Energy-delay Tradeoff in Wireless Network Coding

Jasper Goseling, Richard J. Boucherie, and Jan-Kees van Ommeren

j.goseling@utwente.nl, r.j.boucherie@utwente.nl, j.w.c.vanommeren@utwente.nl

Stochastic Operations Research Group,
Department of Applied Mathematics,
University of Twente,
P.O. Box 217, 7500 AE, Enschede, The Netherlands

May 23, 2011

Abstract

A queueing model for wireless communication network in which network coding is employed is introduced. It is shown that networks with coding are closely related to queueing networks with positive and negative customers. Analytical upper and lower bounds on the energy consumption and the delay are obtained using a Markov reward approach. The tradeoff between minimizing energy consumption and minimizing delay is investigated. Exact expressions are given for the minimum energy consumption and the minimum delay attainable in a network.

1 Introduction

Current wireless networking technology is based on the principle that information is transported in a wireless network by forwarding packets. An important aspect of this is that information from different connections is kept independent. The concept of network coding, introduced in [1], is based on the observation that it can be useful to do additional processing at intermediate nodes and combine different packets. This implies that information from different connections is mixed. We will see that network coding can improve the efficiency of wireless networks. In particular, we will analyze the increase in efficiency offered by network coding in terms of energy consumption and delay.

To illustrate the difference between classical packet forwarding and network coding, we consider a wireless network in which devices A and C need to exchange bits x and y using a relay B. First, as illustrated in Figure 1a, consider the classical case. Four transmissions – seperated in time, frequency or signal space – are required. Most importantly, note that the relay is transmitting twice

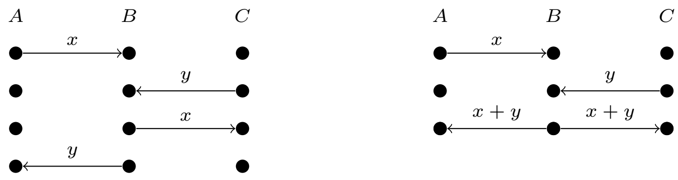

Figure 1: Illustrating network coding.

and that each of the transmissions is useful to only one of the other nodes. Second, consider the network coding case, as depicted in Figure 1b. The relay B computes $z = x + y$, the exclusive or of the bits x and y, and transmits z, which is again a single bit. Node A recovers y by taking the exclusive or of z and x, where we note that x is already known by A. Node C can recover x, by taking the exclusive or of z and y. Hence, the transmission of a single bit by the relay is useful to both other nodes.

From the above example it is clear that, compared to the classical approach, network coding can reduce the number of transmissions in a wireless communication network. Hence, network coding has the potential to reduce the energy consumption of nodes as well as the delay experienced by packets. These two properties are among the most important bottlenecks in wireless networks and it is precisely these aspects that are studied in this paper.

If we consider the energy consumed by the relay in the simple example discussed above, we observe that it can be reduced by 50% by employing network coding. If the problem of transmitting a single bit over a single relay is generalized to transmitting streams of packets in larger networks, the reduction of 50% in energy consumption can still be achieved [31] and even larger savings are possible [17].

The results in [17, 31] depend on the assumption that relay nodes always have packets from all connections to transmit, *i.e.*, queues are saturated. In this paper we develop a queueing model without this assumption and analyze the expected energy consumption and delay. In particular, we will compare network coding with classical network operation.

We model a wireless network in which network coding is employed as a continuous-time Markov process. In particular we will consider the two-dimensional model arising from the queueing process at the relay in the example from Figure 1. In the model we keep two queues, one for packets from each of the sessions. Packets from the two sessions arrive independently. We model the transmission of a combination of packets as the simultaneous departure of two packets, one packet from each queue. One of the results established in this paper is that in order to keep the system stable, it is neccessary to also transmit uncoded packets. In particular, if packets are present in one queue while the other queue is empty, uncoded packets will need to be transmitted. We will, therefore, allow for operating policies in which packets from one queue can be transmitted uncoded if the other queue is empty.

Our queueing model has similar properties as a queueing network with positive and negative customers [15]. However, due to the behavior when one of the queues is empty, it does, in general, not a have a geometric product-form stationary distribution. Hence, exact results on its performance can not readily be obtained and we will resort to finding analytical performance bounds. These bounds will be obtained from Markov reward error bounding techniques [30] by relating the performance to that of a perturbed model with a product-form stationary distribution. Observe that packets arrive at the relay one by one, but that packets depart from two queues simultaneously. This is the typical behaviour of queueing networks with positive and negative customers. We will see that if the operating policy that decides when to transmit an uncoded packet is chosen carefully, our model is a queueing network with positive and negative customers.

The introduction of network coding has led to a surge of research papers, as well as a number of monographs and introductory papers, see, for instance, [9, 12–14, 32] and the references therein. In this section discuss literature related to the contributions made in this paper and refer to the reader to the existing literature for a complete overview on results established in the field.

In recent years there has been significant attention to the role of stochastic arrivals and queueing in relation to network coding. Lun et al. [24] develop a Jackson network model for lossy networks with network coding and a single (multicast) session. Their approach does not seem to generalize to multiple sessions. Sagduyu and Ephremides [25] consider stochastic arrivals in a network coding system with multiple connections and analyze the tradeoff between throughput and energy consumption. They do not consider coding strategies that allow to transmit uncoded packets. Our model provides a natural way of analyzing these strategies. Most other work, for instance [10, 21, 28, 29], is focussing on the delay in unreliable broadcast systems with feedback. Our interest is in the energy consumption in networks without losses. In particular we are interested in the impact of stochastic arrivals on the energy benefits demonstrated in [31].

We make the following contributions:

1. The analogy between queueing networks with coding and queueing networks with positive and negative customers is demonstrated.

2. Stability criteria for network with coding are derived.

3. Analytical bounds on the energy consumption and the delay are given.

4. Exact expressions for the minimum possible energy consumption and minimum possible delay are given.

The remainder of this paper is organized as follows. In Section 2 we specify the continuous-time Markov chain that will be analyzed and the performance measures of interest. Section 3 is devoted to discussing some of the preliminaries that are required later in the paper. In particular we discuss queueing networks with negative customer and Markov reward error bounds. In Section 4 we give neccesary and sufficient conditions for ergodicity of our model. Performance bounds on expected queue size and energy consumption are presented in 5. Numerical examples of the results obtained in the paper are given in Section 6. Finally, in Section 7 we discuss the results presented in this paper and offer suggestions for future work.

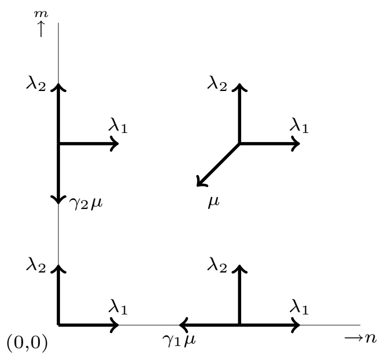

Figure 2: Transition diagram for Q^{γ_1,γ_2}, the Markov process of the coded system.

2 Model and Problem Statement

We consider a single node in a wireless network that is acting as a relay for two sessions and develop two different continuous-time queueing models. The classical case without network coding is covered by the first model. In the second model network coding is used. Packets from both sessions arrive at the node according to independent Poisson processes with rate λ_1 and λ_2. The time required to transmit a packet, *i.e.*, to provide service for a packet, is exponentially distributed with rate μ.

The uncoded system is modelled as a single server operating on a single queue using a FIFO policy. Hence the uncoded system is an M/M/1 queue with arrival rate $\lambda_1 + \lambda_2$ and service rate μ.

In the coded system a seperate queue is kept for each session, leading to a two-dimensional model in which the state variables N and M denote the number of packets contained in each of the queues. Network coding is employed by transmitting linear combinations of two packets, one packet from each queue in a combination. This means that a service completion will reduce the number of packets in both queues by one. If only one queue has a packet it is transmitted uncoded and a service completion will remove only one packet from a queue. Since transmitting an uncoded packet is unfavorable in terms of, for instance, energy consumption, we allow for an operating policy in which uncoded packets will not always be transmitted if opportunity arises.

If there is the opportunity to transmit a packet from the first queue, while the second queue is empty, this packet will be transmitted with probability γ_1. Similarly, packets from the second queue will be transmitted uncoded with probablity γ_2.

The above description leads to a continuous-time Markov chain Q^{γ_1,γ_2} on state space $\mathbb{N}_0^2$ with transition rates q^{γ_1,γ_2} defined as

$$q_{n,m}^{\gamma_1,\gamma_2}(i,j) = \begin{cases} \lambda_1, & \text{if } i = 1, \quad j = 0, \quad n \geq 0 \quad m \geq 0, \\ \lambda_2, & \text{if } i = 0, \quad j = 1, \quad n \geq 0 \quad m \geq 0, \\ \mu, & \text{if } i = -1, \quad j = -1, \quad n > 0, \quad m > 0, \\ \gamma_1\mu, & \text{if } i = -1, \quad j = 0, \quad n > 0, \quad m = 0, \\ \gamma_2\mu, & \text{if } i = 0, \quad j = -1, \quad n = 0, \quad m > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

where $q_{n,m}^{\gamma_1,\gamma_2}(i,j)$ denotes the transition rate from state (n,m) to state $(n+i, m+j)$. To ensure irreducibility and aperiodicity of the chain we assume $\lambda_1 > 0$, $\lambda_2 > 0$ and $\mu > 0$. Remember that γ_1 and γ_2 denote probabilities and take values in the interval $[0,1]$. The transition structure is depicted in the transition diagram of Fig. 2. To simplify the notation in the remainder of the paper we introduce

$$\rho_1 = \frac{\lambda_1}{\mu}, \quad \rho_2 = \frac{\lambda_2}{\mu}, \quad \gamma_1^* = \frac{\rho_1 - \rho_2}{1-\rho_2}, \quad \gamma_2^* = \frac{\rho_2 - \rho_1}{1-\rho_1}. \tag{2}$$

In Section 4 it will become clear that γ_1^* and γ_2^* are useful for expressing stability criteria.

Our interest is in two different steady-state performance measures of Q^{γ_1,γ_2}. To introduce notation, first consider an arbitary cost/reward function $f_{\beta_1,\beta_2} : \mathbb{N}_0^2 \to [0,\infty)$, depending on parameters β_1 and β_2. Let

$$F_{\beta_1,\beta_2}^{\gamma_1,\gamma_2} = \mathbb{E}^{\gamma_1,\gamma_2}[f_{\beta_1,\beta_2}(N,M)], \tag{3}$$

where the expected value is over the stationary distribution of the process Q^{γ_1,γ_2}.

