Weighted sums of orthogonal polynomials related to birth-death processes with killing

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Abstract. We consider sequences of orthogonal polynomials arising in the analysis of birth-death processes with killing. Motivated by problems in this stochastic setting we discuss criteria for convergence of certain weighted sums of the polynomials.

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1 Introduction

A *birth-death process with killing* is a continuous-time Markov chain $\mathcal{X} := \{X(t),\ t \geq 0\}$ taking values in $\{0, 1, 2, \ldots\}$, where 0 is an absorbing state, and with transition rates $q_{ij},\ j \neq i$, satisfying

$$\begin{aligned} &q_{i,i+1} = \lambda_i, \quad q_{i+1,i} = \mu_{i+1}, \quad q_{i0} = \nu_i, \quad i \geq 1, \\ &q_{ij} = 0, \quad i = 0 \ \text{ or } \ |i - j| > 1, \end{aligned} \tag{1}$$

where $\lambda_i > 0$, $\mu_{i+1} > 0$ and $\nu_i \geq 0$ for $i \geq 1$. It will be convenient to let $\lambda_0 = \mu_1 = 0$. The parameters λ_i and μ_i are the *birth* rate and *death* rate, respectively, while ν_i is the rate of absorption (or *killing* rate) in state i. We will assume throughout that $\nu_i > 0$ for at least one state $i \geq 1$. When $\nu_1 > 0$ but $\nu_i = 0$ for all $i > 1$, $\mathcal{X}$ is usually referred to as a (pure) *birth-death process*, ν_1 then being interpreted as the death rate in state 1.

The transition rates of the process $\mathcal{X}$ determine a sequence of polynomials $\{Q_n\}$ through the recurrence relation

$$\begin{aligned} &\lambda_n Q_n(x) = (\lambda_n + \mu_n + \nu_n - x)Q_{n-1}(x) - \mu_n Q_{n-2}(x), \quad n > 1, \\ &\lambda_1 Q_1(x) = \lambda_1 + \nu_1 - x, \quad Q_0(x) = 1. \end{aligned} \tag{2}$$

The sequence $\{Q_n\}$ plays an important role in the analysis of the process $\mathcal{X}$ and will be the main object of study in this paper. We will focus in particular on weighted sums

$$\sum_{n=0}^{\infty} w_n Q_n(x) \tag{3}$$

for certain nonnegative weights w_n depending on the transition rates of the process and certain values of x, since the existence of *quasi-stationary distributions* (see Section 4) for the corresponding birth-death process with killing requires the convergence of such series. Our aim is to collect and supplement a number of results that have appeared in the stochastic literature, and present them from an orthogonal-polynomial perspective. This will also give us the opportunity to rectify some statements in [9] and to supply some new proofs.

In the special case of a (pure) birth-death process relevant weighted sums of the type (3) have been studied in [12] (where the polynomials R_n^* have the role of our Q_n). However, the technique employed there (involving *kernel polynomials*) does not seem to be applicable in the more general setting at hand.

The remainder of this paper is organised as follows. In Section 2 we collect a number of basic properties of the polynomial sequence $\{Q_n\}$. These will enable us to derive in Section 3 some further properties of the polynomials Q_n and, subsequently, to establish criteria for convergence of the series (3) for certain values of w_n and x. In Section 4 we will briefly discuss the relevance of our findings for the analysis of birth-death processes with killing, in particular with regard to the existence of quasi-stationary distributions.

2 Preliminaries

By letting

$$P_0(x) := 1 \quad \text{and} \quad P_n(x) := (-1)^n \lambda_1\lambda_2 \dots \lambda_n Q_n(x), \quad n \geq 1,$$

we obtain the *monic* polynomials corresponding to $\{Q_n\}$ of (2), which satisfy the three-terms recurrence relation

$$\begin{aligned} &P_n(x) = (x - \lambda_n - \mu_n - \nu_n)P_{n-1}(x) - \lambda_{n-1}\mu_n P_{n-2}(x), \quad n > 1, \\ &P_1(x) = x - \lambda_1 - \nu_1, \quad P_0(x) = 1. \end{aligned} \tag{4}$$

