# Necessary conditions for the invariant measure of a random walk to be a sum of geometric terms

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**Abstract**

We consider the invariant measure of homogeneous random walks in the quarter-plane. In particular, we consider measures that can be expressed as an infinite sum of geometric terms. We present necessary conditions for the invariant measure of a random walk to be a sum of geometric terms. We demonstrate that each geometric term must individually satisfy the balance equations in the interior of the state space. We show that the geometric terms in an invariant measure must have a pairwise-coupled structure. We further show that the random walk cannot have transitions to the North, Northeast or East. Finally, we show that for an infinite sum of geometric terms to be an invariant measure at least one coefficient must be negative. This paper extends our previous work for the case of finitely many terms to that of countably many terms.

**Keywords**: random walk; quarter-plane; invariant measure; geometric term; algebraic curve; pairwise-coupled; compensation method
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## 1 Introduction

Random walks for which the invariant measure is a geometric product form are often used to model practical systems. The benefit of using such models

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is that their performance can be easily analyzed with tractable closed-form expressions. However, the class of random walks which have a product form invariant measure is rather limited. Therefore, it is of interest to find larger classes of tractable measures that can be the invariant measures for random walks in the quarter-plane. In previous work we considered the measures that can be expressed as a linear combination of finitely many geometric terms [6]. In the current work we consider the case of countably many terms.

We study random walks in the quarter-plane that are homogeneous in the sense that transition probabilities are translation invariant. Our interest is in finite measures $m(i, j)$ that can be expressed as

$$m(i, j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho, \sigma)\rho^i\sigma^j, \quad \text{with} \quad |\Gamma| = \infty, \tag{1}$$

*i.e.*, $m(i, j)$ can be expressed as an infinite sum of geometric terms.

Contrary to much other work, for instance [7, 9, 11, 12], our interest is not in finding the invariant measure for specific random walks. Instead our interest is in characterizing the fundamental properties of random walks, sets $\Gamma$ and coefficients $\alpha$ that allow an invariant measure to be expressed in form (1). The contributions of the current work are a set of necessary conditions:

1. The elements of $\Gamma$ must individually satisfy the balance equations in the interior of the state space.
2. Consecutive elements of $\Gamma$ must have a common coordinate, *i.e.*, $\Gamma$ must have a structure which we refer to as pairwise-coupled.
3. In the interior of the state space, the random walk has no transitions to the North, Northeast or East.
4. At least one of the coefficients $\alpha(\rho, \sigma)$ in (1) must be negative.

The balance equations in the interior of the state space induce an algebraic curve. A closely related curve arises as the kernel of the boundary value problems studied in [7, 9] and related work. Some of its basic properties were derived in [9]. An important part of the current work consists of studying this algebraic curve in more detail. Section 3 presents new results on the properties of this curve.

Adan et al. [1, 2, 3, 4] use a compensation approach to construct an invariant measure that is an infinite sum of geometric terms. In particular, this is done for the random walk that have no transitions to the North, Northeast and East. In addition the geometric terms in the measure will individually satisfy the interior balance equations and negative weights will occur in the sum. In that sense, the results from [3] complement the necessary conditions that are obtained in the current paper.

It is interesting to point out that the measures that are obtained in [3] can consist of multiple pairwise coupled sets and that these measures, therefore, do not satisfy our condition 2 mentioned above, which states that only a single pairwise coupled set can occur. There is no contradiction between these results, since the measure that is constructed in [3] is not finite. In fact, the only finite measures that have been constructed using the compensation approach consist of a single pairwise coupled set. Note, in addition that the random walks studied in [1, 2, 3, 4] do not satisfy our homogeneity conditions. We discuss these conditions and the relation to the compensation approach in more detail in Section 5.

For the reflected Brownian motion in a wedge, the invariant measure that is a sum of finitely many exponentials was studied by Dieker et al. [8]. They show that for the invariant measure to be a linear combination of finitely many exponential measures, there must be an odd number of terms that have a pairwise-coupled structure. The method for the continuous state space Brownian motion, however, cannot be used for the discrete state space random walk. In [6] we investigated the invariant measure that is a sum of finitely many geometrics for the random walk in the quarter-plane. The necessary conditions derived in the current paper form the natural extension of our results from [6] to the case of countably many terms.

The remainder of this paper is structured as follows. In Section 2 we present the model. An important algebraic curve has been investigated in Section 3. The necessary conditions for the invariant measure of a random walk to be a sum of geometric terms are given in Section 4. The conclusion is drawn in Section 5.

## 2 Model

Consider a two-dimensional random walk $P$ on the pairs $S = \{(i,j), i,j \in \mathbb{N}_0\}$ of non-negative integers. We refer to $\{(i,j)|i > 0, j > 0\}$, $\{(i,j)|i > 0, j = 0\}$, $\{(i,j)|i = 0, j > 0\}$ and $(0,0)$ as the interior, the horizontal axis, the vertical axis and the origin of the state space, respectively. The transition probability from state $(i,j)$ to state $(i+s, j+t)$ is denoted by $p_{s,t}(i,j)$. Transitions are restricted to the adjoining points (horizontally, vertically and diagonally), *i.e.*, $p_{s,t}(k,l) = 0$ if $|s| > 1$ or $|t| > 1$. The process is homogeneous in the sense that for each pair $(i,j)$, $(k,l)$ in the interior (respectively on the horizontal axis and on the vertical axis) of the state space

$$p_{s,t}(i,j) = p_{s,t}(k,l) \quad \text{and} \quad p_{s,t}(i-s, j-t) = p_{s,t}(k-s, l-t), \tag{2}$$

for all $-1 \leq s \leq 1$ and $-1 \leq t \leq 1$. We introduce, for $i > 0$, $j > 0$, the notation $p_{s,t}(i,j) = p_{s,t}$, $p_{s,0}(i,0) = h_s$ and $p_{0,t}(0,j) = v_t$. Note that the first equality of (2) implies that the transition probabilities for each part of

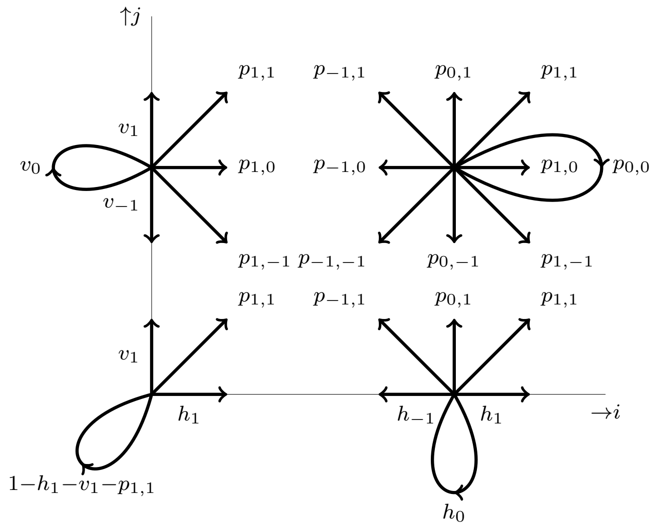



Figure 1: Random walk in the quarter-plane.

the state space are translation invariant. The second equality ensures that also the transition probabilities entering the same part of the state space are translation invariant. The above definitions imply that $p_{1,0}(0,0) = h_1$ and $p_{0,1}(0,0) = v_1$. The model and notations are illustrated in Figure 1.

We assume that the random walk is ergodic,*i.e.*, irreducible, aperiodic and positive recurrent, and has invariant measure $m$, *i.e.*, for $i, j > 0$,

$$m(i,j) = \sum_{s=-1}^{1}\sum_{t=-1}^{1} m(i-s, j-t)p_{s,t}, \tag{3}$$

$$m(i,0) = \sum_{s=-1}^{1} m(i-s,1)p_{s,-1} + \sum_{s=-1}^{1} m(i-s,0)p_{s,0},$$

$$m(0,j) = \sum_{t=-1}^{1} m(1,j-t)p_{-1,t} + \sum_{t=-1}^{1} m(0,j-t)p_{0,t}.$$

The balance at the origin is implied by the balance equation for all other states. We will refer to the above equations as the balance equations in the interior, the horizontal axis and the vertical axis of the state space respectively.