The first performance measure that we consider is the expected energy consumption per unit time. Before introducing the corresponding cost function, consider the cost function

$$c_{\beta_1,\beta_2}(n,m) = \beta_1\mu\mathbb{1}_{\{n>0,m=0\}} + \beta_2\mu\mathbb{1}_{\{n=0,m>0\}} + \mu\mathbb{1}_{\{n>0,m>0\}}, \tag{4}$$

with parameters β_1 and β_2. The energy consumption will be bounded in terms of $C_{\beta_1,\beta_2}^{\hat{\gamma}_1,\hat{\gamma}_2}$, where $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are not necessarily equal to γ_1 and γ_2, respectively. The energy consumed by transmitting a packet is μ per unit time. Therefore, the expected energy consumed per unit time is

$$\begin{aligned} 0, &\quad \text{if } n = 0, m = 0, \\ \gamma_1\mu, &\quad \text{if } n > 0, m = 0, \\ \gamma_2\mu, &\quad \text{if } n = 0, m > 0, \\ \mu, &\quad \text{if } n > 0, m > 0, \end{aligned} \tag{5}$$

where it is taken into account that a packet is transmitted with probability γ_1 (γ_2) if there is a packet in the first (second) queue while the second (first) queue is empty. It follows that the expected energy consumption of Q^{γ_1,γ_2} equals

$$C_{\gamma_1,\gamma_2}^{\gamma_1,\gamma_2} = \mathbb{E}^{\gamma_1,\gamma_2}\left[c_{\gamma_1,\gamma_2}(N,M)\right], \tag{6}$$

which will also be denoted as C^{γ_1,γ_2}, omitting the subscripts.

The second performance measure of interest is the expected delay, *i.e.*, the expected sojourn time of a packet in the system. Without loss of generality we will consider the delay of packets of the first connection and denote it by D^{γ_1,γ_2}. By Little's law it follows that the expected delay is $D^{\gamma_1,\gamma_2} = \mathbb{E}[N]/\lambda_1$, leading to cost function

$$d(n,m) = \frac{1}{\lambda_1} n. \tag{7}$$

Remember, that the uncoded system is an M/M/1 queue with arrival rate $\lambda_1 + \lambda_2$ and service rate μ. Therefore, the expected energy consumption in the uncoded system is

$$C_{\text{uncoded}} = \lambda_1 + \lambda_2 \tag{8}$$

and the expected delay is

$$D_{\text{uncoded}} = \frac{1}{\mu - \lambda_1 - \lambda_2}. \tag{9}$$

3 Preliminaries

Before starting the analysis of the process Q^{γ_1,γ_2} in the next section we will provide some background on the techniques that will be used. First, we discuss queueing networks with positive and negative customers, their relevance for the work at hand and the stationary distribution of such networks. Next, we provide results on Markov reward error bounding techniques.

3.1 Queueing Networks with Positive and Negative Customers

We start this section with an interpretation of Q^{γ_1,γ_2} for the case that $\gamma_1+\gamma_2 = 1$. Under this condition the network can be interpreted as having two dedicated servers, one for each queue, operating at rates $\gamma_1\mu$ and $\gamma_2\mu$. In addition, if a packet is leaving from one of the queues and there is a packet in the other queue, that packet is also removed from the queue. This type of queueing network was first studied by Gelenbe [15]. The networks considered by Gelenbe in [15] are very similar to Jackson networks, with the additional feature that there are two types of customers: positive and negative. Positive customers, upon arriving at a node, require service and are placed in the queue. Negative customers, upon arriving at a queue, do not require service and instead, remove a positive customer from the queue. Upon completing service, there are three possible actions for a positive customer: 1) it leaves the system, 2) it enters another queue in the system as a positive customer, or 3) it enters another queue in the system as a negative customer. The customer chooses randomly, with a fixed probability distribution, which action to take and/or which queue to join. It is shown in [4, 15] that these networks have a product-form stationary distribution. The parameters of the distribution are the solution of a set of polynomial equations that can be given for any network of the form described above. We give the resulting set of equations for the system Q^{γ_1,γ_2} under the condition that $\gamma_1 + \gamma_2 = 1$.

Theorem 1 (Gelenbe [15])**.** *Consider the Markov process Q^{γ_1,γ_2} with $\gamma_1+\gamma_2 = 1$. If the system of equations in σ_1 and σ_2 given by*

$$\gamma_1\sigma_1 + \gamma_2\sigma_1\sigma_2 = \rho_1, \quad \gamma_2\sigma_2 + \gamma_1\sigma_1\sigma_2 = \rho_2, \tag{10}$$

has a unique solution satisfying $0 < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$*, then the stationary distribution* $\pi(n, m)$ *is given by*

$$\pi(n, m) = (1 - \sigma_1)\,\sigma_1^n\,(1 - \sigma_2)\,\sigma_2^m.$$

The above result is a special case of the result by Gelenbe, for a network of two queues with service rates $\gamma_1\mu$ and $\gamma_2\mu$, external arrival of positive customers with rates λ_1 and λ_2, no external arrivals of negative customers, and customers leaving one queue entering the other queue as negative customers.

Remark 1. *The above theorem provides an expression for the stationary distribution under the condition that a unique solution* $0 < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$ *exists. Note that this corresponds exactly to the condition that the process is stable, i.e., ergodic. Neccessary and sufficient for stability are given in [16] in terms of the properties of a fixed point of a continuous function derived from* (10)*. This fixed point can, in general, not be given in explicit form. In Section 4 we will obtain explicit stability criteria for* Q^{γ_1,γ_2} *based on the theory of two dimensional random walks in the positive orthant.*

Remark 2. *The original work of Gelenbe was based on applications in neural networks. Other applications have been reported in, e.g., distributed computing and database systems. The application to commmunication networks is new.*

Remark 3. *The concept of networks with positive and negative customers can be generalized in many directions, see, for instance, [2, 3, 26, 27]. These generalizations include networks in which a negative customer removes positive customers from multiple queues simultaneously. Extensions of the current work to larger networks with more data connections seem possible based on these generalizations.*

3.2 Markov Reward Error Bounds

Since for $\gamma_1 + \gamma_2 \neq 1$ the steady-state distribution of the queueing process can not be obtained in a tractable analytical form, we will use the Markov reward approach to obtain analytical approximations on the performance of Q^{γ_1,γ_2}. This technique, developed by van Dijk [6, 7], is based on relating the steady-state performance of the process to the cummulative reward structure in the discrete-time uniformized process. An introduction to this technique is given in, for instance, [30]. Throughout the remainder of this section we omit the dependence on γ_1 and γ_2 in the notation.

Let $f : \mathbb{N}_0^2 \to [0, \infty)$ be an arbitrary performance measure and denote by $\mathbb{E}\,[f(N, M)]$ the expected performance of Q under the (unknown) stationary distribution π. In addition consider a second Markov process $\bar{Q}$ on the same state space $\mathbb{N}_0^2$, but with different transition rates $\bar{q}$. Finally, consider a second performance measure $\bar{f} : \mathbb{N}_0^2 \to [0, \infty)$. Assume that the stationary distribution for $\bar{Q}$, $\bar{\pi}$ is known. We will approximate $\mathbb{E}\,[f(N, M)]$ in terms of $\bar{\mathbb{E}}\left[\bar{f}(N, M)\right]$, the expected value of the perturbed measure under distribution $\bar{\pi}$.

Assume that Q and $\bar{Q}$ can both be uniformized and let h be a suitable uniformization constant for both processes, *i.e.*, let h satisfy

$$h \leq \left(\sum_{i,j} q_{n,m}(i, j)\right)^{-1} \quad \text{and} \quad h \leq \left(\sum_{i,j} \bar{q}_{n,m}(i, j)\right)^{-1}, \tag{11}$$

for all $(n,m) \in \mathbb{N}_0^2$.

Let P denote the discrete-time Markov process obtained from uniformization of Q. For P the probability of jumping from (n,m) to $(n+i,m+j)$, $p_{n,m}(i,j)$, is defined as

$$p_{n,m}(i,j) = \begin{cases} hq_{n,m}(i,j), & \text{if } (i,j) \neq (0,0), \\ 1 - h\sum_{(i,j)\neq 0} q_{n,m}(i,j), & \text{if } (i,j) = (0,0). \end{cases} \tag{12}$$

On P consider a one step reward $hf(n,m)$ whenever the system is in state (n,m). This leads to expected cummulative reward $F(k,n,m)$, incurred by the uniformized process at time step k when starting from state (n,m) at time 0, defined as

$$F(k+1,n,m) = hf(n,m) + \sum_{i,j} p_{n,m}(i,j)F(k,n+i,m+j), \tag{13}$$

for $k > 0$ and $F(0,n,m) = 0$.

Using the above notation and definitions we are ready to state the results from [30] that will be used to obtain bounds on $\mathbb{E}\left[f(N,M)\right]$.

Theorem 2 (van Dijk [30])**.** *Let Q and $\bar{Q}$ be two continuous-time Markov processes on the same state space $\mathbb{N}_0^2$, with transition rates q and $\bar{q}$, respectively. In addition consider the cost functions f and $\bar{f}$ from $\mathbb{N}_0^2$ to $[0,\infty)$. Suppose that there exists a function $\xi : \mathbb{N}_0^2 \to [0,\infty)$ such that for all $(n,m) \in \mathbb{N}_0^2$ and $k \in \mathbb{N}_0$,*

$$\Bigg| f(n,m) - \bar{f}(n,m) + \sum_{i,j} \left[q_{n,m}(i,j) - \bar{q}_{n,m}(i,j)\right] \cdot \\ \left[F(k,n+i,m+j) - F(k,n,m)\right] \Bigg| \leq \xi(n,m). \tag{14}$$

Then

$$\left|\mathbb{E}\left[f(N,M)\right] - \bar{\mathbb{E}}\left[\bar{f}(N,M)\right]\right| \leq \sum_{n,m} \bar{\pi}(n,m)\xi(n,m). \tag{15}$$

Theorem 3 (van Dijk [30])**.** *Let Q and $\bar{Q}$ be two continuous-time Markov processes on the same state space $\mathbb{N}_0^2$, with transition rates q and $\bar{q}$, respectively. In addition consider the cost functions f and $\bar{f}$ from $\mathbb{N}_0^2$ to $[0,\infty)$. Suppose that for all $(n,m) \in \mathbb{N}_0^2$ and $k \in \mathbb{N}_0$,*

$$f(n,m) - \bar{f}(n,m) + \sum_{i,j} \left[q_{n,m}(i,j) - \bar{q}_{n,m}(i,j)\right] \cdot \\ \left[F(k,n+i,m+j) - F(k,n,m)\right] \leq 0. \tag{16}$$

Then

$$\mathbb{E}\left[f(N,M)\right] \leq \bar{\mathbb{E}}\left[\bar{f}(N,M)\right]. \tag{17}$$

Remark 4. *The key step in applying Theorem 2 or 3 is to bound terms of the form*

$$F(k,n+i,m+j) - F(k,n,m). \tag{18}$$

We will refer to terms of this type as bias terms.