As a consequence (see, for example, Chihara [3, Theorems I.4.4 and II.3.1]) $\{P_n\}$, and hence $\{Q_n\}$, constitutes a sequence of orthogonal polynomials with respect to a probability measure (a positive Borel measure of total mass 1) on $\mathbb{R}$. That is, there exist a (not necessarily unique) probability measure ψ on $\mathbb{R}$ and constants $\rho_j > 0$ such that

$$\rho_j \int_{-\infty}^{\infty} Q_i(x)Q_j(x)\psi(dx) = \delta_{ij}, \quad i, j \geq 0, \tag{5}$$

where δ_{ij} is Kronecker's delta. It can readily be seen that, actually,

$$\rho_0 = 1 \quad \text{and} \quad \rho_n = \frac{\lambda_1\lambda_2 \dots \lambda_n}{\mu_2\mu_3 \dots \mu_{n+1}}, \quad n > 0. \tag{6}$$

The particular form of the parameters in the recurrence relation (4) and our assumption $\nu_i > 0$ for at least one state i allow us to draw more specific conclusions on ψ. Namely, by [6, Theorem 1.3] there exists a probability measure

ψ on the *open* interval $(0,\infty)$ with finite moment of order -1, that is,

$$\int_{(0,\infty)} \frac{\psi(dx)}{x} < \infty, \tag{7}$$

satisfying (5). Moreover, by [6, Theorem 4.1] this measure is the unique probability measure ψ satisfying (5) if and only if

$$\sum_{n=0}^{\infty} \rho_n Q_n^2(0) = \infty. \tag{8}$$

In the terminology of the theory of the moment problem (8) is necessary and sufficient for the *Hamburger moment problem* associated with the polynomials $\{Q_n\}$ to be *determined*. By [6, Theorem 4.1] again, (8) is also necessary and sufficient for (5) to have a unique solution ψ with all its support on the nonnegative real axis, that is, for the *Stieltjes moment problem* associated with $\{Q_n\}$ to be *determined*. We note that these results generalize Karlin and McGregor [11, Theorem 14 and Corollary] (see also Chihara [4, Theorems 2 and 3]), which refer to the pure birth-death case $\nu_1 > 0$ and $\nu_i = 0$ for $i > 1$.

The orthogonality relation (5) implies that the *orthonormal* polynomials $\{p_n\}$ corresponding to $\{Q_n\}$ satisfy $p_n(x) = \sqrt{\rho_n} Q_n(x)$ so, by a renowned result from the theory of moments (Shohat and Tamarkin [13, Corollary 2.7]), we actually have

$$\sum_{n=0}^{\infty} \rho_n Q_n^2(x) < \infty \quad \text{for all} \quad x \in \mathbb{R} \tag{9}$$

if the Hamburger moment problem associated with $\{Q_n\}$ is *indeterminate*. For later use we recall another famous result from the theory of moments ([13, Corollary 2.6]), stating that if the Hamburger moment problem is determined, then

$$\psi(\{x\}) = \left(\sum_{n=0}^{\infty} \rho_n Q_n^2(x) \right)^{-1}, \quad x \in \mathbb{R}, \tag{10}$$

which is to be interpreted as zero if the sum diverges. Hence, if the Hamburger moment problem is determined we have

$$\sum_{n=0}^{\infty} \rho_n Q_n^2(x) < \infty \iff \psi(\{x\}) > 0, \quad x \in \mathbb{R}. \tag{11}$$

It follows that $\psi(\{0\}) = 0$ since determinacy of the Hamburger moment problem is equivalent to (8). Evidently, this is consistent with the fact that there must be an orthogonalizing measure on the *open* interval $(0, \infty)$.

If the Hamburger moment problem associated with $\{Q_n\}$ is indeterminate, then, by Chihara [2, Theorem 5], there is a unique orthogonalizing probability measure for which the infimum of its support is maximal. We will refer to this measure (which happens to be discrete) as the *natural* measure. Evidently, the natural measure has all its mass on the positive real axis.