We are interested in the measures that are the linear combinations of positive geometric terms.

**Definition 1** (Induced measure)**.** *The measure $m$ is called induced by* $\Gamma \subset$

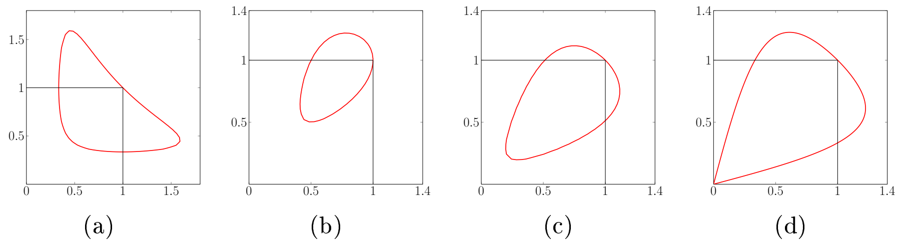



Figure 2: Examples of $Q(x, y) = 0$. (a) $p_{1,0} = p_{0,1} = \frac{1}{5}$, $p_{-1,-1} = \frac{3}{5}$. (b) $p_{1,0} = \frac{1}{5}$, $p_{0,-1} = p_{-1,1} = \frac{2}{5}$. (c) $p_{1,1} = \frac{1}{62}, p_{-1,1} = p_{1,-1} = \frac{10}{31}, p_{-1,-1} = \frac{21}{62}$. (d) $p_{-1,1} = p_{1,-1} = \frac{1}{4}, p_{-1,-1} = \frac{1}{2}$.

$\mathbb{R}^2_+$ *if*

$$m(i,j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\rho^i\sigma^j,$$

*with* $\alpha(\rho,\sigma) \in \mathbb{R}\backslash\{0\}$ *for all* $(\rho,\sigma) \in \Gamma$.

We are interested in finite measures, in which the sum does not depend on the ordering of the terms. Therefore, we assume absolute convergence,

$$\sum_{(\rho,\sigma)\in\Gamma} |\alpha(\rho,\sigma)| \frac{1}{1-\rho}\frac{1}{1-\sigma} < \infty. \tag{4}$$

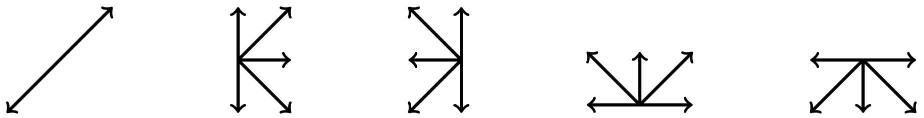
To identify the geometric measures that individually satisfy the balance equations in the interior of the state space, (3), we introduce the polynomial

$$Q(x,y) = xy\left(\sum_{s=-1}^{1}\sum_{t=-1}^{1} x^{-s}y^{-t}p_{s,t} - 1\right),$$

to capture the notion of balance, *i.e.*, $Q(\rho,\sigma) = 0$ implies that $m(i,j) = \rho^i\sigma^j, i,j \in S$ satisfies (3). Several examples of the level set $Q(x,y) = 0$ are displayed in Figure 2. Let $Q$ denote the set of real $(x,y)$ satisfying $Q(x,y) = 0$, *i.e.*,

$$Q = \Big\{(x,y) \in \mathbb{R}^2 \mid Q(x,y) = 0\Big\}. \tag{5}$$

We are only interested in the properties of the algebraic curve $Q(x,y) = 0$ in $\mathbb{R}^2_+$, where $\mathbb{R}^2_+ = \{(x,y)|x \geq 0, y \geq 0\}$. We denote the subset of $Q(x,y) = 0$ in $\mathbb{R}^2_+$ by $Q^+$. Moreover, we denote the boundary of $\mathbb{R}^2_+$ by $R_B$ where $R_B = \{(x,y) \in \mathbb{R}^2_+|xy = 0\}$. Due to the requirement of a finite measure, we will be interested in $U = \{(x,y)|(x,y) \in (0,1)^2\}$. In addition to $U$, we introduce $\bar{U} = [0,1)^2$. The properties of the algebraic curve $Q(x,y) = 0$ will be studied first in Section 3.



(a) (b) (c) (d) (e)

Figure 3: Singular random walks: non-zero transitions in the interior of the state space.

## 3 Algebraic curve $Q$ in $\mathbb{R}^2$

In this section we will analyze the algebraic curve $Q$ in $\mathbb{R}^2$. Fayolle et al. [9] have extensively studied the algebraic curve that is defined through $xy(\sum_{s=-1}^{1}\sum_{t=-1}^{1} x^s y^t p_{s,t} - 1) = 0$ and arises from studying the generating function of the invariant measure. We study the polynomial $Q(x, y)$ that arises naturally from the balance equations. It is clear that the properties that are derived in [9] can be related to the properties of $Q(x, y)$ by considering $\tilde{p}_{i,j} = p_{-i,-j}$. In this section we will present some of the results from [9] that will be useful in the sequel as well as a number of new results. The results from [9] are mostly algebraic of nature. The new results that we present deal with the geometry of $Q$.

First, following [9] we define the notion of a singular random walk.

**Definition 2** (Singular random walk)**.** *Random walk $P$ is called singular if the associated polynomial $Q(x, y)$ is either reducible or of degree 1 in at least one of the variables.*

In the remainder we will often distinguish between singular and non-singular random walks. In addition we will analyze the singularities of curve $Q$, which is defined in the usual way below. Note that the two notions are not related; it is, for instance, possible to have a non-singular random walk for which $Q$ has a singularity.

**Definition 3** (Singularity of $Q$)**.** *Point $(x, y) \in Q$ is a singularity of multiplicity $m$, $m > 1$, iff at $(x, y)$ all partial derivatives of $Q(x, y)$ of order less than $m$ vanish and at least one partial derivative of order $m$ is non-zero.*

The following result characterizes singular random walks in terms of their transition probabilities.

**Lemma 1** (Lemma 2.3.2 [9])**.** *Random walk $P$ is* singular *if, and only if, one of the following conditions holds:*

1. *There exists $(i, j) \in \mathbb{Z}^2$, $|i| \leq 1$, $|j| \leq 1$, such that only $p_{i,j}$ and $p_{-i,-j}$ are different from 0 (see Figure 3(a) and the other three cases obtained by rotation);*

2. *There exists $i$, $|i| = 1$, such that for any $j$, $|j| \leq 1$, $p_{i,j} = 0$ (see Figure 3(b) and Figure 3(c));*

3. *There exists $j$, $|j| = 1$, such that for any $i$, $|i| \leq 1$, $p_{i,j} = 0$ (see Figure 3(d) and Figure 3(e)).*

The algebraic results that we use from [9] are defined in terms of the branch points of the multi-valued algebraic functions $X(y)$ and $Y(x)$ which are defined through

$$Q(X(y), y) = Q(x, Y(x)) = 0,$$

for $x, y \in \mathbb{C}$. First, observe that by reordering the terms in $Q(x, y) = 0$ we get

$$(\sum_{s=-1}^{1} y^{-s+1} p_{-1,s})x^2 + (\sum_{s=-1}^{1} y^{-s+1} p_{0,s} - y)x + (\sum_{s=-1}^{1} y^{-s+1} p_{1,s}) = 0. \quad (6)$$

Therefore, the branch points of $X(y)$ are the roots of $\Delta_x(y) = 0$ where $\Delta_x(y)$ is defined as

$$\Delta_x(y) = (\sum_{s=-1}^{1} y^{-s+1} p_{0,s} - y)^2 - 4(\sum_{s=-1}^{1} y^{-s+1} p_{-1,s})(\sum_{s=-1}^{1} y^{-s+1} p_{1,s}). \quad (7)$$

