Controllability and Stability of 3D Heat Conduction Equation in a Submicroscale Thin Film

Hanif Heidari, Hans Zwart and Alaeddin Malek

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Abstract

We obtain a closed form analytic solution for the Dual Phase Lagging equation. This equation is a linear, time-independent partial differential equation modeling the heat distribution in a thin film. The spatial domain is of micrometer and nanometer geometries. We show that the solution is described by a semigroup, and obtain a basis of eigenfunctions. The closure of the set of eigenvalues contains an interval, and so the theory on Riesz spectral operator of Curtain and Zwart cannot be applied directly. The exponential stability and the approximate controllability is shown.

Keywords: Thin film, Dual phase lagging, Semigroup

1 Introduction

The time-dependent initial-boundary value partial differential equations (PDEs) of second order are widely used to describe continuous heat conduction problems in macroscopic regions. The microscopic heat flux equation developed from physical and mathematical reasoning is different from the traditional heat equation [9, 10]. It is a third-order PDE that contains mixed derivatives with respect to time and space. The heat transfer equation at the microscale was derived by Qui and Tien [8], based on the hypothesis that energy input is absorbed by electrons and lattice in a substance. Energy balance applied to an elemental volume at location r and time t must include the contributions of the energy storage of the electron gas and the lattice. Chiffell [1] and some other researchers (for example, see [12]) proposed an equation similar to energy equation without including the electron energy storage. Some numerical methods were applied to solve the microscopic heat flux equation for example see [5, 6] and references there in. However there are less exact solution to DPL equation. In this paper closed analytical solution of the dual-phase-lagging differential equation is proposed using semigroup theory. The contribution of this paper is twofold. On the one hand, it gives a semigroup formulation for DPL equation. This is on the other hand that gives a closed analytical form of the solution for DPL equation. In section 2 DPL equation is formulated as an abstract differential equation. In lack of internal heat sources a solution of DPL equation for homogeneous boundary conditions is proposed in Section 3.1. The exact solution of the general DPL equation (non-homogeneous boundary conditions and in the presence of internal heat sources) are considered in Subsections 3.1 and 3.2. Section 4 concludes the paper.

2 Semigroup formulation

We consider the physical domain to be a thin film, which its thickness at the nano or micro scale, i.e.,

$$\Omega = \{(x, y, z) \mid 0 \leq x \leq l, 0 \leq y \leq h, 0 \leq z \leq \epsilon\}$$

and, ϵ is of the order to 0.01nm or 0.01μm. If all the thermophysical material properties are assumed to be constant, the dual-phase-lagging heat conduction equation given by, [5]:

$$\frac{1}{\alpha}(\frac{\partial u}{\partial t} + \tau_q \frac{\partial^2 u}{\partial t^2}) = \nabla^2 u + \tau_q(\frac{\partial^3 u}{\partial t x^2} + \frac{\partial^3 u}{\partial t y^2}) + \tau_u \frac{\partial^3 u}{\partial t z^2} + s, \tag{2.1}$$

where α is thermal diffusivity of the material, $u(x, y, z, t)$ is temperature at position (x, y, z) and time t, τ_q and τ_u are the time lags of the heat flux and temperature gradient, respectively, and s represents the internal heat sources. The parameters α, τ_q and τ_u are positive constants, [6]. The initial conditions are assumed to be of the general form:

$$\begin{aligned} u(x, y, x, 0) &= f_1(x, y, z) \\ \frac{\partial u}{\partial t}(x, y, z, 0) &= f_2(x, y, z) \end{aligned} \tag{2.2}$$

which f_1 and f_2 are real-valued functions. The boundary conditions are given by

$$\begin{aligned} \frac{\partial u}{\partial t}(0,y,z,t) &= 0 \qquad & \frac{\partial u}{\partial t}(l,y,z,t) &= 0 \\ \frac{\partial u}{\partial t}(x,0,z,t) &= 0 \qquad & \frac{\partial u}{\partial t}(x,h,z,t) &= 0 \\ \frac{\partial u}{\partial t}(x,y,0,t) &= 0 \qquad & \frac{\partial u}{\partial t}(x,y,\epsilon,t) &= 0 \end{aligned} \tag{2.3}$$

for $t > 0$.

The system of equations (2.1), (2.2) and (2.3) can be transformed to an abstract differential equation. As state space we choose the energy space $\mathcal{H}$, which is a Hilbert space $H_0^1(\Omega) \times L^2(\Omega)$ with the inner product

$$\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rangle_e = \frac{1}{2} \int_\Omega \frac{\alpha}{\tau_q} \nabla u_1 \cdot \nabla w_1 + u_2 w_2 \; dX, \tag{2.4}$$

where $dX = dxdydz$.

On this state space we write (2.1), (2.2) and (2.3) as

$$\begin{cases} \frac{d}{dt} \begin{pmatrix} u \\ u_t \end{pmatrix} = A \begin{pmatrix} u \\ u_t \end{pmatrix} + Bs \\ \begin{pmatrix} u \\ u_t \end{pmatrix} |_{t=0} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \end{cases} \tag{2.5}$$

where $u_t = \frac{\partial u}{\partial t}$, $B = \begin{pmatrix} 0 \\ \frac{\alpha}{\tau_q} \end{pmatrix}$ and A is given by

$$A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ \alpha \operatorname{div}(u_3) - \frac{1}{\tau_q} u_2 \end{pmatrix}, \tag{2.6}$$

where $u_3 = \frac{1}{\tau_q} \nabla u_1 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla u_2$, and

$$D(A) = \{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H_0^1(\Omega) \oplus H_0^1(\Omega) \mid u_3 \in D(\operatorname{div}) \}. \tag{2.7}$$

Lemma 2.1. Let A and it's domain given by (2.6) and (2.7), respectively. The adjoint of A is given by

$$A^* \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ \alpha \operatorname{div}(v_3) - \frac{1}{\tau_q} v_2 \end{pmatrix} \tag{2.8}$$

with the following domain

$$D(A^*) = \{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in H_0^1(\Omega) \oplus H_0^1(\Omega) \mid v_3 \in D(\operatorname{div}) \}. \tag{2.9}$$

where $v_3 = -\frac{1}{\tau_q} \nabla v_1 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla v_2$.

