

The Invariant Measure of Homogeneous Markov Processes in The Quarter-Plane: Representation in Geometric Terms

Yanting Chen^{*}, Richard J. Boucherie^{*}, and Jasper Goseling^{*}

^{*}Stochastic Operations Research Group, Department of Applied Mathematics, University of Twente

December 13, 2011

Abstract

We consider the invariant measure of a homogeneous continuous-time Markov process in the quarter-plane. The basic solutions of the global balance equation are the geometric distributions. We first show that the invariant measure can not be a finite linear combination of basic geometric distributions, unless it consists of a single basic geometric distribution. Second, we show that a countable linear combination of geometric terms can be an invariant measure only if it consists of pairwise-coupled terms. As a consequence, we obtain a complete characterization of all countable linear combinations of geometric distributions that may yield an invariant measure for a homogeneous continuous-time Markov process in the quarter-plane.

keywords: Invariant measure, Continuous-time Markov process, Quarter-plane, Geometric product form, Random walk.

^{*}P.O. Box 217, 7500 AE Enschede, The Netherlands. Email:{Y.Chen, R.J.Boucherie, J.Goseling}@utwente.nl

1 Introduction

Homogeneous continuous-time Markov processes in the quarter-plane have translation invariant transition rates, except for the rates along the horizontal and vertical boundaries. In literature, many examples exist of such processes with geometric invariant measure, including Jackson networks and queuing networks with negative customers, see [1] for an overview. The compensation approach of Adan [2, 3] has revealed that Markov processes without transitions to the east, north and northeast can have an invariant measure that is a countable linear combination of geometric terms.

The present paper provides a complete characterization of all countable linear combinations of geometric distributions that may yield an invariant measure for homogeneous Markov processes in the quarter-plane. In particular, our contributions are as follows. First we show that the invariant measure cannot be a finite linear combination of geometric distributions, unless it consists of a single geometric distribution. Second, we show that a countable linear combination of geometric terms can be an invariant measure only if it consists of pairwise-coupled terms, *i.e.*, only if each 2-dimensional geometric distribution in the sequence shares a parameter with the previous term in the sequence.

Alternative approaches to analyzing the invariant measure of Markov processes in the quarter-plane are available in the literature. Most notably, generating functions have been used for the analysis of a variety of such problems, see, *e.g.*, [4, 5]. A general theory is provided in [6, 7], in which the invariant measure is characterized via transforms. However, an explicit expression for the invariant measure is usually hard to obtain. The present paper characterizes invariant measures of a tractable form, *i.e.*, linear combinations of geometric terms.

The remainder of this paper is structured as follows. In Section 2 we present the model and some definitions. The main results of this paper are given in Section 3.

2 Model and Definitions

2.1 Model

Consider a two-dimensional continuous-time Markov process on the pairs (i, j) of non-negative integers. We refer to $\{(i, j) | i > 0, j > 0\}$, $\{(i, j) | i > 0, j = 0\}$, $\{(i, j) | i = 0, j > 0\}$ and $(0, 0)$ as the interior, the horizontal axis, the vertical axis and the origin of the state space, respectively. The transition rate from (i, j) to $(i + s, j + t)$ is denoted by $q_{s,t}(i, j)$. The process is homogeneous in the sense that for any (i, j) and (k, l) in the interior of the state space

$$q_{s,t}(i, j) = q_{s,t}(k, l) \quad \text{and} \quad q_{s,t}(i - s, j - t) = q_{s,t}(k - s, l - t). \quad (1)$$

for all s and t . Moreover, (1) holds for all pairs (i, j) , (k, l) on the horizontal and vertical axis respectively. Note that the first equality of (1) implies that the rates of transitions leaving from each part of the state space are translation invariant. The second equality ensures that also rates entering the same part of the state space are translation invariant.¹ Transitions are restricted to adjoining points (horizontally, vertically and diagonally), *i.e.*, $q_{s,t}(k, l) = 0$ if $|s| > 1$ or $|t| > 1$. We introduce, for $i > 0$, $j > 0$, the notation $q_{s,t}(i, j) = q_{s,t}$, $q_{s,0}(i, 0) = h_s$ and $q_{0,t}(0, j) = v_t$. Finally, let $-q_{0,0}$, $-h_0$ and $-v_0$ denote the outgoing rates in the interior, at the horizontal axis and at the vertical axis respectively. The model and notation are illustrated in Figure 1. We will refer to this type of process as a *homogeneous Markov process*. In the remainder of this paper, if not explicitly stated, we assume that a Markov process is ergodic.

2.2 Candidate geometric measures

Our interest is homogeneous Markov processes with an finite invariant measure that can be expressed as a linear combination of geometric measures. In particular, we will consider geometric measures that satisfy the balance equations in the interior of the space, *i.e.*, measures of the form $\rho^i \sigma^j$, with

¹Note that this is a stronger notion of homogeneity than considered in, for instance, [3].