In similar fashion, by rewriting $Q(x, y) = 0$ into

$$(\sum_{t=-1}^{1} x^{-t+1} p_{t,-1})y^2 + (\sum_{t=-1}^{1} x^{-t+1} p_{t,0} - x)y + (\sum_{t=-1}^{1} x^{-t+1} p_{t,1}) = 0, \quad (8)$$

it follows that the branch points of $Y(x)$ are the roots of $\Delta_y(x) = 0$ where $\Delta_y(x)$ is defined as

$$\Delta_y(x) = (\sum_{t=-1}^{1} x^{-t+1} p_{t,0} - x)^2 - 4(\sum_{t=-1}^{1} x^{-t+1} p_{t,-1})(\sum_{t=-1}^{1} x^{-t+1} p_{t,1}). \quad (9)$$

Next, we present two lemmas that fully characterizes the location of the branch points of $Y(x)$ and $X(y)$ in terms of the transition probabilities of the random walk. These results provide us with the opportunity to connect the geometry of $Q$ with the interior transition probabilities. The first lemma presented below follows from Lemmas 2.3.8–2.3.10 of [9]. The result readily follows if one takes into account that in the current paper we consider only ergodic random walks, whereas [9] also allows for non-ergodic random walks.

**Lemma 2** (Lemma 2.3.8–2.3.10 [9])**.** *For all non-singular random walks such that $M_y \neq 0$, $Y(x)$ has four real branch points. Moreover, $Y(x)$ has two branch points $x_1$ and $x_2$ (resp. $x_3$ and $x_4$) inside (resp. outside) the unit circle.*

*For the pair* $(x_3, x_4)$*, the following classification holds:*

1. *if* $p_{-1,0} > 2\sqrt{p_{-1,-1}p_{-1,1}}$*, then* $x_3$ *and* $x_4$ *are positive;*
2. *if* $p_{-1,0} = 2\sqrt{p_{-1,-1}p_{-1,1}}$*, then one point is infinite and the other is positive, possibly infinite;*
3. *if* $p_{-1,0} < 2\sqrt{p_{-1,-1}p_{-1,1}}$*, then one point is positive and the other is negative.*

*Similarly, for the pair* $(x_1, x_2)$*,*

1. *if* $p_{1,0} > 2\sqrt{p_{1,-1}p_{1,1}}$*, then* $x_1$ *and* $x_2$ *are positive;*
2. *if* $p_{1,0} = 2\sqrt{p_{1,-1}p_{1,1}}$*, then one point is* $0$ *and the second is non-negative;*
3. *if* $p_{1,0} < 2\sqrt{p_{1,-1}p_{1,1}}$*, then one point is positive and the other is negative.*

*For all non-singular random walks for which* $M_y = 0$*, one of the branch points of* $Y(x)$ *is equal to* $1$*. In addition,*

1. *if* $M_x < 0$*, then two other branch points have a modulus bigger than* $1$ *and the remaining one has a modulus less than* $1$*;*
2. *if* $M_x > 0$*, then two branch points are less than* $1$ *and the modulus of the remaining one is bigger than* $1$*.*

*Furthermore, the positivity conditions are the same as the case when* $M_y \neq 0$*. This lemma is true also for* $X(y)$*, up to a proper symmetric change of the parameters.*

The next lemma deals with multiplicity of the branch points.

**Lemma 3** (Lemma 2.3.10 [9]). *The branch points of* $X(y)$ *and* $Y(x)$ *with multiplicity* $2$ *occur only at* $0$*,* $1$ *and* $\infty$*.*

The final result of [9] that we need characterizes ergodicity of a random walk in terms of the drift in the interior and along the boundaries of the state space. Let us first define the drift of the random walk $P$ as $M, M'$ and $M''$.

$$M = (M_x, M_y) = (\sum_{t=-1}^{1} p_{1,t} - \sum_{t=-1}^{1} p_{-1,t}, \sum_{s=-1}^{1} p_{s,1} - \sum_{s=-1}^{1} p_{s,-1}),$$

$$M' = (M'_x, M'_y) = (h_1 + p_{1,1} - h_{-1} - p_{-1,1}, \sum_{s=-1}^{1} p_{s,1}),$$

$$M'' = (M''_x, M''_y) = (\sum_{t=-1}^{1} p_{1,t}, v_1 + p_{1,1} - v_{-1} - p_{1,-1}).$$

**Theorem 1** (Theorem 1.2.1 [9]). *Consider an irreducible aperiodic random walk $P$, when $M \neq 0$, the random walk is ergodic if and only if one of the following three conditions holds:*

1. $$\begin{cases} M_x < 0, M_y < 0, \\ M_x M_y' - M_y M_x' < 0, \\ M_y M_x'' - M_x M_y'' < 0; \end{cases}$$

2. $M_x < 0, M_y \geq 0, M_y M_x'' - M_x M_y'' < 0;$

3. $M_x \geq 0, M_y < 0, M_x M_y' - M_y M_x' < 0.$

It will be shown that when $M = 0$, which means the drift of the random walk is zero, the algebraic curve $Q^+$ reduces to a single point $(1, 1)$. Therefore, it is impossible to have sum of geometric terms invariant measure for the random walks with zero drift. Hence, the random walk with zero drift is of minor importance in our paper. The remainder of this section is related to new results on the geometry of $Q$. First, we investigate the possible intersection of $Q$ and $R_B$ where $R_B$ is the boundary of the first quadrant.

**Lemma 4.** *Consider a non-singular random walk $P$. If $(x, y) \in Q \cap \mathbb{R}_+^2$ then either $x > 0$ and $y > 0$ or $x = y = 0$, i.e., $Q$ cannot cross $R_B$ except in the origin.*

*Proof.* If $(x, y)$ is the intersection of $Q$ and $x = 0$, then $y$ must be the root of the following quadratic equation,

$$p_{1,-1} y^2 + p_{1,0} y + p_{1,1} = 0. \tag{10}$$

We now show that the roots of (10) are non-positive by considering all possible choices of $p_{1,-1}, p_{1,0}$ and $p_{1,1}$. If $p_{1,-1} \neq 0$, then (10) has either no root or two non-positive roots by investigating the relations of the roots using Vieta's formulas. If $p_{1,-1} = 0$ and $p_{1,0} \neq 0$, then (10) has one non-positive root. If $p_{1,-1} = p_{1,0} = 0$ and $p_{1,1} \neq 0$, then (10) has no root. The random walk with $p_{1,-1} = p_{1,0} = p_{1,1} = 0$ is excluded because it is singular. In similar fashion it follows that $Q(x, y) = 0$ can only intersect $y = 0$ when $x \leq 0$. Therefore, the only possible intersection of $Q$ and $R_B$ is the origin. □

Now we characterize the number of connected components in the first quadrant.

**Lemma 5.** *Consider a non-singular random walk $P$ with non-zero drift. The connected components are closed. The algebraic curve $Q$ has one and only one connected component in $\mathbb{R}_+^2$. Moreover, this connected component has non-empty intersection with the unit square $U$.*

*Proof of Lemma 5.* First observe that because of the continuity of functions $\Delta_x(y)$ and $\Delta_y(x)$ defined in (7) and (9), respectively, the connected components are closed. This means, for each line in $\mathbb{R}^2$ that the multiplicity of the intersection of this line and a connected component is either two or zero. It can readily be verified that $(1,1)$, $(1, \frac{\sum_{s=-1}^{1} p_{s,1}}{\sum_{s=-1}^{1} p_{s,-1}})$ and $(\frac{\sum_{t=-1}^{1} p_{1,t}}{\sum_{t=-1}^{1} p_{-1,t}}, 1)$ are on $Q$. Therefore, there is at least one connected component in the first quadrant.

