# Designing Cyclic Appointment Schedules for Outpatient Clinics with Scheduled and Unscheduled Patient Arrivals

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**Abstract** We present a methodology to design appointment systems for outpatient clinics and diagnostic facilities that offer both walk-in and scheduled service. The developed blueprint for the appointment schedule prescribes the number of appointments to plan per day and the moment on the day to schedule the appointments. The method consists of two models that are linked by an algorithm; one for the day process that governs scheduled and unscheduled arrivals on the day and one for the access process of scheduled arrivals. Appointment schedules that balance the waiting time at the facility for unscheduled patients and the access time for scheduled patients, are calculated iteratively using the outcomes of the two models. The method is of general nature and can therefore also be applied to scheduling problems in other sectors than health care.

**Keywords** Health Care Management; Service Operations; Production Planning and Scheduling; Queuing Theory; Stochastic Methods

## 1 Introduction

Developing appointment schedules for service facilities that process both scheduled and unscheduled arrivals is challenging, as it requires planning and scheduling on different time scales. A well-designed appointment system comprises an efficient day appointment schedule and provides timely access. This paper is motivated by challenges faced by hospital outpatient clinics that serve patients on a walk-in basis. Most of these clinics also have a limited number of appointment slots. There are various organizational (e.g., fixed slots for patients

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in a care pathway, patients with long travel time to the hospital, children) and medical (e.g., local anesthesia or contrast fluid required) reasons to give a patient an appointment. In this paper, we introduce a method to design appointment schedules for such facilities.

Advantages of a walk-in system are a higher level of accessibility and more freedom for patients to choose the date and time of their hospital visit. Disadvantages are a possible highly variable demand and as a consequence low utilization and high waiting time (the time between the physical arrival at the facility and the start of consultation and/or treatment). The advantage of an appointment system is that workload can be dispersed, while it has the disadvantage of a potentially long access time (the time between the day of the appointment request and the appointment date). Since prolonged access times result in a delay of treatment, deterioration of health condition is a serious risk (Murray and Berwick 2003). Allowing patients to walk in effectively reduces access times to zero, and thus increases quality of care. In addition, health care facilities typically aim to guarantee a certain service level with respect to the access time for patients with an appointment.

The challenge in a mixed system is thus to balance access time for appointment patients and waiting time for walk-in patients. To achieve this, we develop a methodology that schedules appointments when the expected walk-in demand is low. To smoothen the system, in periods of high demand part of the walk-in patients is offered an appointment at a later moment. Of course, this is undesirable since it increases access time and may involve an additional clinic visit. Walk-in demand (Ashton et al. 2004, Cochran and Roche 2009) and demand for appointments requests (Williams et al. 2010) are often cyclic; therefore, we develop a cyclic appointment schedule. Appointment scheduling has received considerable attention in the literature (Section 2), as opposed to the development of models that relate access and waiting time (Gupta and Denton 2008).

Our contribution is a methodology that incorporates unscheduled and scheduled arrivals and maximizes the number of unscheduled patients served on the day of arrival, while satisfying a pre-specified access time norm for scheduled patients. We model the unscheduled arrivals with a stochastic non-stationary arrival process and incorporate balking behaviour. The scheduled patients have priority, may not show up, and appointment requests are assumed to arrive according to a cyclic pattern. To account for the cyclic arrivals, the appointment schemes we develop are also cyclic, where the cycle is a repeating sequence of days. The cycle length can, for instance, be a week or a month. The cyclic appointment schedule (CAS) specifies a capacity cycle (the maximum number of patients that can be scheduled on each day of the cycle) and a day schedule (the maximum number of patients to be scheduled per time slot on each day). Access time and waiting time are measured on different time scales, since access time is counted between days and waiting time during a day.

To facilitate the two time scales, our approach consists of decomposing the appointment planning process and the service process during the day. For both processes we propose an analytical evaluation model. The first model determines the access time for scheduled patients for any given capacity cycle. The second model determines the expected number of unscheduled jobs that cannot be seen on the day of arrival. The two models are linked by an iterative algorithm that stops when the CAS is found in which the fraction of unscheduled jobs seen on the day of arrival is maximized, given that the restriction on the access time is satisfied. A numerical example of a small problem instance demonstrates the potential of the methodology. In this example complete enumeration is applied to find optimal day schedules. Our future research will aim at incorporating heuristics to quickly find (close to) optimal day schedules, so that larger problem sizes can be tackled. Finding an optimal day schedule is not straightforward and a field of research on its own (Cayirli and Veral 2003, Gupta and Denton 2008).

This paper is organized as follows. Section 2 provides a literature review. In Section 3, we give an introduction to the methodology and provide a formal problem description. Sections 4-6 present the access and day process evaluation models and the algorithm. Section 7 describes the numerical example, followed by the discussion and conclusions in Section 8.

## 2 Literature

In many service facilities customers are requested to make an appointment. There is a substantial body of literature focusing on the design of appointment systems. Health care is the most prevalent application area and hence also most considered in the literature (see the surveys by Cayirli and Veral (2003) and Gupta and Denton

(2008)). Appointment systems can be regarded as a combination of two distinct queueing systems. The first queueing system concerns customers making an appointment and waiting until the day the appointment takes place. The second queueing system concerns the process of a service session during a particular day. We denote these two queueing processes as the ‘access process’ and the ‘day process’. The remainder of this section provides an overview of the literature relevant for the present work and is structured as follows: (1) appointment scheduling, (2) access time models, and (3) integrating the access process and the day process.

### 2.1 Appointment scheduling

Appointment scheduling concerns designing blueprints for day-appointment schedules with typical objectives as minimizing customer waiting time, and maximizing resource utilization or minimizing resource idle time. A large part of the literature focuses on scheduling a given number of appointments on a particular day (e.g. Liao et al. 1993, Liu and Liu 1998, Vanden Bosch et al. 1999, Kaandorp and Koole 2007). The extent to which various aspects that impact the performance of an appointment schedule are incorporated varies, such as customer punctuality (e.g. Lehaney et al. 1999), customers not showing up (’no-shows’) (e.g. Ho and Lau 1992, Kaandorp and Koole 2007), lateness of the server at the start of a service session (e.g. Liu and Liu 1998), service interruptions (e.g. Lehaney et al. 1999) and the variance of service duration (e.g. Ho and Lau 1992).

Research techniques employed in appointment scheduling can be divided in analytical and simulation-based approaches, of which the latter is most widely applied (Cayirli and Veral 2003). In the day process we aim for an analytical approach, namely finite time Markov chain analysis. Related examples with health care applications are Pegden and Rosenshine (1990), Liao et al. (1993), Vanden Bosch et al. (1999), Kaandorp and Koole (2007) and Hassin and Mendel (2008), although these references do not consider unscheduled customers.

Often, a homogeneous customer population is assumed (Creemers 2009). Some studies however, focus on service systems with various customer types. Differentiation between customer types is identified as a consequence of distinct service requirements (e.g. Klassen and Rohleder 1996, Wang 1999, Vanden Bosch et al. 1999, Vanden Bosch and Dietz 2000, Cayirli et al. 2008). Also, distinct priority levels may be a reason for patient type differentiation. An example can be found in Patrick and Puterman (2007), where service slots are premarked for various scheduled customer classes. In this paper, customer type differentiation arises from distinct arrival processes.

The effect of mixed arrival processes is studied in Green et al. (2006a), Kolisch and Sickinger (2008) and Sickinger and Kolisch (2009). Here, scheduled outpatients, unscheduled inpatients and emergency patients are taken into account. Patients without an appointment are either emergency patients who require non-preemptive priority or inpatients available for ‘call-in’ at any time during the day. These unscheduled patients are assumed to arrive according to an equal arrival rate throughout the day. In our case, we consider walk-in patients without priority who cannot be called in during the day. Moreover, we consider non-stationary arrivals to incorporate the expected peak behavior of walk-in demand. Studies that do incorporate non-priority unscheduled arrivals similar to the unscheduled arrivals in this paper are Reilly et al. (1978), Swisher et al. (2001), Su and Shih (2003), Ashton et al. (2004), Cayirli et al. (2006), LaGanga and Lawrence (2007), Cayirli et al. (2008); however, in all cases a simulation approach is employed. Also, these studies do not incorporate balking behavior of unscheduled customers.

### 2.2 Access time models

As our approach consists of a decomposition, isolated access time models are also of interest. The access process we consider is discrete-time and cyclical in both the arrival and service processes. Various access time models based on continuous-time queueing models are available. Examples are the $M(t)|M|s(t)$ queue Green and Soares (2007) and the adapted $M|M|s$ queue that models time-dependent demand Green et al. (2001). The latter method is also applied to a health care problem in Green et al. (2006b). To preserve the discrete-time nature we take as starting point the generating function approach for slotted queueing models in discrete time Bruneel and Wuyts (1994). A survey on discrete-time queueing systems is presented in Bruneel (1993).

Models to evaluate the length of hospital waiting lists are introduced in Worthington (1987), and further studied in for example Goddard and Tavakoli (2008). In these models homogeneous appointment request arrivals are assumed. In polling models, multiple queues are served by one server in cyclic order (see Takagi (1988) for an overview). However, cyclic arrival rates and cyclic service capacity have not yet been incorporated in polling models.

### 2.3 Linking the access and the day process

We found only a few examples that jointly consider the access and day process. In Ramakrishnan et al. (2005) the authors propose a two time scale model for the Emergency Department (ED) – Ward patient flow. The fast time scale of the ED is modeled by a continuous time Markov chain, while the slower time scale of the wards is modeled by a discrete time Markov chain. In Vanden Bosch and Dietz (2000) and Klassen and Rohleder (2004), appointment schedules ranging over a horizon of several days are evaluated. The aim is to minimize the patient's waiting and the doctor's idle time, but the patient's access time is not studied in detail.