Remark 5. *Clearly, if a function* $\xi : \mathbb{N}_0^2 \to [0, \infty)$ *can be found such that both* (14) *and* (16) *hold then Theorems 2 and 3 can be combined to*

$$-\sum_{n,m} \bar{\pi}(n,m)\xi(n,m) \leq \mathbb{E}\left[f(N,M)\right] - \bar{\mathbb{E}}\left[\bar{f}(N,M)\right] \leq 0. \tag{19}$$

Remark 6. *If, in* (16), $\leq$ *is replaced by* $\geq$ *then Theorem 3 will provide a lower bound on* $\mathbb{E}[f(N,M)]$ *instead of an upper bound, i.e.,* $\mathbb{E}[f(N,M)] \geq \bar{\mathbb{E}}[f(N,M)]$.

4 Stability

The continuous-time Markov process Q^{γ_1,γ_2} is stable if and only if the corresponding uniformized (discrete-time) process is stable. The uniformized process is a two-dimensional homogeneous random walk in the positive quadrant, a type of process that has been extensively studied; see, for instance, [5, 11] and the references therein. Necessary and sufficient conditions for ergodicity are given in [11]. In this section we present this result and apply it to Q^{γ_1,γ_2}. The stability conditions are given for the continuous-time process directly, without explicitly introducing the uniformized process.

For the moment consider an arbitrary Markov process in the positive quadrant, with transition rates q, for which the uniformized process is a homogeneous random walk. The stability conditions can be expressed in terms of the drift of the process. Let $\nabla = (\nabla_1, \nabla_2)$, $\nabla^h = (\nabla_1^h, \nabla_2^h)$ and $\nabla^v = (\nabla_1^v, \nabla_2^v)$ denote the drift in the interior, at the horizontal axis and at the vertical axis, respectively. More precisely, let

$$\nabla = \left(\sum_{i,j} i \cdot q_{n,m}(i,j), \sum_{i,j} j \cdot q_{n,m}(i,j) \right), \tag{20}$$

$$\nabla^h = \left(\sum_{i,j} i \cdot q_{n,0}(i,j), \sum_{i,j} j \cdot q_{n,0}(i,j) \right), \tag{21}$$

$$\nabla^v = \left(\sum_{i,j} i \cdot q_{0,m}(i,j), \sum_{i,j} j \cdot q_{0,m}(i,j) \right), \tag{22}$$

where $n > 0$, $m > 0$. Note that, since the process is homogeneous, the above definitions do not depend on the choice of n and m.

The necessary and sufficient conditions for ergodicity as given in [11] are valid under the technical condition that $q_{n,m}(i,j) > 0$ only if $-1 \leq i \leq 1$ and $-1 \leq j \leq 1$, *i.e.*, in the interior of the state space only short jumps are possible. Since this condition is satisfied for Q^{γ_1,γ_2} we will be able to apply the result.

Theorem 4 (Malyshev [11, pp. 3])**.** *Consider a Markov process in the positive quadrant, with transition rates q, for which the uniformized process is a homogeneous random walk. Let the transition rates satisfy* $q_{n,m}(i,j) > 0$, $n > 0$, $m > 0$ *only if* $-1 \leq i \leq 1$ *and* $-1 \leq j \leq 1$. *This process is ergodic if and only* *if one of the following three conditions is satisfied*

$$\text{(i)} \quad \begin{cases} \nabla_1 < 0, \\ \nabla_2 < 0, \\ \nabla_1 \cdot \nabla_2^h - \nabla_2 \cdot \nabla_1^h < 0, \\ \nabla_1 \cdot \nabla_1^v - \nabla_1 \cdot \nabla_2^v < 0, \end{cases} \tag{23}$$

$$\text{(ii)} \quad \nabla_1 \geq 0, \quad \nabla_2 < 0, \quad \nabla_1 \cdot \nabla_2^h - \nabla_2 \cdot \nabla_1^h < 0, \tag{24}$$

$$\text{(iii)} \quad \nabla_1 < 0, \quad \nabla_2 \geq 0, \quad \nabla_2 \cdot \nabla_1^v - \nabla_1 \cdot \nabla_2^v < 0. \tag{25}$$

Note that the above result is presented in [11] for discrete-time random walks. An extension to homogeneous continuous-time Markov processes is readilly obtained by considering the uniformized process. The drift parameters of the original and uniformized process are the same up to a scaling factor. This scaling factor is determined by the uniformized parameter and the same for all vectors. The sign of the expressions in (23)–(25) is therefore the same for the continuous-time and uniformized processes.

By applying the above theorem to Q^{γ_1,γ_2} we obtain necessary and sufficient stability criteria. These criteria demonstrate the purpose of introducing γ_1^* and γ_2^* in (2) as $\gamma_1^* = (\rho_1 - \rho_2)/(1-\rho_2)$ and $\gamma_2^* = (\rho_2 - \rho_1)(1-\rho_1)$. Remember that γ_1 and γ_2 denote probabilities and hence lie in the interval $[0, 1]$.

Theorem 5. *The process Q^{γ_1,γ_2} is ergodic if and only if $\rho_1 < 1$, $\rho_2 < 1$, $\gamma_1 > \gamma_1^*$ and $\gamma_2 > \gamma_2^*$.*

Proof. We have

$$\begin{aligned} \nabla_1 &= \lambda_1 - \mu, & \nabla_2 &= \lambda_2 - \mu, \\ \nabla_1^h &= \lambda_1 - \gamma_1\mu, & \nabla_2^h &= \lambda_2, \\ \nabla_1^v &= \lambda_1, & \nabla_2^v &= \lambda_2 - \gamma_2\mu. \end{aligned} \tag{26}$$

The conditions in the theorem correspond to case (i) of Theorem 4. We only need to show that cases (ii) and (iii) can not occur. For case (ii) suppose that $\nabla_1 \geq 0$ and $\nabla_2 < 0$. Now, in order to have $\nabla_1 \cdot \nabla_2^h - \nabla_2 \cdot \nabla_1^h < 0$ we need

$$\begin{aligned} \gamma_1 &> \frac{\lambda_1}{\mu} - \frac{\lambda_2}{\mu} \cdot \frac{\lambda_1 - \mu}{\lambda_2 - \mu} \\ &\geq 1, \end{aligned} \tag{27}$$

which is never satisfied since γ_1 denotes a probability and is at most one by definition. In similar fashion case (iii) would require $\gamma_2 > 1$. □ □

Remark 7. *One of the performance measures of interest is the expected energy consumption in Q^{γ_1,γ_2}. Intuitively, to minimize energy consumption, γ_1 and γ_2 should be chosen as small as possible. An obvious choice is $\gamma_1 = \gamma_2 = 0$, but Theorem 5 shows that in that case Q^{γ_1,γ_2} is not stable. In the remainder we discuss how this result can be obtained without resorting to the theory developed in [11]. Note that it was also observed in [25] that a process with $\gamma_1 = \gamma_2 = 0$ is not stable.*

Consider an alternative representation of Q^{γ_1,γ_2} in which the state variables are N, the number of packets in the first queue, and $K = N - M$, the difference *between the number of packets in the first and the second queue. Observe, that if $\gamma_1 = \gamma_2 = 0$, the only changes to K occur from arrivals of packets, i.e., K increases by one with rate λ_1 and decreases with one with rate λ_2. The corresponding discrete-time process, obtained after uniformization, is a random walk on $\mathbb{Z}$ and is not ergodic. If $\lambda_1 = \lambda_2$, the process is null recurrent.*

5 Performance

5.1 The Perturbed Process

All performance bounds in this section will be obtained by perturbing some of the transition rates along the boundary of the state space. More precisely, we obtain bounds on the performance of our process of interest Q^{γ_1,γ_2} in terms of the performance of the perturbed process $\bar{Q}^{\alpha,1-\alpha}$, where $0 \leq \alpha \leq 1$ is a free parameter and where the bar notation is used to emphasize the role of the second process. Note that for the perturbed process we have the following transition rates

$$\bar{q}^{\alpha,1-\alpha}_{n,m}(i,j) = \begin{cases} \alpha\mu, & \text{if } i=-1, \quad j=0, \quad n>0, \quad m=0, \\ (1-\alpha)\mu, & \text{if } i=0, \quad j=-1, \quad n=0, \quad m>0, \\ q^{\gamma_1,\gamma_2}_{n,m}(i,j), & \text{otherwise.} \end{cases} \tag{28}$$

The effect of the perturbation is that along the vertical axis the rate towards the origin is changed from $\gamma_1\mu$ to $\alpha\mu$. Along the vertical axis the rate towards the origin changes from $\gamma_2\mu$ to $(1-\alpha)\mu$. In order to apply Theorem 2 or 3 we need to obtain the sign of the LHS of (16) or a bound on the LHS of (14), respectively. Since, $q_{n,m}(i,j) = \bar{q}_{n,m}(i,j)$ unless $n > 0$, $m = 0$, $i = -1$ and $j = 0$ or $n = 0$, $m > 0$, $i = 0$ and $j = -1$, we only need to obtain bounds on the following two bias terms

$$F(k,n,0) - F(k,n-1,0), \tag{29}$$
$$F(k,0,m) - F(k,0,m-1). \tag{30}$$

These bounds will be given for the specific performance measures of interest in Subsections 5.2 and 5.3.

The parameter α can be chosen freely, but the process $\bar{Q}^{\alpha,1-\alpha}$ should be ergodic. The next theorem states that given λ_1, λ_2 and μ satisfying $\lambda_1 < \mu$ and $\lambda_2 < \mu$, a suitable α always exists. Moreover it gives the stationary distribution of $\bar{Q}^{\alpha,1-\alpha}$ as given by Theorem 1.