Proof. For $\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix} \right) \in D(A)$, we have that

$$\begin{aligned} \left\langle A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e &= \left\langle \begin{pmatrix} u_2 \\ \alpha \operatorname{div}(u_3) - \frac{1}{\tau_q} u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e & (2.10) \\ &= \frac{1}{2} \int_\Omega \frac{\alpha}{\tau_q} \nabla u_2 \cdot \nabla v_1 + (\alpha \operatorname{div}(u_3) - \frac{1}{\tau_q} u_2) v_2 dX. & (2.11) \end{aligned}$$

We know that $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix} \right) \in D(A^*)$ if and only if for all $\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix} \right) \in D(A)$ we can write (2.10) as

$$\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rangle_e = \frac{1}{2} \int_\Omega \frac{\alpha}{\tau_q} \nabla u_1 \cdot \nabla w_1 + u_2 w_2 dX \tag{2.12}$$

for some $(w_1, w_2) \in H_0^1(\Omega) \times L^2(\Omega)$.

It is easy to see $\left(\begin{smallmatrix} u_1 \\ 0 \end{smallmatrix} \right) \in D(A)$ if and only if $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. For this element in $D(A)$, equation (2.10) becomes $\int_\Omega \frac{\alpha}{\tau_q} \operatorname{div}(\nabla u_1) v_2 dX$. This can be written as (2.12) if and only if $v_2 \in H_0^1(\Omega)$. Hence, if $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix} \right) \in D(A^*)$, then $v_2 \in H_0^1(\Omega)$. Using this we can write (2.10) for general $\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix} \right) \in D(A)$ as

$$\begin{aligned} \left\langle A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e =& \frac{1}{2} \int_\Omega \frac{\alpha}{\tau_q} \nabla u_2 \cdot \nabla v_1 - \alpha u_3 \cdot \nabla v_2 - \frac{1}{\tau_q} u_2 v_2 dX \\ =& \frac{1}{2} \int_\Omega \frac{\alpha}{\tau_q} \nabla u_2 \cdot \nabla v_1 - (\frac{\alpha}{\tau_q} \nabla u_1 \cdot \nabla v_2 + \\ & \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla u_2 \cdot \nabla v_2) - \frac{1}{\tau_q} u_2 v_2 dX. \qquad (2.13) \end{aligned}$$

We define $v_3 \in L^2(\Omega)$ as $v_3 = -\frac{1}{\tau_q}\nabla v_1 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla v_2$ and write (2.13) as

$$\frac{1}{2}\int_\Omega -\frac{\alpha}{\tau_q}\nabla u_1 \cdot \nabla v_2 - \nabla u_2 \cdot (\alpha v_3) - \frac{1}{\tau_q} u_2 v_2 dX. \tag{2.14}$$

Equation (2.14) can be written in the form (2.12) if and only if $v_3 \in D(\text{div})$. Hence, the domain of A^* is given by (2.9), and if $\binom{v_1}{v_2} \in D(A^*)$, then

$$\left\langle A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e = \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} -v_2 \\ \alpha \operatorname{div}(v_3) - \frac{1}{\tau_q} v_2 \end{pmatrix} \right\rangle_e \tag{2.15}$$

Thus we have proved the assertion. □

Using this lemma, it is not hard to show that A generates a contraction semigroup on $\mathcal{H}$.

Theorem 2.2. The operator A as defined in (2.6) and (2.7) is the infinitesimal generator of a strongly continuous contraction semigroup on $\mathcal{H}$.

Proof. We check that both A and A^* are dissipative on $\mathcal{H}$. Then the result follows from Lumer-Phillips Theorem [7].

$$\begin{aligned} \left\langle A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle_e &= \left\langle \begin{pmatrix} u_2 \\ \alpha \operatorname{div}(u_3) - \frac{1}{\tau_q} u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle_e \\ &= \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_2 \cdot \nabla u_1 + (\alpha \operatorname{div}(u_3) - \frac{1}{\tau_q} u_2) u_2 dX \\ &= \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_1 \cdot \nabla u_2 - \alpha u_3 \cdot \nabla u_2 - \frac{1}{\tau_q} u_2^2 dX \\ &= \frac{1}{2}\int_\Omega -\alpha \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla u_2 \right) \cdot \nabla u_2 - \frac{1}{\tau_q} u_2^2 dX, \end{aligned} \tag{2.16}$$

where we used integration by parts and the fact that u_1 and u_2 are zero at the boundary. Since the right hand side of (2.16) is less than or equal to zero, we see that A is dissipative on $\mathcal{H}$.

The proof that A^* is dissipative on $\mathcal{H}$ is done in a similar way.

$$\begin{aligned} \left\langle A^* \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e &= \left\langle \begin{pmatrix} -v_2 \\ \alpha \operatorname{div}(v_3) - \frac{1}{\tau_q} v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e \\ &= \frac{1}{2}\int_\Omega -\frac{\alpha}{\tau_q}\nabla v_2 \cdot \nabla u_1 + (\alpha \operatorname{div}(v_3) - \frac{1}{\tau_q} v_2) v_2 dX \\ &= \frac{1}{2}\int_\Omega -\frac{\alpha}{\tau_q}\nabla v_1 \cdot \nabla v_2 - \alpha v_3 \cdot \nabla v_2 - \frac{1}{\tau_q} v_2^2 dX. \end{aligned} \tag{2.17}$$

Hence by substituting v_3 in relation (2.17) we get the results as follows $\frac{1}{2}\int_\Omega -\alpha \left(\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{smallmatrix} \right) \nabla v_2 \right) \cdot \nabla v_2 - \frac{1}{\tau_q} v_2^2 dX \leq 0.$ □

3 Solution derivation

In this section we find the solution of the abstract differential equation (2.5). At first, the solution is obtained in the case $s = 0$. We obtain the solution by showing that the normalized eigenfunctions of A form a Riesz basis in $\mathcal{H}$, and thus the solution can be written with respect to this basis.