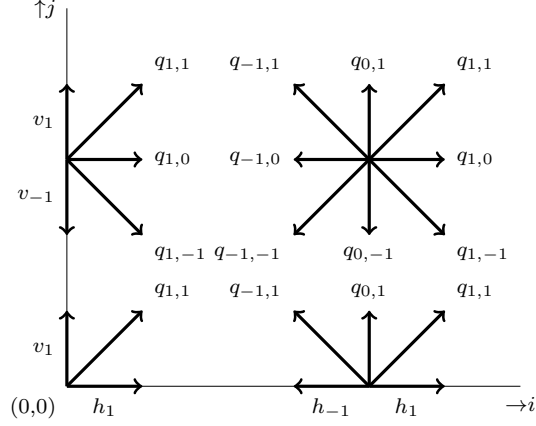


Figure 1: Homogeneous Markov process.

$(\rho, \sigma) \in C$,

$$C = \left\{ (\rho, \sigma) \in (0, 1)^2 \mid \sum_{s=-1}^1 \sum_{t=-1}^1 \rho^{-s} \sigma^{-t} q_{s,t} = 0 \right\}. \quad (2)$$

If m is a linear combination of terms $\Gamma \subset C$ we say that m is induced by Γ , see Figure 2(a).

Definition 1 (Induced measure). *Measure m is called induced by $\Gamma \subset C$ if*

$$m(i, j) = \sum_{(\rho, \sigma) \in \Gamma} \alpha(\rho, \sigma) \rho^i \sigma^j, \quad \Gamma \subset C,$$

with $\alpha(\rho, \sigma) > 0$ for all $(\rho, \sigma) \in \Gamma$.

We assume that m is finite. In this case, $\sum_{(\rho, \sigma) \in \Gamma} \alpha(\rho, \sigma) (1 - \rho)^{-1} (1 - \sigma)^{-1} < \infty$. We restrict our attention to Γ of finite and countably infinite cardinality.

2.3 Uncoupled partitions

Different ways of partitioning set Γ will be introduced here. These partitions play an essential role in the analysis later on.

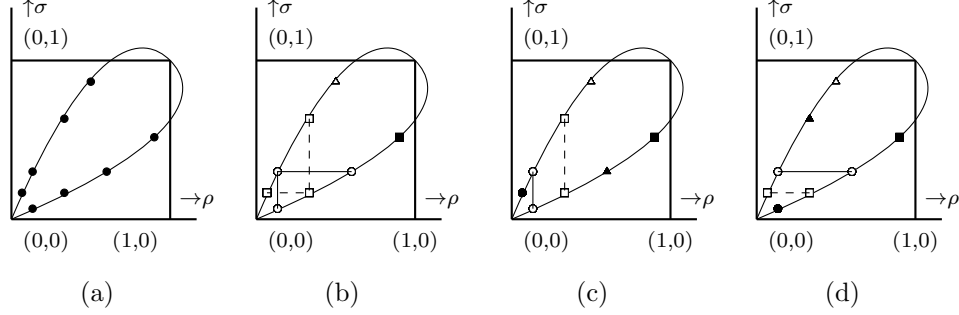


Figure 2: Partitions of Γ . In (a) an example of curve C and subset Γ are given. Figures (b), (c) and (d) give the uncoupled, horizontally uncoupled and vertically uncoupled partitions of Γ respectively, where different components are marked by different symbols.

Definition 2 (Uncoupled partition). *A partition $\{\Gamma_1, \Gamma_2, \dots\}$ of Γ is 1) horizontally uncoupled if $(\rho, \sigma) \in \Gamma_i$ and $(\tilde{\rho}, \tilde{\sigma}) \in \Gamma_j$ for $i \neq j$, implies that $\rho \neq \tilde{\rho}$; is 2) vertically uncoupled if $(\rho, \sigma) \in \Gamma_i$ and $(\tilde{\rho}, \tilde{\sigma}) \in \Gamma_j$ for $i \neq j$, implies that $\sigma \neq \tilde{\sigma}$; and is 3) uncoupled if it is both horizontally and vertically uncoupled.*

We call the partition with the largest number of components a maximal partition.