The advanced (or open) access methodology Murray and Berwick (2003) also considers two time scales. With advanced access, a clinic leaves a fraction of appointment slots vacant for patients that request an appointment on the same day or within a couple of days. As many patients as possible are scheduled on the day they make an appointment request. One should determine the optimal ratio between the reserved capacity for long-term and same-day appointments (Dobson et al. 2011). This principle is slightly adapted in Liu et al. (2010), where the demand for short term appointments is distributed over several days, to smooth the daily load of the system. The aim of the advanced access methodology is to minimize access time ("do today's work today"). Note that in an advanced access clinic patients do announce themselves in advance and make a (same-day) appointment, contrary to the type of unscheduled patients we consider, who just show up. Models that study the advanced access methodology usually focus on capacity distribution (e.g. Dobson et al. 2011, Qu et al. 2007, Qu and Shi 2009).

Formulating a model to design an appointment schedule considering two time scales is usually done using simulation techniques (e.g. (e.g. Kopach et al. 2007)). An analytic approach is presented in Patrick et al. (2008), where the effect of capacity allocation among competing patient classes on access time targets is studied using techniques from Markov Decision Modeling and Mathematical Programming. An approach related to ours, although without the presence of walk-in patients, is given in Creemers and Lambrecht (2010). The authors consider a service facility, and first develop a vacation queuing system to determine the access time. Subsequently an appointment system is developed that calculates the waiting time at the facility.

## 3 Formal Problem Description

This section defines all modeling assumptions, defines the CAS, formally states the research goal and gives an overview of the proposed approach. Then, Sections 4 and 5 present two models to respectively evaluate the access time to the facility and the day schedule performance. In Section 6, the two models are connected by an algorithm, through which the best CAS is computed. Since our approach is generically applicable, we also present the methodology in the generic terms: a facility that serves scheduled and unscheduled jobs.

***Assumptions.*** A facility consisting of $R$ resources is operational during $T$ time slots of length $h$, during each day in a cycle of $D$ days. Two types of jobs have to be served: scheduled and unscheduled jobs. Service takes one time slot. Scheduled jobs are given a specific date and time immediately when an appointment is requested. In addition, when the facility is temporarily congested, unscheduled jobs are also offered an appointment: if the service of an unscheduled job cannot start within $g$ time slots after arrival, it will leave the facility and an appointment will be planned for another day. We will refer to such jobs as *deferred* unscheduled jobs, or just

Table 1: Notation introduced in Section 3

| Symbol | Description |
|---|---|
| $R$ | Number of resources |
| $T$ | Number of time slots during a day |
| $t$ | Time slot index ($t = 1, \dots, T$) |
| $h$ | Length of a time slot |
| $D$ | Cycle length in days |
| $d$ | Day index ($d = 1, \dots, D$) |
| $g$ | Patience of an unscheduled job, expressed in the number of slots a job is willing to wait |
| $\lambda^d$ | Initial appointment request arrival rate on day $d$ |
| $\chi_t^d$ | Unscheduled job arrival rate on day $d$ during time interval $(t-1, t]$ |
| $c_t^d$ | Maximum number of appointments to schedule in slot $t$ on day $d$ |
| $C^d$ | Appointment schedule on day $d$, $C^d = (c_1^d, \dots, c_T^d)$ |
| $C$ | Cyclic appointment schedule, $C = (C^1, \dots, C^D)$ |
| $k^d$ | Maximum number of appointments to schedule on day $d$ |
| $K$ | Capacity cycle, $K = (k^1, \dots, k^D)$ |
| $F$ | $\mathbb{E}$[Fraction of unscheduled jobs to serve at day of arrival during one cycle] |
| $S(y)$ | Access time service level: fraction of jobs with access time not greater than $y$ |
| $(y, S^{norm}(y))$ | Access time service level requirement: fraction of jobs with access time not greater than $y$ is at least $S(y)$ |
| $\phi^d$ | Distribution of the number of deferred jobs on day $d$ |
| $\gamma^d$ | Total appointment request arrival distribution on day $d$ |
| $\nu^d$ | Expected number of deferred jobs on day $d$ |

deferred jobs. The first available appointment slot for scheduled and deferred jobs is always the next day at the earliest. All appointments, both scheduled jobs and deferred unscheduled jobs, are scheduled according to a First Come First Served (FCFS) principle.

We assume a non-stationary Poisson process for the arrivals of appointment requests, with $\lambda^1, \dots, \lambda^D$ the arrival rates for different days in the cycle. Next, during each day in the cycle, we assume a non-stationary Poisson arrival process for unscheduled job arrivals, with slot-dependent arrival rates: $\chi_t^d$ for day $d = 1, \dots, D$ and time slot $t = 1, \dots, T$. Table 1 summarizes the notation introduced in this section.

***Cyclic appointment schedule.*** To effectively counterbalance the non-stationarity at both the daily and cyclic (i.e. weekly, biweekly or monthly) level, we aim to design an appointment schedule that is cyclic. We introduce the CAS $C = (C^1, \dots, C^D)$, with $C^d = (c_1^d, \dots, c_T^d)$, where $c_t^d$ specifies the maximum number of jobs that may be scheduled in slot $t$ on day $d$.

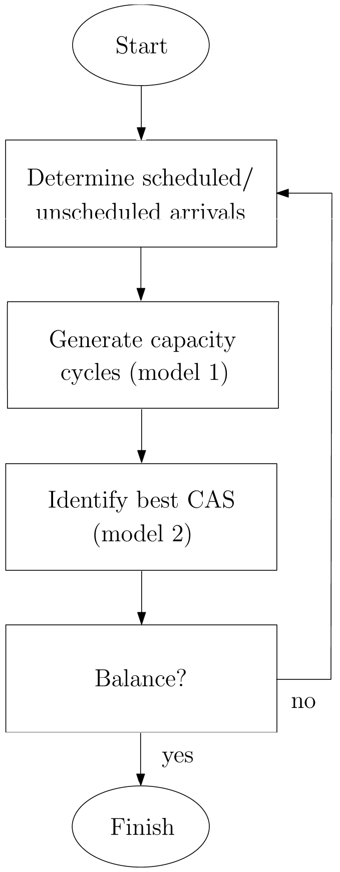
To find an adequate appointment schedule, we propose a decomposition. First, we introduce the concept of a capacity cycle $K = (k^1, \dots, k^D)$, where $k^d$ prescribes the maximum number of jobs to schedule for day $dD$ in the cycle. Second, given the capacity cycle $K$, the day plan is specified. In order to match the capacity cycle $K$, the day plan $C^d$ should be such that $k^d = \sum_{t=1}^{T} c_t^d$.

***Goal.*** An effective strategy balances the opportunities (1) for unscheduled jobs to be served on the same day without long waiting time and (2) for scheduled jobs to be served within an acceptable access time. To this end, we define the best policy as the cyclic appointment schedule in which the expected fraction of unscheduled jobs served on the day of arrival, $F$, is maximized, while for scheduled jobs the access time service level, $S(y)$, defined as the percentage of jobs that is served within $y$ days, is above a pre-specified norm $S^{norm}(y)$. The value of the vector $(y, S^{norm}(y))$ is chosen by the facility.

***Approach.*** The best CAS is determined by employing an iterative algorithm that effectively utilizes our decomposition of the CAS in the capacity cycle and the day plan. Figure 1 provides an overview of the algorithm.

In each iteration, first, capacity cycles are generated with at most $R \cdot T$ appointments per day, for which the access time service level norm will be satisfied. All jobs requesting an appointment are taken into account – thus both scheduled jobs and deferred unscheduled jobs. We derive the distribution of the number of deferred

Fig. 1: The algorithm




unscheduled jobs $\phi^d$, so that the distribution of the total number of appointment requests on day $d$ is the sum of a Poisson distribution with parameter $\lambda^d$ and the distribution $\phi^d$. To assess whether specific capacity cycles with arrival distribution $\gamma^d$ satisfy the access time norm $S^{norm}(y)$, a cyclic slotted queuing model is proposed (Model I, presented in Section 4).

Next, for each capacity cycle generated in the first step, the best day schedule is determined. Given the queue length probabilities resulting from Model I and the unscheduled job arrival rates, $\chi_t^d$, for each day the $k^d$ appointments are distributed over the $T$ time slots, such that the number of deferred unscheduled jobs is minimized. To achieve this, a Markov reward model is presented (Model II, Section 5), which is used to calculate the performance of a specific day schedule.

Then, the capacity cycle that achieves the lowest expected number of deferred unscheduled jobs over the entire cycle is chosen as the best cycle. If the expected numbers of deferred unscheduled jobs $\nu^d$ did not change significantly since the last iteration, the algorithm stops. If not, the entire process is repeated. A detailed description of the algorithm is given in Section 6.

## 4 Model I: Access Time Evaluation

In this section, a cyclic slotted queuing model is presented that allows for an evaluation of the access time for scheduled jobs, given an arbitrary capacity cycle. To this purpose, we focus on the backlog, $B^d$, at the start of each day $d$. We define the backlog as the number of jobs for which a request for an appointment has already been made, while the appointment itself has not yet taken place. We formulate a Lindley type equation to characterize the backlog, and use a probability-generating function approach to derive expressions for the distribution of the backlog at the start of each day in the cycle. From the backlog distribution, we will derive the access time distribution. A summary of the notation used in this section is given in Table 2.

***Lindley type equation.*** Consider day $d$. During the day, a maximum number of jobs, $k^d$, is served, and a number of new jobs, $A^d$, arrives. At the start of day $d$, there is a backlog $B^d$. Since it is not possible to make an appointment on the day of arrival itself, the backlog at the start of the next day equals the backlog on day $d$ minus the number of jobs served on day $d$ plus the number of jobs that arrived on day $d$. This can be formalized in the following Lindley type equation:

$$B^{d+1} = (B^d - k^d)^+ + A^d,$$

where $(x)^+ = x$ if $x > 0$, and 0 otherwise.