It is well known (see, for example, [3, Section II.4]) that the polynomials Q_n have real zeros $x_{n1} < x_{n2} < \ldots < x_{nn},\ n \geq 1$, which are closely related to $\mathrm{supp}(\psi)$, the support of the orthogonalizing probability measure ψ, where ψ, if not uniquely determined by (5), should be interpreted as the natural measure. In particular we have

$$\xi := \lim_{n\to\infty} x_{n1} = \inf \mathrm{supp}(\psi) \geq 0, \tag{12}$$

where the limit exists since the sequence $\{x_{n1}\}$ is (strictly) decreasing (see, for example, [3, Theorem I.5.3]). Considering that

$$(-1)^n P_n(x) = \lambda_1\lambda_2\ldots\lambda_n Q_n(x) = (x_{n1} - x)(x_{n2} - x)\ldots(x_{nn} - x),$$

it now follows that

$$y < x \leq \xi \iff Q_n(y) > Q_n(x) > 0 \quad \text{for all} \quad n > 0, \tag{13}$$

a result that will be used later on.

The quantity ξ (which happens to be the *decay parameter* of the associated birth-death process with killing $\mathcal{X}$) plays an important part in what follows, and it will be useful to relate ξ to the parameters in the recurrence relation (2). From [7, Theorem 7] we obtain the bound

$$\xi \geq \inf_{i\geq 1} \left\{ \lambda_i + \mu_i + \nu_i - a_{i+1} - \frac{\lambda_{i-1}\mu_i}{a_i} \right\} \tag{14}$$

for any sequence $(a_1, a_2, \ldots)$ of positive numbers. Choosing $a_i = \lambda_{i-1}$ for $i > 1$ it follows in particular that

$$\xi \geq \inf_{i\geq 1} \nu_i. \tag{15}$$

In [3, Corollary to Theorem IV.2.1] one finds the simple upper bound

$$\xi \leq \inf_{i\geq 1}\{\lambda_i + \mu_i + \nu_i\}, \tag{16}$$

while more refined upper bounds are given in [7]. Similar inequalities hold true for $\sigma := \inf \operatorname{supp}(\psi)'$, the infimum of the *derived* set of the support of the (natural) orthogonalizing measure. (See [3, Section II.4] for the relation between σ and the zeros of the polynomials $\{Q_n\}$.) In particular, by [7, Theorem 9] we have

$$\sigma \geq \liminf_{i\to\infty}\left\{\lambda_i + \mu_i + \nu_i - a_{i+1} - \frac{\lambda_{i-1}\mu_i}{a_i}\right\} \tag{17}$$

for any sequence $(a_1, a_2, \ldots)$ of positive numbers. Again choosing $a_i = \lambda_{i-1}$ for $i > 1$ it follows that

$$\sigma \geq \liminf_{i\to\infty} \nu_i. \tag{18}$$

Since ξ must be an isolated point in $\operatorname{supp}(\psi)$ if $\xi < \sigma$, we can now conclude the following.

Lemma 1 If $\xi < \liminf_{i\to\infty}\nu_i$, then ξ is an isolated point in the support of the (natural) orthogonalizing measure.

We note that as a consequence of this lemma

$$\xi < \liminf_{i\to\infty}\nu_i \quad \Longrightarrow \quad \xi > 0, \tag{19}$$

since ψ is a measure on the positive real axis.

Drawing near the end of our preliminaries we note the useful relation

$$\lambda_n\rho_{n-1}(Q_n(x) - Q_{n-1}(x)) = \sum_{j=0}^{n-1}(\nu_{j+1} - x)\rho_j Q_j(x), \quad n > 0, \tag{20}$$

which follows easily by induction from (2). Hence we can write, for all $x \in \mathbb{R}$,

$$Q_n(x) = 1 + \sum_{k=0}^{n-1}(\lambda_{k+1}\rho_k)^{-1}\sum_{j=0}^{k}(\nu_{j+1} - x)\rho_j Q_j(x), \quad n > 0, \tag{21}$$

and, in particular,

$$Q_n(0) = 1 + \sum_{k=0}^{n-1}(\lambda_{k+1}\rho_k)^{-1}\sum_{j=0}^{k}\nu_{j+1}\rho_j Q_j(0) \geq 1, \quad n > 0. \tag{22}$$