We begin by calculating the eigenvalues and eigenfunctions of A.

$$A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Leftrightarrow \begin{cases} u_2 = \lambda u_1 \\ \alpha \operatorname{div}(u_3) - \frac{1}{\tau_q} u_2 = \lambda u_2 \end{cases} \tag{3.1}$$

Therefore, $u_2 = \lambda u_1$ and

$$\alpha \operatorname{div}[\frac{1}{\tau_q} \nabla u_1 + \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{smallmatrix} \right) \nabla u_2] = (\lambda + \frac{1}{\tau_q}) u_2 \Leftrightarrow$$

$$\alpha \operatorname{div}[\frac{1}{\tau_q} \nabla u_1 + \lambda \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{smallmatrix} \right) \nabla u_1] = \lambda(\lambda + \frac{1}{\tau_q}) u_1 \tag{3.2}$$

which is equivalent to

$$\begin{cases} u_1 \in H_0^1(\Omega) \cap H^2(\Omega) \\ (\frac{\alpha}{\tau_q} + \lambda\alpha)(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}) + (\frac{\alpha}{\tau_q} + \lambda \frac{\alpha\tau_u}{\tau_q}) \frac{\partial^2 u_1}{\partial z^2} = (\lambda \frac{1}{\tau_q} + \lambda^2) u_1 \end{cases} \tag{3.3}$$

We want to find all solutions of (3.3). Therefore, we first obtain a set of solutions. It is easily seen that $\varphi_{nmk}(x, y, z) = \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi y}{h}) \sin(\frac{k\pi z}{\epsilon})$ lies in $H_0^1(\Omega)$. Furthermore, it satisfies (3.2) if and only if λ_{nmk} satisfies

$$\lambda_{nmk}^2 + (\alpha[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + \frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2] + \frac{1}{\tau_q}) \lambda_{nmk} + \tag{3.4}$$
$$\frac{\alpha}{\tau_q} \left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + (\frac{k\pi}{\epsilon})^2 \right] = 0.$$

The solution of above equation is denoted as follows:

$$\lambda_{+nmk} = \frac{1}{2}(-b + \sqrt{\Delta}) \quad n \in \mathbb{N}, m \in \mathbb{N}, k \in \mathbb{N} \tag{3.5}$$

$$\lambda_{-nmk} = \frac{1}{2}(-b - \sqrt{\Delta}) \quad n \in \mathbb{N}, m \in \mathbb{N}, k \in \mathbb{N}, \tag{3.6}$$

where $b=\alpha[(\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+\frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2]+\frac{1}{\tau_q}$ and $\Delta=b^2-\frac{4\alpha}{\tau_q}[(\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2]$.

For $\lambda_{\pm nmk}$ defined by (3.5) and (3.6), it is easy to see that

$$\varphi_{\pm nmk}(x,y,z)=\begin{pmatrix}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon})\\ \lambda_{\pm nmk}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon})\end{pmatrix} \tag{3.7}$$

lies in the domain of A, and satisfies $A\varphi_{\pm nmk}=\lambda_{\pm nmk}\varphi_{\pm nmk}$. Hence, $\varphi_{\pm nmk}$ is an eigenfunction of A. If $n\neq\tilde{n}$, or $m\neq\tilde{m}$, or $k\neq\tilde{k}$, then

$$\langle\varphi_{\pm nmk},\varphi_{\pm\tilde{n}\tilde{m}\tilde{k}}\rangle_e=0. \tag{3.8}$$

Furthermore, in the inner product of our state space, we have

$$\begin{aligned}\langle\varphi_{+nmk},\varphi_{-nmk}\rangle_e=&\frac{\alpha lh\epsilon}{16\tau_q}\left[(\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right]+\\ &\frac{1}{2}\int_\Omega\lambda_{+nmk}\lambda_{-nmk}\sin^2(\frac{n\pi x}{l})\sin^2(\frac{m\pi y}{h})\sin^2(\frac{k\pi z}{\epsilon})dX=\\ &\frac{lh\epsilon}{16}\left(\frac{\alpha}{\tau_q}\left((\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right)+\lambda_{+nmk}\lambda_{-nmk}\right)\\ &=\frac{lh\epsilon}{8}\left(\frac{\alpha}{\tau_q}\left((\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right)\right)\end{aligned} \tag{3.9}$$

and

$$\langle\varphi_{+nmk},\varphi_{+nmk}\rangle_e=\frac{lh\epsilon}{16}\left(\frac{\alpha}{\tau_q}\left((\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right)+\lambda_{+nmk}^2\right) \tag{3.10}$$

$$\langle\varphi_{-nmk},\varphi_{-nmk}\rangle_e=\frac{lh\epsilon}{16}\left(\frac{\alpha}{\tau_q}\left((\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right)+\lambda_{-nmk}^2\right) \tag{3.11}$$

Lemma 3.1. The normalized set of of eigenvectors $\{\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|},\frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|},n,m,k\in\mathbb{N}\}$ form a Riesz basis of $\mathcal{H}=H_0^1(\Omega)\times L^2(\Omega)$.

Proof. It is well-known that $\{\sqrt{\frac{8}{lh\epsilon}}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}),n,m,k\in\mathbb{N}\}$ form an orthonormal basis of $L^2(\Omega)$. Similarly, we have that the vectors $\{\frac{1}{\sqrt{\mu_{nmk}}}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}),n,m,k\in\mathbb{N}\}$, with

$$\mu_{nmk}=\frac{lh\epsilon}{8}\left((\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right) \tag{3.12}$$

form an orthonormal basis of $H_0^1(\Omega)$.

Let $w = \binom{w_1}{w_2} \in \mathcal{H}$. There exist $\{c_{1,nmk}\}_{n,m,k\in\mathbb{N}}$ and $\{c_{2,nmk}\}_{n,m,k\in\mathbb{N}}$ in ℓ^2 such that

$$w_1(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} c_{1,nmk}\frac{1}{\sqrt{\mu_{nmk}}}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}) \quad (3.13)$$

$$w_2(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} c_{2,nmk}\sqrt{\frac{8}{lh\epsilon}}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}). \quad (3.14)$$

Using the normalized eigenfunctions, we see that we can write (3.13), (3.14) as

$$w = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} d_{+nmk}\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|} + d_{-nmk}\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|}. \quad (3.15)$$

with

$$\begin{cases} \frac{d_{+nmk}}{\|\varphi_{+nmk}\|} + \frac{d_{-nmk}}{\|\varphi_{-nmk}\|} = \frac{c_{1,nmk}}{\sqrt{\mu_{nmk}}} \\ \lambda_{+nmk}\frac{d_{+nmk}}{\|\varphi_{+nmk}\|} + \lambda_{-nmk}\frac{d_{-nmk}}{\|\varphi_{-nmk}\|} = c_{2,nmk}\sqrt{\frac{8}{lh\epsilon}}. \end{cases} \quad (3.16)$$

This we write in a matrix notation

$$\begin{pmatrix} c_{1nmk} \\ \sqrt{\frac{8}{lh\epsilon}}c_{2nmk} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\mu_{nmk}}}{\|\varphi_{+nmk}\|} & \frac{\sqrt{\mu_{nmk}}}{\|\varphi_{-nmk}\|} \\ \frac{\lambda_{+nmk}}{\|\varphi_{+nmk}\|} & \frac{\lambda_{-nmk}}{\|\varphi_{-nmk}\|} \end{pmatrix} \begin{pmatrix} d_{+nmk} \\ d_{-nmk} \end{pmatrix}. \quad (3.17)$$

The set $\{\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|},\ \frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}, n,m,k \in \mathbb{N}\}$ form a Riesz basis of $\mathcal{H} = H_0^1(\Omega)\times L^2(\Omega)$ if and only if $\{d_{\pm nmk}\}_{nmk} \in \ell^2$ whenever $\{c_{\pm nmk}\}_{nmk} \in \ell^2$.