***A Generating function approach.*** Using an approach based on generating functions (Bruneel and Wuyts 1994), we derive expressions for the distribution of the backlog at the start of each day in the cycle. The transition probabilities for going from state $B^d = i$ to state $B^{d+1} = i'$ are given by:

$$\mathbb{P}\left(B^{d+1} = i' | B^d = i\right) = \begin{cases} \mathbb{P}\left(A^d = i'\right) & \text{if} \quad i - k^d \leq 0 \\ \mathbb{P}\left(A^d = i' - i + k^d\right) & \text{if} \quad i - k^d > 0. \end{cases}$$

Let $\pi_j^d$ denote the stationary probability that at the start of day $d$, the backlog equals $j$ jobs. Furthermore, let $a_j^d$ denote the probability that $A^d = j$. Note that the underlying probability distribution does not necessarily has to be Poisson. The stationary probabilities can be computed recursively, under the condition that the capacity for scheduled jobs is larger than the average demand, i.e. $\sum_d \mathbb{E}[A^d] < \sum_d k^d$, since otherwise we would be dealing with an unstable system. For $d = 1, \ldots, D, j \geq 0$ we obtain:

$$\pi_j^{d+1} = a_j^d \sum_{i=0}^{k^d-1} \pi_i^d + \sum_{q=0}^{j} a_{j-q}^d \pi_{k^d+q}^d . \tag{1}$$

We multiply both sides of (1) with the complex number $z^j$, where $|z| \leq 1$, and $z^j$ denotes $z$ raised to the power $j$, as opposed to index $d$ in $\pi_j^d$, $a_j^d$ and $k^d$. The summation of both sides of the resulting equation over $j$ yields the probability-generating function for $\pi^{d+1}$:

$$P_{B^{d+1}}(z) = \sum_{j=0}^{\infty} \pi_j^{d+1} z^j = \sum_{j=0}^{\infty} \left( a_j^d \sum_{i=0}^{k^d-1} \pi_i^d + \sum_{q=0}^{j} a_{j-q}^d \pi_{k^d+q}^d \right) z^j.$$

From this we obtain:

$$P_{B^{d+1}}(z) = \sum_{j=0}^{\infty} \pi_j^{d+1} z^j = P_{A^d(z)} z^{-k^d} P_{B^d(z)} + P_{A^d(z)} z^{-k^d} \sum_{i=0}^{k^d-1} \pi_i^d \left( z^{k^d} - z^i \right).$$

Table 2: Notation introduced in Section 4

| Symbol | Description |
|---|---|
| $B^d$ | Backlog at start of day $d$ |
| $P_{B^d}(z)$ | Generating function of $B^d$ |
| $A^d$ | Number of appointment requests arriving at day $d$ |
| $a_j^d$ | Appointment request arrival probabilities, $\mathbb{P}\left(A^d = j\right)$ |
| $P_{A^d}(z)$ | Generating function of $A^d$ |
| $\pi_j^d$ | Stationary backlog probabilities, $\mathbb{P}\left(B^d = j\right)$ |
| $k$ | Total number of available appointment slots in a capacity cycle, $k = \sum_d k^d$ |
| $\mathbb{E}[W^d]$ | $\mathbb{E}$[Access time for an appointment request arriving at day $d$] |
| $\mathbb{E}[W]$ | $\mathbb{E}$[Access time for an arbitrary appointment request] |

Rearranging terms and changing the order of summation leads to the probability generating function of $B^d$:

$$P_{B^d}(z) = \frac{\sum_{i=1}^{D}\sum_{q=0}^{k^{d+D-i}-1}(z^{k^{d+D-i}} - z^q)\pi_q^{d+D-i}\left[\prod_{s=d}^{d+D-i-1} z^{k^s}\prod_{r=0}^{i-1} P_{A^{d+D-r-1}}(z)\right]}{\prod_{g=1}^{D} z^{k^g} - \prod_{h=1}^{D} P_{A^h}(z)},$$

where, since we consider days in a repeating cycle, we define:

$$d := \begin{cases} D & ,d \ mod \ D = 0 \\ d \ mod \ D & ,\text{otherwise.} \end{cases}$$

The generating functions uniquely determine the stationary probabilities $\pi_j^d$, $j = 0, \ldots, k^d - 1$, $d = 1, \ldots, D$. To calculate these probabilities, we build upon the approach given in Adan et al. (2006). Define $k$ as the total number of available appointment slots in a capacity cycle, i.e. $k = \sum_{d=1}^{D} k^d$. Then, the denominator of $P_{B^d}(z)$ has $k-1$ zeros inside the unit disk; this can be shown by using Rouché's theorem (Kleinrock 1975). All generating functions, including $P_{B^d}(z)$, are bounded for $|z| \leq 1$, and therefore the zeros of the denominator are also zeros of the numerator (Bruneel and Wuyts 1994). Thus we obtain $k-1$ equations, and use $P_{B^d}(1) = 1$ to secure the last equation. The $k-1$ zeros of the denominator of $P_{B^d}(z)$ can be found by solving:

$$\prod_{g=1}^{D} z^{k^g} - \prod_{h=1}^{D} P_{A^h}(z) = 0. \tag{2}$$

The solutions of (2) also represent zeros of the numerator. Together with the normalizing equation $P_{B^d}(1) = 1$, $P_{B^d}(z)$ is completely defined for $d = 1, \ldots, D$. Note that now only the backlog probabilities for $j = 0, \ldots, k^d - 1$, have been derived. The remaining backlog probabilities are calculated directly using (1).

***Performance measures.*** The access time distribution can be directly derived from the backlog probabilities, since appointment requests are served according to the FCFS principle. The FCFS service order and the impossibility of making an appointment request for the day of arrival results in an access time of at least one day. Several performance measures can be derived. Of particular interest are the probability distribution of the access time, the expected access time and the access time service level.

*1. The probability distribution of the access time.* First we derive the conditional access time probability that the access time for a client arriving at day $d$ exceeds $y$ days, given that the backlog at the start of day $d$ equals $b$ clients. As argued, for $y = 0$, we have that

$$\mathbb{P}[W^d > y | B^d = b] = 1 \quad \forall b.$$

For $y > 0$, we have that

$$\mathbb{P}[W^d > y | B^d = b] = \begin{cases} 1 & \text{if} \quad b \geq \sum_{i=0}^{y} k^{d+i} \\ \frac{\sum_{j=s+1}^{\infty}(j-s)\cdot\mathbb{P}[A^d = j]}{\mathbb{E}[A^d]} & \text{otherwise,} \end{cases} \tag{3}$$

where $s$ represents the number of jobs arrived on day $d$ that will be served within $y$ days:

$$s = \min\left\{\sum_{i=1}^{y} k^{d+i}, \sum_{i=0}^{y} k^{d+i} - b\right\}.$$

We can explain formula (3) as follows. First, when the backlog $b$ outnumbers the available capacity in $y$ days, the conditional probability that the access time exceeds $y$ days equals 1. Otherwise, all arrivals beyond the number $s$ will wait for more than $y$ days. There are $j - s$ such arrivals. Then, the probability that the access time for a client arriving at day $d$ exceeds $y$ days, equals

$$\mathbb{P}[W^d > y] = \sum_{b=0}^{\infty} \mathbb{P}[W^d > y | B^d = b] \cdot \mathbb{P}[B^d = b].$$

*2. The expected access time.* Analogously, the expected access time for an appointment request that arrives on day $d$ is computed with:

$$\mathbb{E}[W^d | B^d = b] = \sum_{y=0}^{\infty} \mathbb{P}[W^d > y | B^d = b],$$

and thus

$$\mathbb{E}[W^d] = \sum_{b=0}^{\infty} \mathbb{E}[W^d|B^d = b] \cdot \mathbb{P}[B^d = b],$$

and

$$\mathbb{E}[W] = \sum_{d=1}^{D} \mathbb{E}[W^d] \frac{\mathbb{E}[A^d]}{\sum_{q=1}^{D} \mathbb{E}[A^q]} .$$

*3. The access time service level.* Using the access time probability distribution, we determine the fraction of scheduled jobs for which the access time does not exceed $y$. We define this as follows:

$$S(y) = \sum_{d=1}^{D} \left(1 - \mathbb{P}[W^d > y]\right) \frac{\mathbb{E}[A^d]}{\sum_{q=1}^{D} \mathbb{E}[A^q]}.$$

## 5 Model II: day process evaluation

In this section, we present a model to evaluate the performance of a single day in the CAS. Recall that the CAS consists of a capacity cycle, $K = (k^1, \dots, k^D)$, that prescribes the maximum number of jobs that can be scheduled for day $d$. Using model I, we were able to evaluate the access time performance of a given capacity cycle. Below, we evaluate the day process of a given appointment schedule, by formulating a Markov reward process.

Note that although day appointment schedule $C^d$ is open for scheduling appointments, there may be less backlog than the $k^d = \sum_t c_t^d$ available appointment slots. Therefore, we introduce the notation $\widetilde{C}^d$ to represent the *realized* day planning, which is the schedule we evaluate. Now, $\widetilde{C}^d = \left(\widetilde{c}_1^d, \dots, \widetilde{c}_T^d\right)$ expresses the actually utilized appointment slots. Since appointments are planned on a FCFS basis, the realized appointment day schedule, $\widetilde{C}^d$, will always be a truncated version of the day schedule, $C^d$. Of course, unoccupied appointment slots can be used for unscheduled jobs.

Since we will consider the day performance on a day-by-day basis, in the remainder of this section we drop the superscript $d$ for notational convenience. Table 3 provides a summary of the notation introduced in this section.