Evidently, $Q_n(0)$ is increasing in n. Moreover, by [10, Lemma 1] we have $\lim_{n\to\infty} Q_n(0) = \infty$ if and only if

$$\sum_{k=0}^{\infty}(\lambda_{k+1}\rho_k)^{-1}\sum_{j=0}^{k}\nu_{j+1}\rho_j = \infty, \tag{23}$$

which happens to be a necessary and sufficient condition for *absorption* of the associated birth-death process with killing (see [10, Theorem 1]). Another condition on the parameters of the process that will play a role in what follows is

$$\sum_{k=0}^{\infty}(\lambda_{k+1}\rho_k)^{-1}\sum_{j=k+1}^{\infty}\rho_j = \infty. \tag{24}$$

This condition is equivalent to the *unkilled* process (the pure birth-death process obtained by setting all killing rates equal to zero) having a *natural* or *exit boundary* at infinity. For interpretations and more information we refer to Anderson [1, Section 8.1].

3 Results

As announced in the Introduction we will focus in this section on criteria for convergence of the series $\sum w_n Q_n(x)$ for certain weights w_n and certain values of x. Specifically, we will focus on the weights $w_n = \rho_n$ and $w_n = \nu_{n+1}\rho_n$. As far as the argument x is concerned we are primarily interested in the case $x = \xi$, but will present our findings for $x \leq \xi$ whenever possible. Concrete results will be obtained conditional on $\xi < \liminf_{i\to\infty}\nu_i$ or $\xi > \limsup_{i\to\infty}\nu_i$. We recall from (13) that $Q_n(x) > 0$ for all n if $x \leq \xi$, a result that will be used repeatedly. Note also that, by (13) again, convergence of $\sum \rho_n Q_n(y)$ implies convergence of $\sum \rho_n Q_n(x)$ if $y < x \leq \xi$.

We start off by giving some auxiliary lemmas. The first contains a sufficient condition for monotonicity of the sequence $\{Q_n(x)\}_{n\geq N}$ for N sufficiently large, and hence for the existence of $Q_\infty(x) := \lim_{n\to\infty} Q_n(x)$.

Lemma 2 Let $x \leq \xi$. If $x < \liminf_{i\to\infty}\nu_i$ or $x > \limsup_{i\to\infty}\nu_i$, then the (positive) sequence $\{Q_n(x)\}_{n\geq N}$ is monotone for N sufficiently large.

Proof If $x \leq \xi$ and $x < \liminf_{i\to\infty} \nu_i$ we have $(\nu_{n+1} - x)\rho_n Q_n(x) > 0$ for n sufficiently large. Hence, by (20),

$$\lambda_{n+1}\rho_n(Q_{n+1}(x) - Q_n(x)) > \lambda_n\rho_{n-1}(Q_n(x) - Q_{n-1}(x)),$$

so that

$$Q_n(x) \geq Q_{n-1}(x) \quad \Longrightarrow \quad Q_m(x) > Q_{m-1}(x), \quad m > n,$$

for n sufficiently large, implying monotonicity of the sequence $\{Q_n(x)\}_{n\geq N}$ for N sufficiently large.

A similar proof leads to the same conclusion if $x > \limsup_{i\to\infty} \nu_i$. □

Our second auxiliary lemma concerns the the polynomials

$$D_n(x) := \lambda_n\rho_{n-1}(Q_{n-1}(x) - Q_n(x)), \quad n \geq 1. \tag{25}$$

Lemma 3 Let $x \leq \xi$, and $x < \liminf_{i\to\infty} \nu_i$ or $x > \limsup_{i\to\infty} \nu_i$.

(i) The limit $D_\infty(x) := \lim_{n\to\infty} D_n(x)$ exists (allowing for $\pm\infty$).