This holds if and only if the matrix in (3.17) is (uniformly) bounded and (uniformly) bounded invertible. Using (3.12), (3.10), and (3.11), we see that

$$\mu_{nmk} \le \frac{\alpha}{2\tau_q}\|\varphi_{+nmk}\|^2, \qquad \mu_{nmk} \le \frac{\alpha}{2\tau_q}\|\varphi_{-nmk}\|^2 \qquad \text{and}$$

$$\lambda_{+nmk}^2 \le \frac{16}{lh\epsilon}\|\varphi_{+nmk}\|^2, \qquad \lambda_{-nmk}^2 \le \frac{16}{lh\epsilon}\|\varphi_{-nmk}\|^2.$$

So the coefficients of the matrix in (3.17) are (uniformly) bounded, which implies that the same holds for the matrix.

Since $\lambda_{+nmk} \neq \lambda_{-nmk}$, we have that for all n, m, and k the matrix is invertible. Now we investigate its limit behaviour. We have that, see (3.5)

$$\begin{aligned} -\lambda_{+nmk} &= b - \sqrt{\Delta} = \frac{b^2 - \Delta}{b + \sqrt{\Delta}} \\ &= \frac{\frac{32\alpha}{\tau_q lh\epsilon}\mu_{nmk}}{b + \sqrt{\Delta}} \leq \frac{32\alpha}{\tau_q lh\epsilon}\frac{\mu_{nmk}}{b}. \end{aligned}$$

From this it is easily seen that λ_{+nmk} is bounded. Since this is bounded $\frac{\lambda_{+nmk}}{\|\varphi_{+nmk}\|}$ converges to zero for $n, m, k \to \infty$. Furthermore, we obtain that, see (3.10)

$$\inf_{n,m,k} \frac{\mu_{nmk}}{\|\varphi_{+nmk}\|^2} > 0$$

and

$$\inf_{n,m,k} \frac{\lambda_{-nmk}}{\|\varphi_{-nmk}\|} = \inf_{n,m,k} \frac{b - \lambda_{+nmk}}{\|\varphi_{-nmk}\|} > 0.$$

So we see that the diagonal of the matrix in (3.17) is bounded away from zero, wherea sthe lower triagular element converges to zero. Together with the boundedness of all the elements, we conclude that this matrix is (uniformly) boundedly invertible.

Hence we coclude that $\{\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|}, \frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}, n, m, k \in \mathbb{N}\}$ form a Riesz basis of $\mathcal{H}$. □

Since the normalized eigenfunctions $\{\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|}, \frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}, n, m, k \in \mathbb{N}\}$ form a Riesz basis of $\mathcal{H}$, we have that they are all the eigenfunctions. Thus the solutions of (3.3) are found.

With this it is easy to derive the formula's for the C_0-semigroup. Consider the system (2.5), and let $f = \binom{f_1}{f_2} = \sum_{n,m,k=1}^{\infty} d_{+nmk}\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|} + d_{-nmk}\frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}$. Following [2, Chapter 2], we have that the solution of (2.5) with $s = 0$ is given by

$$\begin{pmatrix} u \\ u_t \end{pmatrix} = \sum_{n,m,k=1}^{\infty} d_{+nmk}e^{\lambda_{+nmk}t}\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|} + d_{-nmk}e^{\lambda_{-nmk}t}\frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}. \tag{3.18}$$

We can find the coefficients d's by using the biorthonormal sequence of our Riesz-basis, i.e., the $\psi_{\pm nmk} \in \mathcal{H}$ such that

$$\langle \psi_{\pm nmk}, \frac{\varphi_{\pm pqr}}{\|\varphi_{\pm pqr}\|}\rangle_e = \begin{cases} 1 & \text{signs are the same}, n = p, m = q, k = r \\ 0 & \text{otherwise} \end{cases}$$

Since $d_{\pm nmk} = \langle f, \psi_{\pm nmk} \rangle_e$, we have that the semigroup generated by A is

$$T(t)f = \sum_{n,m,k=1}^{\infty} \langle f, \psi_{+nmk} \rangle_e e^{\lambda_{+nmk}t} \frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|} + \langle f, \psi_{-nmk} \rangle_e e^{\lambda_{-nmk}t} \frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}. \quad (3.19)$$

3.1 Inhomogeneous case

In the inhomogeneous case there exists a heat source and the term s in differential equation (2.1) is nonzero.

The function $\begin{pmatrix} 0 \\ \frac{\alpha}{\tau_q} s \end{pmatrix}$ can be represented as follows

$$\begin{pmatrix} 0 \\ \frac{\alpha s}{\tau_q} \end{pmatrix} = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (\beta_{+nmk}(t)\varphi_{+nmk} - \beta_{+nmk}(t)\varphi_{-nmk}) \quad (3.20)$$

From (3.19) and (3.20), the solution of (2.1) is as follows:

$$\begin{pmatrix} u \\ u_t \end{pmatrix} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} [\exp(\lambda_{+nmk}t) d_{+nmk}\varphi_{+nmk} \int_0^t \exp(\lambda_{+nmk}(t-v))\beta_{+nmk}(v)\varphi_{nmk} dv] + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} [\exp(\lambda_{-nmk}t) d_{-nmk}\varphi_{-nmk} + \int_0^t \exp(\lambda_{-nmk}(t-v))\beta_{-nmk}(v)\varphi_{-nmk} dv] \quad (3.21)$$