***Assumptions.*** For clarity of presentation, some of the assumptions introduced in Section 3 are repeated. During one day the facility of $R$ resources is operational during $T$ intervals of length $h$. Two types of jobs have to be served: scheduled and unscheduled jobs. Service always takes one time slot of length $h$. At the beginning of each

Table 3: Notation introduced in Section 5

| Symbol | Description |
|---|---|
| $\widetilde{C}$ | Realized schedule under CAS $C$, $\widetilde{C} = (\widetilde{C}^1, \dots, \widetilde{C}^D), \widetilde{C}^d = \left(\widetilde{c}_1^d, \dots, \widetilde{c}_T^d\right)$ |
| $q$ | $\mathbb{P}$(No-show of a scheduled job) |
| $e_t$ | Number of slots available for unscheduled jobs in the next $g$ intervals after time $t$ |
| $p_t^s(s)$ | $\mathbb{P}$(Number of scheduled jobs arriving at the start of slot $t = s$) |
| $p_t^u(u)$ | $\mathbb{P}$(Number of unscheduled jobs arriving during interval $(t-1,t] = u$) |
| $\mathbb{P}\left[(s,u)_{t+1} \mid (k,l)_t\right]$ | Transition probability from state $(t,k,l)$ to state $(t+1,s,u)$ |
| $Q_t(s,u)$ | $\mathbb{P}$(Number of scheduled, unscheduled jobs waiting at start of slot $t = s,u$) |
| $\nu_t$ | $\mathbb{E}$[Number of deferred jobs in time interval $(0,t]$] |
| $\nu$ | $\mathbb{E}$[Total number of deferred jobs] |
| $\phi_t$ | Distribution of the number of deferred jobs in time interval $(t-1,t]$ |
| $\phi$ | Distribution of the total number of deferred jobs |

time slot, a service can start. If there are both scheduled and unscheduled jobs, scheduled jobs are given priority. Overtime is not allowed.

Scheduled jobs arrive on time, according to the schedule $\widetilde{C}$. In addition, we allow for no-shows, that is, the probability that a scheduled job actually arrives at the facility equals $1-q$, so that $q$ represents the probability that a job does not show up.

Unscheduled jobs arrive at the facility according to an inhomogeneous Poisson process with slot-dependent arrival rate $\chi_t$. If the service of an unscheduled job cannot start within $g$ time slots after arriving, it will leave the facility and an appointment will be planned for another day. We assume that the facility has no pre-knowledge about potential no-shows. Therefore, an unscheduled job arriving during interval $(t-1,t]$ will stay if –and only if– the number of unscheduled jobs already waiting is strictly smaller than the minimum number of service slots during the upcoming $g$ intervals that are not utilized by scheduled jobs. The number of time slots anticipated to be available for unscheduled jobs during the upcoming $g$ intervals is denoted by $e_t$:

$$e_t = \sum_{j=t}^{\min\{t+g-1,T\}} (R-\widetilde{c}_j). \tag{4}$$

***States.*** The state of the system is denoted by the tuple $(t,s,u)$, which specifies that at the beginning of time slot $t$, $s$ scheduled and $u$ unscheduled jobs are present.

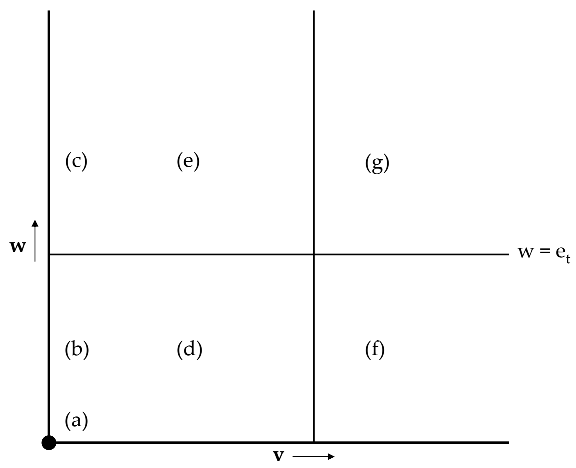
***Transition probabilities.*** Let $p_t^s(s)$ denote the probability that $s$ scheduled jobs arrive at the beginning of time slot $t$. Since each no-show is assumed to occur independently, these probabilities are calculated as follows:

$$p_t^s(s) = \begin{cases} \binom{\widetilde{c}_t}{s}(1-q)^s(q)^{\widetilde{c}_t-s} & ,0 \le s \le \widetilde{c}_t \\ 0 & ,s > \widetilde{c}_t. \end{cases}$$

Let $p_t^u(u)$ denote the probability that $u$ unscheduled jobs arrive during time interval $(t-1,t]$. As specified, $p_t^u(u)$ is Poisson distributed with slot dependent parameter $\chi_t$. Note that $\chi_1$ represents the arrival rate of unscheduled jobs that arrive before the opening time of the facility. Furthermore, note that any distribution function $p_t^u$ can be used in the day process evaluation model. Therefore, for model I the assumption of a Poisson arrival process is not strictly required.

Let $\mathbb{P}[(s,u)_{t+1} \mid (v,w)_t]$ denote the transition probability of jumping from state $(t,v,w)$ to $(t+1,s,u)$. Below we specify these transition probabilities for all possible events. In Figure 2, the state space for an arbitrary time slot $t$ is displayed in which the seven different possible events (a)-(g) are indicated. The events can be separated in three groups: first, cases (a)-(c) in which no scheduled job is served ($v=0$), second, cases (d) and (e) in which

Fig. 2: Day process state space and events



both scheduled and unscheduled jobs are served ($v < R$), and third, cases (f) and (g) in which only scheduled jobs are served ($v \geq R$). In the expressions below, $\mathbb{1}_A$ represents the indicator function; $\mathbb{1}_A = 1$ if condition $A$ is satisfied, and 0 otherwise.

Case (a). $v = w = 0$; no job served:

$$\mathbb{P}\left[(s,u)_{t+1} \mid (v,w)_t\right] = p^s_{t+1}(s)p^u_{t+1}(u).$$

Case (b). $v = 0, 0 < w \leq e_t$; unscheduled job(s) served:

$$\mathbb{P}\left[(s,u)_{t+1} \mid (v,w)_t\right] = p^s_{t+1}(s)p^u_{t+1}(u-w+\min\{R,w\})\,\mathbb{1}_{(u \geq w-\min\{R,w\})}.$$

Case (c). $v = 0, w > e_t$; unscheduled job(s) served, unscheduled job(s) abandoned:

$$\mathbb{P}\left[(s,u)_{t+1} \mid (v,w)_t\right] = p^s_{t+1}(s)p^u_{t+1}(u-e_t+R)\,\mathbb{1}_{(u \geq e_t-R)}.$$

Case (d). $v < R, w \leq e_t$; scheduled and unscheduled job(s):

$$\mathbb{P}\left[(s,u)_{t+1} \mid (v,w)_t\right] = p^s_{t+1}(s)p^u_{t+1}(u-w+\min\{(R-v),w\})\,\mathbb{1}_{(u \geq w-\min\{(R-v),w\})}.$$

Case (e). $v < R, w > e_t$; scheduled and unscheduled job(s) served, unscheduled job(s) abandoned:

$$\mathbb{P}\left[(s,u)_{t+1} \mid (v,w)_t\right] = p^s_{t+1}(s)p^u_{t+1}(u-e_t+R-v)\,\mathbb{1}_{(u \geq e_t-R+v)}.$$

Case (f). $v \geq R, w \leq e_t$; scheduled job(s) served:

$$\mathbb{P}\left[(s,u)_{t+1} \mid (v,w)_t\right] = p^s_{t+1}(s-v+R)p^u_{t+1}(u-w)\,\mathbb{1}_{(s \geq v-R)}\,\mathbb{1}_{(u \geq w)}.$$

Case (g). $v \geq R, w > e_t$; scheduled job(s) served, unscheduled job(s) abandoned:

$$\mathbb{P}\left[(s,u)_{t+1} \mid (v,w)_t\right] = p^s_{t+1}(s-v+R)p^u_{t+1}(u-e_t)\,\mathbb{1}_{(s \geq v-R)}\,\mathbb{1}_{(u \geq e_t)}.$$

***Performance measures.*** Let $Q_t(s,u)$ denote the probability that at the start of slot $t$ there are $s$ scheduled and $u$ unscheduled jobs present. $Q_t(s,u)$ can be calculated as follows:

$$Q_1(s,u) = p^s_1(s) \cdot p^u_1(u).$$

For $t = 2,...,T$:

$$Q_{t+1}(s,u) = \sum_v \sum_w Q_t(v,w)\mathbb{P}\left[(s,u)_{t+1} \mid (v,w)_t\right].$$

The expected number of deferred jobs $\nu = \nu_T$ is calculated accordingly:

$$\nu_1 = \sum_{s=0}^{\infty} \sum_{u=e_1+1}^{\infty} (u-e_1) \cdot Q_1(s,u).$$

For $t = 2,...,T$:

$$\nu_t = \nu_{t-1} + \sum_{s=0}^{\infty} \sum_{u=e_t+1}^{\infty} (u-e_t) \cdot Q_t(s,u).$$

The distribution of the number of deferred jobs, $\phi$, can be calculated as follows. For $t = 1,\ldots,T$:

$$\phi_t(j) = \begin{cases} \sum_{s=0}^{\infty} \sum_{u=0}^{e_t} Q_t(s,u) & , j = 0 \\ \sum_{s=0}^{\infty} Q_t(s, e_t+j) & , j > 0, \end{cases}$$

and

$$\phi = \phi_1 * \ldots * \phi_T,$$

where $*$ denotes the discrete convolution function.

*Remark 1* Clearly, other performance measures that might be of interest, such as waiting time and utilization indicators, can also be calculated. Since in the algorithm of the next section, we will minimize the number of deferred jobs, we restricted ourselves here to the calculation of this performance measure.

## 6 Algorithm: Balancing Scheduled and Unscheduled Arrivals

The algorithm presented in this section links the access and day process. Models I and II are used iteratively to maximize the number of unscheduled jobs served during the day of arrival, given the pre-specified access time service level norm. As mentioned before, unscheduled jobs that cannot be served within $g$ time slots receive an appointment. The algorithm determines the optimal size of this group of deferred jobs by gradually increasing its size during each iteration. Table 4 summarizes the notation presented in this section.