Lemma 1. *Among all the horizontally uncoupled partitions, there exists a unique maximal horizontally uncoupled partition.*

Proof. Assume that $\{\Gamma_i\}_{i=1}^H$ and $\{\Gamma'_i\}_{i=1}^H$ are different maximal horizontally uncoupled partitions of Γ . W.l.o.g. $\Gamma_1 \cap \Gamma'_1 \neq \emptyset$ and $\Gamma_1 \setminus \Gamma'_1 \neq \emptyset$. Consider $(\rho, \sigma) \in \Gamma_1 \setminus \Gamma'_1$ and $(\tilde{\rho}, \tilde{\sigma}) \in \Gamma_1 \cap \Gamma'_1$. If $\rho = \tilde{\rho}$, then $\{\Gamma'_i\}_i$ is not a horizontally uncoupled partition. If $\rho \neq \tilde{\rho}$, then $\{\Gamma_i\}_i$ is not maximal. \square

Existence of maximal unique (vertically) uncoupled partitions follows similarly. Examples of maximal uncoupled partition, horizontally, vertically uncoupled partitions can be found in Figures 2(b), 2(c) and 2(d) respectively. We denote the number of components in the maximal horizontally uncoupled partition by H and the components themselves as Γ_i^h , $i = 1, \dots, H$. The common horizontal coordinate of set Γ_i^h is denoted by $\varrho(\Gamma_i^h)$. The maximal vertically uncoupled partition has V components, Γ_j^v , $j = 1, \dots, V$,

where elements of Γ_j^v have common vertical coordinate $\zeta(\Gamma_j^v)$. The maximal uncoupled partition is denoted by $\{\Gamma_k^u\}_{k=1}^U$. The elements of this partition can be obtained by taking the union of elements from $\{\Gamma_i^h\}_{i=1}^H$ or $\{\Gamma_j^v\}_{j=1}^V$. For $k = 1, \dots, U$, let $I_k \subset \{1, \dots, H\}$ and $J_k \subset \{1, \dots, V\}$ be such that $\Gamma_k^u = \bigcup_{i \in I_k} \Gamma_i^h = \bigcup_{j \in J_k} \Gamma_j^v$. Using the maximal uncoupled partition, we can introduce measures m_k , defined as

$$m_k(i, j) = \sum_{(\rho, \sigma) \in \Gamma_k^u} \alpha(\rho, \sigma) \rho^i \sigma^j. \quad (3)$$

This allows us to write $m(i, j) = \sum_{k=1}^U m_k(i, j)$.

Remark: Note that H , V and U are not necessarily finite. With a slight abuse of notation we will write expressions involving for instance $\sum_{k=1}^H$, or $k = 1, \dots, H$, also when H is infinite, *i.e.*, the number of components is countably infinite. The same assumption holds for V .

3 Analysis

We consider the structure and cardinality of Γ . In Section 3.1 we consider the number of uncoupled components in Γ . In Section 3.2 we study the structure of Γ in more detail. Finally, in Section 3.3 we consider the cardinality of Γ .

3.1 Number of uncoupled components in Γ

The following theorem is the first main result. It states that an invariant measure can not be induced by a set Γ of which the uncoupled partition contains multiple components.

Theorem 1. *Consider a homogeneous Markov process P and its invariant measure m . If m is induced by $\Gamma \subset C$, then $U = 1$, *i.e.*, the maximal uncoupled partition $\{\Gamma_1^u, \dots, \Gamma_U^u\}$ of Γ consists of a single component.*

The proof of the theorem is deferred to the end of this section. We first

introduce some additional notation. For any set $\Gamma_i \subset \Gamma$ let

$$B^h(\Gamma_i) = \sum_{(\rho, \sigma) \in \Gamma_i} \alpha(\rho, \sigma) \sum_{s=-1}^1 (\rho^{1-s} h_s + \rho^{1-s} \sigma q_{s,-1}), \quad (4)$$

$$B^v(\Gamma_i) = \sum_{(\rho, \sigma) \in \Gamma_i} \alpha(\rho, \sigma) \sum_{t=-1}^1 (\sigma^{1-t} v_t + \rho \sigma^{1-t} q_{-1,t}). \quad (5)$$

Note that $B^h(\Gamma)$ and $B^v(\Gamma)$ are the balance equations for the measure induced by Γ at the horizontal and vertical boundary respectively.

Lemma 2. *Consider a homogeneous Markov process P and any finite measure m induced by some $\Gamma \subset C$. The sequences $\{B^h(\Gamma_i^h)\}_{i=1}^H$ and $\{B^v(\Gamma_i^v)\}_{i=1}^V$ are absolutely convergent.*

Proof. W.l.o.g. we will only prove that the sequence $\{B^h(\Gamma_i^h)\}_{i=1}^H$ is absolutely convergent.

$$\begin{aligned} \sum_{i=1}^H |B^h(\Gamma_i^h)| &= \sum_{i=1}^H \left| \sum_{(\rho, \sigma) \in \Gamma_i^h} \alpha(\rho, \sigma) \sum_{s=-1}^1 (\rho^{1-s} h_s + \rho^{1-s} \sigma q_{s,-1}) \right| \\ &\leq \sum_{i=1}^H \sum_{(\rho, \sigma) \in \Gamma_i^h} \alpha(\rho, \sigma) \sum_{s=-1}^1 (\rho^{1-s} |h_s| + \rho^{1-s} \sigma |q_{s,-1}|) \\ &\leq M \sum_{(\rho, \sigma) \in \Gamma} \alpha(\rho, \sigma) \frac{1}{1-\rho} \frac{1}{1-\sigma} \\ &< \infty, \end{aligned}$$

where the last equality follows from the fact that m is a finite measure, see also Definition 1. \square

Note that Lemma 2 does not require m to be the invariant measure of P , it can be any finite measure. The following lemma is a key element for the proof of Theorem 1.