(ii) If $0 < D_\infty(x) \leq \infty$, then there exist constants $c > 0$ and $N \in \mathbb{N}$ such that

$$Q_n(x) \geq c\sum_{k=n}^{\infty}(\lambda_{k+1}\rho_k)^{-1}, \quad n \geq N, \tag{26}$$

and, for any nonnegative sequence $\{\tau_n\}$,

$$\sum_{n=N}^{\infty} \tau_n Q_n(x) \geq c\sum_{n=N}^{\infty}(\lambda_{n+1}\rho_n)^{-1}\sum_{k=N}^{n}\tau_k. \tag{27}$$

(iii) If $-\infty \leq D_\infty(x) < 0$, then there exist constants $c > 0$ and $N \in \mathbb{N}$ such that

$$Q_n(x) > c\sum_{k=N}^{n-1}(\lambda_{k+1}\rho_k)^{-1}, \quad n > N, \tag{28}$$

and, for any nonnegative sequence $\{\tau_n\}$,

$$\sum_{n=N}^{\infty} \tau_n Q_n(x) \geq c\sum_{n=N}^{\infty}(\lambda_{n+1}\rho_n)^{-1}\sum_{k=n+1}^{\infty}\tau_k. \tag{29}$$

Proof In view of (20) $D_n(x)$ can be represented as

$$D_n(x) = \sum_{j=0}^{n-1} (x - \nu_{j+1}) \rho_j Q_j(x). \tag{30}$$

So, under the conditions of the lemma, the sequence $\{D_n(x)\}_{n \geq N}$ is monotone for N sufficiently large, implying the existence of the limit.

To prove statement (ii) we note that $0 < D_\infty(x) \leq \infty$ implies the existence of constants $c > 0$ and $n \in \mathbb{N}$ such that $D_n(x) > c$ for all $n > N$. Hence

$$Q_n(x) > Q_{n+1}(x) + c(\lambda_{n+1}\rho_n)^{-1}, \quad n \geq N,$$

and (26) follows by induction. Multiplying both sides of (26) by τ_n, summing over all $n \geq N$ and interchanging summation signs on the right-hand side subsequently yields (27).

Statement (iii) is proven similarly. □

Our first theorem gives a sufficient condition for convergence of the series (3) with $w_n = \rho_n$.

Theorem 1 If $\xi \geq x > \limsup_{i \to \infty} \nu_i$, then

$$\sum_{n=0}^{\infty} (\lambda_{n+1}\rho_n)^{-1} = \infty \implies \sum_{n=0}^{\infty} \rho_n Q_n(x) < \infty. \tag{31}$$

Proof Let $\xi \geq x > \limsup_{i \to \infty} \nu_i$ and suppose $\sum \rho_n Q_n(x) = \infty$. Then, in view of (30), $D_n(x) \geq 1$ for n sufficiently large. But by (21) and (30) we have

$$\sum_{n=0}^{k} (\lambda_{n+1}\rho_n)^{-1} D_{n+1}(x) = 1 - Q_{k+1}(x) < 1$$

for all k, so that $\sum (\lambda_{n+1}\rho_n)^{-1}$ must converge. □

We will see in Section 4 that convergence results for $\sum \rho_n Q_n(\xi)$ are relevant in particular when (23) prevails, which happens to be a condition under which we can prove a converse of Theorem 1, and more.

Theorem 2 Let (23) be satisfied. If $\xi \geq x > \limsup_{i \to \infty} \nu_i$, then

$$\sum_{n=0}^{\infty} (\lambda_{n+1}\rho_n)^{-1} < \infty \implies \sum_{n=0}^{\infty} \nu_{n+1} \rho_n Q_n(x) = \sum_{n=0}^{\infty} \rho_n Q_n(x) = \infty. \tag{32}$$

Proof Lemma 2 tells us that the sequence $\{Q_n(x)\}_{n\geq N}$ is monotone for N sufficiently large if $x > \limsup_{i\to\infty} \nu_i$, so that $Q_\infty(x)$ exists and $0 \leq Q_\infty(x) \leq \infty$. The conditions (23) and $\sum(\lambda_{n+1}\rho_n)^{-1} < \infty$ imply $\sum \nu_{n+1}\rho_n = \infty$. So if $0 < Q_\infty(x) \leq \infty$, then $\sum \nu_{n+1}\rho_n Q_n(x) = \infty$, whence $\sum \rho_n Q_n(x) = \infty$ and we are done. Let us therefore assume that, for n sufficiently large, $Q_n(x)$ decreases to 0 and hence $D_n(x) > 0$. Since $x > \nu_n$ for n sufficiently large, the representation (30) shows that $D_n(x)$ is increasing for n sufficiently large, so we must have $0 < D_\infty(x) \leq \infty$. Subsequently choosing $\tau_n = \nu_{n+1}\rho_n$ and applying Lemma 3 (ii), we conclude with (23) that