In the first iteration, the expected number of deferred jobs is set to zero. Then, the best scheduling cycle (using Model I) with accompanying appointment schedule (using Model II) is determined, given the appointment request arrival processes with rate $\lambda^d$ and that of unscheduled job arrivals with rate $\chi_t^d$. If the expected number of jobs that has to be deferred under the best policy is significantly greater than in the previous iteration, then apparently the reserved capacity for appointments was not sufficient. In this case, the algorithm starts a new iteration. The distribution of the number of deferred jobs on day $d$ in iteration $n$ is denoted by $\phi^d(n)$, and the expected number by $\nu^d(n)$.

In the subsequent iteration, to account for the jobs that were deferred, the distribution of appointment request arrivals $\gamma^d(n)$ is set to

$$\gamma^d(n) = P(\lambda^d) * \phi^d(n-1),$$

where $P(\lambda^d)$ denotes the Poisson distribution with parameter $\lambda^d$. As such, the appointment requests generated by deferred jobs are taken into account on the day of occurrence in the previous iteration. Then, a new best policy is calculated. As more appointment slots are reserved, this may result in more deferred jobs than in the previous iteration. This iterative procedure is repeated until on each day in the cycle, a balance is found between the anticipated extra demand for appointments from deferred unscheduled jobs (which was $\nu^d(n-1)$) and the realized deferred unscheduled jobs (which is $\nu^d(n)$); expressed formally, the algorithm terminates if, for some small $\varepsilon$,

$$|\nu^d(n) - \nu^d(n-1)| < \varepsilon \quad ,d = 1,\dots,D.$$

It is important to note that we aim for balance on a day-by-day basis. Balance just on a cycle basis ($|\sum_d \nu^d(n) - \nu^d(n-1)| < \varepsilon$) is not sufficient, since only in the case that $\left|\nu^d(n) - \nu^d(n-1)\right| < \varepsilon, d = 1,\dots,D$, it is guaranteed that the appointment requests of deferred jobs occur in the way that was anticipated. Only then we can assure that in the access time calculations, we account for the deferred jobs on the day they occur, since the access time calculations that use $\phi^d(n-1)$, based upon which the capacity cycle is designed, are still valid for $\phi^d(n)$ in this case.

We now specify the procedure used to find an optimal policy within each iteration. First, by applying Model I, all capacity cycles fulfilling the specified access time service level norm are generated. So, given $\gamma^d(n)$, all capacity cycles $K = (k^1,\dots,k^D)$ satisfying $S^{norm}(y)$ are generated. Suppose that $m$ different capacity cycles

Table 4: Notation introduced in section 6

| Symbol | Description |
|---|---|
| $n$ | Iteration counter |
| $\phi^d(n)$ | Distribution of the number of deferred jobs on day $d$ in iteration $n$ |
| $\nu^d(n)$ | Expected number of deferred jobs on day $d$ in iteration $n$ |
| $\gamma^d(n)$ | Total appointment request arrival distribution on day $d$ in iteration $n$ |
| $\varepsilon$ | Precision of the algorithm's stop criterion |
| $K(n_f)$ | Capacity cycle option $f$ consisting of $(k^1(n_f),\dots,k^D(n_f))$ in iteration $n$ |
| $C(n_f)$ | The best CAS given capacity cycle $K(n_f)$ |
| $\bar{\pi}_j^d(n_f)$ | The probability that in iteration $n$ under capacity cycle $K(n_f)$ $j$ appointment reservations are utilized by appointments on day $d$ |
| $\nu_C^*(n_f)$ | $\mathbb{E}$[Total number of deferred jobs in iteration $n$ under capacity cycle $K(n_f)$ and CAS $C$] |
| $\nu_{C^d\|j}^d(n_f)$ | $\mathbb{E}$[Number of deferred jobs on day $d$ in iteration $n$ under capacity cycle $K(n_f)$ and CAS $C$ when $j$ appointment slots are utilized by scheduled jobs] |

satisfy the norm, then denote these options for iteration $n$ by $K(n_f) = (k^1(n_f), \ldots, k^D(n_f))$, $f = 1, \ldots, m$. From these options, the best capacity cycle is selected, which is the capacity cycle that minimizes the expected number of deferred jobs. To do this, for each scheduling cycle option $K(n_f)$, the best CAS $C(n_f)$ is determined.

The best CAS's are determined by applying Model II as follows. First, observe that although in a capacity cycle $K(n_f)$ there are $k^d(n_f)$ appointment slots reserved on day $d$, not all of these reserved slots are necessarily utilized by scheduled jobs. Since appointments are planned according to the FCFS principle, we know from the queue length probability vectors $\pi^d(n_f)$ of Model I, the probabilities of utilizing the first $j$ out of the $k^d(n_f)$ reservations under capacity cycle $K(n_f)$. Let us denote these probabilities by $\bar{\pi}^d_j(n_f)$:

$$\bar{\pi}^d_j(n_f) = \begin{cases} \pi^d_j(n_f) & , j = 0, \ldots, k^d(n_f) - 1 \\ \sum_{q=k^d(n_f)}^{\infty} \pi^d_q(n_f) & , j = k^d(n_f). \end{cases}$$

By evaluating each day appointment schedule for $d = 1, \ldots, D$, $f = 1, \ldots, m$ and $j = 0, \ldots, k^d(n_f)$, the best CAS is determined for each capacity cycle $K(n_f)$, so by complete enumeration. Denote the expected total number of deferred jobs in cycle $K(n_f)$ under appointment schedule $C$ by $\nu_C(n_f)$. With $\nu^*(n_f)$ defined as the expected total number of deferred jobs in cycle $K(n_f)$, under the best CAS the best cyclic appointment schedules are those that minimize:

$$\nu^*(n_f) = \min_C \nu_C(n_f) = \min_C \sum_{d=1}^{D} \sum_{j=0}^{k^d(n_f)} \bar{\pi}^d_j(n_f)\, \nu^d_{C^d|j}(n_f),$$

where $\nu^d_{C^d|j}(n_f)$ denotes the expected number of deferred jobs on day $d$ under capacity cycle $K(n_f)$ and cyclic appointment schedule $C$, if $j$ appointment slots are utilized by scheduled jobs. Note that $C^d|j$ is a truncated version of $C^d$, in exactly the same way that $\widetilde{C}^d$ was defined in Section 5. Now, the final step is to select the capacity cycle $K(n_f)$ and accompanying CAS, which is the CAS with the lowest expected number of deferred jobs, namely:

$$\nu^*(n) = \min_f \nu^*(n_f) \quad , f^*(n) = arg \min_f \nu^*(n_f) \quad , C^*(n) = arg \min_C \nu_C(n_{f^*}).$$

Figure 3 displays the complete algorithm in pseudocode.

*Remark 2 (Convergence)* For the system to be stable we require that $\sum_d \lambda^d + \sum_d \sum_t \chi^d_t < R \cdot T$, so that total demand does not exceed capacity. In addition, we would like to determine the conditions under which the algorithm will converge. Therefore, first observe that since the unscheduled job arrival rate $\chi^d_t$ is fixed and the first iteration starts with no deferred jobs, i.e. $\nu^d(0) = 0$, in each iteration it is not possible to choose the CAS

Fig. 3: The algorithm

| | |
|---|---|
| *Step 1:* *specify input* | *Specify:* $R, T, D, g, q, S^{norm}(y), \varepsilon$; $\forall d : \lambda^d; \forall d, t : \chi^d_t$. |
| *Step 2:* *initialize algorithm* | $n := 1; \forall d : \nu^d(1) := 0, \gamma^d(1) := P(\lambda^d)$. |
| *Step 3:* *determine feasible cycles* | *Given* $\gamma^d(n)$, *determine all* $K(n_f), f = 1, \ldots, m$, *such that* $S(y) \geq S^{norm}(y)$. $\forall f, d$ : *store* $\pi^d(n_f)$. |
| *Step 4:* *choose best cycle* | *Determine* $\nu^*(n), f^*(n)$ *and* $C^*$. |
| *Step 5:* *assess current solution* | *If* $\forall d : \lvert \nu^d(n) - \nu^d(n-1) \rvert < \varepsilon$, *then stop, else proceed to step 6.* |
| *Step 6:* *adjust deferrals* | $\forall d : \nu^d(n+1) := \nu^d(n), \phi^d(n+1) := \phi^d(n)$, $\gamma^d(n+1) := P(\lambda^d) + \phi^d(n+1)$; $n := n+1$ *and return to step 3.* |

such that $\sum_d \nu^d(n) < \sum_d \nu^d(n-1)$. The total expected number of deferred jobs $\sum_d \nu^d(n)$ is thus monotonically non-decreasing. Also, if the access time norm $S^{norm}(y)$ is set such that it can be satisfied if all jobs are planned, we ensure that in each iteration it is possible to find feasible capacity cycles, i.e. capacity cycles for which $S(y) \geq S^{norm}(y)$. However, convergence of the algorithm is not assured. Although not likely for practical instances, it cannot be guaranteed that the algorithm does not run into the situation that it keeps jumping between points for which the total expected number of deferred jobs does not change, but without day-by-day balance, i.e. $\left|\sum_d \nu^d(n) - \nu^d(n-1)\right| < \varepsilon$, and not $|\nu^d(n) - \nu^d(n-1)| < \varepsilon$, for all $d$. If such a case occurs, an additional rule to act as a tie-breaker is required. We extensively tested the algorithm by evaluating fifteen different instances (see Section 7). Convergence was obtained for all instances, also in the cases for which we tried to force the jumping behavior.