Lemma 3. *Consider a measure m and homogeneous Markov process P . Let m be induced by $\Gamma \subset C$. Then m is the invariant measure of P if and only if for all $1 \leq i \leq H$, $1 \leq j \leq V$, $B^h(\Gamma_i^h) = 0$ and $B^v(\Gamma_j^v) = 0$.*

Proof. Since m is the invariant measure of P , it satisfies the balance equations at state $(i, 0)$. Therefore,

$$\begin{aligned} 0 &= \sum_{k=-1}^1 [m(i-k, 0)h_k + m(i-k, 1)q_{k,-1}] \\ &= \sum_{(\rho, \sigma) \in \Gamma} \alpha(\rho, \sigma) \sum_{k=-1}^1 [\rho^{i-k}h_k + \rho^{i-k}\sigma q_{k,-1}] \end{aligned} \quad (6)$$

$$\begin{aligned} &= \sum_{s=1}^H \varrho(\Gamma_s^h)^{i-1} \sum_{(\rho, \sigma) \in \Gamma_s^h} \alpha(\rho, \sigma) \sum_{k=-1}^1 [\rho^{1-k}h_k + \rho^{1-k}\sigma q_{k,-1}] \\ &= \sum_{s=1}^H \varrho(\Gamma_s^h)^{i-1} B^h(\Gamma_s^h). \end{aligned} \quad (7)$$

The exchange of summations is justified by Lemma 2.

Suppose that H is finite. From (7) it follows that $B^h(\Gamma_i^h)$, $1 \leq i \leq H$, satisfy a Vandermonde type system of equations. Moreover, from the properties of a maximal horizontally uncoupled partition, the coefficients $\varrho(\Gamma_i^h)$ are all distinct. It follows that $B^h(\Gamma_i^h) = 0$, $1 \leq i \leq H$. For countably infinite H we resort to [8, Theorem 1], which can be applied based on Lemma 2. Using the same reasoning it follows that $B^v(\Gamma_i^v) = 0$, $1 \leq i \leq V$, finishing one direction of the proof.

Validity of the other direction can be readily verified by observing that, if $B^h(\Gamma_i^h) = 0$, then $\sum_{i=1}^H B^h(\Gamma_i^h) = 0$ and the balance equation for $(i, 0)$, $i > 0$ is satisfied. Using the same reasoning balance at the vertical axis is satisfied. Balance in the interior is satisfied by the assumption that m is induced by $\Gamma \subset C$. Finally, balance in the origin is implied by balance in other parts of the state space. \square

Proof of Theorem 1. We will show that the measures m_k , $k = 1, \dots, U$, satisfy all balance equations. Let measure m_k be induced by Γ_k . By definition of C this implies that all m_k , $k = 1, \dots, U$ satisfy the balance equations in the interior of the state space. For the balance equation for m_k at state $(i, 0)$

at the horizontal boundary we obtain

$$\begin{aligned}
& \sum_{s=-1}^1 [m_k(i-s, 0)h_s + m_k(i-s, 1)q_{s,-1}] \\
&= \sum_{s=-1}^1 \left[\sum_{(\rho, \sigma) \in \Gamma_k^u} \alpha(\rho, \sigma) \rho^{i-s} h_s + \sum_{(\rho, \sigma) \in \Gamma_k^u} \alpha(\rho, \sigma) \rho^{i-s} \sigma q_{s,-1} \right] \\
&= \sum_{(\rho, \sigma) \in \Gamma_k^u} \alpha(\rho, \sigma) \sum_{s=-1}^1 [\rho^{i-s} h_s + \rho^{i-s} \sigma q_{s,-1}] \\
&= \sum_{l \in I_k} \varrho(\Gamma_l^h)^{i-1} \sum_{(\rho, \sigma) \in \Gamma_l^h} \alpha(\rho, \sigma) \sum_{s=-1}^1 [\rho^{1-s} h_s + \rho^{1-s} \sigma q_{s,-1}] \\
&= \sum_{l \in I_k} \varrho(\Gamma_l^h)^{i-1} B^h(\Gamma_l^h) \\
&= 0.
\end{aligned}$$

By Lemma 2, the interchange of the summations leading to the second equality is valid. The last equality follows from Lemma 3.