The ergodic conditions for the random walk with non-zero drift from Theorem 1 imply that one of the following requirements must be satisfied,

$$0 < \frac{\sum_{s=-1}^{1} p_{s,1}}{\sum_{s=-1}^{1} p_{s,-1}} < 1, \quad 0 < \frac{\sum_{t=-1}^{1} p_{1,t}}{\sum_{t=-1}^{1} p_{-1,t}} < 1. \tag{11}$$

Without loss of generality, let us assume $0 < \frac{\sum_{s=-1}^{1} p_{s,1}}{\sum_{s=-1}^{1} p_{s,-1}} < 1$. If $(1,1)$ and $(1, \frac{\sum_{s=-1}^{1} p_{s,1}}{\sum_{s=-1}^{1} p_{s,-1}})$ are on different connected components, then these two components must be completely contained in $x \leq 1$ and $x \geq 1$ respectively, due to the closedness of the component and degree of $Q$. However, in that case, two of the branch points of $Y(x)$ are equal to 1 which contradicts Lemma 2. Therefore, $(1,1)$ and $(1, \frac{\sum_{s=-1}^{1} p_{s,1}}{\sum_{s=-1}^{1} p_{s,-1}})$ are on the same connected component.

We have $\Delta_y(1) \neq 0$, due to the continuity of $\Delta_y(x)$, the connected component containing $(1,1)$ must extend to both the inside and outside of the unit square $U$. If $Q$ contains only one connected component, this connected component must be in $\mathbb{R}_+^2$ because $Q$ can only intersect $R_B$ at the origin.

It still remains to show that $Q$ contains a single connected component in $\mathbb{R}_+^2$. If $Q$ would contain multiple connected components in $\mathbb{R}_+^2$, by using the property that the multiplicity of the intersections of any horizontal or vertical line and $Q(x,y) = 0$ is at most two, we can conclude there are three branch points either inside or outside the unit square, which contradicts Lemma 2. Therefore, there is one and only one connected component of $Q$ in the first quadrant and it has non-empty intersection with $U$. □

*Remark*: When the drift of the random walk is zero, the unique connected component in the first quadrant reduces to point $(1,1)$, see Section 6.5 [9].

Now we know that $Q^+$, the intersection of $Q(x,y) = 0$ and $\mathbb{R}_+^2$, is a connected component. Denote the branch points of $Y(x)$ and $X(y)$ on $Q^+$ by $x_l, x_r$ with $x_l < x_r$ and $y_b, y_t$ with $y_b < y_t$ respectively. Let $y_l, y_r, x_b, x_t$ satisfy$(x_l, y_l), (x_r, y_r), (x_t, y_t), (x_b, y_b) \in Q$, see Figure 4(a). We will refer to $(x_l, y_l), (x_r, y_r), (x_t, y_t), (x_b, y_b)$ as branch points of $Q^+$. From Lemma 2, we know that $0 \leq x_l \leq 1 \leq x_r$, $0 \leq y_b \leq 1 \leq y_t$. Since we are only interested in finite measures, we only consider $Q^+$ in $\bar{U}$. Recall from Section 2 that

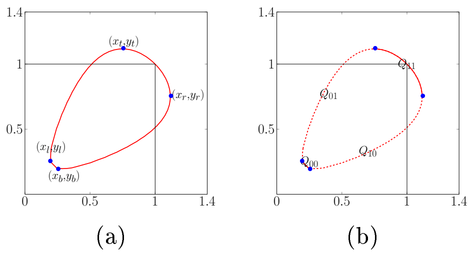


Figure 4: $Q^+$ for the random walk from Figure 2(c). (a) Branch points of $Q^+$. (b) Partition of $Q^+$.

$\bar{U} = [0,1)^2$. Lemma 5 states that $Q_U^+ = Q^+ \cap U$ is a non-empty set for an ergodic random walk with non-zero drift. We first start the analysis of $Q^+$.

**Definition 4** (Partition of $Q^+$). *The partition $\{Q_{00}, Q_{01}, Q_{10}, Q_{11}\}$ of $Q^+$ is defined as follows: $Q_{00}$ is the part of $Q$ connecting $(x_l, y_l)$ and $(x_b, y_b)$; $Q_{10}$ is the part of $Q$ connecting $(x_b, y_b)$ and $(x_r, y_r)$; $Q_{01}$ is the part of $Q$ connecting $(x_l, y_l)$ and $(x_t, y_t)$; $Q_{11}$ is the part of $Q$ connecting $(x_r, y_r)$ and $(x_t, y_t)$.*

An example of the partition of $Q^+$ is illustrated in Figure 4(b). The monotonicity of $X(y)$ and $Y(x)$ plays an crucial role in analyzing the pairwise-couple set with infinite cardinality, therefore we analyze it here.

**Lemma 6.** *Consider $(x, y) \in Q_{i,1-i}$ and $(\tilde{x}, \tilde{y}) \in Q_{i,1-i}$ where $i = 0, 1$, if $\tilde{x} > x$, then $\tilde{y} > y$. Consider $(x, y) \in Q_{i,i}$ and $(\tilde{x}, \tilde{y}) \in Q_{i,i}$ where $i = 0, 1$, if $\tilde{x} > x$, then $\tilde{y} < y$.*

*Proof.* We only consider $Q_{01}$. The proofs for the other cases follow analogously. Assume $(x, y), (\tilde{x}, \tilde{y}) \in Q_{01}$ and $\tilde{x} > x$, if $\tilde{y} \leq y$, then there exists $(\rho, \sigma), (\tilde{\rho}, \tilde{\sigma}) \in Q_{01}$ such that $\tilde{\sigma} = \sigma$. However, there also exists another geometric term $(\hat{\rho}, \hat{\sigma}) \in Q_{10} \cup Q_{11}$ such that $\hat{\sigma} = \sigma$. This contradicts that the multiplicity of the intersection of any horizontal line and $Q$ is at most two, which completes the proof. □

Next, we turn our attention to the singularity of $Q$, as defined in Definition 3. We will see below that if one set from the partition of $Q^+$ is empty, then the curve $Q^+$ will have a singularity. The singularity plays an important role in the analysis later.

**Lemma 7.** *For all non-singular random walks with non-zero drift, $(x, y)$ is a singularity of $Q^+$ if and only if it is a crunode of order 2 and $x$ and $y$ are branch points of multiplicity 2 of $Y(x)$ and $X(y)$ respectively.*

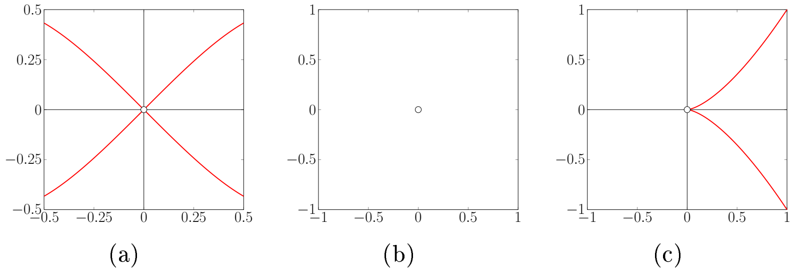



Figure 5: Types of real double point. (a) crunode. (b) acnode. (c) ordinary cusp.

*Proof.* We prove by contradiction that it is not possible to have singularity of order larger than 2. Suppose that $(\tilde{x}, \tilde{y}) \in Q^+$ is a singularity of order larger than 2. From Lemma 4 it follows that we need to consider the cases i) $\tilde{x} > 0$ and $\tilde{y} > 0$, and ii) $(\tilde{x}, \tilde{y}) = (0, 0)$. If $\tilde{x} > 0$ and $\tilde{y} > 0$ it follows from

$$\frac{\partial^2 Q(x,y)}{\partial^2 x} = \sum_{t=-1}^{1} p_{-1,t} y^{-t+1} = 0, \quad \text{and} \quad \frac{\partial^2 Q(x,y)}{\partial^2 y} = \sum_{s=-1}^{1} p_{s,-1} x^{-s+1} = 0,$$

that $p_{-1,1} = p_{-1,0} = p_{-1,-1} = p_{0,-1} = p_{1,-1} = 0$, which leads to a non-ergodic random walk. For $(\tilde{x}, \tilde{y}) = (0, 0)$ it follows from

$$\frac{\partial^2 Q(x,y)}{\partial x \partial y} = 4xyp_{-1,-1} + 2xp_{-1,0} + 2yp_{0,-1} + p_{0,0} - 1 = 0,$$

that $p_{00} = 1$, which leads to a random walk that is not irreducible. This concludes the proof that a singularity has at most order 2.