Theorem 6. *The system $\bar{Q}^{\alpha,1-\alpha}$ is ergodic iff*

$$\gamma_1^* < \alpha < 1 - \gamma_2^*, \tag{31}$$

in which case it has steady-state distribution

$$\bar{\pi}_\alpha(n,m) = [1-\sigma_1(\alpha)]\,\sigma_1(\alpha)^n\,[1-\sigma_2(\alpha)]\,\sigma_2(\alpha)^m, \tag{32}$$

where $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$ are the unique solution of

$$\alpha\sigma_1(\alpha) + (1-\alpha)\sigma_1(\alpha)\sigma_2(\alpha) = \rho_1, \quad (1-\alpha)\sigma_2(\alpha) + \alpha\sigma_1(\alpha)\sigma_2(\alpha) = \rho_2, \tag{33}$$

satisfying $0 < \sigma_1(\alpha) < 1$ and $0 < \sigma_2(\alpha) < 1$. Given $\rho_1 < 1$ and $\rho_2 < 1$ it is always possible to choose α such that $\bar{Q}^{\alpha,1-\alpha}$ is ergodic.

Proof. The stability condition (31) follows directly from Theorem 5; the stationary distribution (32) follows from Theorem 1. For the last statement note that besides condition (31), we need $0 \le \alpha \le 1$. Therefore, we need to prove that

$$(\gamma_1^*, 1-\gamma_2^*) \cap [0,1] \neq \emptyset. \tag{34}$$

First we show that $(\gamma_1^*, 1-\gamma_2^*) \neq \emptyset$ by proving that

$$\gamma_1^* = \frac{\rho_1 - \rho_2}{1-\rho_2} < \frac{1-\rho_2}{1-\rho_1} = 1 - \gamma_2^*. \tag{35}$$

for $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. This follows directly by rewriting the inequality as

$$0 < \left(\rho_1 - \frac{1}{2}\right)^2 + \left(\rho_2 - \frac{1}{2}\right)^2 + \left(\rho_1\rho_2 + \frac{1}{2}\right). \tag{36}$$

Finally, (34), follows from the observation that $\gamma_1^* < 1$ and $1-\gamma_2^* > 0$. □ □

Remark 8. *The above theorem gives the stationary distribution in implicit form, i.e., $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$ are given as the solutions of a system of quadratic equations. The explicit solution is*

$$\sigma_i(\alpha) = \frac{-b_i + \sqrt{b_i^2 - 4a_ic_i}}{2a_i}, \tag{37}$$

where

$$a_1 = \alpha, \qquad b_1 = 1 - \alpha + \rho_2\frac{1-\alpha}{\alpha} - \rho_1, \quad c_1 = -\rho_1\frac{1-\alpha}{\alpha}, \tag{38}$$

$$a_2 = 1-\alpha, \quad b_2 = \alpha + \rho_1\frac{\alpha}{1-\alpha} - \rho_2, \qquad c_2 = -\rho_2\frac{\alpha}{1-\alpha}. \tag{39}$$

The values of $\sigma_i(\alpha)$, $i = 1,2$ as given in (37) *are the only solutions of* (33) *that satisfy $0 < \sigma_i(\alpha) < 1$, $i = 1,2$. The other solutions to* (33), $\tilde{\sigma}_i(\alpha) = (-b_i - \sqrt{b_i^2 - 4a_ic_i})/(2a_i)$, $i = 1,2$, *do not satisfy this additional constraint. Observe that $\tilde{\sigma}_1(1) = 0$ and $\tilde{\sigma}_2(0) = 0$. From $b_i^2 - 4a_ic_i \neq 0$, $i = 1,2$ and continuity of $\sigma_i(\alpha)$, $i = 1,2$ in the process parameters it follows that* (37) *gives the desired roots.*

5.2 Delay

The first performance measure of interest is D^{γ_1,γ_2}, the expected delay of packets of the first session. We established in Section 2 that $D^{\gamma_1,\gamma_2} = \mathbb{E}[N]/\lambda_1$.

Our first result gives the exact distribution of N if $\gamma_1 = 1$.

Theorem 7. *Let $P(N = n)$ denote the probability that $N = n$ in steady state. If $\gamma_1 = 1$ then*

$$P(N = n) = (1-\rho_1)\rho_1^n. \tag{40}$$

Proof. Consider the one-dimensional continuous Markov process in N, the number of packets in the first queue. The transition rates of this process are independent of the number of packets in the second queue. The process is equivalent to a M/M/1 queue with load ρ_1. □ □

We will make use of the comparison result of Theorem 3. The next technical lemma, the proof of which is deferred to Appendix A, provides the required signs of the bias terms. Remember from Subsection 3.2 that $D^{\gamma_1,\gamma_2}(k,n,m)$ denotes the expected cummulative reward incurred by the uniformized process at time step k when starting from state (n,m) at time 0, under reward function d.

Lemma 1. *Let $d : \mathbb{N}_0^2 \to [0,\infty)$, $d(n,m) = n/\lambda_1$. For all $(n,m) \in \mathbb{N}_0^2$ and $k \in \mathbb{N}_0$*

$$D^{\gamma_1,\gamma_2}(k,n+1,m) - D^{\gamma_1,\gamma_2}(k,n,m) \geq 0, \tag{41}$$

$$D^{\gamma_1,\gamma_2}(k,n,m+1) - D^{\gamma_1,\gamma_2}(k,n,m) \leq 0. \tag{42}$$

The first use of the above lemma is in establishing a monotonicity result.

Theorem 8. *The expected delay of packets of the first session of Q^{γ_1,γ_2} is monotone in γ_1 and γ_2. More precisely,*

$$D^{\gamma_1,\gamma_2} \leq D^{\tilde{\gamma}_1,\gamma_2}, \quad \text{if } \gamma_1 > \tilde{\gamma}_1, \tag{43}$$

$$D^{\gamma_1,\gamma_2} \geq D^{\gamma_1,\tilde{\gamma}_2}, \quad \text{if } \gamma_2 > \tilde{\gamma}_2. \tag{44}$$

Proof. Follows directly from Lemma 1 and Theorem 3 by observing that d does not depend on γ_1 and γ_2. □ □

Let

$$D^* = \inf\{D^{\gamma_1,\gamma_2}; \gamma_1 > \gamma_1^*, \gamma_2 > \gamma_2^*\}. \tag{45}$$

From Theorems 7 and 8 we directly obtain the value of D^*.

Corollary 1. *The minimum delay is $(\mu - \lambda_1)^{-1}$, i.e., $D^* = (\mu - \lambda_1)^{-1}$.*

Proof. From Theorem 8 it follows that the minimum of D^{γ_1,γ_2} is attained at $\gamma_1 = 1$. The result follows directly from Theorem 7. □ □

Next, we provide bounds on D^{γ_1,γ_2}. The fact that $D^{\gamma_1,\gamma_2}(k,n+1,0) - D^{\gamma_1,\gamma_2}(k,n,0)$ and $D^{\gamma_1,\gamma_2}(k,0,m+1) - D^{\gamma_1,\gamma_2}(k,0,m)$ have different signs provides the opportunity to obtain both upper and lower bounds on D^{γ_1,γ_2} using only the comparison result of Theorem 3. In particular, we can either choose $\alpha = \gamma_1$ or $\alpha = 1 - \gamma_2$ for the perturbed system, leading to an upper respectively a lower bound, or vice versa depending on the value of $\gamma_1 + \gamma_2$ as will become clear in Theorem 9. Since the perturbed system needs to be stable, *i.e.*, $\gamma_1^* \leq \alpha \leq 1 - \gamma_2^*$ from Theorem 6, it is not possible to obtain both upper and lower comparison bounds for all values of γ_1 and γ_2.

Theorem 9. *The expected delay of packets of the first session of Q^{γ_1,γ_2} is bounded as*

$$D^{\gamma_1,\gamma_2} \geq \frac{\sigma_1(\gamma_1)}{\lambda_1\left[1 - \sigma_1(\gamma_1)\right]}, \qquad \text{if } \gamma_1 + \gamma_2 \geq 1 \text{ and } \gamma_1 < 1 - \gamma_2^*, \tag{46}$$

$$D^{\gamma_1,\gamma_2} \leq \frac{\sigma_1(1-\gamma_2)}{\lambda_1\left[1 - \sigma_1(1-\gamma_2)\right]}, \quad \text{if } \gamma_1 + \gamma_2 \geq 1 \text{ and } \gamma_2 < 1 - \gamma_1^*, \tag{47}$$

$$D^{\gamma_1,\gamma_2} \leq \frac{\sigma_1(\gamma_1)}{\lambda_1\left[1 - \sigma_1(\gamma_1)\right]}, \qquad \text{if } \gamma_1 + \gamma_2 \leq 1 \text{ and } \gamma_1 < 1 - \gamma_2^*, \tag{48}$$

$$D^{\gamma_1,\gamma_2} \geq \frac{\sigma_1(1-\gamma_2)}{\lambda_1\left[1 - \sigma_1(1-\gamma_2)\right]}, \quad \text{if } \gamma_1 + \gamma_2 \leq 1 \text{ and } \gamma_2 < 1 - \gamma_1^*. \tag{49}$$

Proof. First consider $\alpha = \gamma_1$, which from condition (31) in Theorem 6 can be used only if $\gamma_1 < 1 - \gamma_2^*$. The performance of the perturbed system is given by

$$\bar{\mathbb{E}}^{\gamma_1, 1-\gamma_1}[d(N, M)] = \bar{\mathbb{E}}^{\gamma_1, 1-\gamma_1}[N/\lambda_1] = \frac{\sigma_1(\gamma_1)}{\lambda_1 [1 - \sigma_1(\gamma_1)]}. \tag{50}$$

To decide whether the above expression provides an upper or a lower bound note that

$$q_{0,m}^{\gamma_1,\gamma_2}(0,-1) - \bar{q}_{0,m}^{\gamma_1,1-\gamma_1}(0,-1) \geq 0, \quad \text{if } \gamma_1 + \gamma_2 \geq 1,$$
$$q_{0,m}^{\gamma_1,\gamma_2}(0,-1) - \bar{q}_{0,m}^{\gamma_1,1-\gamma_1}(0,-1) \leq 0, \quad \text{if } \gamma_1 + \gamma_2 \leq 1.$$

The above inequalities, together with Lemma 1 and Theorem 3, lead to the bounds given in (46) and (48).