$$\sum_{n=N}^{\infty} \nu_{n+1}\rho_n Q_n(x) \geq c \sum_{k=N}^{\infty} (\lambda_{k+1}\rho_k)^{-1} \sum_{j=N}^{k} \nu_{j+1}\rho_j = \infty,$$

which establishes the theorem. □

We will see in Section 4 that the question of whether $\sum \nu_{n+1}\rho_n Q_n(x)$ and $\xi \sum \rho_n Q_n(x)$ are equal – answered in the affirmative in the setting of the previous theorem – plays an crucial role in the application we have in mind. Under the additional condition (24) we can also prove equality in the setting of Theorem 1.

Theorem 3 Let (24) be satisfied. If $\xi \geq x > \limsup_{i\to\infty} \nu_i$, then

$$\sum_{n=0}^{\infty} (\lambda_{n+1}\rho_n)^{-1} = \infty \implies \sum_{n=0}^{\infty} \nu_{n+1}\rho_n Q_n(x) = \xi \sum_{n=0}^{\infty} \rho_n Q_n(x) < \infty. \quad (33)$$

Proof If $\sum(\lambda_{n+1}\rho_n)^{-1} = \infty$ then (26) cannot prevail, so we must have $-\infty \leq D_\infty(x) \leq 0$ by Lemma 3. Assuming $-\infty \leq D_\infty(x) < 0$ we can choose $\tau_n = \rho_n$ and conclude from Lemma 3 (iii) that $\sum \rho_n Q_n(x) = \infty$, which, however, contradicts Theorem 1. So we must have $D_\infty(x) = 0$, which, together with (30) and Theorem 1, establishes the result. □

Now turning to the case $\xi < \liminf_{i\to\infty} \nu_i$, we first observe the following. If the Hamburger moment problem associated with $\{Q_n\}$ is determined we have, in view of (11) and (12),

$$x < \xi \implies \sum_{n=0}^{\infty} \rho_n Q_n^2(x) = \infty. \quad (34)$$

However, when $x = \xi$ the sum may be finite. A sufficient condition for finiteness is given in the next lemma.

Lemma 4 If $\xi < \liminf_{i\to\infty} \nu_i$, then $\sum_{n=0}^{\infty} \rho_n Q_n^2(\xi) < \infty$.

Proof If the Hamburger moment problem associated with $\{Q_n\}$ is indeterminate the conclusion is always true in view of the result stated around (9). Otherwise, by (11), it suffices to show that $\psi(\{\xi\}) > 0$, but this follows from Lemma 1. □

Considering that $Q_n(\xi) > 0$ for all n, we can now state a sufficient condition for convergence of the series (3) with $w_n = \rho_n$ and $x = \xi$.

Theorem 4 If $\xi < \liminf_{i\to\infty} \nu_i$, then $\sum_{n=0}^{\infty} \rho_n Q_n(\xi) < \infty$.

Proof Let $\xi < \liminf_{i\to\infty} \nu_i$ and suppose $\sum \rho_n Q_n(\xi) = \infty$. Then

$$\sum_{j=0}^{n} (\nu_{j+1} - \xi)\rho_j Q_j(\xi) \to \infty \quad \text{as} \quad n \to \infty,$$

so that, by (20), $Q_n(\xi)$ increases in n for n sufficiently large. But then we would also have $\sum \rho_n Q_n^2(\xi) = \infty$, which is impossible in view of Lemma 4. So $\sum \rho_n Q_n(\xi)$ must converge. □

With a view to the application described in the next section we are, as before, interested in the question of whether $\sum \nu_{n+1}\rho_n Q_n(x)$ and $\xi \sum \rho_n Q_n(x)$ are equal. Our final result gives a sufficient condition.