4 Stability

Since the coefficients of the quadratic polynomial (3.4) are positive, all solutions of (3.4) have negative real parts. Furthermore, if we write

$$\begin{aligned}
\Delta =& b^2 - 4\frac{\alpha}{\tau_q}\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + (\frac{k\pi}{\epsilon})^2\right] \\
=&\alpha^2\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + \frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2\right]^2 + \\
&2\frac{\alpha}{\tau_q}\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + \frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2\right] + \frac{1}{\tau_q^2} - 4\frac{\alpha}{\tau_q}\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + (\frac{k\pi}{\epsilon})^2\right] \\
=&\alpha^2\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + \frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2\right]^2 - \\
&2\frac{\alpha}{\tau_q}\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + \frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2\right] + \frac{1}{\tau_q^2} + \frac{4\alpha}{\tau_q}(\frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2) - 4\frac{\alpha}{\tau_q}(\frac{k\pi}{\epsilon})^2 \\
=&\left(\alpha\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + (\frac{k\pi}{\epsilon})^2\right] - \frac{1}{\tau_q}\right)^2 + \frac{4\alpha}{\tau_q}(\frac{k\pi}{\epsilon})^2(\frac{\tau_u}{\tau_q} - 1). \qquad (4.1)
\end{aligned}$$

If $\tau_u \geq \tau_q$, then (4.1) implies that $\Delta > 0$ for all $n, m, k \in \mathbb{N}$. If $\tau_u \ll \tau_q$, then (4.1) may be negative for $n = m = k = 1$. However, since the first term grows like k^4, whereas the last grows like as k^2 there can only the finitely many triple (n, m, k) for which (4.1) is negative. We assume that $\Delta > 0$ for all $n, m, k \in \mathbb{N}$ and so we assume that all eigenvalues are real and simple.

Here, we would to prove that the spectrum of DPL equation is bounded and away from zero. For doing this the following notations are introduced: $(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 = F$, $(\frac{k\pi}{\epsilon})^2 = G$.

So, we must prove the boundedness of following equation (see Eq.(3.5)).

$$-\alpha(F + \frac{\tau_u}{\tau_q}G) - \frac{1}{\tau_q} + \sqrt{(\alpha(F + \frac{\tau_u}{\tau_q}G) + \frac{1}{\tau_q})^2 - \frac{4\alpha}{\tau_q}(F + G)} = \lambda_{+nmk} \qquad (4.2)$$

$$\lambda_{+nmk} = [\sqrt{([\alpha(F + \frac{\tau_u}{\tau_q}G)] + \frac{1}{\tau_q})^2 - \frac{4\alpha}{\tau_q}(F+G)} - (\frac{1}{\tau_q} + \alpha(F + \frac{\tau_u}{\tau_q}G))]$$
$$\times \frac{\sqrt{([\alpha(F + \frac{\tau_u}{\tau_q}G)] + \frac{1}{\tau_q})^2 - \frac{4\alpha}{\tau_q}(F+G)} + (\frac{1}{\tau_q} + \alpha(F + \frac{\tau_u}{\tau_q}G))}{\sqrt{([\alpha(F + \frac{\tau_u}{\tau_q}G)] + \frac{1}{\tau_q})^2 - \frac{4\alpha}{\tau_q}(F+G)} + (\frac{1}{\tau_q} + \alpha(F + \frac{\tau_u}{\tau_q}G))} =$$
$$\frac{-\frac{4\alpha}{\tau_q}(F+G)}{\sqrt{([\alpha(F + \frac{\tau_u}{\tau_q}G)] + \frac{1}{\tau_q})^2 - \frac{4\alpha}{\tau_q}(F+G)} + (\frac{1}{\tau_q} + \alpha(F + \frac{\tau_u}{\tau_q}G))} < 0 \quad (4.3)$$

By considering the above relation, there exist three ways that the element zero maybe lies in spectrum. The ways are F, G or F and G together tends to infinity. But, we have the following results

$$\lim_{F\to\infty} \lambda_{+n,m,k} = -\frac{1}{\tau_q} \quad (4.4)$$
$$\lim_{G\to\infty} \lambda_{+n,m,k} = -\frac{1}{\tau_u} \quad (4.5)$$

Therefore, it remains to analysis $\lim_{F,G\to\infty} \lambda_{+nmk}$. Let $F = \xi G$ where ξ is a real positive constant. The equation (4.3) can be re written as follows

$$\lim_{F,G\to\infty} \lambda_{+nmk} = \frac{-\frac{4\alpha}{\tau_q}(\xi+1)G}{\sqrt{(\alpha(\xi + \frac{\tau_u}{\tau_q})G + \frac{1}{\tau_q})^2 - \frac{4\alpha}{\tau_q}(1+\xi)G} + (\frac{1}{\tau_q} + \alpha(\xi + \frac{\tau_u}{\tau_q})G)} =$$
$$\frac{-\frac{2\alpha}{\tau_q}(\xi+1)}{\alpha(\xi + \frac{\tau_u}{\tau_q})} = \frac{-2(\xi+1)}{\tau_q\xi + \tau_u} \quad (4.6)$$

Equations (4.4)-(4.6) prove that the eigenvalues of DPL equation are bounded above and away from zero.

4.1 Continuity of spectrum

In this section, It will be shown that the closure of spectrum of matrix A has a continuous part. Therefore, the base $\{\phi_{nmk}\}_{\pm nmk}$ is not a Riesz-spectral system.

Let $\mathbb{P} = \{p \in [0,\infty) \mid \exists \textit{ sequence } (n,m,k) \in \mathbb{N}^3 \textit{ such that } \frac{G}{F} \to p\}$. It is easy that $\mathbb{P}$ is dense in $[0,\infty)$. Furthermore

$$\lim_{F,G\to\infty, \frac{G}{F}\to p} \lambda_{+nmk} = \frac{-2(1+p)}{\tau_q + p\tau_u} \quad (4.7)$$

The spectrum A is a closed set in $\mathbb{C}$. Thus for all $p \in \mathbb{P}$ we have the following

$$\frac{-2(1+p)}{\tau_q + p\tau_u} \in \sigma(A) \tag{4.8}$$

where $\sigma(A)$ is the spectrum of A. The set $\mathbb{P}$ is dense in $[0,\infty)$, thus the interval between $\frac{-1}{\tau_q}$ and $\frac{-1}{\tau_u}$ lies in spectrum of A.

