22, 18] and the spiral permutations of [18] may be a promising starting point for such an investigation; cf. Section 7 of [22].

# 6 Bibliography — Publications Related to Queneau Numbers

The items in this bibliography have been ordered chronologically.

1. A. Tavera, Arnaut Daniel et la spirale, *Subsidia Pathaphysica* Troisième et nouvelle série (1963) No. 1, 73–78.

2. R. Queneau, Note complémentaire sur la sextine, *Subsidia Pathaphysica* Troisième et nouvelle série (1963) No. 1, 79–80.

3. R. Queneau, *Bâtons, chiffres et lettres* (1965), Gallimard, Paris.

4. J. Roubaud, Un problème combinatoire posé par la poésie lyrique des troubadours, *Math. Sci. Humaines/Math. Soc. Sci.* **27** (1969) 5–12.

5. M. Bringer, Sur un problème de R. Queneau, *Math. Sci. Humaines/Math. Soc. Sci.* **27** (1969) 13–20. • This is an important paper: it contains many interesting mathematical results and a partial characterization of the Queneau numbers.

6. C.W. Carroll & W.F. Orr, On the generalization of the sestina, *Delta (Waukesha)* **5** (1975) 32–44. • This hard to access and frequently ignored, important paper contains the first set of necessary and sufficient conditions on a natural number to be a Queneau number. Unfortunately, the authors were clearly unaware of [5] and a bit sloppy in the treatment of their minus signs.

7. J. Roubaud, $N$-ines, autrement dit quenines, *La bibliothèque Oulipienne* no. 5/65 (1993).

8. J. Roubaud, $N$-ines, autrement dit quenines (encore), *La bibliothèque Oulipienne* no. 5/66 (1993).

9. J. Roubaud, Réflexions historiques et combinatoires sur la $n$-ine autrement dit quenine, *La bibliothèque Oulipienne* 5/66 (2000) 99–124 [Contribution à la réunion 395 de l'Oulipo, le 17 septembre 1993]. • This contains/extends both [7] and [8]. It collects all that is known of Queneau numbers at that time —except the complete characterization of [6]— and provides an erroneous characterization of $P(Q)$ which has been corrected in [13]; consequently it wrongly stated that 141 is a Queneau number; cf. [10].

10. G. Esposito-Farèse, Nombres de Queneau, `http://www.gef.free.fr/oulipo7.html` or `http://www.iap.fr/users/esposito/oulipo7.html` (2000). • It is shown that J. Roubaud's statement that $141 \in P(Q)$ in [9] is incorrect.

11. J. Roubaud, Battement de Monge, *La bibliothèque Oulipienne* **158** (2006).

12. M. Audin, Mathématiques et littérature — un article avec des mathématiques et de la littérature, *Math. Sci. Humaines/Math. Soc. Sci.* **178** (2007) 63-86.

13. J.-G. Dumas, Caractérisation des quenines et leur représentation spirale, *Math. Sci. Humaines/Math. Soc. Sci.* **184** (2008) 9–23. • Although this paper does not refer to [6], it is devoted to refined characterizations of the Queneau numbers, thereby extending the main results of [5] and [6]. These characterizations may also be considered as a correction to the erroneous ones in [9].

14. M. Audin, Poésie, spirales, et battements de cartes, *Images des mathématiques* **18** (2009), `http://images.math.cnrs.fr/Poesie-spirales-et-battements-de.html`.

15. M. Audin, L'Oulipo et les mathématiques – une description (2009), `http://www-irma.u-strasbg.fr/~maudin/ExposeRennes.pdf`.

16. P.R.J. Asveld, Permuting operations on strings: their permutations and their primes (2009), TR-CTIT-09-26, Dept. of Comp. Sci., Twente University of Technology, Enschede, The Netherlands, `http://eprints.eemcs.utwente.nl/15655`. • This report is the predecessor of [21] and [22]; these latter two references contain all the material of this report apart from §§7.3–7.5. Conjectures 7.2(2)–(3) have been established by J.-G. Dumas in [18]. Consequently, in the meantime this report became rather obsolete.