In similar fashion it follows that the balance equations at the vertical boundary are satisfied. As a consequence, we have shown that m_1, \dots, m_U are invariant measures of P . When $U > 1$, this contradicts to the fact that there is a unique invariant measure for ergodic Markov process P . \square

3.2 Structure of Γ

In this section the structure of set Γ will be discussed. From Theorem 1 it follows if the number of components in the maximal uncoupled partition is greater than one, then a measure induced by Γ cannot be the invariant measure of homogeneous Markov process P . In this section we investigate the measure induced by a set with one uncoupled component. To this end, we introduce the notion of a pairwise-coupled set.

Definition 3 (Pairwise-coupled). *A countable ordered subset Γ of C , $\Gamma = \{(\rho_k, \sigma_k), k = 1, 2, 3, \dots\}$ is a pairwise-coupled set if and only if one of the following is true.*

- 1) $\rho_1 = \rho_2, \sigma_1 > \sigma_2, \rho_2 > \rho_3, \sigma_2 = \sigma_3, \rho_3 = \rho_4, \sigma_3 > \sigma_4, \dots$,

- 2) $\rho_1 > \rho_2, \sigma_1 = \sigma_2, \rho_2 = \rho_3, \sigma_2 > \sigma_3, \rho_3 > \rho_4, \sigma_3 = \sigma_4, \dots$,
3) $\rho_1 < \rho_2, \sigma_1 = \sigma_2, \rho_2 = \rho_3, \sigma_2 > \sigma_3, \rho_3 < \rho_4, \sigma_3 = \sigma_4, \dots$,
4) $\rho_1 = \rho_2, \sigma_1 > \sigma_2, \rho_2 < \rho_3, \sigma_2 = \sigma_3, \rho_3 = \rho_4, \sigma_3 > \sigma_4, \dots$.

Pairwise coupling allows the explicit characterization of the structure of set Γ that is required if the measures induced by Γ are the invariant measure of homogeneous Markov process P . The following corollary is a simple application of Theorem 1.

corollary 1. *Consider a homogeneous Markov process P and its invariant measure m . If m is induced by $\Gamma \subset C$, then Γ is a pairwise-coupled set.*

3.3 Cardinality of Γ

From Sections 3.1 and 3.2 we know that Γ consists of a single component and is hence pairwise-coupled. The next theorem characterizes the cardinality of this component.

Theorem 2. *Consider a homogeneous Markov process P and its invariant measure m . If m is induced by $\Gamma \subset C$, then Γ can contain either one or countably many elements.*

The proof of this theorem follows from Lemma 4 and 7, that deal with the cases of $|\Gamma| = 2$ and $2 < |\Gamma| < \infty$, respectively.

Lemma 4. *Consider a homogeneous Markov process P and its invariant measure m . If m is induced by a pairwise-coupled set $\Gamma \subset C$, then $|\Gamma| \neq 2$.*

Proof. Suppose that

$$m(i, j) = \alpha(\rho, \sigma)\rho^i\sigma^j + \alpha(\rho, \tilde{\sigma})\rho^i\tilde{\sigma}^j, \quad (8)$$

where $(\rho, \sigma) \in C$ and $(\rho, \tilde{\sigma}) \in C$.

It follows from the definition of C that σ and $\tilde{\sigma}$ are the roots of the following quadratic equation in x ,

$$\sum_{k=-1}^1 \sum_{s=-1}^1 \rho^{-s} q_{s,k} x^{1-k} = 0. \quad (9)$$

Note that the maximal vertically uncoupled partition of $\{(\rho, \sigma), (\rho, \tilde{\sigma})\}$ consists of the two singleton components $\{(\rho, \sigma)\}$ and $\{(\rho, \tilde{\sigma})\}$. It follows

from Lemma 3 that $B^v(\{(\rho, \sigma)\}) = B^v(\{(\rho, \tilde{\sigma})\}) = 0$. Therefore, σ and $\tilde{\sigma}$ are the roots of

$$\sum_{s=-1}^1 (\rho q_{-1,s} + v_s) x^{1-s} = 0. \quad (10)$$

From a comparison of the coefficients of (9) and (10) it follows that either *a*) one of the roots will be 1, contradicting the definition of C , or *b*) the transition rates of P are such that P is not irreducible and hence not ergodic. Hence, m as defined in (8) can not be the invariant measure of P . Using the same argument it follows that a form $\alpha(\rho, \sigma)\rho^i\sigma^j + \alpha(\rho, \tilde{\sigma})\rho^i\tilde{\sigma}^j$ cannot be the invariant measure of P . \square

Before proving the final lemma, *i.e.*, that Γ satisfying $2 < |\Gamma| < \infty$ can not induce an invariant measure, we introduce a final piece of notation and some technical results that will help in the presentation of the remaining proofs. Observe that $B^h(\Gamma_i^h) = 0$ is a linear relation in h_1 and h_{-1} . Let $b^h(\Gamma_i^h)$ be defined as

$$B^h(\Gamma_i^h) = 0 \iff b^h(\Gamma_i^h) = \left(1 - \frac{1}{\varrho(\Gamma_i^h)}\right) h_1 + (1 - \varrho(\Gamma_i^h)) h_{-1}. \quad (11)$$