5 Controllability

Consider Eq. 2.1 with initial and homogeneous boundary conditions as defined in equations 2.2 and 2.3 respectively. Let $s : \Omega_1 \times (0,T) \longrightarrow \mathbb{R}$ be control function, such that $\Omega_1 \subset \Omega$. The controllability problem problem can be stated as follows:

*Statement*1: Let the final time T, $\delta > 0$ and desired state $\begin{pmatrix} h(x) \\ r(x) \end{pmatrix} \in \mathcal{H}$ are given. Do exist a control function $s(x,t)$ such that

$$\| \begin{pmatrix} u(x,T) \\ u_t(x,T) \end{pmatrix} - \begin{pmatrix} h(x) \\ r(x) \end{pmatrix} \|_{\mathcal{H}} \leq \delta. \tag{5.1}$$

Lemma 5.1. *Statement*1 is equivalent to *Statement*2 as follows

*Statement*2: There exist a function $g(x,t)$ and a constant N such that the control problem is equivalent to find $s(x,t)$ such that $u(x,t) = g(x,t)$ in the space $X = span\{\varphi_{1,nmk} \mid 1 \leq n,m,k \leq N\}$.

Proof. Define $g(x,t) = h(x) + (t-T)r(x)$. It is obvious that $g(x,t)$ satisfies desired final state. Furthermore, there exist a constant number N and sequences $\{c_{\pm nmk}\}_{nmk}$, $\{e_{\pm nmk}\}_{nmk}$ such that

$$\| \begin{pmatrix} u(x,T) \\ u_t(x,T) \end{pmatrix} - \sum_{\substack{i=-N \\ i\neq 0}}^{N} \sum_{\substack{j=-N \\ j\neq 0}}^{N} \sum_{\substack{k=-N \\ k\neq 0}}^{N} c_{nmk}\varphi_{nmk} \| \leq \frac{\delta}{2} \tag{5.2}$$

$$\| \begin{pmatrix} h(x) \\ h(x) \end{pmatrix} - \sum_{\substack{i=-N \\ i\neq 0}}^{N} \sum_{\substack{j=-N \\ j\neq 0}}^{N} \sum_{\substack{k=-N \\ k\neq 0}}^{N} e_{nmk}\varphi_{nmk} \| \leq \frac{\delta}{2}. \tag{5.3}$$

Equations (5.2) and (5.2) result (5.1). ☐

The following lemma will be used to prove the existence and find control functions.

Lemma 5.2. The $\{\sin(nx)\}_{n=1}^{N}$ are linearly independent in every domain that contains a continuous part of $\Omega_1 \subset \mathbb{R}$.

Proof. For simplicity, we consider only two terms as follows

$$f(x) = \sum_{n=1}^{2} p_k \sin(kx) \tag{5.4}$$

$f(x)$ can be represented as follows

$$f(x) = C1 \exp(A1t)x_0 \tag{5.5}$$

where

$$A1 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad C1 = \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ p_1 \\ 0 \\ p_2 \end{pmatrix}$$

and for each k

$$A_k = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \quad C_k = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

It should be proved that if $f(x) = 0$ in some interval then $p_1 = p_2 = 0$. This is equivalent that the system $(C1, A1)$ is observable. It is enough to show that rank $\begin{pmatrix} C1 \\ mI - A1 \end{pmatrix} = N \quad \forall m \in \sigma(A1)$ by using Hautus test. The kth or $k+1$th eigenvalues of $A1$ and corresponding eigenvector have the following form (k is odd)

$$e_k = +ki \qquad m_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ kth \\ k+1th \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.6}$$

where e_k denote the eigenvalue and m_k denote the eigenvector.

Consider kth or $k+1$th eigenvalue of $A1$ and its corresponding eigenvector m_k. We know that only non zero components of the vector m_k are placed in

kth and $k+1$th component of m_k. This means that, if we call the kth and $k+1$th component of m_k by v_k and v_{k+1} respectively, then $e_kI - A1$ loose the space SP =span$\{D \in \mathbb{R}^n \mid \frac{D_k}{D_{k+1}} = \frac{v_k}{v_{k+1}}\}$. But the kth component of $C1$ is not zero and $k+1$th component of $C1$ is zero. This implies that rank $\begin{pmatrix} C1 \\ mI - A1 \end{pmatrix} = N \quad \forall m \in \sigma(A1) = N$. □

Corollary 5.3. Let P_N be a projection that project every function into N dimensional space. If $f : \mathcal{R}^3 \to R$ be given, then there exists $\bigcup_{i,j,k} r_{i,j,k}$, such that

$$P_N f(x,y,z) = \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{k=1}^{N} r_{m,n,k} \sin(nx)\sin(my)\sin(kz). \tag{5.7}$$

Theorem 5.4. If $\Omega_1 \subset \Omega$ be such that it contains an interval in $X \times Y \times Z$, then DPL equation is approximately controllable.

Proof. Here for simplicity, without loose of generality, only x direction is considered. It can be easily extend to higher dimensions. Let $g(x,t)$ be desired trajectory. If we consider the following representation for $g(x,t)$

$$g(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n$$

then, unknown control function should be found such that

$$\int_0^t \exp(\lambda_n(t-s))b_n(q)dq = a_n(t) \quad n = 1,2,...N. \tag{5.8}$$

where

$$b_n(q) = \int_{\Omega_1} s(q,x)\varphi_n(x)dx. \tag{5.9}$$

From equations (5.8) and (5.9), one can find that

$$b_n(t) = -\lambda_n a_n(t) + \frac{d}{dt}a_n(t) \quad n = 1, \cdots N. \tag{5.10}$$

So, the control problem is reduced to find $s(q,x)$ such that equation (5.9) satisfied for $n = 1, \cdots, N$, where $b_n(q)$ is defined in (5.10). By using previous lemma, $s(q,x)$ can be represented as follows in any N dimensional space

$$s(q,x) = \sum_{n=1}^{N} d_n \sin(\frac{n\pi x}{l}). \tag{5.11}$$

Therefore, the coefficients $d_1, \cdots, d_n$, should be found such that

$$\digamma_N \bar{d_N} = \bar{b_N}, \tag{5.12}$$

where,

$$\digamma_N = [a_{i,j}]_{N\times N}, \quad a_{i,j} = \int_{\Omega_1} \sin(\frac{i\pi x}{l}) \sin(\frac{j\pi x}{l}) dx, \quad \bar{b_N} = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}, \quad \bar{d_N} = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix}.$$

By considering previous lemma, the matrix $\digamma$ is full rank. Therefore, the system (5.12) has a unique solution. The unique solution with Lemma (5.1) conclude approximate controllability. □

In the following, it will be proved that the DPL equation is not exactly controllable by using the following version of Hautus test [3].