Next consider $\alpha = 1 - \gamma_2$. The ergodicity condition (31) reduces to $\gamma_2 < 1 - \gamma_1^*$. Finally, the inequalities

$$q_{n,0}^{\gamma_1,\gamma_2}(-1,0) - \bar{q}_{n,0}^{\gamma_1,1-\gamma_1}(-1,0) \geq 0, \quad \text{if } \gamma_1 + \gamma_2 \geq 1,$$
$$q_{n,0}^{\gamma_1,\gamma_2}(-1,0) - \bar{q}_{n,0}^{\gamma_1,1-\gamma_1}(-1,0) \leq 0, \quad \text{if } \gamma_1 + \gamma_2 \leq 1,$$

lead to bounds (47) and (49). □ □

Conjecture 1. *For all values $(n, m) \in \mathbb{N}_0^2$, $|D^{\gamma_1,\gamma_2}(k, n+1, 0) - D^{\gamma_1,\gamma_2}(k, n, 0)|$ and $|D^{\gamma_1,\gamma_2}(k, 0, m+1) - D^{\gamma_1,\gamma_2}(k, 0, m)|$ are bounded uniformly in k.*

Remark 9. *The bounds of Theorem 9 are only valid for specific ranges of values for γ_1 and γ_2. If Conjecture 1 holds it is possible to obtain error bounds on the expected delay using Theorem 2. The bounds would be valid for all parameter ranges and possibly better in the ranges already covered by Theorem 9.*

5.3 Energy Consumption

The second performance measure for which we obtain bounds is the expected energy consumption $C^{\gamma_1,\gamma_2} = C_{\gamma_1,\gamma_2}^{\gamma_1,\gamma_2} = \mathbb{E}^{\gamma_1,\gamma_2}[c_{\gamma_1,\gamma_2}(N, M)]$ with cost function c_{γ_1,γ_2} defined in (4). In addition to the sign of the bias terms we are also able to obtain bounds on their size. The proof of the next result is deferred to Appendix A.

Lemma 2. *Let $c_{\gamma_1,\gamma_2} : \mathbb{N}_0^2 \to [0, \infty)$,*

$$c_{\gamma_1,\gamma_2}(n, m) = \gamma_1 \mu \mathbb{1}_{\{n>0, m=0\}} + \gamma_2 \mu \mathbb{1}_{\{n=0, m>0\}} + \mu \mathbb{1}_{\{n>0, m>0\}}.$$

For all $(n, m) \in \mathbb{N}_0^2$ and $k \in \mathbb{N}_0$

$$0 \leq C_{\gamma_1,\gamma_2}^{\gamma_1,\gamma_2}(k, n+1, m) - C_{\gamma_1,\gamma_2}^{\gamma_1,\gamma_2}(k, n, m) \leq 1, \tag{51}$$

$$0 \leq C_{\gamma_1,\gamma_2}^{\gamma_1,\gamma_2}(k, n, m+1) - C_{\gamma_1,\gamma_2}^{\gamma_1,\gamma_2}(k, n, m) \leq 1. \tag{52}$$

The signs of the bias terms can be used to establish the following monotonicity result.

Theorem 10. *The energy consumption of Q^{γ_1,γ_2} is monotone in γ_1 and γ_2. More precisely,*

$$C^{\gamma_1,\gamma_2}_{\gamma_1,\gamma_2} \geq C^{\tilde{\gamma}_1,\gamma_2}_{\tilde{\gamma}_1,\gamma_2}, \quad \textit{if } \gamma_1 > \tilde{\gamma}_1, \tag{53}$$

$$C^{\gamma_1,\gamma_2}_{\gamma_1,\gamma_2} \geq C^{\gamma_1,\tilde{\gamma}_2}_{\gamma_1,\tilde{\gamma}_2}, \quad \textit{if } \gamma_2 > \tilde{\gamma}_2. \tag{54}$$

Proof. We use Theorem 3 to compare Q^{γ_1,γ_2} under cost function c_{γ_1,γ_2} with $Q^{\tilde{\gamma}_1,\gamma_2}$ under cost function $c_{\tilde{\gamma}_1,\gamma_2}$, where $\gamma_1 > \tilde{\gamma}_1$. Note that contrary to the monotonicity result of Theorem 8, the reward function in the perturbed model is different from the original reward function. Using Lemma 2 we obtain, for $n > 0$

$$\sum_{i,j} [q_{n,0}(i,j) - \bar{q}_{n,0}(i,j)] [C(k,n+i,j) - C(k,n,0)] =$$
$$(\gamma_1 - \tilde{\gamma}_1)\mu [C(k,n-1,0) - C(k,n,0)] \geq -(\gamma_1 - \tilde{\gamma}_1)\mu. \tag{55}$$

Therefore,

$$c_{\gamma_1,\gamma_2}(n,0) - c_{\tilde{\gamma}_1,\gamma_2}(n,0) + \sum_{i,j} [q_{n,0}(i,j) - \bar{q}_{n,0}(i,j)] \cdot$$
$$[C(k,n+i,j) - C(k,n,0)] \geq 0, \tag{56}$$

and (53) follows from Theorem 3. Monotonicity in γ_2 follows in similar fashion. □ □

Let

$$C^* = \inf\{C^{\gamma_1,\gamma_2}; \gamma_1 > \gamma_1^*, \gamma_2 > \gamma_2^*\}. \tag{57}$$

In Appendix B we prove the following result, which gives an exact expression for C^*.

Theorem 11.

$$C^* = \max\{\lambda_1, \lambda_2\}. \tag{58}$$

Since we have the signs as well as a bound on the value of the bias terms we can employ both the comparison result of Theorem 3 and the error bound result from Theorem 2. Some care needs to be taken in chosing the value of α for the perturbed model $\bar{Q}^{\alpha,1-\alpha}$. We will see that by restricting α to the range $[\min\{\gamma_1, 1-\gamma_2\}, \max\{\gamma_1, 1-\gamma_2\}]$, it is possible to employ Theorem 3.

Theorem 12. *Let* $\min\{\gamma_1, 1-\gamma_2\} \leq \tilde{\alpha} \leq \max\{\gamma_1, 1-\gamma_2\}$. *Then*

$$\bar{C}^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} - \delta^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} \mathbb{1}_{\{\gamma_1+\gamma_2 \geq 1\}} \;\leq\; C^{\gamma_1,\gamma_2}_{\gamma_1,\gamma_2} \;\leq\; \bar{C}^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} + \delta^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} \mathbb{1}_{\{\gamma_1+\gamma_2 \leq 1\}}, \tag{59}$$

where

$$\bar{C}^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} = \gamma_1\mu\sigma_1(\tilde{\alpha})[1-\sigma_2(\tilde{\alpha})] + \gamma_2\mu[1-\sigma_1(\tilde{\alpha})]\sigma_2(\tilde{\alpha}) + \mu\sigma_1(\tilde{\alpha})\sigma_2(\tilde{\alpha}), \tag{60}$$

$$\delta^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} = \left|\tilde{\alpha} - \gamma_1\right|\mu\sigma_1(\tilde{\alpha})[1-\sigma_2(\tilde{\alpha})] + \left|1-\tilde{\alpha}-\gamma_2\right|\mu[1-\sigma_1(\tilde{\alpha})]\sigma_2(\tilde{\alpha}). \tag{61}$$

Proof. The expected energy consumption of Q^{γ_1,γ_2} will be bounded in terms of

$$\bar{C}^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} = \mathbb{E}^{\tilde{\alpha},1-\tilde{\alpha}}[c_{\gamma_1,\gamma_2}(N,M)], \tag{62}$$

the value of which can be easily computed based on the stationary distribution of $\bar{Q}^{\tilde{\alpha},1-\tilde{\alpha}}$ given in Theorem 6.

If $\gamma_1+\gamma_2 \le 1$ then $\gamma_1-\tilde{\alpha} \le 0$ and $\gamma_2-(1-\tilde{\alpha}) \le 0$. Therefore, by Lemma 2 and Theorem 3, $C^{\gamma_1,\gamma_2}_{\gamma_1,\gamma_2} \ge \bar{C}^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2}$. If, on the other hand, $\gamma_1+\gamma_2 \ge 1$, then $\gamma_1-\tilde{\alpha} \ge 0$ and $\gamma_2-(1-\tilde{\alpha}) \ge 0$, and it follows that $C^{\gamma_1,\gamma_2}_{\gamma_1,\gamma_2} \le \bar{C}^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2}$.

It remains to show that

$$\bar{C}^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} - \delta^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} \;\le\; C^{\gamma_1,\gamma_2}_{\gamma_1,\gamma_2} \;\le\; \bar{C}^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} + \delta^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2}. \tag{63}$$

We will use Theorem 2 with $\bar{c} = c_{\gamma_1,\gamma_2}$ and hence are required to find a function $\xi : \mathbb{N}_0^2 \to [0,\infty)$ that satisfies

$$\left| \sum_{i,j} \left[q^{\gamma_1,\gamma_2}_{n,m}(i,j) - \bar{q}^{\tilde{\alpha},1-\tilde{\alpha}}_{n,m}(i,j) \right] \left[C(k,n+i,m+j) - C(k,n,m) \right] \right| \le \xi(n,m) \tag{64}$$

and

$$\sum_{n,m} \bar{\pi}(n,m)\xi(n,m) = \delta^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2}. \tag{65}$$

From Lemma 2 and the definitions of Q^{γ_1,γ_2} and $\bar{Q}^{\tilde{\alpha},1-\tilde{\alpha}}$ it follows that

$$\xi(n,m) = \begin{cases} \left|\gamma_1-\tilde{\alpha}\right|, & \text{if } n>0, m=0, \\ \left|\gamma_2-(1-\tilde{\alpha})\right|, & \text{if } n=0, m>0, \\ 0, & \text{otherwise.} \end{cases} \tag{66}$$

satisfies (64). A simple computation, using the product-form distribution $\bar{\pi}(\tilde{\alpha})$, shows that (65) is also satisfied. □ □

Remark 10. *We have limited $\tilde{\alpha}$ to the interval $[\min\{\gamma_1, 1-\gamma_2\}, \max\{\gamma_1, 1-\gamma_2\}]$, making sure that Theorem 3 can be used. Obviously, $\tilde{\alpha}$ also needs to satisfy $0 \le \tilde{\alpha} \le 1$ and $\gamma_1^* < \tilde{\alpha} < 1-\gamma_2^*$. It is readily verified that there always exists an $\tilde{\alpha}$ that satisfies all constraints, i.e., Theorem 12 provides upper and lower bounds for all values of the process parameters.*

Remark 11. *Outside the interval $[\min\{\gamma_1, 1-\gamma_2\}, \max\{\gamma_1, 1-\gamma_2\}]$ there are values of $\hat{\alpha}$ that still satisfy $\gamma_1^* < \hat{\alpha} < 1-\gamma_2^*$. For these values of $\hat{\alpha}$, Theorem 3 can not be used, but Theorem 2 is still valid. This would lead to bounds of the form*

$$\bar{C}^{\hat{\alpha},1-\hat{\alpha}}_{\gamma_1,\gamma_2} - \delta^{\hat{\alpha},1-\hat{\alpha}}_{\gamma_1,\gamma_2} \;\le\; C^{\gamma_1,\gamma_2}_{\gamma_1,\gamma_2} \;\le\; \bar{C}^{\hat{\alpha},1-\hat{\alpha}}_{\gamma_1,\gamma_2} + \delta^{\hat{\alpha},1-\hat{\alpha}}_{\gamma_1,\gamma_2}. \tag{67}$$

For clarity of exposition bounds of this type have been omitted in the current work.