Analogously we define $b^v(\Gamma_i^v)$ as

$$B^v(\Gamma_i^v) = 0 \iff b^v(\Gamma_i^v) = \left(1 - \frac{1}{\varsigma(\Gamma_i^v)}\right) v_1 + (1 - \varsigma(\Gamma_i^v)) v_{-1}. \quad (12)$$

The following technical result will greatly simplify the presentation of our final proofs of the cases when the measure is induced by Γ satisfies $2 < |\Gamma| < \infty$.

Lemma 5. *If $\tilde{\sigma} > \sigma$ and $\tilde{\rho} > \rho$ then*

$$\begin{aligned} b^h(\{(\rho, \sigma), (\rho, \tilde{\sigma})\}) &> b^h(\{(\rho, \sigma)\}), & b^h(\{(\rho, \sigma), (\rho, \tilde{\sigma})\}) &< b^h(\{(\rho, \tilde{\sigma})\}), \\ b^v(\{(\rho, \sigma), (\tilde{\rho}, \sigma)\}) &> b^v(\{(\rho, \sigma)\}), & b^v(\{(\rho, \sigma), (\tilde{\rho}, \sigma)\}) &< b^v(\{(\tilde{\rho}, \sigma)\}). \end{aligned}$$

Proof. From the definition in (11) it follows that

$$\begin{aligned} b^h(\{(\rho, \sigma), (\rho, \tilde{\sigma})\}) &= \frac{\alpha(\rho, \sigma)\sigma + \alpha(\rho, \tilde{\sigma})\tilde{\sigma}}{\alpha(\rho, \sigma) + \alpha(\rho, \tilde{\sigma})} (\rho q_{-1,-1} + q_{0,-1} + \frac{1}{\rho} q_{1,-1}) - \\ &\quad q_{1,1} - q_{0,1} - q_{-1,1}, \end{aligned}$$

$$b^h(\{(\rho, \sigma)\}) = \sigma(\rho q_{-1,-1} + q_{0,-1} + \frac{1}{\rho} q_{1,-1}) - q_{1,1} - q_{0,1} - q_{-1,1},$$

and

$$b^h(\{(\rho, \tilde{\sigma})\}) = \tilde{\sigma}(\rho q_{-1,-1} + q_{0,-1} + \frac{1}{\rho} q_{1,-1}) - q_{1,1} - q_{0,1} - q_{-1,1}.$$

From the above the first row of inequalities follow directly. The remaining inequalities follow directly from (12). \square

The second technical lemma that we will need is readily verified and stated without proof.

Lemma 6. *If $t_1(1 - \rho) + t_2(1 - \tilde{\rho}) \geq 0$, $t_1(1 - 1/\rho) + t_2(1 - 1/\tilde{\rho}) \geq 0$ and $\tilde{\rho} > \rho$, then $t_1 \leq 0$ and $t_2 \geq 0$.*

The final result, together with Lemma 4 it provides the proof of Theorem 2 is the following lemma.

Lemma 7. *Consider a homogeneous Markov process P , if $2 < |\Gamma| < \infty$, then no measure induced by Γ can be in the invariant measure of P .*

Proof of Lemma 7. Now we need to show the the measure of the form $m(i, j) = \sum_{l=1}^n \rho_l^i \sigma_l^j$ with $n < \infty$ can not be an invariant measure. W.l.o.g. we can assume that $\rho_1 = \rho_2$, $\sigma_1 > \sigma_2$ and $\sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_n$. Moreover, $\varsigma(\Gamma_j^V)$ is strictly decreasing with j . We deal with two cases separately. The first case is $\rho_2 > \rho_3$, $\rho_3 \geq \rho_4 \geq \dots \geq \rho_n$ and $\varrho(\Gamma_i^H)$ is strictly decreasing with i . The second case is $\rho_2 < \rho_3$, $\rho_3 \leq \rho_4 \leq \dots \leq \rho_n$ and $\varrho(\Gamma_i^H)$ is strictly increasing with i .