## 7 Numerical Experiments

The algorithm was coded with the CodeGear Delphi programming language and tested on an Intel 2.2 Ghz PC with 4Gb of RAM. We tested the algorithm on a variety of fifteen scenarios, each with different characteristics. To demonstrate our methodology, we choose to present one of the numerical experiments in this section. First, we present the input parameters. Second, we discuss the evolution of the algorithm, and finally, we show the end results for the case study.

***Input parameters.*** We consider a facility with one resource, and employs a cycle with length $D = 5$ days, where each day consists of $T = 8$ slots. The initial demand per day for appointment requests is given by $(\lambda^1, \ldots, \lambda^5) = (5, 0, 2, 0, 7)$. The arrival rates of unscheduled jobs $\chi_t^d$ are given in Table 5. These arrival rates are chosen such that different days in the cycle represent different unscheduled arrival patterns, as also illustrated by Figure 4. The access time service level norm is set such that 95% of the jobs that are eventually scheduled are served within two cycles or less, $(y, S^{norm}(y)) = (10, 0.95)$. Furthermore, we assume that unscheduled jobs are willing to wait for a maximum of two time slots, i.e. $g = 2$, and for computational convenience we assume that the number of deferred jobs on day $d$, $\phi^d$, is Poisson distributed. For simplicity, we also assume that all scheduled jobs show up, i.e. $q = 0$. The stop criterion of the algorithm applies the threshold $\varepsilon = 0.0001$. Table 6 provides an overview of the input parameters. Note that the total expected demand for scheduled jobs per cycle is 14, and the total expected demand for unscheduled jobs per cycle is 22. Since there are $D \cdot T = 40$ time slots available within a cycle, the utilization of the system is 90%.

***Execution of the algorithm.*** The algorithm was executed and the results obtained from each iteration are displayed in Table 7. In the first iteration the number of deferred unscheduled jobs is positive on each day of the cycle, $\nu^d(1) > 0, d = 1, \ldots, D$. The total number of deferred jobs is $\sum_d \nu^d(1) = 4.055$. Therefore, the deferred jobs are added to the scheduled arrival stream and a new iteration is started. This procedure is repeated until after iteration 14, balance is obtained for each day, i.e. $|\nu^d(n) - \nu^d(n-1)| < \varepsilon, d = 1, \ldots, D$. From Figure 5 and 6 it is seen that (as described in Remark 2, Section 6) the total number of deferred jobs is monotonically non-decreasing, while deferrals on the day level are both increasing and decreasing. The fluctuations are substantial in the first iterations and the system stabilizes already after six iterations.

Table 5: Unscheduled job arrival rates per slot per day

| $\chi_t^d$ | $t$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
| 1 | 0.30 | 0.60 | 1.00 | 1.40 | 1.40 | 1.00 | 0.55 | 0.25 | 6.50 |
| 2 | 1.10 | 1.00 | 0.90 | 0.80 | 0.70 | 0.60 | 0.50 | 0.40 | 6.00 |
| 3 | 0.15 | 0.30 | 0.45 | 0.60 | 0.60 | 0.45 | 0.30 | 0.15 | 3.00 |
| 4 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.80 |
| 5 | 0.30 | 0.90 | 1.50 | 1.00 | 0.30 | 0.75 | 0.65 | 0.30 | 5.70 |

Table 6: Overview of the input parameters

| Parameter | Description | Value |
|---|---|---|
| $D$ | Cycle length | 5 |
| $T$ | Number of time slots | 8 |
| $\lambda^1,\ldots,\lambda^5$ | Appointment request arrival rates | $5,0,2,0,7$ |
| $(y,S^{norm}(y))$ | Service level norm | $(10,0.95)$ |
| $g$ | Unscheduled job patience | 2 |
| $q$ | No-show probability | 0 |
| $\varepsilon$ | Algorithm precision | 0.0001 |

This behavior is also reflected by the dynamics of the capacity cycles found. The total number of reserved slots for appointment slots develops as follows: $(16,19,21,21,21,22,\ldots,22)$. Again, although the total number of reserved slots $\sum_d k^d$ is monotonically non-decreasing, for a specific day $k^d$ may also decrease. For example, the capacity cycles of iterations 3–5 all have a total capacity of 21, but the capacity cycle obtained in the third iteration is changed in iteration 4 so that one appointment is shifted from day 5 to day 3. This change is reversed in iteration 5. The final capacity cycle is already obtained in iteration 6. The only purpose of iteration 7–14 is to obtain the desired balance in the daily deferrals. Note that this is a direct result of the magnitude of $\varepsilon$. If $\varepsilon$ had been set larger, the algorithm would have stopped earlier.

***Results*** Table 8 presents the final results for the numerical example. The percentage of unscheduled jobs served on the day of arrival is 69%, so $F = 0.69$. This fraction is composed by fractions $F^1,\ldots,F^D$ that differ from day to day ($F^d = (\sum_t \chi_t^d - \nu^d)/\sum_t \chi_t^d$). For example, since day 4 is a quiet day with respect to unscheduled job arrivals, it is completely filled with appointments. Only if no appointment request is made in one of the reserved slots, an unscheduled job can be served. Apparently, it pays off to serve on average only 7% of the unscheduled jobs directly on day 4 in the cycle. This is a result of the fact that only 3.6% of the unscheduled jobs arrive on day 4, and that accordingly appointments are preferably planned on this day. The deferred unscheduled jobs stream per day and the expected number of unscheduled jobs served on the day of arrival are displayed in Table 8, which also reflects that on day 4 a small amount of unscheduled jobs is directly served but also relatively few jobs are deferred. The realized service level $S(10) = 0.962$ is well above the defined service level norm of 0.95.

Fig. 4: Graphical representation of the appointment request arrival rates per slot per day

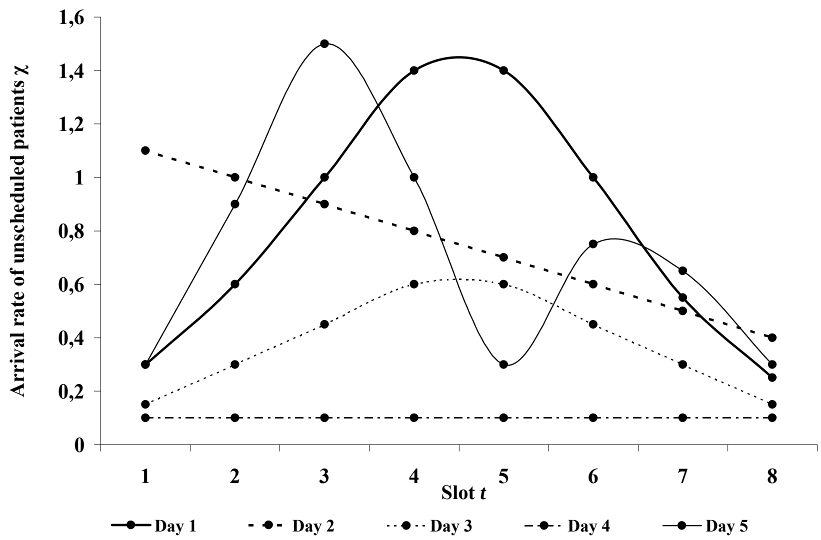


Table 7: Results per iteration step of the algorithm

| Iteration | Day | Tot. app. req. rate | Deferral rate | | Difference | Capacity cycle | CAS |
|---|---|---|---|---|---|---|---|
| $n$ | $d$ | $\gamma^d$ | $\nu^d(n-1)$ | $\nu^d(n)$ | $\lvert\nu^d(n-1)-\nu^d(n-1)\rvert$ | $k^d$ | $C^d$ |
| 1 | 1 | 5 | 0 | 1.133 | 1.133 | 1 | (1,0,0,0,0,0,0,0) |
| | 2 | 0 | 0 | 0.865 | 0.865 | 1 | (1,0,0,0,0,0,0,0) |
| | 3 | 2 | 0 | 0.547 | 0.547 | 4 | (1,1,0,1,0,0,1,0) |
| | 4 | 0 | 0 | 0.637 | 0.637 | 8 | (1,1,1,1,1,1,1,1) |
| | 5 | 7 | 0 | 0.873 | 0.873 | 2 | (1,1,0,0,0,0,0,0) |
| 2 | 1 | 6.133 | 1.133 | 1.456 | 0.323 | 2 | (1,1,0,0,0,0,0,0) |
| | 2 | 0.865 | 0.865 | 1.296 | 0.431 | 2 | (1,0,0,0,0,0,1,0) |
| | 3 | 2.547 | 0.547 | 0.549 | 0.002 | 4 | (1,1,0,1,0,0,1,0) |
| | 4 | 0.637 | 0.637 | 0.736 | 0.099 | 8 | (1,1,1,1,1,1,1,1) |
| | 5 | 7.873 | 0.873 | 1.371 | 0.498 | 3 | (1,1,0,0,0,0,1,0) |
| 3 | 1 | 6.456 | 1.456 | 1.456 | 0.000 | 2 | (1,1,0,0,0,0,0,0) |
| | 2 | 1.296 | 1.296 | 1.296 | 0.000 | 2 | (1,0,0,0,0,0,1,0) |
| | 3 | 2.549 | 0.549 | 0.952 | 0.403 | 5 | (1,1,1,0,0,1,0,1) |
| | 4 | 0.736 | 0.736 | 0.715 | 0.021 | 8 | (1,1,1,1,1,1,1,1) |
| | 5 | 8.371 | 1.371 | 1.752 | 0.381 | 4 | (1,1,0,0,0,1,1,0) |
| 4 | 1 | 6.456 | 1.456 | 1.456 | 0.000 | 2 | (1,1,0,0,0,0,0,0) |
| | 2 | 1.296 | 1.296 | 1.296 | 0.000 | 2 | (1,0,0,0,0,0,1,0) |
| | 3 | 2.952 | 0.952 | 1.498 | 0.546 | 6 | (1,1,1,0,1,0,1,1) |
| | 4 | 0.715 | 0.715 | 0.742 | 0.027 | 8 | (1,1,1,1,1,1,1,1) |
| | 5 | 8.752 | 1.752 | 1.402 | 0.350 | 3 | (1,1,0,0,0,0,1,0) |
| 5 | 1 | 6.456 | 1.456 | 1.456 | 0.000 | 2 | (1,1,0,0,0,0,0,0) |
| | 2 | 1.296 | 1.296 | 1.296 | 0.000 | 2 | (1,0,0,0,0,0,1,0) |
| | 3 | 3.498 | 1.498 | 0.954 | 0.544 | 5 | (1,1,1,0,0,1,0,1) |
| | 4 | 0.742 | 0.742 | 0.771 | 0.029 | 8 | (1,1,1,1,1,1,1,1) |
| | 5 | 8.402 | 1.402 | 2.049 | 0.647 | 4 | (1,1,0,0,1,0,1,0) |
| 6 | 1 | 6.456 | 1.456 | 1.456 | 0.000 | 2 | (1,1,0,0,0,0,0,0) |
| | 2 | 1.296 | 1.296 | 1.296 | 0.000 | 2 | (1,0,0,0,0,0,1,0) |
| | 3 | 2.954 | 0.954 | 1.495 | 0.541 | 6 | (1,1,1,0,1,0,1,1) |
| | 4 | 0.771 | 0.771 | 0.721 | 0.050 | 8 | (1,1,1,1,1,1,1,1) |
| | 5 | 9.049 | 2.049 | 1.794 | 0.255 | 4 | (1,1,0,0,0,1,1,0) |
| | | ⋮ | | | | ⋮ | |
| 14 | 1 | 6.456 | 1.456 | 1.456 | 0.000 | 2 | (1,1,0,0,0,0,0,0) |
| | 2 | 1.296 | 1.296 | 1.296 | 0.000 | 2 | (1,0,0,0,0,0,1,0) |
| | 3 | 3.497 | 1.497 | 1.497 | 0.000 | 6 | (1,1,1,0,1,0,1,1) |
| | 4 | 0.743 | 0.743 | 0.743 | 0.000 | 8 | (1,1,1,1,1,1,1,1) |
| | 5 | 8.897 | 1.897 | 1.897 | 0.000 | 4 | (1,1,0,0,0,1,1,0) |