17. P.R.J. Asveld, Some families of permutations and their primes (2009), TR-CTIT-09-27, Dept. of Comp. Sci., Twente University of Technology, Enschede, The Netherlands, `http://eprints.eemcs.utwente.nl/15678`. • This report consists of computer generated data: subsets of Queneau numbers and related integer sequences, the cycle structure of the relevant families of permutations, etc. It might be of interest to researchers looking for patterns at a lower level (e.g., the structure of families of "similar" permutations with a cycle structure different from a single cycle).

18. J.-G. Dumas, Les rayons des permutations spirales, *Math. Sci. Humaines/Math. Soc. Sci.* **192** (2010) 5–27. • Apart from interesting generalizations and variations of the principal spiral permutation this contribution contains a further refinement of the characterizations in [13].

19. V. Vallet, Entre mathématique et littérature: les nombres de Queneau, *L'Ouvert* **118** (2010) 19–37. • The mathematical part of this paper surveys the main results of [5] and [13], but it does not refer to [6].

20. M.P. Saclolo, How a medieval troubadour became a mathematical figure, *Notices Amer. Math. Soc.* **58** (2011) no. 5, 682–687, correction/addition *Notices Amer. Math. Soc.* **58** (2011) no. 7, 895. • A splendid introduction and overview of the subject including all material till 2010; the only omission is a reference to [6]. A must read for everyone interested in the history and the properties of Queneau numbers.

21. P.R.J. Asveld, Permuting operations on strings and their relation to prime numbers, *Discr. Appl. Math.* **159** (2011) 1915-1932, • This paper relates the Queneau numbers to shuffling a deck of cards, to the Josephus problem and to Archimedes' spirals. Emphasis is on characterizing (some subsets of) the set of Queneau numbers using the twist operation and related permuting operations on strings.

22. P.R.J. Asveld, Permuting operations on strings and the distribution of their prime numbers, *Discr. Appl. Math.* **161** (2013) 1868–1881 • Subsets of Queneau numbers and the sets of their associated prime numbers are studied in this report, resulting in densities. For a limited preview, see Section 5 above. The mean results rely either on Artin's conjecture (on primitive roots) or on the Generalized Riemann Hypothesis.

## Additional References

23. R.L. Graham, D.E. Knuth & O. Patashnik, *Concrete Mathematics* (1989), Addison-Wesley, Reading, MA.

24. G.H. Hardy & E.M. Wright, *An Introduction to the Theory of Numbers* (1938), Fourth edition (1959), Oxford University Press, Oxford, UK.

25. C. Hooley, On Artin's conjecture, *J. Reine Angew. Math.* **225** (1967) 209–220.

26. M. Jantzen, The power of synchronizing operations on strings, *Theoret. Comput. Sci.* **14** (1981) 127–154.

27. M. Jantzen, On twist-closed trios: a new morphic characterization of r.e. sets, *Found. of Comput. Sci.* 97, Lect. Notes in Comput. Sci. **1337** (1997) Springer, Berlin, pp. 143–152.

28. M. Jantzen, Hierarchies of principal twist-closed trios, *STACS* 98, Lect. Notes in Comput. Sci. **1373** (1998) Springer, Berlin, pp. 344–355.

29. M. Jantzen & A. Kurgansky, Refining the hierarchy of blind multicounter languages and twist-closed trios, *Inform. Comput.* **185** (2003) 158–181.

30. P. Moree, On primes in arithmetic progression having a prescribed primitive root II, *Funct. Approx. Comment. Math.* **39** (2008), part 1, 133–144.

31. M.R. Murty, Artin's conjecture for primitive roots, *Math. Intell.* **10** (1988) 59–67.

32. P. Schumer, The Josephus problem; once more around, *Math. Mag.* **75** (2002) 12–17.

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34. P. Stevenhagen. The correction factor in Artin's primitive root conjecture, *J. Théor. Nombres Bordeaux* **15** (2003) 383-391.

35. S.Y. Yan, *Number Theory for Computing* (2000), Springer-Verlag, Berlin – Heidelberg – New York.