Remark 12. *The bounds given in Theorem 12 depend on $\tilde{\alpha}$. In practice one will want to use the $\tilde{\alpha}$ that provides the tightest bound. By distinguishing between $\gamma_1+\gamma_2 \le 1$ and $\gamma_1+\gamma_2 \ge 1$ as well as lower and upper bounds, we obtain optimization problems that can be readily solved. For the upper bounds in the case that $\gamma_1+\gamma_2 \le 1$ we need to find, for instance,*

$$\min\left\{ \bar{C}^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} + \delta^{\tilde{\alpha},1-\tilde{\alpha}}_{\gamma_1,\gamma_2} : \gamma_1 \le \tilde{\alpha} \le 1-\gamma_2 \right\}. \tag{68}$$

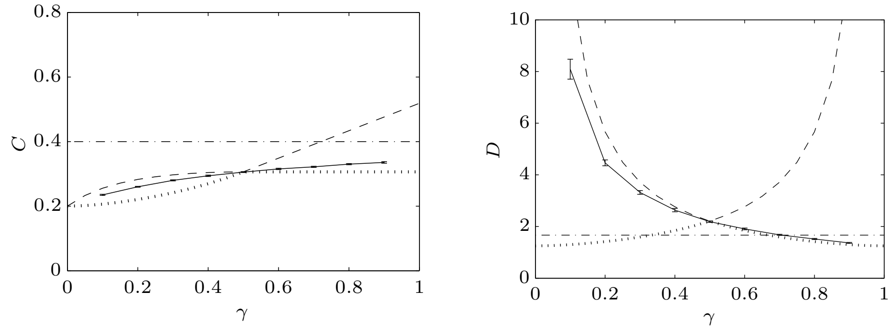

Figure 3: Performance of a symmetric system $Q^{\gamma,\gamma}$ under low load ($\mu = 1$, $\lambda_1 = \lambda_2 = 0.2$, $\gamma_1 = \gamma_2 = \gamma$.) Depicted are the analytical lower (dotted lines) and upper bounds (dashed lines), the simulation results (solid lines), and the performance of the corresponding uncoded system (dashed-dotted lines).

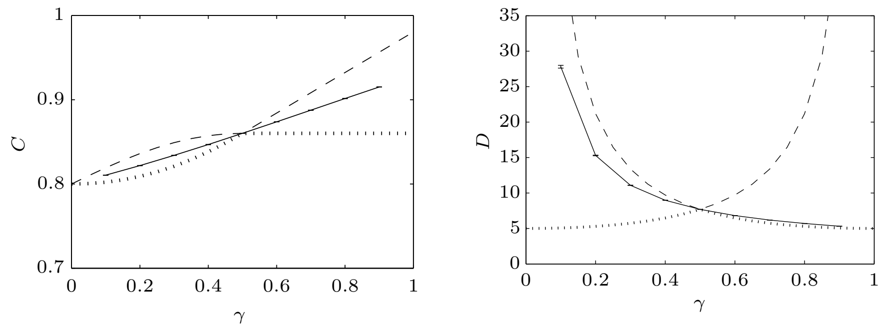

Figure 4: Performance of a symmetric system $Q^{\gamma,\gamma}$ under high load ($\mu = 1$, $\lambda_1 = \lambda_2 = 0.8$, $\gamma_1 = \gamma_2 = \gamma$.) Depicted are the analytical lower (dotted lines) and upper bounds (dashed lines), and simulation result (solid lines).

This corresponds to finding the minimum of the function

$$\begin{aligned}
&\gamma_1 \mu \sigma_1(\tilde{\alpha}) \left[1 - \sigma_2(\tilde{\alpha})\right] + \gamma_2 \mu \left[1 - \sigma_1(\tilde{\alpha})\right] \sigma_2(\tilde{\alpha}) + \mu \sigma_1(\tilde{\alpha}) \sigma_2(\tilde{\alpha}) \\
&\qquad + (\tilde{\alpha} - \gamma_1) \mu \sigma_1(\tilde{\alpha}) \left[1 - \sigma_2(\tilde{\alpha})\right] + (1 - \tilde{\alpha} - \gamma_2) \mu \left[1 - \sigma_1(\tilde{\alpha})\right] \sigma_2(\tilde{\alpha}), \quad (69)
\end{aligned}$$

in the interval $\gamma_1 \leq \tilde{\alpha} \leq 1 - \gamma_2$, *which is tedious, but simple calculus.*

6 Numerical Examples

We provide some examples of application of Theorems 9 and 12 for specific parameter values. In addition to the values of the analytical bounds we provide numerical results obtained through simulation. The simulation results have been obtained within 99% confidence intervals. These confidence intervals are depicted in all figures, but sometimes so small that they are not visible.

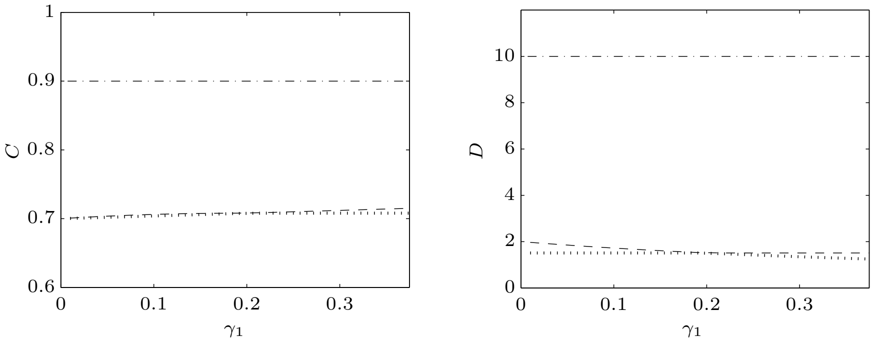

Figure 5: Performance of an asymmetric system ($\mu = 1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.7$, $\gamma_2 = 0.8$.) Depicted are the analytical lower (dotted line) and upper bound (dashed line), and the performance in an uncoded system (dashed-dotted line).

Throughout this section we assume $\mu = 1$. Remember that the performance of the uncoded system is given in (8) and (9).

First we consider a symmetric system with $\lambda_1 = \lambda_2 = \lambda$ and $\gamma_1 = \gamma_2 = \gamma$. We fix λ and consider the performance of $Q^{\gamma,\gamma}$ as a function of γ. We consider $Q^{\gamma,\gamma}$ under two scenario's. The first scenario is that of a relatively low load of $\lambda = 0.2$, the results of which are depicted in Figure 3. In addition to the analytical bounds and the simulation result, we provide the performance of the uncoded system. It is interesting to note that the delay in the coded system is only smaller than that of the uncoded system for large values of γ. For these values of γ the energy savings are significantly smaller than the 50% percent that are theoretically possible. The second scenario that we consider for $Q^{\gamma,\gamma}$ is that of a relatively high load of $\lambda = 0.8$. The results of which are depicted in Figure 4. Note, that for $\lambda = 0.8$, the uncoded system is not stable.

Next, we consider a system in which $\lambda_1 < \lambda_2$. In particular, we consider $\lambda_1 = 0.2$, $\lambda_2 = 0.7$ and $\gamma_2 = 0.8$, and analyze the influence of γ_1. In Figure 5 we give the analytical performance bounds of $Q^{\gamma_1,0.8}$ and the performance of the uncoded system. Since, the upper and lower bounds nearly coincide, we have omitted the simulation results. It can be observed that the coded system is performing significantly better, in terms of energy consumption as well as delay.

Finally, we consider a system in which $\lambda_1 > \lambda_2$. In particular, we consider $\lambda_1 = 0.7$, $\lambda_2 = 0.2$ and $\gamma_2 = 0$, and analyze the influence of γ_1. In Figure 6 we give the analytical performance bounds of $Q^{\gamma_1,0}$ and the performance of the uncoded system.

7 Discussion

We have provided a queueing analysis of a wireless network in which network coding is employed. We have compared the energy consumption and the delay in the network with that of a network in which network coding is not used by deriving analytical upper and lower bounds on the performance of the coded system. It is shown that different operating policies can be used to tradeoff

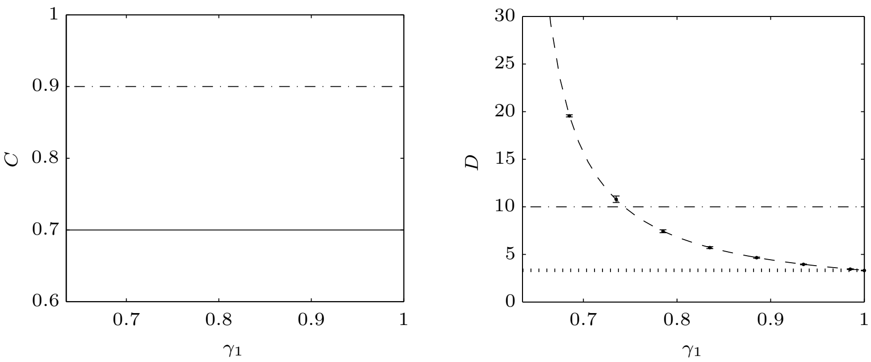

Figure 6: Performance of an asymmetric system ($\mu = 1$, $\lambda_1 = 0.7$, $\lambda_2 = 0.2$, $\gamma_2 = 0$.) The upper and lower bounds on the energy consumption coincide and are depicted in a solid line. The lower and upper bound on the delay are depicted in a dotted and dashed line, respectively. The simulation result of the delay nearly coincides with the upper bound, hence only the confidence intervals are depicted. The performance of the uncoded system is depicted in dashed-dotted lines.

energy consumption against delay. Exact results have been obtained for the minimum possible energy consumption and the minimum possible delay.