The resulting capacity cycle is $K = (2,2,6,8,4)$, with corresponding day schedules which we discuss one-by-one below. Note that to achieve the service level norm it is required to reserve a buffer capacity of 1.11 to account for variability in appointment request arrivals, since 22 appointment slots are reserved while the average total number of jobs to schedule within a cycle is $\sum_d(\lambda^d + \nu^d) = 14 + 6.89 = 20.89$. Apparently, the service level norm is achieved with only 5% buffer capacity, thus reserved capacity for appointments can be used efficiently.

The realized expected load per day, denoted by $L^1, \dots, L^D$, is a result of the capacity cycle, the probabilities that the reserved appointment slots are utilized by appointment requests and the expected number of unscheduled jobs served on day of arrival $\sum_t \chi_t^d - \nu^d$. It turns out that the load is balanced throughout the cycle where each day has a realized load between 6.7 and 7.7.

Finally, we discuss the resulting day schedules, to explain the moments on which the appointments are planned (see also Figure 7).

Fig. 5: Graphical representation of the evolution of the deferral rates per day

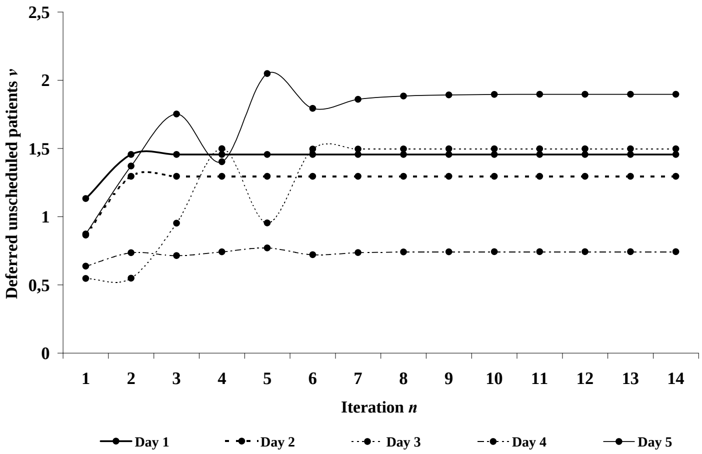
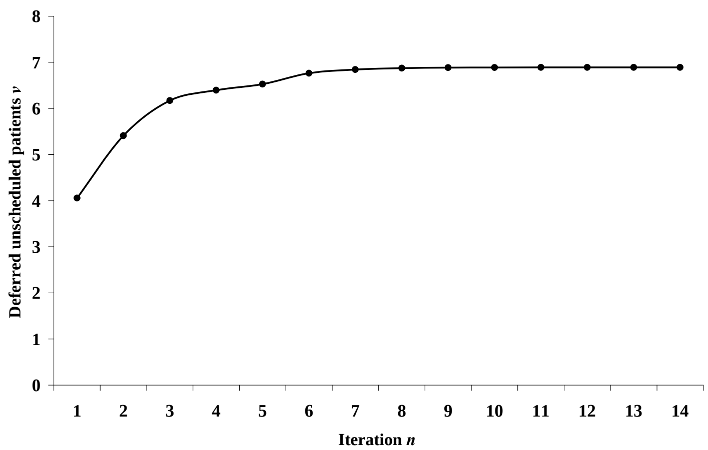



Fig. 6: Graphical representation of the evolution of the total deferral rate




Day 1, $C^1 = (1,1,0,0,0,0,0,0)$. Although the lowest unscheduled arrival rate occurs at end of the day, the appointments are planned at the beginning of the day. Since unscheduled jobs are willing to wait 2 time slots, a peak in arrivals has an impact until two slots afterwards. If appointments were planned at the end of the day, there is no possibility to serve arriving unscheduled jobs, while when planning appointments at slots at the beginning of the day, early unscheduled arrivals can be served in the third time slot.

Day 2, $C^2 = (1,0,0,0,0,0,1,0)$. Again, the tendency to plan appointments early shows up. But, the drop in unscheduled arrivals is such that it is worthwhile to plan one appointment at the end of the day. However, again although the lowest arrival rate occurs in the latest time slot, the appointment is planned one slot before, to be able to serve an unscheduled job arriving during interval $(T-3, T-1]$.

Day 3, $C^3 = (1,1,1,0,1,0,1,1)$. The demand for unscheduled jobs is relatively low. Therefore, only two slots are left open in which no appointment is planned. These are planned during the peak hours of unscheduled arrivals. However, the open slots are not planned consecutively, so to spread the possibilities for unscheduled job service.

Day 4, $C^4 = (1,1,1,1,1,1,1,1)$. As described before, this is a quiet day for unscheduled jobs. Therefore, all slots are reserved for scheduled jobs. However, note that not always are all reserved slots used for appointments; in 88% of the cases all reserved slots on day 4 are utilized for scheduled jobs.

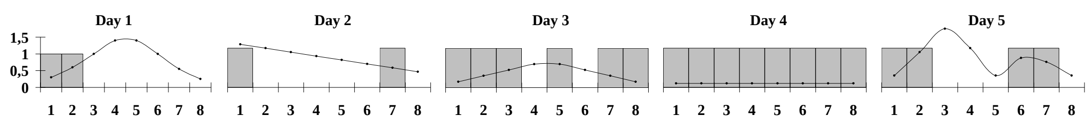
Table 8: End results for the case study

| Indicator | Description | Value |
|---|---|---|
| $F$ | Fraction unscheduled directly served | 0.69 |
| $F^1,\dots,F^5$ | Daily fraction unscheduled directly served | $0.78, 0.78, 0.50, 0.07, 0.67$ |
| $S(10)$ | Service level scheduled jobs | 0.962 |
| $\nu^1,\dots,\nu^D$ | Deferral rate per day | $1.46, 1.30, 1.50, 0.74, 1.90$ |
| $\sum_t \chi_t^1 - \nu^1,\dots,\sum_t \chi_t^D - \nu^D$ | Unscheduled job service rate per day | $5.04, 4.70, 1.50, 0.06, 3.80$ |
| $L^1,\dots,L^D$ | Realized utilization per day | $7.04, 6.70, 7.48, 7.71, 7.06$ |
| $K$ | Capacity cycle | $(2, 2, 6, 8, 4)$ |
| $C^1$ | CAS day 1 | $(1, 1, 0, 0, 0, 0, 0, 0)$ |
| $C^2$ | CAS day 2 | $(1, 0, 0, 0, 0, 0, 1, 0)$ |
| $C^3$ | CAS day 3 | $(1, 1, 1, 0, 1, 0, 1, 1)$ |
| $C^4$ | CAS day 4 | $(1, 1, 1, 1, 1, 1, 1, 1)$ |
| $C^5$ | CAS day 5 | $(1, 1, 0, 0, 0, 1, 1, 0)$ |

Day 5, $C^4 = (1, 1, 0, 0, 0, 1, 1, 0)$. The appointments are planned around the unscheduled arrival peaks. It is remarkable that the two later appointments do not occur exactly during the off-peak hours but later, which can also be explained by the aforementioned delayed impact of unscheduled arrival peaks.

The final conclusion is that the resulting CAS and its performance is the outcome of the complex interaction between the scheduled job arrival rates $\lambda^d$, the unscheduled jobs arrival patterns $\chi_t^d$, and the service level requirement $S^{norm}(y)$. For example, if $S^{norm}(y)$ is set tighter, it is to be expected that the resulting capacity cycle more closely resembles the total arrival rates for appointment requests $\gamma^d$. Also, since there would be less flexibility to spread the appointments, the fraction of unscheduled jobs served on the day of arrival, $F$, would decrease.