For the first case we consider the relations

$$\begin{aligned} (1 - 1/\rho_1) h_1 + (1 - \rho_1) h_{-1} &= b^h(\Gamma_1^h), \\ (1 - 1/\rho_n) h_1 + (1 - \rho_n) h_{-1} &= b^h(\Gamma_n^h), \\ (1 - 1/\sigma_1) v_1 + (1 - \sigma_1) v_{-1} &= b^v(\Gamma_1^v), \\ (1 - 1/\sigma_n) v_1 + (1 - \sigma_n) v_{-1} &= b^v(\Gamma_n^v), \end{aligned} \tag{13}$$

which by Lemma 3 are required to hold if m is the invariant measure of P . We will construct s_1 , s_2 , t_1 and t_2 that satisfy

$$\begin{aligned} (1 - 1/\rho_1) s_1 + (1 - 1/\rho_n) s_2 &\geq 0, \\ (1 - \rho_1) s_1 + (1 - \rho_n) s_2 &\geq 0, \\ (1 - 1/\sigma_1) t_1 + (1 - 1/\sigma_n) t_2 &\geq 0, \\ (1 - \sigma_1) t_1 + (1 - \sigma_n) t_2 &\geq 0 \end{aligned} \tag{14}$$

and

$$b^h(\Gamma_1^h)s_1 + b^h(\Gamma_H^h)s_2 + b^v(\Gamma_1^v)t_1 + b^v(\Gamma_V^v)t_2 < 0. \quad (15)$$

By Farkas' Lemma this leads to a contradiction to (13).

The s_1, s_2, t_1 and t_2 are constructed by considering the auxiliary measure $\bar{m} = \alpha(\rho_1, \sigma_1)\rho_1^i\sigma_1^j + \alpha(\rho_n, \sigma_n)\rho_n^i\sigma_n^j$ and the homogeneous Markov process \bar{P} , that has the same transition rates as P in the interior of the state space and rates $\bar{h}_1, \bar{h}_{-1}, \bar{v}_1$ and \bar{v}_{-1} along the boundaries. We now consider the relations

$$\begin{aligned} (1 - 1/\rho_1)\bar{h}_1 + (1 - \rho_1)\bar{h}_{-1} &= b^h(\{(\rho_1, \sigma_1)\}), \\ (1 - 1/\rho_n)\bar{h}_1 + (1 - \rho_n)\bar{h}_{-1} &= b^h(\{(\rho_n, \sigma_n)\}), \\ (1 - 1/\sigma_1)\bar{v}_1 + (1 - \sigma_1)\bar{v}_{-1} &= b^v(\{(\rho_1, \sigma_1)\}), \\ (1 - 1/\sigma_n)\bar{v}_1 + (1 - \sigma_n)\bar{v}_{-1} &= b^v(\{(\rho_n, \sigma_n)\}). \end{aligned} \quad (16)$$

If (16) would hold for $\bar{h}_1 = h_1, \bar{h}_{-1} = h_{-1}, \bar{v}_1 = v_1$ and $\bar{v}_{-1} = v_{-1}$, \bar{m} would be the invariant measure of P which contradicts the assumption that m is the invariant measure of P . However, if \bar{P} is ergodic (16) can not hold due to Theorem 1. If \bar{P} is not ergodic, there is no finite invariant measure and (16) can not hold either. Therefore, (16) is not satisfied for any non-negative $\bar{h}_1, \bar{h}_{-1}, \bar{v}_1$ and \bar{v}_{-1} . By Farkas' Lemma, there exist s_1, s_2, t_1 and t_2 that satisfy (14) and

$$b^h(\{(\rho_1, \sigma_1)\})s_1 + b^h(\{(\rho_n, \sigma_n)\})s_2 + b^v(\{(\rho_1, \sigma_1)\})t_1 + b^v(\{(\rho_n, \sigma_n)\})t_2 < 0.$$

Note, that from Lemma 5 it follows that $b^h(\{\Gamma_1^h\}) < b^h(\{(\rho_1, \sigma_1)\})$, $b^h(\{\Gamma_H^h\}) \geq b^h(\{(\rho_n, \sigma_n)\})$, $b^v(\{\Gamma_1^v\}) = b^v(\{(\rho_1, \sigma_1)\})$ and $b^v(\{\Gamma_V^v\}) \geq b^v(\{(\rho_n, \sigma_n)\})$. Also, from Lemma 6 it follows that $s_1 \geq 0, s_2 \leq 0, t_1 \geq 0, t_2 \leq 0$. Therefore, s_1, s_2, t_1 and t_2 satisfy (15). This concludes the proof of the first case.

For the second case we consider the relations

$$\begin{aligned} (1 - 1/\sigma_1)v_1 + (1 - \sigma_1)v_{-1} &= b^v(\Gamma_1^v), \\ (1 - 1/\sigma_2)v_1 + (1 - \sigma_2)v_{-1} &= b^v(\Gamma_2^v), \end{aligned} \quad (17)$$

that are necessary for m to be the invariant measure and obtain a contradiction by constructing t_1 and t_2 that satisfy