The queueing model of the network configuration that we have studied in this work has similar properties as queueing networks with positive and negative customers. We believe that other wireless network configurations can be modelled using generalizations [3, 26] of networks for positive and negative customers.

A Proof of Lemmas 1 and 2

In this section we prove the Lemmas 1 and 2 together, by deriving expressions for an arbitrary cost function f and specializing these expressions for the cost functions under consideration, c_{γ_1,γ_2} and d. In the remainder of this section we omit dependence on γ_1 and γ_2 from the notation. Remember that $F(k, n, m)$ denotes the expected cummulative cost incurred by the uniformized process at time step k when starting from state (n, m). We introduce the following notation

$$\Delta_{F,1}^{k+1}(n, m) = F(k, n + 1, m) - F(k, n, m), \tag{70}$$

$$\Delta_{F,2}^{k+1}(n, m) = F(k, n, m + 1) - F(k, n, m). \tag{71}$$

The inequalities that need to proven for all $k \in \mathbb{N}_0$ and $(n, m) \in \mathbb{N}_0^2$

$$0 \leq \Delta_{C,1}^{k}(n, m) \leq 1, \tag{72}$$

$$0 \leq \Delta_{C,2}^{k}(n, m) \leq 1, \tag{73}$$

$$\Delta_{D,1}^{k}(n, m) \geq 0, \tag{74}$$

$$\Delta_{D,2}^{k}(n, m) \leq 0. \tag{75}$$

We will not prove (73) explicitly, since it follows from (72) by symmetry of the system and the reward function c. The other inequalities will be established using induction over k. For $k = 0$ all inequalities hold, since by definition $F(0, n, m) = 0$. In the remainder of this section we prove the inductive step for various values of n and m.

A.1 Interior ($n > 0$, $m > 0$)

First, consider the interior of the state space, *i.e.*, $n > 0$, $m > 0$. We have

$$\Delta^{k+1}_{F,1}(n, m) = hf(n+1, m)-hf(n, m)+h\lambda_1\Delta^k_{F,1}(n+1, m)+h\lambda_2\Delta^k_{F,1}(n, m+1)+ \\ h\mu\Delta^k_{F,1}(n-1, m-1) + (1-h\lambda_1-h\lambda_2-h\mu)\Delta^k_{F,1}(n, m), \quad (76)$$

$$\Delta^{k+1}_{F,2}(n, m) = hf(n, m)-hf(n, m+1)+h\lambda_1\Delta^k_{F,2}(n+1, m)+h\lambda_2\Delta^k_{F,2}(n, m+1)+ \\ h\mu\Delta^k_{F,2}(n-1, m-1) + (1-h\lambda_1-h\lambda_2-h\mu)\Delta^k_{F,2}(n, m). \quad (77)$$

Specializing (76) for the cost function $d : \mathbb{N}_0^2 \to [0, \infty)$, $d(n, m) = n/\lambda_1$, gives

$$\Delta^{k+1}_{D,1}(n, m) \geq h \geq 0, \quad (78)$$

proving (41) for $n > 0$, $m > 0$. For $\Delta^{k+1}_{D,2}(n, m)$ it follows directly from (77) and the induction hypothesis that $\Delta^{k+1}_{D,2}(n, m) \leq 0$.

For the cost function c, the inequality $\Delta^{k+1}_{C,1}(n, m) \geq 0$ follows directly from (76). The corresponding upper bound also follows from (76) and is given by

$$\Delta^{k+1}_{C,1}(n, m) \leq h\lambda_1 + h\lambda_2 - h\mu + (1 - h\lambda_1 - h\lambda_2 - \mu) \leq 1. \quad (79)$$

A.2 Horizontal axis ($n > 0$, $m = 0$)

We proceed by considering the horizontal axis, $n > 0$ and $m = 0$. Again the starting point is the following pair of equations for a general cost function.

$$\Delta^{k+1}_{F,1}(n, 0) = hf(n+1, 0) - hf(n, 0) + h\lambda_1\Delta^k_{F,1}(n+1, 0) + h\lambda_2\Delta^k_{F,1}(n, 1)+ \\ h\gamma_1\mu\Delta^k_{F,1}(n-1, 0) + (1-h\lambda_1-h\lambda_2-h\gamma_1\mu)\Delta^k_{F,1}(n, 0), \quad (80)$$

$$\begin{aligned}\Delta^{k+1}_{F,2}(n, 0) &= hf(n, 1) - hf(n, 0) + h\lambda_1\Delta^k_{F,2}(n+1, 0) + h\lambda_2\Delta^k_{F,2}(n, 1)+ \\ &\quad h\mu F(k, n-1, 0) - h\gamma_1\mu F(k, n-1, 0)+ \\ &\quad (1-h\lambda_1-h\lambda_2-h\mu)F(k, n, 1) - (1-h\lambda_1-h\lambda_2-h\gamma_1\mu)F(k, n, 0) \\ &= hf(n, 1) - hf(n, 0) + h\lambda_1\Delta^k_{F,2}(n+1, 0) + h\lambda_2\Delta^k_{F,2}(n, 1)- \\ &\quad h\mu(1-\gamma_1)\Delta^k_{F,1}(n-1, 0) + (1-h\lambda_1-h\lambda_2-h\mu)\Delta^k_{F,2}(n, 0).\end{aligned} \quad (81)$$

It follows directly from the induction hypothesis, together with (80) and (81), that $\Delta^{k+1}_{D,1}(n, 0) \geq 0$ and $\Delta^{k+1}_{S,D}(n, 0) \leq 0$ respectively.

For the cost function c, $\Delta^{k+1}_{C,1}(n, 0) \geq 0$ is immediate. The corresponding upper bound follows from

$$\Delta^{k+1}_{C,1}(n, 0) \leq h\lambda_1 + h\lambda_2 + h\gamma_1\mu + (1 - h\lambda_1 - h\lambda_2 - h\gamma_1\mu) \leq 1. \quad (82)$$

A.3 Vertical axis ($n = 0$, $m > 0$)

Next step is the vertical axis, $n = 0$ and $m > 0$, for which we start with

$$\begin{aligned}\Delta_{F,1}^{k+1}(0,m) = {} & hf(1,m) - hf(0,m) + h\lambda_1\Delta_{F,1}^{k}(1,m) + h\lambda_2\Delta_{F,1}^{k}(0,m+1) + \\ & h\mu F(k,0,m-1) - h\gamma_2\mu F(k,0,m-1) + \\ & (1-h\lambda_1-h\lambda_2-h\mu)F(k,1,m) - (1-h\lambda_1-h\lambda_2-h\gamma_2\mu)F(k,0,m) \\ = {} & hf(1,m) - hf(0,m) + h\lambda_1\Delta_{F,1}^{k}(1,m) + h\lambda_2\Delta_{F,1}^{k}(0,m+1) - \\ & h\mu(1-\gamma_2)\Delta_{F,2}^{k}(0,m-1) + (1-h\lambda_1-h\lambda_2-h\mu)\Delta_{F,1}^{k}(0,m),\end{aligned} \tag{83}$$

$$\begin{aligned}\Delta_{F,2}^{k+1}(0,m) = {} & hf(0,m+1) - hf(0,m) + h\lambda_1\Delta_{F,2}^{k}(1,m) + h\lambda_2\Delta_{F,2}^{k}(0,m+1) + \\ & h\gamma_2\mu\Delta_{F,2}^{k}(0,m-1) + (1-h\lambda_1-h\lambda_2-h\gamma_2\mu)\Delta_{F,2}^{k}(0,m).\end{aligned} \tag{84}$$

For the cost function d, we obtain $\Delta_{D,1}^{k+1}(0,m) \geq 0$ from (83) directly. Also, the upper bound $\Delta_{D,2}^{k+1}(0,m) \leq 0$ follows directly from (84).

The bounds for the cost function c are obtained as

$$\Delta_{C,1}^{k+1}(0,m) \geq h(1-\gamma_2)\mu - h(1-\gamma_2)\mu \geq 0, \tag{85}$$

and

$$\begin{aligned}\Delta_{C,1}^{k+1}(0,m) \leq {} & h(1-\gamma_2)\mu + h\lambda_1 + h\lambda_2 + (1-h\lambda_1-h\lambda_2-h\mu) \\ = {} & 1 - h\gamma_2\mu \\ \leq {} & 1,\end{aligned} \tag{86}$$

from (83).

A.4 Origin ($n = 0$ and $m = 0$)

The final step in the proof of Lemmas 1 and 2 is to consider the origin, $n = 0$ and $m = 0$.

$$\begin{aligned}\Delta_{F,1}^{k+1}(0,0) = {} & hf(1,0) - hf(0,0) + h\lambda_1\Delta_{F,1}^{k}(1,0) + h\lambda_2\Delta_{F,1}^{k}(0,1) + h\gamma_1\mu F(k,0,0) + \\ & (1-h\lambda_1-h\lambda_2-h\gamma_1\mu)F(k,1,0) - (1-h\lambda_1-h\lambda_2)F(k,0,0) \\ = {} & hf(1,0) - hf(0,0) + h\lambda_1\Delta_{F,1}^{k}(1,0) + h\lambda_2\Delta_{F,1}^{k}(0,1) + \\ & (1-h\lambda_1-h\lambda_2-h\gamma_1\mu)\Delta_{F,1}^{k}(0,0).\end{aligned} \tag{87}$$

$$\begin{aligned}\Delta_{F,2}^{k+1}(0,0) = {} & hf(0,1) - hf(0,0) + h\lambda_1\Delta_{F,2}^{k}(1,0) + h\lambda_2\Delta_{F,2}^{k}(0,1) + h\gamma_2\mu F(k,0,0) + \\ & (1-h\lambda_1-h\lambda_2-h\gamma_2\mu)F(k,1,0) - (1-h\lambda_1-h\lambda_2)F(k,0,0) \\ = {} & hf(0,1) - hf(0,0) + h\lambda_1\Delta_{F,2}^{k}(1,0) + h\lambda_2\Delta_{F,2}^{k}(0,1) + \\ & (1-h\lambda_1-h\lambda_2-h\gamma_2\mu)\Delta_{F,2}^{k}(0,0).\end{aligned} \tag{88}$$

Again,the bound $\Delta_{D,1}^{k+1}(0,0) \geq 0$ follows directly. Also $\Delta_{D,2}^{k+1}(0,0) \leq 0$ is immediate.