Next, we demonstrate that if $(x, y)$ is a singularity, then $x$ and $y$ are branch points of $Y(x)$ and $X(y)$ respectively. By combining $Q(x, y) = 0$ with

$$\frac{\partial Q(x,y)}{\partial x} = 2x\Big(\sum_{t=-1}^{1} p_{-1,t} y^{-t+1}\Big) + \Big(\sum_{t=-1}^{1} p_{0,t} y^{-t+1} - y\Big) = 0$$

we obtain

$$\sum_{t=-1}^{1} p_{-1,t} y^{-t+1} x^2 = \sum_{t=-1}^{1} p_{1,t} y^{-t+1},$$

which means $x$ is the root of $\Delta_y(x) = 0$, defined in (9) and therefore a branch point of $Y(x)$. Similarly, it follows from $Q(x, y) = 0$ and $\partial Q(x, y)/\partial y = 0$ that $y$ is a branch point of $X(y)$.

Now, we are ready to prove that a singularity $(x, y)$ is a crunode. We refer the reader to any source on algebraic curves for a classification on singularities, for instance [10]. An illustration of all possible singularities of order 2 is given in Figure 5. Note, that the figure does not include a ramphoid cusp, since it has order larger than 2. A singularity cannot be an ordinary cusp, because $x$ and $y$ are branch points of $Y(x)$ and $X(y)$ respectively. Moreover, $(x, y)$ is not an acnode because $Q_U^+$ is non-empty due to Lemma 5. Therefore, a singularity is a crunode.

The final result in this lemma follows from the observation that if $x$ and $y$ are branch points of $Y(x)$ and $X(y)$ respectively and $(x, y)$ is a crunode then $x$ and $y$ must have multiplicity two. □

**Lemma 8.** *The algebraic curve $Q$ has a singularity in $\bar{U}$ if and only if $p_{0,1} = p_{1,1} = p_{1,0} = 0$, in which case it is located in the origin.*

*Proof.* Lemma 7 states that $(x, y)$ is a singularity of $Q^+$ if and only if it is a crunode of order 2 and $x$ and $y$ are branch points of multiplicity 2 of $Y(x)$ and $X(y)$ respectively. Therefore, we only need to consider $(x, y)$ where $x$ and $y$ are the multiple roots of $\Delta_y(x) = 0$ and $\Delta_x(y) = 0$ respectively. A multiple root of $\Delta_y(x) = 0$ and $\Delta_x(y) = 0$ can only occur at $x = 0, 1$ or $\infty$ and $y = 0, 1$ or $\infty$, respectively, due to Lemma 3. Therefore, $x = 0$ and $y = 0$ must be multiple roots of $\Delta_y(x) = 0$ and $\Delta_x(y) = 0$, respectively, if there is a singularity in $\bar{U}$. We know from [9, Lemma 2.3.10] that $\Delta_y(x) = 0$ has the multiple root at 0 if and only if one of the following holds:

$$p_{-1,0} = p_{-1,1} = p_{0,1} = 0, \tag{12}$$

$$p_{1,0} = p_{1,1} = p_{0,1} = 0, \tag{13}$$

$$p_{-1,-1} = p_{0,-1} = p_{1,-1} = 0 \tag{14}$$

and $\Delta_x(y) = 0$ has the multiple root at 0 if and only if one of the following holds:

$$p_{0,-1} = p_{1,-1} = p_{1,0} = 0, \tag{15}$$

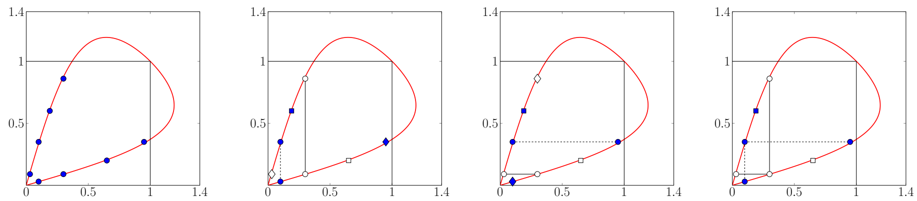
$$p_{0,1} = p_{1,1} = p_{1,0} = 0, \tag{16}$$

$$p_{-1,-1} = p_{-1,0} = p_{-1,1} = 0. \tag{17}$$

Conditions (14) and (17) are excluded because either of them will lead to a singular random walk. The combinations of conditions (12) and (15), (12) and (16), (13) and (15) will lead to singular random walks as well. Therefore, the algebraic curve $Q$ has a singularity in $\bar{U}$ if and only if $p_{0,1} = p_{1,1} = p_{1,0} = 0$, in which case it is located in the origin. □

# 4 Constraints to invariant measures and random walks

In this section, we will first characterize the structure of the candidate set of geometric terms which may lead to an invariant measure. Then we will



(a) (b) (c) (d)

Figure 6: Partitions of set $\Gamma$. (a) curve $Q^+$ of Figure 2(d) and $\Gamma \subset Q^+$ as dots. (b) horizontally uncoupled partition with 6 sets. (c) vertically uncoupled partition with 6 sets. (d) uncoupled partition with 4 sets. Different sets are marked by different symbols.

provide necessary conditions on the transition probabilities of a random walk to allow for an infinite sum of geometric terms to constitute the invariant measure. Finally, we demonstrate that it is necessary to have at least one negative coefficient in an invariant measure that is an infinite sum of geometric terms.

### 4.1 Uncoupled partitions

The proofs in this and subsequent sections are based on the notion of uncoupled partitions, which is introduced first.

**Definition 5** (Uncoupled partition)**.** *A partition* $\{\Gamma_1, \Gamma_2, \cdots\}$ *of* $\Gamma$ *is* horizontally uncoupled *if* $(\rho, \sigma) \in \Gamma_p$ *and* $(\tilde{\rho}, \tilde{\sigma}) \in \Gamma_q$ *for* $p \neq q$*, implies that* $\tilde{\rho} \neq \rho$*,* vertically uncoupled *if* $(\rho, \sigma) \in \Gamma_p$ *and* $(\tilde{\rho}, \tilde{\sigma}) \in \Gamma_q$ *for* $p \neq q$*, implies* $\tilde{\sigma} \neq \sigma$*, and* uncoupled *if it is both horizontally and vertically uncoupled.*

We call a partition with the largest number of sets a maximal partition.

**Lemma 9** (Lemma 1 [6])**.** *The maximal vertically uncoupled partition, the maximal horizontally uncoupled partition and the maximal uncoupled partition are unique.*

Examples of a maximal horizontally uncoupled partition, of a maximal vertically uncoupled partition and of a maximal uncoupled partition can be found in Figure 6. Let $H$ denote the number of elements in the maximal horizontally uncoupled partition and $\Gamma_p^h$, $p = 1, \dots, H$, the sets themselves, where elements of $\Gamma_p^h$ have common horizontal coordinate $\rho(\Gamma_p^h)$. The maximal vertically uncoupled partition has $V$ sets, $\Gamma_q^v$, $q = 1, \cdots, V$, where elements of $\Gamma_q^v$ have common vertical coordinate $\sigma(\Gamma_q^v)$. The maximal uncoupled partition is denoted by $\{\Gamma_k^u\}_{k=1}^U$. The $H, V, U$ are allowed to be infinite.

### 4.2 Constraints to set $\Gamma$

Before we analyze the structure of the candidate set of geometric terms which may lead to an invariant measure, we present some results from [5] in our notation for a special case.

**Theorem 2** (Theorem 1, Lemma 1 [5]). *Consider a real measure $\mu(\rho)$ with the bounded compact support $K$. If*

$$\int P(\rho)\, d\mu(\rho) = 0 \tag{18}$$

*for all polynomials $P$, then $\mu = 0$.*

We first show that only the sum of geometric terms chosen from $Q_U^+$ may lead to the invariant measure for a random walk.