$$(1 - 1/\sigma_1)t_1 + (1 - 1/\sigma_2)t_2 \geq 0, \quad (18)$$

$$(1 - \sigma_1)t_1 + (1 - \sigma_2)t_2 \geq 0, \quad (19)$$

$$b^v(\Gamma_1^v)t_1 + b^v(\Gamma_1^v)t_2 < 0. \quad (20)$$

The auxiliary measure that is used is $\tilde{m}(i, j) = \alpha(\rho_1, \sigma_1)\rho_1^i\sigma_1^j + \alpha(\rho_2, \sigma_2)\rho_2^i\sigma_2^j$. Observe that $\rho_1 = \rho_2$ and that the corresponding relations are

$$\begin{aligned} (1 - 1/\rho_1) h_1 + (1 - \rho_1)h_{-1} &= b^h(\{(\rho_1, \sigma_1), (\rho_2, \sigma_2)\}), \\ (1 - 1/\sigma_1) v_1 + (1 - \sigma_1)v_{-1} &= b^v(\{(\rho_1, \sigma_1)\}), \\ (1 - 1/\sigma_2) v_1 + (1 - \sigma_2)v_{-1} &= b^v(\{(\rho_2, \sigma_2)\}). \end{aligned}$$

From Farkas' Lemma and Lemma 4 it follows that there exist s_1, t_1 and t_2 that satisfy (18), (19) and

$$b^h(\{(\rho_1, \sigma_1), (\rho_2, \sigma_2)\})s_1 + b^v(\{(\rho_1, \sigma_1)\})t_1 + b^v(\{(\rho_2, \sigma_2)\})t_2 \leq 0, \quad (21)$$

where $s_1 = 0$, since it satisfies $(1 - 1/\rho_1)s_1 \geq 0$ and $(1 - \rho_1)s_1 \geq 0$. Moreover, since, $b^v(\Gamma_1^v) = b^v(\{(\rho_1, \sigma_1)\})$ and, by Lemma 5, we have $b^v(\Gamma_2^v) > b^v(\{(\rho_2, \sigma_2)\})$. Moreover, by Lemma 6 we have, $t_1 \geq 0, t_2 \leq 0$. Then it follows that t_1, t_2 satisfy (20). This concludes the proof of the second case. \square

To summarize the contributions of the present paper, we combine Theorems 1 and 2 into the following corollary.

corollary 2. *Consider a homogeneous Markov process in the quarter-plane and its invariant measure m . If m is of the form*

$$m(i, j) = \sum_{(\rho, \sigma) \in \Gamma} \alpha(\rho, \sigma)\rho^i\sigma^j,$$

then Γ is pairwise-coupled and has either one element or countably many.

Note that if Γ has one element, then m as geometric product form and many examples exist in literature, see also [9]. The existence of invariant measures with countably many terms has been demonstrated in [2].

4 Acknowledgments

The authors wish to thank Anton Stoorvogel for some useful discussions and his pointer to [8]. Yanting Chen gratefully acknowledges support through a CSC scholarship.

References

- [1] R. J. Boucherie and N. M. van Dijk, Eds., *Queueing Networks: A Fundamental Approach*. Springer Verlag, 2010.
- [2] I. J.-B. F. Adan, J. Wessels, and W. H. M. Zijm, “Analysis of the asymmetric shortest queue problem,” *Queueing Systems Theory Appl.*, vol. 8, no. 1, pp. 1–58, 1991.
- [3] ———, “A compensation approach for two-dimensional Markov processes,” *Adv. in Appl. Probab.*, vol. 25, no. 4, pp. 783–817, 1993. [Online]. Available: <http://dx.doi.org/10.2307/1427792>
- [4] G. Fayolle and R. Iasnogorodski, “Two coupled processors: the reduction to a Riemann-Hilbert problem,” *Z. Wahrsch. Verw. Gebiete*, vol. 47, no. 3, pp. 325–351, 1979. [Online]. Available: <http://dx.doi.org/10.1007/BF00535168>
- [5] L. Flatto and H. P. McKean, “Two queues in parallel,” *Comm. Pure Appl. Math.*, vol. 30, no. 2, pp. 255–263, 1977.
- [6] J. W. Cohen and O. J. Boxma, *Boundary value problems in queueing system analysis*, ser. North-Holland Mathematics Studies. Amsterdam: North-Holland Publishing Co., 1983, vol. 79.
- [7] G. Fayolle, R. Iasnogorodski, and V. Malyshev, *Random walks in the quarter-plane: algebraic methods, boundary value problems and applications*. Springer Verlag, 1999, vol. 40.
- [8] L. Brown, A. Shields, and K. Zeller, “On absolutely convergent exponential sums,” *Transactions of the American Mathematical Society*, vol. 96, no. 1, pp. 162–183, 1960.
- [9] N. Bayer and R. J. Boucherie, “On the structure of the space of geometric product-form models,” *Probab. Engrg. Inform. Sci.*, vol. 16, no. 2, pp. 241–270, 2002. [Online]. Available: <http://dx.doi.org/10.1017/S0269964802162073>