A necessary condition for exact controllability of exponentially stable systems (2.5) is existence a constant $M > 0$ such that for every $\varrho \in \mathbb{C}_-$ and every $x \in D(A)$ the following relation holds

$$\| (\varrho I - A)x \|^2 + |Re\varrho| \| Bx \|^2 > M|Re\varrho|^2 \| x \|^2 . \tag{5.13}$$

Here $\mathbb{C}_-$ denotes the open left half plane.

Lemma 5.5. Hautus test is necessary and sufficient condition for exact controllability of DPL equation.

Proof. The eigenvalues and eigenfunctions of A is as same as A^*, also B is restricted operator from Ω to Ω_1. So $\| B \| = \| B^* \|$ in $L^2(\Omega)$.

So, the Hautus test for (A^*, B^*) is equivalent to Hautus test for (A, B). □

The Hautus test for $\frac{\varphi_{\pm nmk}}{\|\varphi_{\pm nmk}\|}$ is as follows:

$$|\varrho - \lambda_{\pm nmk}|^2 + |Re\varrho| k_{\pm nmk} \geq M(Re\varrho)^2 \tag{5.14}$$

where $k_{\pm nmk} = \| B\frac{\varphi_{\pm nmk}}{\|\varphi_{\pm nmk}\|} \|^2$. Since, we assumed that all eigenvalues of A are real, it is enough to consider only real negative ϱ.

The relation (5.14) can be simplified as follows:

$$\varrho^2 - 2\lambda_{\pm nmk}\varrho + \lambda_{\pm nmk}^2 + (-\varrho)k_{\pm nmk} \geq M\varrho^2 \quad \textit{Since } \varrho < 0$$
$$(1 - M)\varrho^2 - (2\lambda_{\pm nmk} + k_{\pm nml})\varrho + \lambda_{\pm nmk}^2 \geq 0 \tag{5.15}$$

the relation (5.15) is a second order equation with respect to ϱ. Here, there exist three cases for roots of equation (5.15) as follows:
(a) Two distinct real roots
(b) One double root
(c) No real roots.

Two distinct real roots:

$$(2\lambda_{\pm nmk} + k_{\pm nmk})^2 - 4\lambda_{\pm nmk}^2(1-M) > 0 \Rightarrow m > \frac{4\lambda_{\pm nmk}k_{\pm nmk} + k_{\pm nmk}^2}{-4\lambda_{\pm nmk}^2} \quad (5.16)$$

The following conditions are necessary for holding the relation (5.15) at each $s < 0$ in the case (a).

$$(a1) \; : \quad M < 1 \Rightarrow \frac{4\lambda_{\pm nmk}k_{\pm nmk} + k_{\pm nmk}^2}{-4\lambda_{\pm nmk}^2} < 1 \Rightarrow (2\lambda_{\pm nmk} + k_{\pm nmk})^2 (5.17)$$

$$(a2) \; : \quad k_{\pm nmk} > -2\lambda_{\pm nmk} \quad (5.18)$$

Since $k_{\pm nmk} < 1$ and some of eigenvalues $\lambda_{\pm nmk}$ are greater than 0.5, the relation (5.18) is contradiction. Therefore, Hautus test does not satisfy in case (a).

One double real root:
This is impossible.

No real root:

$$(2\lambda_{\pm nmk} + k_{\pm nmk})^2 - 4\lambda_{\pm nmk}^2(1-M) < 0 \Rightarrow \frac{(2\lambda_{\pm nmk} + k_{\pm nmk})^2}{4\lambda_{\pm nmk}^2} < 1 - M \Rightarrow$$

$$M < 1 - \frac{(2\lambda_{\pm nmk} + k_{\pm nmk})^2}{4\lambda_{\pm nmk}^2} \Rightarrow k_{\pm nmk} < -4\lambda_{\pm nmk} \quad (5.19)$$

Also $1 - M > 0$. Therefore, by choosing M very close to zero, each element of the set $\{\varphi_{\pm nmk}\}_{nmk}$ satisfies Hautus test. Since $\{\varphi_{\pm nmk}\}_{nmk}$ forms Riesz basis, the exact controllability of DPL equation is proved. Here, it is shown that the control region Ω_1 can be selected very small that it is valuable in controller design.

5.1 Boundary Controllability

In this subsection, It will be shown that DPL equation is not approximately boundary controllable. The results are shown for the set of boundary conditions (5.20), but for any two component of boundary conditions (2.3), similar results will be obtained.

Consider the system (2.1) – (2.3) with following changes:

$$\begin{cases} s = 0, \\ \frac{\partial u}{\partial t}(x, y, 0, t) = f_1(x, y, t) \qquad \frac{\partial u}{\partial t}(x, y, \epsilon, t) = f_2(x, y, t) \end{cases} \tag{5.20}$$

where f_1 and f_2 are control functions.

The system (2.1) – (2.3) with assumption (5.20) can be transformed to the following abstract differential equation

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ u_t \end{pmatrix} \\ \begin{pmatrix} u \\ u_t \end{pmatrix} |_{t=0} = \begin{pmatrix} m \\ g \end{pmatrix} \\ G\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \end{cases} \tag{5.21}$$

where the 2×2 matrix $\mathcal{L}$ and the operator G are in the form:
$\mathcal{L}\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) = \begin{pmatrix} u_2 \\ \alpha \operatorname{div}(u_3) - \frac{1}{\tau_q} u_2 \end{pmatrix}$ and $G\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} Q(x, y, 0) \\ Q(x, y, \epsilon) \end{pmatrix}$.