**Theorem 3.** *If the invariant measure for a random walk in the quarter-plane is induced by $\Gamma \subset \mathbb{R}_+^2$, then $\Gamma \subset Q_U^+$.*

*Proof.* Since $m$ satisfies balance (3) in the interior of the state space, *i.e.*, $(i, j)$ with $i > 0$ and $j > 0$,

$$\sum_{(\rho,\sigma)\in\Gamma} \rho^i \sigma^j [\alpha(\rho,\sigma)(1 - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \rho^{-s}\sigma^{-t} p_{s,t})] = 0.$$

Let $j = 1$, we will get an infinite system of equations when $i = 1, 2, \cdots,$

$$\sum_{p=1}^{\infty} (\rho(\Gamma_p^h))^{i-1} [\sum_{(\rho,\sigma)\in\Gamma_p^h} \alpha(\rho,\sigma)(\rho\sigma - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \rho^{1-s}\sigma^{1-t} p_{s,t})] = 0. \tag{19}$$

Similarly, let $i = 1$, we will get another infinite system of equations when $j = 1, 2, \cdots,$

$$\sum_{q=1}^{\infty} (\sigma(\Gamma_q^v))^{j-1} [\sum_{(\rho,\sigma)\in\Gamma_q^v} \alpha(\rho,\sigma)(\rho\sigma - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \rho^{1-s}\sigma^{1-t} p_{s,t})] = 0.$$

The absolute convergence of $\{\alpha(\rho_k,\sigma_k)(\rho_k\sigma_k - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \rho_k^{1-s}\sigma_k^{1-t} p_{s,t})\}$, where $k = 1, 2, \cdots$ and $(\rho_k, \sigma_k) \in \Gamma$ will be shown here. Because of assumption (4), *i.e.*, the absolute convergence of the terms of which the sum is $m$, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} |\alpha(\rho_k,\sigma_k)(\rho_k\sigma_k - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \rho_k^{1-s}\sigma_k^{1-t} p_{s,t})| \\
<&B \sum_{k=1}^{\infty} |\alpha(\rho_k,\sigma_k)| \frac{1}{1-\rho_k}\frac{1}{1-\sigma_k} \\
<&\infty,
\end{aligned}$$

where $B$ is a finite positive constant. We define a real measure $\mu$ as

$$\mu(\rho) = \begin{cases} \sum_{(\rho,\sigma)\in\Gamma_p^h} \alpha(\rho,\sigma)(\rho\sigma - \sum_{s=-1}^{1}\sum_{t=-1}^{1}\rho^{1-s}\sigma^{1-t}p_{s,t}) \\ \quad \text{for} \quad \rho \in \{\rho(\Gamma_1^h), \rho(\Gamma_2^h), \cdots\}, \\ 0 \quad \text{otherwise.} \end{cases}$$

We can now write (19) as

$$\sum_{p=1}^{\infty}(\rho(\Gamma_p^h)^{i-1}\mu(\rho(\Gamma_p^h))) = \int (\rho(\Gamma_p^h))^{i-1}\, d\mu(\rho(\Gamma_p^h)) = 0,$$

for $i = 1, 2, 3\cdots$. This indicates that

$$\int P(\rho)\, d\mu(\rho) = 0$$

for all $P(\rho) = \rho^j$ where $j = 0, 1, 2\cdots$. Hence $\int P(\rho)\, d\mu(\rho) = 0$ for all polynomials. Moreover, for $p = 1, 2, \cdots$, the sequence $\sum_{(\rho,\sigma)\in\Gamma_p^h} \alpha(\rho,\sigma)(\rho\sigma - \sum_{s=-1}^{1}\sum_{t=-1}^{1}\rho^{1-s}\sigma^{1-t}p_{s,t})$ is absolutely convergent. Hence, the compact support of this sequence is a bounded interval. Therefore, by using Theorem 2, $\mu = 0$, *i.e.*, $\sum_{(\rho,\sigma)\in\Gamma_p^h} \alpha(\rho,\sigma)(\rho\sigma - \sum_{s=-1}^{1}\sum_{t=-1}^{1}\rho^{1-s}\sigma^{1-t}p_{s,t}) = 0$, for $p = 1, 2, \cdots$. In similar fashion, we obtain for $q = 1, 2, \cdots$, that $\sum_{(\rho,\sigma)\in\Gamma_q^v} \alpha(\rho,\sigma)(\rho\sigma - \sum_{s=-1}^{1}\sum_{t=-1}^{1}\rho^{1-s}\sigma^{1-t}p_{s,t}) = 0$, These two solutions guarantee that $\rho_k\sigma_k - \sum_{s=-1}^{1}\sum_{t=-1}^{1}\rho_k^{1-s}\sigma_k^{1-t}p_{s,t} = 0$, for $k = 1, 2, 3\cdots$, which is the interior balance equation for the measure $m = \rho_k^i\sigma_k^j$, hence $Q(\rho,\sigma) = 0$ for $(\rho,\sigma) \in \Gamma$. □

We now make two observations on the structure of $\Gamma \subset Q_U^+$ for which the maximal uncoupled partition consists of only one set. Firstly, for any $(\rho,\sigma) \in \Gamma$ there always exist either $(\rho,\tilde{\sigma}) \in \Gamma$, with $\tilde{\sigma} \neq \sigma$ or $(\tilde{\rho},\sigma) \in \Gamma$ with $\tilde{\rho} \neq \rho$. Secondly, the degree of $Q(\rho,\sigma)$ is at most two in each variable. This means, for instance, that if $(\rho,\sigma) \in \Gamma$ and $(\rho,\tilde{\sigma}) \in \Gamma$, $\tilde{\sigma} \neq \sigma$, then there does not exist $(\rho,\hat{\sigma}) \in \Gamma$, where $\hat{\sigma} \neq \sigma$ and $\hat{\sigma} \neq \tilde{\sigma}$. By repeating the above two arguments for other elements in $\Gamma$ it follows that $\Gamma$ must have a pairwise-coupled structure. An example of such a set is $\Gamma = \{(\rho_k,\sigma_k), k = 1, 2, 3\cdots\}$, where

$$\rho_1 = \rho_2, \sigma_1 > \sigma_2, \rho_2 > \rho_3, \sigma_2 = \sigma_3, \rho_3 = \rho_4, \sigma_3 > \sigma_4, \cdots. \tag{20}$$

The above discussion leads to the definition of a pairwise-coupled set in terms of the number of sets in a maximal uncoupled partition.

**Definition 6** (Pairwise-coupled set)**.** *A set $\Gamma \subset Q_U^+$ is pairwise-coupled if and only if the maximal uncoupled partition of $\Gamma$ contains only one set.*

We are now ready to show that if there are multiple sets in the maximal uncoupled partition of $\Gamma$, then the measure induced by this $\Gamma$ cannot be the invariant measure. We first introduce some additional notation. For any set $\Gamma_p^h$ from the maximal horizontally uncoupled partition of $\Gamma$, let

$$B^h(\Gamma_p^h) = \sum_{(\rho,\sigma)\in\Gamma_p^h} \alpha(\rho,\sigma)[\sum_{s=-1}^{1} \left(\rho^{1-s}h_s + \rho^{1-s}\sigma p_{s,-1}\right) - \rho].$$

For any set $\Gamma_q^v$ from the maximal vertically uncoupled partition of $\Gamma$, let

$$B^v(\Gamma_q^v) = \sum_{(\rho,\sigma)\in\Gamma_q^v} \alpha(\rho,\sigma)[\sum_{t=-1}^{1} \left(\sigma^{1-t}v_t + \rho\sigma^{1-t}p_{-1,t}\right) - \sigma].$$