Theorem 5 Let (23) and (24) be satisfied. If $\xi < \liminf_{i\to\infty} \nu_i$, then

$$\xi \sum_{n=0}^{\infty} \rho_n Q_n(\xi) = \sum_{n=0}^{\infty} \nu_{n+1}\rho_n Q_n(\xi) < \infty.$$

Proof Theorem 4 tells us that $\sum \rho_n Q_n(\xi) < \infty$ under the conditions of the theorem. Assuming $0 < D_\infty(\xi) \le \infty$, we can choose $\tau_n = \nu_{n+1}\rho_n$ and conclude from Lemma 3 (*ii*) that $\sum \nu_{n+1}\rho_n Q_n(\xi) = \infty$, as a consequence of (23). But this is impossible, since it would imply $D_\infty(\xi) = -\infty$, in view of (30).

Next assuming $-\infty \leq D_\infty(\xi) < 0$, we can choose $\tau_n = \rho_n$ and apply Lemma 3 (*iii*). But in view of (24) this would lead us to the false conclusion that $\sum \rho_n Q_n(\xi) = \infty$. So we must have $D_\infty(\xi) = 0$ and the result follows by (30). □

4 Application

A *quasi-stationary distribution* for the birth-death process with killing $\mathcal{X}$ of the Introduction is a proper probability distribution $m := (m_j,\ j \geq 1)$ over the nonabsorbing states such that the state probabilities at time t, conditional on the process being in one of the nonabsorbing states at time t, do not vary with t when m is chosen as initial distribution. It is known (see, e.g. [5]) that a quasi-stationary distribution can only exist when eventual absorption at state 0 is certain, that is, (23) is satisfied, and $\xi > 0$. Under these circumstances a necessary and sufficient condition for a probability distribution to be a quasi-stationary distribution for $\mathcal{X}$ is given in the next theorem.

Theorem 6 [5, Theorem 6.2] Let $\mathcal{X}$ be a birth-death process with killing satisfying (23) and $\xi > 0$. Then the distribution $(m_j,\ j \geq 1)$ is a quasi-stationary distribution for $\mathcal{X}$ if and only if there is a real number $x,\ 0 < x \leq \xi$, such that both

$$m_j = \frac{\rho_{j-1} Q_{j-1}(x)}{\sum_{n=0}^{\infty} \rho_n Q_n(x)}, \quad j \geq 1, \tag{35}$$

and

$$x \sum_{n=0}^{\infty} \rho_n Q_n(x) = \sum_{n=0}^{\infty} \nu_{n+1} \rho_n Q_n(x) < \infty. \tag{36}$$

Combining this result with the Theorems 2, 3, 4 and 5 of the previous section yields the following two theorems.

Theorem 7 Let $\mathcal{X}$ be a birth-death process with killing satisfying (23), (24) and $\xi > \limsup_{i \to \infty} \nu_i$. Then a quasi-stationary distribution for $\mathcal{X}$ exists if and only if $\sum (\lambda_{n+1}\rho_n)^{-1} = \infty$, in which case $(m_j,\ j \geq 1)$ defined by (35) constitutes a quasi-stationary distribution for every $x,\ 0 < x \leq \xi$.

Theorem 8 Let $\mathcal{X}$ be a birth-death process with killing satisfying (23), (24) and $\xi < \liminf_{i\to\infty} \nu_i$. Then $(m_j,\ j \geq 1)$ defined by (35) with $x = \xi$ constitutes a quasi-stationary distribution for $\mathcal{X}$.

These theorems should me compared with Theorem 2 and Theorem 1, respectively, of [9]. The proofs of the latter results use the equality

$$\xi \sum_{n=0}^{\infty} \rho_n Q_n(\xi) = \sum_{n=0}^{\infty} \nu_{n+1} \rho_n Q_n(\xi), \tag{37}$$

which is claimed in [10, Theorem 2] to be true under all circumstances (allowing for the value ∞). Unfortunately, there is a gap in the proof of [10, Theorem 2], which raises doubts on the unconditional validity of (37), and therefore on the conclusions that have been drawn in [9, Theorem 1 and Theorem 2] on the basis of (37). The Theorems 7 and 8 show, however, that adding the (mild) condition (24) is sufficient for these conclusions to remain valid. Moreover, while [9, Theorem 2] states only the existence of a quasi-stationary distribution under the conditions of Theorems 7, the latter theorem actually establishes the existence of an infinite family of quasi-stationary distributions.

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