Here,

$$D(\mathcal{L}) = \{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\Omega) \oplus H^1(\Omega) \mid u_3 \in D(\operatorname{div}), u_1(x, y, z, t) \mid_{x=0,l} = 0, u_1(x, y, z, t) \mid_{y=0,h} = 0\}. \tag{5.22}$$

We would to transfer system (5.21) to a another system as follows:

$$\left\{ \frac{d}{dt}\begin{pmatrix} P \\ Q \end{pmatrix} = A\begin{pmatrix} P \\ Q \end{pmatrix} + B\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right. \tag{5.23}$$

where the matrix A is defined in Section 2, matrix B is unknown in (5.23) and should be achieved. It is better that matrix B^* achieved because of controllability analysis. The following equality is used for finding B^* (see [11].)

$$\left\langle \mathcal{L}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_e - \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, A^*\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_e = \left\langle G\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, B^*\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle \tag{5.24}$$

We have:

$$\left\langle \mathcal{L}\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}w_1\\w_2\end{pmatrix}\right\rangle_e = \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_2\cdot\nabla w_1 + \alpha u_3 w_2 - \frac{1}{\tau_q}u_2w_2 =$$
$$\frac{1}{2}\int_x\int_y [\frac{\alpha\tau_u}{\tau_q}(\frac{\partial}{\partial z}u_2)w_2] \,|_{z=0}^{\epsilon}\, dxdy + \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_2\cdot\nabla w_1 -$$
$$\frac{\alpha}{\tau_q}\nabla u_1\cdot\nabla w_2 - \alpha(\frac{\partial w_2}{\partial x}\frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial y}\frac{\partial u_2}{\partial y}) - \frac{\alpha\tau_u}{\tau_q}\frac{\partial w_2}{\partial z}\frac{\partial u_2}{\partial z}d\Omega \quad (5.25)$$

$$\left\langle \begin{pmatrix}u_1\\u_2\end{pmatrix}, A^*\begin{pmatrix}w_1\\w_2\end{pmatrix}\right\rangle_e = \frac{1}{2}\int_\Omega -\frac{\alpha}{\tau_q}\nabla u_1\cdot\nabla w_2 + \alpha\,\mathrm{div}(w_3)u_2 - \frac{1}{\tau_q}w_2u_2 d\Omega =$$
$$\frac{1}{2}\int_x\int_y [\frac{\alpha\tau_u}{\tau_q}(\frac{\partial}{\partial z}w_2)u_2 - \frac{\partial w_1}{\partial z}u_2] \,|_{z=0}^{\epsilon}\, dxdy + \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_2\cdot\nabla w_1 -$$
$$\frac{\alpha}{\tau_q}\nabla u_1\cdot\nabla w_2 - \alpha(\frac{\partial w_2}{\partial x}\frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial y}\frac{\partial u_2}{\partial y}) - \frac{\alpha\tau_u}{\tau_q}\frac{\partial w_2}{\partial z}\frac{\partial u_2}{\partial z}d\Omega. \quad (5.26)$$

Therefore

$$\left\langle \mathcal{L}\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}w_1\\w_2\end{pmatrix}\right\rangle_e - \left\langle \begin{pmatrix}u_1\\u_2\end{pmatrix}, A^*\begin{pmatrix}w_1\\w_2\end{pmatrix}\right\rangle_e$$
$$= \frac{-1}{2}\int_x\int_y [\frac{\alpha\tau_u}{\tau_q}(\frac{\partial}{\partial z}w_2)u_2 - \frac{\partial w_1}{\partial z}u_2] \,|_{z=0}^{\epsilon}\, dxdy$$
$$= -\frac{1}{2\epsilon}\int_\Omega [\frac{\alpha\tau_u}{\tau_q}(\frac{\partial}{\partial z}w_2)u_2 - \frac{\partial w_1}{\partial z}u_2] \,|_{z=0}^{\epsilon}\, d\Omega$$
$$= \langle G\left(u_1u_2\right), \frac{-1}{2\epsilon}\begin{pmatrix}\int_\Omega \frac{\partial w_1}{\partial z}(0) - \frac{\alpha\tau_u}{\tau_q}\frac{\partial}{\partial z}w_2(0)d\Omega\\ \int_\Omega \frac{\partial w_1}{\partial z}(\epsilon) - \frac{\alpha\tau_u}{\tau_q}\frac{\partial}{\partial z}w_2(\epsilon)d\Omega\end{pmatrix}\rangle. \quad (5.27)$$

So the equations (5.24)-(5.27) imply

$$B^*\begin{pmatrix}w_1\\w_2\end{pmatrix} = \frac{-1}{2\epsilon}\begin{pmatrix}\int_\Omega \frac{\partial w_1}{\partial z}(0) - \frac{\alpha\tau_u}{\tau_q}\frac{\partial}{\partial z}w_2(0)d\Omega\\ \int_\Omega \frac{\partial w_1}{\partial z}(\epsilon) - \frac{\alpha\tau_u}{\tau_q}\frac{\partial}{\partial z}w_2(\epsilon)d\Omega\end{pmatrix}$$

For the controllability of the system (5.23), we will analyze observability of the following dual system

$$\begin{cases}\frac{d}{dt}\begin{pmatrix}P1\\Q1\end{pmatrix} = A^*\begin{pmatrix}P1\\Q1\end{pmatrix}\\ y = B^*\begin{pmatrix}P1\\Q1\end{pmatrix}\end{cases} \quad (5.28)$$

In the following, it will bw shown that the system (5.28) is not approximately observable because there exist some functions $\psi(x,y,z)$ such that $B^*\psi(x,y,z) = 0$.

$$B^*\psi(x,y,z) = B^*\begin{pmatrix} \sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}) \\ \lambda_{n,m,k}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}) \end{pmatrix} =$$

$$\begin{pmatrix} \int_{x=0}^{l}\int_{y=0}^{h}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})(1-\frac{\alpha\tau_u\lambda_{n,m,k}}{\tau_q})dxdy \\ \lambda_{n,m,k}(-1)^k\int_{x=0}^{l}\int_{y=0}^{h}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})(1-\frac{\alpha\tau_u\lambda_{n,m,k}}{\tau_q})dxdy \end{pmatrix} \quad (5.29)$$

The equation (5.29) results that the system (5.21) is not approximately controllable when n or m be even.

6 Conclusion

The aim of this study is to apply semigroup theory on dual-phase-lagging equation and constructing it solution in various initial and boundary conditions. The semigroup methods provide a closed form solution of dual-phase-lagging equation that is valuable for analyzing the dynamical system governed by dual-phase-lagging equation. The effect of each initial and boundary conditions in behavior of dynamical system can be analyzed by existing this closed analytical form while this not true for some numerical methods such as finite difference. References [13, 10] argue in analytical solution of DPL equation. The internal source s is not considered in [13]. Analytical solution of DPL with source term is considered in [10], but it is not efficient in computational point of new. It seems the analytical solution that obtained by using semigroup method gives a closed form analytical solution with a fewer computations effort.

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