**Theorem 4.** *Consider the random walk $P$, if the invariant measure $m$ of $P$ is $m(i,j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\rho^i\sigma^j$, then $\Gamma$ is pairwise-coupled and for all $p,q \in \{1,2,3,\cdots\}$, $B^h(\Gamma_p^h) = 0$ and $B^v(\Gamma_q^v) = 0$.*

*Proof.* Since $m$ is the invariant measure of $P$, $m$ satisfies the balance equations at state $(i,0)$ for $i = 1,2,3\cdots$. Therefore,

$$\begin{aligned} 0 &= \sum_{s=-1}^{1} \left[m(i-s,0)h_s + m(i-s,1)p_{s,-1}\right] - m(i,0) \\ &= \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)[\sum_{s=-1}^{1} \left(\rho^{i-s}h_s + \rho^{i-s}\sigma p_{s,-1}\right) - \rho^i] \\ &= \sum_{p=1}^{\infty} \rho(\Gamma_p^h)^i \sum_{(\rho,\sigma)\in\Gamma_p^h} \alpha(\rho,\sigma)[\sum_{s=-1}^{1} \left(\rho^{-s}h_s + \rho^{-s}\sigma p_{s,-1}\right) - 1] \\ &= \sum_{p=1}^{\infty} \rho(\Gamma_p^h)^{i-1} B^h(\Gamma_p^h). \end{aligned} \tag{21}$$

We now show the absolute convergence of sequence $\{B^h(\Gamma_p^h)\}$ for $p = 1,2,3\cdots$. Because of the assumption that $m(i,j)$ is a finite measure, we have

$$\begin{aligned} &\sum_{p=1}^{\infty} |B^h(\Gamma_p^h)| \\ \leq &\sum_{k=1}^{\infty} \alpha(\rho_k,\sigma_k)[\sum_{s=-1}^{1} \left(\rho_k^{1-s}h_s + \rho_k^{1-s}\sigma p_{s,-1}\right) - \rho_k] \\ < &\infty, \end{aligned}$$

the last inequality holds due to the same reasoning that is in the proof of Theorem 3. If we define a real measure $\mu$ as

$$\mu(\rho) = \begin{cases} B^h(\Gamma_p^h) & \text{if} \quad \rho = \rho(\Gamma_p^h), \\ 0 \quad \text{otherwise.} \end{cases}$$

We can now write (21) as

$$\sum_{p=1}^{\infty} (\rho(\Gamma_p^h)^{i-1} \mu(\rho(\Gamma_p^h))) = \int (\rho(\Gamma_p^h))^{i-1} \, d\mu(\rho(\Gamma_p^h)) = 0,$$

for $i = 1, 2, 3 \cdots$. This indicates that

$$\int P(\rho) \, d\mu(\rho) = 0$$

for all $P(\rho) = \rho^j$ where $j = 0, 1, 2 \cdots$. Hence $\int P(\rho) \, d\mu(\rho) = 0$ for all polynomials. Moreover, the sequence $\{B^h(\Gamma_p^h)\}$ for $p = 1, 2, \cdots$ is absolute convergent. Therefore the compact support of this sequence is a bounded interval. Hence, by using Theorem 2 we have $\mu = 0$, which means $B^h(\Gamma_p^h) = 0$ for $p = 1, 2, \cdots$. Similarly, we can obtain that $B^v(\Gamma_q^v) = 0$ for $q = 1, 2, \cdots$. These two results imply that $\Gamma$ is a pairwise-coupled set. □

### 4.3 Constraint to random walks

In this section, we characterize the random walks of which the invariant measure may be an infinite sum of geometric terms. We will show that the existence of transitions to north, northeast or east plays an essential role in distinguishing such kind of random walks.

The next result explains that it is not possible to have an invariant measure that is an infinite sum of geometric terms for singular random walks.

**Theorem 5.** *If the invariant measure of a singular random walk is of the form* $m(i, j) = \sum_{(\rho,\sigma) \in \Gamma} \rho^i \sigma^j$*, then* $|\Gamma| = 1$.

*Proof.* If the invariant measure of a singular random walk is of the form $m(i, j) = \sum_{(\rho,\sigma) \in \Gamma} \rho^i \sigma^j$, then $|\Gamma| \leq 2$ by using Theorem 4 and Lemma 1. Furthermore, we have shown in [6] that the linear combination of two geometric terms cannot be the invariant measure for any random walk, which completes the proof. □

We now focus on non-singular random walks. The next theorem applies Lemma 8 to non-singular random walks.

**Theorem 6.** *Let the invariant measure of the non-singular random walk* $P$ *be* $m(i, j) = \sum_{(\rho,\sigma) \in \Gamma} \alpha(\rho, \sigma) \rho^i \sigma^j$ *with* $|\Gamma| = \infty$*. Then* $p_{1,0} = p_{1,1} = p_{0,1} = 0$ *and* $\Gamma$ *has a unique accumulation point in the origin.*

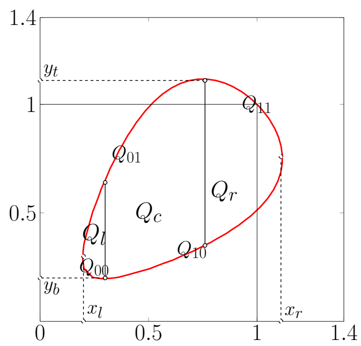



Figure 7: Different partition of $Q^+$ for the random walk in Figure 2(c).

*Proof.* We will demonstrate that in the absence of singularities it is not possible to have an invariant measure $m(i,j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\rho^i\sigma^j$ with $|\Gamma| = \infty$. More precisely, we show that $\Gamma$ must have a singularity of $Q$ as an accumulation point. The result of the theorem then follows, by Lemma 8, that states the algebraic curve $Q$ has a singularity in $\bar{U}$ only if $p_{1,0} = p_{1,1} = p_{0,1} = 0$, in which case it is in the origin.

Suppose that $Q^+$ does not contain any singularities and that $m(i,j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\rho^i\sigma^j$ with $|\Gamma| = \infty$ is the invariant measure of $P$. In the remainder of this proof we will obtain a contradiction by showing that at least one of the terms $(\rho,\sigma) \in \Gamma$ will be outside the unit square and that the measure $m(i,j)$ can, therefore, not be finite.

Assume $x_b \leq x_t$, another possible case can be analyzed similarly. To simplify the presentation we introduce additional notation. Let $\{Q_l, Q_c, Q_r\}$ denote a partition of $Q^+$, where

$$\begin{aligned}
Q_l &= \left\{(x,y) \in Q^+ \;\middle|\; x \leq x_b\right\},\\
Q_c &= \left\{(x,y) \in Q^+ \;\middle|\; x_b < x \leq x_t\right\},\\
Q_r &= \left\{(x,y) \in Q^+ \;\middle|\; x > x_t\right\}.
\end{aligned}$$

Moreover, denote the two pieces of $Q_c$ by $Q_c^t$ and $Q_c^b$ satisfying $\tilde{y} > y$ if $(x,\tilde{y}) \in Q_c^t$ and $(x,y) \in Q_c^b$. Since there are no singularities $Q_l, Q_c$ and $Q_r$ are all non-empty. In addition, we let $\{\Gamma_1,\ldots,\Gamma_K\}$ denote a partition of $\Gamma$, where the elements of $\Gamma_i$ are denoted as $\Gamma_i = \{(\rho_{i,1},\sigma_{i,1}),\ldots,(\rho_{i,L(i)},\sigma_{i,L(i)})\}$ and each $\Gamma_i$ satisfies

$$\begin{aligned}
\rho_{i,1} &> \rho_{i,2}, & \sigma_{i,1} &= \sigma_{i,2},\\
\rho_{i,2} &= \rho_{i,3}, & \sigma_{i,2} &> \sigma_{i,3},\\
\rho_{i,3} &> \rho_{i,4}, & \sigma_{i,3} &= \sigma_{i,4},\\
&\vdots & &\vdots\\
\rho_{i,L(i)-1} &> \rho_{i,L(i)}, & \sigma_{i,L(i)-1} &= \sigma_{i,L(i)},
\end{aligned} \tag{22}$$

In addition the partition $\{\Gamma_1, \ldots, \Gamma_K\}$ is maximal in the sense that no $\Gamma_i \cup \Gamma_j$, $i \neq j$ satisfies (22). In the remainder we will show that $K < \infty$ and that $L(i) < \infty$ for all $i = 1, \ldots, K$, leading to a contradiction to the assumption that $|\Gamma| = \infty$.