Finally, for the cost function c, we obtain $\Delta_{C,1}^{k+1}(0,0) \geq h\gamma_1\mu \geq 0$ and

$$\Delta_{C,1}^{k+1}(0,0) \leq h\gamma_1\mu + h\lambda_1 + h\lambda_2 + (1-h\lambda_1-h\lambda_2-h\gamma_1\mu) \leq 1 \tag{89}$$

from (87).

B Proof of Theorem 11

We need to show that

$$C^* = \inf\{C^{\gamma_1,\gamma_2}_{\gamma_1,\gamma_2}; \gamma_1 > \gamma_1^*, \gamma_2 > \gamma_2^*\} = \max\{\lambda_1, \lambda_2\}. \tag{90}$$

First, assume that $\lambda_1 > \lambda_2$. Under this assumption $\gamma_1^* > 0$ and $\gamma_2^* < 0$. By Theorem 10 it follows that

$$C^* = \lim_{\gamma_1 \to \gamma_1^*} C^{\gamma_1,0}_{\gamma_1,0}. \tag{91}$$

Let $0 < \epsilon < 1 - \gamma_1^*$ and consider the sequence of processes $\{Q^{\gamma_1(l),0}\}_{l\in\mathbb{N}}$, $\gamma_1(l) = \gamma_1^* + \epsilon^l$. For each $l \in \mathbb{N}$ we give an approximation on $C^{\gamma_1(l),0}_{\gamma_1(l),0}$, the energy consumption of $Q^{\gamma_1(l),0}$. In particular, we show that $C^{\gamma_1(l),0}_{\gamma_1(l),0} \to \lambda_1$ as $l \to \infty$. Therefore, consider the sequence of perturbed processes $\{\bar{Q}^{\gamma_1(l),1-\gamma_1(l)}\}_{l\in\mathbb{N}}$. It follows from Theorem 12 that

$$\bar{C}^{\gamma_1(l),1-\gamma_1(l)}_{\gamma_1(l),0} - \delta^{\gamma_1(l),1-\gamma_1(l)}_{\gamma_1(l),0} \le C^{\gamma_1(l),0}_{\gamma_1(l),0} \le \bar{C}^{\gamma_1(l),1-\gamma_1(l)}_{\gamma_1(l),0} + \delta^{\gamma_1(l),1-\gamma_1(l)}_{\gamma_1(l),0}, \tag{92}$$

where $\delta^{\gamma_1(l),1-\gamma_1(l)}_{\gamma_1(l),0}$ is defined in Theorem 12. It is readily verified that

$$\sigma_1(\gamma_1(l)) \longrightarrow 1, \qquad \text{and} \qquad \sigma_2(\gamma_1(l)) \longrightarrow \rho_2, \tag{93}$$

as $l \to \infty$. Therefore, $\delta^{\gamma_1(l),1-\gamma_1(l)}_{\gamma_1(l),0}$ vanishes as $l \to \infty$. From (92) and (93) it follows that

$$C^* = \lim_{l\to\infty} \bar{C}^{\gamma_1(l),1-\gamma_1(l)}_{\gamma_1(l),0} = \gamma_1^*\mu(1-\rho_2) + \mu\rho_2 = \lambda_1. \tag{94}$$

Next, we need to consider the cases $\lambda_1 < \lambda_2$ and $\lambda_1 = \lambda_2$. In similar fashion as the first case it follows for $\lambda_1 < \lambda_2$ that

$$C^* = \lim_{\gamma_2 \to \gamma_2^*} C^{0,\gamma_2}_{0,\gamma_2} = \lim_{\gamma_2 \to \gamma_2^*} \bar{C}^{1-\gamma_2,\gamma_2}_{0,\gamma_2} = \lambda_2 \tag{95}$$

and for $\lambda_1 = \lambda_2 = \lambda$ that

$$C^* = \lim_{\gamma\to 0} C^{\gamma,\gamma}_{\gamma,\gamma} = \lim_{\gamma\to 0} \bar{C}^{\gamma,1-\gamma}_{\gamma,\gamma} = \lambda. \tag{96}$$

□

References

[1] Ahlswede, R., Cai, N., Li, S.Y.R., Yeung, R.W.: Network information flow. IEEE Trans. Inf. Theory **46**(4), 1204–1216 (2000)

[2] Bayer, N., Boxma, O.J.: Wiener-hopf analysis of an M/G/1 queue with negative customers and of a related class of random walks. Queueing Systems **23**(1), 301–316 (1996)

[3] Boucherie, R.J., Chao, X.: Queueing networks with string transitions of mixed vector additions and vector removals. Journal of Systems Science and Complexity **14**(4), 337–355 (2001)

[4] Boucherie, R.J., van Dijk, N.M.: Local balance in queueing networks with positive and negative customers. Annals of Operations Research **48**(5), 463–492 (1994)

[5] Cohen, J.W., Boxma, O.J.: Boundary value problems in queueing system analysis. North-Holland (1983)

[6] van Dijk, N.M.: Perturbation theory for unbounded Markov reward processes with applications to queueing. Advances in Applied Probability **20**(1), 99–111 (1988)

[7] van Dijk, N.M., Putterman, M.L.: Perturbation theory for Markov reward processes with applications to queueing systems. Advances in Applied Probability **20**(1), 79–98 (1988)

[8] Dougherty, R., Freiling, C., Zeger, K.: Insufficiency of linear coding in network information flow. IEEE Trans. Inf. Theory **51**(8), 2745–2759 (2005)

[9] Effros, M., Koetter, R., Médard, M.: Breaking network logjams. Scientific American Magazine **296**(6), 78–85 (2007)

[10] Eryilmaz, A., Ozdaglar, A., Médard, M.: On delay performance gains from network coding. In: Proc. 40th Annual Conference on Information Sciences and Systems (CISS) (2006)

[11] Fayolle, G., Iasnogorodski, R., Malyshev, V.: Random walks in the quarter plane: algebraic methods, boundary value problems, and applications. Springer Verlag (1999)

[12] Fragouli, C., Le Boudec, J.Y., Widmer, J.: Network coding: an instant primer. Computer Communication Review **36**(1), 63–68 (2006)

[13] Fragouli, C., Soljanin, E.: Network coding applications. Foundations and Trends® in Networking **2**(2), 135–269 (2007)

[14] Fragouli, C., Soljanin, E.: Network coding fundamentals. Foundations and Trends® in Networking **2**(1), 1–133 (2007)

[15] Gelenbe, E.: Product-form queueing networks with negative and positive customers. Journal of Applied Probability **28**(3), 656–663 (1991)

[16] Gelenbe, E., Schassberger, R.: Stability of product form g-networks. Probability in the Engineering and Informational Sciences **6**(3), 271–276 (1992)

[17] Goseling, J., Matsumoto, R., Uyematsu, T., Weber, J.H.: Lower bounds on the maximum energy benefit of network coding for wireless multiple unicast. Eurasip Journal on Wireless Communications and Networking (2010). Special issue on network coding for wireless networks

[18] Ho, T., Médard, M., Koetter, R., Karger, D., Effros, M., Shi, J., Leong, B.: A random linear network coding approach to multicast. IEEE Trans. Inf. Theory **52**(10), 4413–4430 (2006)

[19] Jaggi, S., Sanders, P., Chou, P., Effros, M., Egner, S., Jain, K., Tolhuizen, L.: Polynomial time algorithms for multicast network code construction. IEEE Trans. Inf. Theory **51**(6), 1973–1982 (2005)

[20] Katti, S., Rahul, H., Hu, W., Katabi, D., Médard, M., Crowcroft, J.: XORs in the air: practical wireless network coding. In: Proc. of ACM SIGCOMM, pp. 243–254 (2006)

[21] Keller, L., Drinea, E., Fragouli, C.: Online broadcasting with network coding. In: Proc. of the Fourth Workshop on Network Coding, Theory and Applications, pp. 1–6 (2008)

[22] Koetter, R., Médard, M.: An algebraic approach to network coding. IEEE/ACM Trans. Netw. **11**(5), 782–795 (2003)

[23] Li, S., Yeung, R., Cai, N.: Linear network coding. IEEE Trans. Inf. Theory **49**(2), 371–381 (2003)

[24] Lun, D.S., Médard, M., Koetter, R., Effros, M.: On coding for reliable communication over packet networks. Physical Communication **1**(1), 3–20 (2008)

[25] Sagduyu, Y., Ephremides, A.: Network coding in wireless queueing networks: Tandem network case. In: Proc. of IEEE International Symp. on Information Theory, pp. 192–196 (2006)

[26] Serfozo, R., Yang, B.: Markov network processes with string transitions. The Annals of Applied Probability **8**(3), 793–821 (1998)

[27] Shin, Y.W.: Multi-server retrial queue with negative customers and disasters. Queueing Systems **55**(4), 223–237 (2007)

[28] Shrader, B., Ephremides, A.: A queueing model for random linear coding. In: Proc. of Military Communications Conference, pp. 1–7 (2007)

[29] Sundararajan, J., Shah, D., Médard, M.: ARQ for network coding. In: Proc. of IEEE International Symp. on Information Theory, pp. 1651–1655 (2008)

[30] Van Dijk, N.M.: Error bounds and comparison results: The markov reward approach for queueing networks. In: R.J. Boucherie, N.M. Van Dijk (eds.) Queueing Networks: A Fundamental Approach, *International Series in Operations Research & Management Science*, vol. 154. Springer (2011)

[31] Wu, Y., Chou, P.A., Kung, S.Y.: Information exchange in wireless networks with network coding and physical-layer broadcast. In: Proc. 39th Annual Conference on Information Sciences and Systems (CISS) (2005)

[32] Yeung, R.W., Cai, N.: Network coding theory. Foundations and Trends® in Communications and Information Theory **2**(4 and 5), 241–381 (2006)