Fig. 7: The CAS versus the unscheduled job arrival rates




## 8 Discussion and Conclusion

In this paper we have outlined a methodology to develop an appointment schedule for facilities with scheduled and unscheduled arrival streams. The methodology consists of two separate models, one to evaluate the access and the other to evaluate the day process. The two models are linked by an iterative algorithm. An advantage of this modular approach is that the models and the algorithm can be updated separately, so that a high level of flexibility is obtained.

This paper focused on developing a methodology that incorporates the key characteristics of a mixed system and an effective communication between the two time scales of the access and day process. Achieving numerical efficiency will be our next challenge. For the problem instance in Section 7, the CAS was found using complete enumeration. Our work is currently aimed at incorporating heuristics so that larger, more realistic instances can be evaluated. The model structure of the day process suggests that local search techniques are worth exploring (see e.g. Kaandorp and Koole (2007), Vanden Bosch et al. (1999), Vanden Bosch and Dietz (2000).

Some extensions can readily be incorporated in our approach. Management is free to choose the service level norm for the access time. As such, the resulting appointment schedules can be compared for several service levels. Also, different choices for the time jobs are willing to wait ('job patience') could be studied or overbooking to anticipate for no-shows. Furthermore, the access time for scheduled jobs and the fraction of unscheduled jobs who cannot be served on the day of arrival are outcomes of model I and model II respectively, and serve as input for the algorithm. Of course, other model outcomes could be chosen as well. Finally, to incorporate for example planned maintenance of a service facility, the number of available slots in the day process can easily be amended by closing slots. Worthwhile to consider would also be to introduce stochastic service times and job patience in the day process. This might be a better reflection of reality, in particular in health care applications. Last but not least, our focus will be on practical issues in the implementation of the methodology in health care settings in Leiden University Medical Center and Academic Medical Center Amsterdam.

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## References

Adan, I., J.S.H. Van Leeuwaarden, E.M.M. Winands. 2006. On the application of Rouché's theorem in queueing theory. *Operations Research Letters* **34**(3) 355–360.

Ashton, R., L. Hague, M. Brandreth, D. Worthington, S. Cropper. 2004. A simulation-based study of a NHS Walk-in Centre. *Journal of the Operational Research Society* **56**(2) 153–161.

Bruneel, H. 1993. Performance of discrete-time queueing systems. *Computers & Operations Research* **20**(3) 303–320.

Bruneel, H., I. Wuyts. 1994. Analysis of discrete-time multiserver queueing models with constant service times. *Operations Research Letters* **15**(5) 231–236.

Cayirli, T., E. Veral. 2003. Outpatient scheduling in health care: a review of literature. *Production and Operations Management* **12**(4) 519–549.

Cayirli, T., E. Veral, H. Rosen. 2006. Designing appointment scheduling systems for ambulatory care services. *Health Care Management Science* **9**(1) 47–58.

Cayirli, T., E. Veral, H. Rosen. 2008. Assessment of patient classification in appointment system design. *Production and Operations Management* **17**(3) 338–353.

Cochran, J.K., K.T. Roche. 2009. A multi-class queuing network analysis methodology for improving hospital emergency department performance. *Computers & Operations Research* **36**(5) 1497–1512.

Creemers, S. 2009. Appointment-Driven Queueing Systems. Ph.D. thesis, Katholieke Universiteit Leuven.

Creemers, S., M. Lambrecht. 2010. Queueing models for appointment-driven systems. *Annals of Operations Research* **178**(1) 155–172.

Dobson, G., S. Hasija, E.J. Pinker. 2011. Reserving capacity for urgent patients in primary care. *Production and Operations Management* **20**(3) 456–473.

Goddard, J., M. Tavakoli. 2008. Efficiency and welfare implications of managed public sector hospital waiting lists. *European Journal of Operational Research* **184**(2) 778–792.

Green, L.V., P.J. Kolesar, J. Soares. 2001. Improving the SIPP approach for staffing service systems that have cyclic demands. *Operations Research* **49**(4) 549–564.

Green, L.V., S. Savin, B. Wang. 2006a. Managing patient service in a diagnostic medical facility. *Operations Research* **54**(1) 11–25.

Green, L.V., J. Soares. 2007. Computing time-dependent waiting time probabilities in $M(t)/M/s(t)$ queueing systems. *Manufacturing & Service Operations Management* **9**(1) 54–61.

Green, L.V., J. Soares, J.F. Giglio, R.A. Green. 2006b. Using queueing theory to increase the effectiveness of emergency department provider staffing. *Academic Emergency Medicine* **13**(1) 61–68.

Gupta, D., B. Denton. 2008. Appointment scheduling in health care: Challenges and opportunities. *IIE Transactions* **40**(9) 800–819.

Hassin, R., S. Mendel. 2008. Scheduling arrivals to queues: A single-server model with no-shows. *Management Science* **54**(3) 565–572.

Ho, C.J., H.S. Lau. 1992. Minimizing total cost in scheduling outpatient appointments. *Management Science* **38**(12) 1750–1764.

Kaandorp, G.C., G. Koole. 2007. Optimal outpatient appointment scheduling. *Health Care Management Science* **10**(3) 217–229.

Klassen, K.J., T.R. Rohleder. 1996. Scheduling outpatient appointments in a dynamic environment. *Journal of Operations Management* **14**(2) 83–101.

Klassen, K.J., T.R. Rohleder. 2004. Outpatient appointment scheduling with urgent clients in a dynamic, multi-period environment. *International Journal of Service Industry Management* **15**(2) 167–186.

Kleinrock, L. 1975. *Queueing systems, volume 1: theory*. John Wiley & Sons, London, UK.

Kolisch, R., S. Sickinger. 2008. Providing radiology health care services to stochastic demand of different customer classes. *OR Spectrum* **30**(2) 375–395.

Kopach, R., P.C. DeLaurentis, M. Lawley, K. Muthuraman, L. Ozsen, R. Rardin, H. Wan, P. Intrevado, X. Qu, D. Willis. 2007. Effects of clinical characteristics on successful open access scheduling. *Health Care Management Science* **10**(2) 111–124.

LaGanga, L.R., S.R. Lawrence. 2007. Clinic overbooking to improve patient access and increase provider productivity*. *Decision Sciences* **38**(2) 251–276.

Lehaney, B., S.A. Clarke, R.J. Paul. 1999. A case of an intervention in an outpatients department. *Journal of the Operational Research Society* **50**(9) 877–891.

Liao, C.J., C.D. Pegden, M. Rosenshine. 1993. Planning timely arrivals to a stochastic production or service system. *IIE Transactions* **25**(5) 63–73.

Liu, L., X. Liu. 1998. Dynamic and static job allocation for multi-server systems. *IIE Transactions* **30**(9) 845–854.

Liu, N., S. Ziya, V.G. Kulkarni. 2010. Dynamic scheduling of outpatient appointments under patient no-shows and cancellations. *Manufacturing & Service Operations Management* **12**(2) 347–364.

Murray, M., D.M. Berwick. 2003. Advanced access: reducing waiting and delays in primary care. *Journal of the American Medical Association* **289**(8) 1035–1040.

Patrick, J., ML Puterman. 2007. Improving resource utilization for diagnostic services through flexible inpatient scheduling: A method for improving resource utilization. *Journal of the Operational Research Society* **58**(2) 235–245.

Patrick, J., M.L. Puterman, M. Queyranne. 2008. Dynamic multi-priority patient scheduling for a diagnostic resource. *Operations Research* **56**(6) 1507–1525.

Pegden, C.D., M. Rosenshine. 1990. Scheduling arrivals to queues. *Computers & Operations Research* **17**(4) 343–348.

Qu, X., R.L. Rardin, J.A.S. Williams, D.R. Willis. 2007. Matching daily healthcare provider capacity to demand in advanced access scheduling systems. *European Journal of Operational Research* **183**(2) 812–826.

Qu, X., J. Shi. 2009. Effect of two-level provider capacities on the performance of open access clinics. *Health Care Management Science* **12**(1) 99–114.

Ramakrishnan, M., D. Sier, P.G. Taylor. 2005. A two-time-scale model for hospital patient flow. *IMA Journal of Management Mathematics* **16**(3) 197.

Reilly, T.A., V.P. Marathe, B.E. Fries. 1978. A delay-scheduling model for patients using a walk-in clinic. *Journal of Medical Systems* **2**(4) 303–313.

Sickinger, S., R. Kolisch. 2009. The performance of a generalized Bailey–Welch rule for outpatient appointment scheduling under inpatient and emergency demand. *Health Care Management Science* **12**(4) 408–419.

Su, S., C.L. Shih. 2003. Managing a mixed-registration-type appointment system in outpatient clinics. *International Journal of Medical Informatics* **70**(1) 31–40.

Swisher, J.R., S.H. Jacobson, J.B. Jun, O. Balci. 2001. Modeling and analyzing a physician clinic environment using discrete-event (visual) simulation. *Computers & Operations Research* **28**(2) 105–125.

Takagi, H. 1988. Queuing analysis of polling models. *ACM Computing Surveys (CSUR)* **20**(1) 5–28.

Vanden Bosch, P.M.V., D.C. Dietz. 2000. Minimizing expected waiting in a medical appointment system. *IIE Transactions* **32**(9) 841–848.

Vanden Bosch, P.M.V., D.C. Dietz, J.R. Simeoni. 1999. Scheduling customer arrivals to a stochastic service system. *Naval Research Logistics* **46**(5) 549–559.

Wang, P.P. 1999. Sequencing and scheduling N customers for a stochastic server. *European Journal of Operational Research* **119**(3) 729–738.

Williams, P., G. Tai, Y. Lei. 2010. Simulation based analysis of patient arrival to health care systems and evaluation of an operations improvement scheme. *Annals of Operations Research* 1–17.

Worthington, DJ. 1987. Queueing models for hospital waiting lists. *Journal of the Operational Research Society* **38**(5) 413–422.