First, we prove $L(i) < \infty$ by demonstrating that

$$|\Gamma_i \cap Q_l| \leq 1, \quad |\Gamma_i \cap Q_c| < \infty, \quad |\Gamma_i \cap Q_r| \leq 1.$$

Suppose $|\Gamma_i \cap Q_r| \geq 2$, then there exists $(\rho, \sigma)$ and $(\tilde{\rho}, \tilde{\sigma})$ on $Q_{11}$ or $Q_{10}$ satisfying $\tilde{\sigma} = \sigma$, which contradicts to Lemma 6. Therefore, $|\Gamma_i \cap Q_r| \leq 1$. Similarly, we have $|\Gamma_i \cap Q_l| \leq 1$. Next, observe that since $Q^+$ does not have any singularities, there exists a constant $D > 0$ such that the distance of $Q_c^t$ and $Q_c^b$ is at least $D$. This implies that for three consecutive elements in $|\Gamma_i \cap Q_c|$,

$$\rho_{i,j} > \rho_{i,j+1}, \qquad \sigma_{i,j} = \sigma_{i,j+1},$$
$$\rho_{i,j+1} = \rho_{i,j+2}, \quad \sigma_{i,j+1} > \sigma_{i,j+2},$$

we have $\sigma_{i,j+2} \leq \sigma_{i,j} - D$. Therefore, $|\Gamma_i \cap Q_c| < \infty$.

Next, we prove $K < \infty$. More precisely, we will show that $K \leq 2$. Without loss of generality, we assume $K = 3$ and $|\Gamma_i| \geq 2$. Observe that $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ forms a pairwise-coupled set. Using from the above that $|\Gamma_i| < \infty$ for $i = 1, 2, 3$, we must have $\rho_{1,L(1)} = \rho_{2,L(2)}$ with $\rho_{1,L(1)}, \rho_{2,L(2)} \in Q_l$ and $\rho_{2,1} = \rho_{3,1}$ with $\rho_{2,1}, \rho_{3,1} \in Q_r$ after a proper ordering of $\{\Gamma_1, \Gamma_2, \Gamma_3\}$. Moreover, one of $\rho_{2,1}$ and $\rho_{3,1}$ must be on $Q_{11}$. However, together $y_t \geq 1$ and $x_r \geq 1$ due to Lemma 2 and the monotonicity of $Q_{11}$ from Lemma 6, we conclude that $Q_{11}$ is outside of $U$, which contradicts that $\Gamma \subset Q_U^+$ from Theorem 3. $\square$

### 4.4 Constraints to the coefficients

The last section was devoted to finding the constraint to the random walk in which the pairwise-coupled set with infinite cardinality could be obtained. In this section, we show that it is necessary to have a geometric term with negative coefficient in the linear combination of infinite geometric terms.

**Theorem 7.** *If the invariant measure of the non-singular random walk $P$ is $m(i,j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\rho^i\sigma^j$, where $\Gamma \subset Q_U^+$, $|\Gamma| = \infty$ and $\alpha(\rho,\sigma) \in \mathbb{R}\backslash\{0\}$, then at least one $\alpha(\rho,\sigma)$ is negative.*

*Proof.* We know from Theorem 6 that $p_{1,0} = p_{1,1} = p_{0,1} = 0$ in random walk $P$. Without loss of generality we consider $\Gamma$ which is of the form given in (20) and assume that $\alpha(\rho_1, \sigma_1) = 1$. Since measure induced by $\Gamma$ is the invariant measure, it follows from Theorem 4 that $B^h(\Gamma_p^h) = 0$ and $B^v(\Gamma_q^v) = 0$ for all $p, q \in \{1, 2, 3, \cdots\}$. Note that $B^h\{(\rho_1, \sigma_1), (\rho_2, \sigma_2)\} = 0$ indicates that $\alpha(\rho_2, \sigma_2)$ is uniquely determined by

$$\alpha(\rho_2, \sigma_2) = -\frac{T_1}{T_2}\alpha(\rho_1, \sigma_1),$$

where

$$T_i = (1 - \frac{1}{\rho_i})h_1 + (1 - \rho_i)h_{-1} + \sum_{s=-1}^{1} p_{s,1} - \sigma_i(\sum_{s=-1}^{1} \rho_i^{-s} p_{s,-1}).$$

Next, $\alpha(\rho_3, \sigma_3)$ follows from $B^v\{(\rho_2, \sigma_2), (\rho_3, \sigma_3)\} = 0$. In similar fashion, for $k \in \{1, 2, 3, \cdots\}$,

$$\alpha(\rho_{2k}, \sigma_{2k}) = -\frac{T_{2k-1}}{T_{2k}}\alpha(\rho_{2k-1}, \sigma_{2k-1}) \quad \text{where} \quad \rho_{2k} = \rho_{2k-1}, \sigma_{2k} < \sigma_{2k-1}.$$

The following two facts allow us to show that there exists a positive integer $N$ such that $\frac{T_{2k-1}}{T_{2k}} > 0$ when $k > N$. First, we know $p_{1,0} = p_{1,1} = p_{0,1} = 0$ and $\lim_{k\to\infty} \rho_k = 0$ from Theorem 6. Secondly, the ergodic random walk with no drift to northeast requires $h_1 + p_{1,-1} \neq 0$. Note

$$\frac{T_{2k-1}}{T_{2k}} = \frac{(1 - \frac{1}{\rho_{2k-1}})h_1 + (1 - \rho_{2k-1})h_{-1} + p_{-1,1} - \sigma_{2k-1}(\sum_{s=-1}^{1} \rho_{2k-1}^{-s} p_{s,-1})}{(1 - \frac{1}{\rho_{2k}})h_1 + (1 - \rho_{2k})h_{-1} + p_{-1,1} - \sigma_{2k}(\sum_{s=-1}^{1} \rho_{2k}^{-s} p_{s,-1})},$$

By using L'Hospital's rule, we can conclude

$$\lim_{k\to\infty} \frac{T_{2k-1}}{T_{2k}} = \begin{cases} \frac{h_1 + \sigma_{2k-1} p_{1,-1}}{h_1 + \sigma_{2k} p_{1,-1}}, & \text{if } h_1 \neq 0, \quad p_{1,-1} \neq 0; \\ 1, & \text{if } h_1 \neq 0, \quad p_{1,-1} = 0; \\ \frac{\sigma_{2k-1}}{\sigma_{2k}}, & \text{if } h_1 = 0, \quad p_{1,-1} \neq 0. \end{cases}$$

The non-negativity of $\frac{T_{2k-1}}{T_{2k}}$ when $k$ is large enough completes the proof. □

# 5 Conclusion

In this paper, we have obtained necessary conditions for the invariant measure of a random walk to be an infinite sum of geometric terms. Firstly, Theorem 3 says that each geometric term in the linear combination must satisfy the interior balance equation. Secondly, Theorem 4 indicates that only a measure induced by a pairwise-coupled set may yield an invariant measure. Thirdly, Theorem 6 shows that the invariant measure may be an infinite sum of geometric terms only if there are no transitions to the North, Northeast or East. Finally, Theorem 7 requires at least one of the coefficients in the linear combination to be negative.

As mentioned in the introduction, the random walks studied by Adan et al. for the compensation approach do not satisfy our homogeneity conditions. In particular, the second condition in (2) is not satisfied. We believe that our results also apply for this larger class of models. This would imply that the general structure of multiple pairwise coupled set that is provided by the compensation approach would reduce to a single pairwise coupled set if one insists on having a finite measure. The extension of the results from the current paper to the more general class of random walks is, unfortunately, outside the scope of the current work.

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