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# On robustness and dynamics in (un)balanced coalitional games

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**Abstract**

We build upon control theoretic concepts like robustness and dynamics to better accommodate all the situations where the coalitions' values are uncertain and subject to changes over time. The proposed robust dynamic framework provides an alternative perspective on the study of sequences of coalitional games or interval valued games. For a sequence of coalitional games, either balanced or unbalanced, we analyze the key roles of instantaneous and average games. Instantaneous games are obtained by freezing the coalitions' values at a given time and come into play when coalitions' values are known. On the other hand, average games are derived from averaging the coalitions' values up to a given time and are key part of our analysis when coalitions' values are unknown. The main theoretical contribution of our paper is a design method of allocation rules that return solutions in the core and/or $\epsilon$-core of the instantaneous and average games. Theoretical results are then specialized to a simulated example to shed light on the impact of the design method and on the performance of the resulting allocation rules.

**Keywords** cooperative games, dynamic games, unbalancedness.

## 1 Introduction

In this work, we investigate robustness and dynamics in coalitional games. We build upon these two concepts to better accommodate all the situations where the coalitions' values are uncertain and subject to changes over time. More precisely, *robustness* derives from assuming that the values of the coalitions are not known as dealt with in the recent literature on stochastic coalitional games [9, 10]. We deviate from this stochastic framework since we assume that coalitions' values are unknown but bounded within a priori known polytopic sets. So, differently from [9, 10], no stochastic properties are available and the coalitions' values are not random processes. The term *dynamics* relates to the time-varying nature of the coalitions' values, as captured first in [6, 7]. However, our approach differs from [6, 7]

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mainly in the assumption that no relation exists between coalitions' values at consecutive times except the fact that they always belong to given sets.

The motivation of our research comes from the idea that the proposed robust dynamic framework provides a complementary perspective on the studies of sequences of coalitional games, as introduced by [8], and of interval valued games, described in [1]. A first attempt along this line of research is [3], where we translate sequences of balanced coalitional games into a (dynamic) system theoretic framework. In the current paper, we extend this framework to unbalanced games and revisit results in [3] using a new language and new concepts. Thus, notions like control strategy, $\epsilon$-stabilizability, and robustness [2] will be tightly linked to allocation rules, $\epsilon$-core, and balancedness [8].

Our analysis and results are divided in two parts, namely balanced games, and unbalanced games. In the first part concerning balanced games, we start by providing necessary and sufficient conditions for the sequence of games to be balanced. Such conditions involve sets inclusions and concern either sets of allowable allocations or sets of coalitions' values. Next, we propose a design method for allocations belonging to the core of the instantaneous game assuming that coalitions' values are known. This game is obtained by freezing the coalitions' values at a given time. If the coalitions' values are not known, we turn to consider the average games, that is, the games obtained by averaging the coalitions' values up to a given time. In this case, we can guarantee that allocation vectors deviate from the core of the average games of at most a given tolerance $\epsilon$. That is, the allocations belong to the $\epsilon$-core of the average game. We end the first part with results on the infinite horizon version of the problem. After averaging coalitions' values up to infinity, average allocations are shown to belong to the core of the average game. To summarize, lack of information on coalitions' values prevents allocations to be in the core if we look at each instantaneous game. Nevertheless the allocation rule is able to compensate such a lack of information on the long run if the average allocation converges to the core of the average game.

The second set of results concerns unbalanced games. Dropping the balancedness assumption, we are no longer able to guarantee any tolerance $\epsilon$, which means that the allocation vector can deviate any large enough value from the core of the average game. In other words, the allocation vector is no longer guaranteed to be in the $\epsilon$-core of the average game. However, we can still prove that the average allocation converges in probability to the core of the average game.

This paper is organized as follows. In Section 2 we introduce the problem. In Sections 3 and 4 we elaborate on respectively the balanced and unbalanced cases. Section 5 contains a numerical example to illustrate the results. Finally, Section 6 provides conclusions.

## 2 Problem statement

The problem statement consists in three different parts each one giving rise to one of the three following subsections. We start by introducing the sequence of games we focus on. Then, we generalize the framework to include fast periodical allocations. Finally, we comment on the nature of bounds on allocations and turn to the dynamic aspects of our problem.

### 2.1 Sequence of games

Let $N = \{1, \ldots, n\}$ be a set of $n$ players. A coalition $S$ is a nonempty subset of the player set $N$. Let the inclusion $S \subseteq N$ mean that $S$ is a coalition. Denote by $m = 2^n - 1$ the number of coalitions. A cooperative game, or game in coalitional form, is a pair $< N, v >$, where $v$ is the characteristic function assigning a value $v(S)$ to each coalition $S$. Such a game is called balanced if and only if the core is nonempty. The core can be expressed as below, with $\Delta^n$ being the simplex in $\mathbb{R}^n$:

$$C(v) = \{a \in \mathbb{R}^n : \frac{a}{v(N)} \in \Delta^n, \sum_{i \in S} a_i \geq v(S) \,\text{for all}\, S \subseteq N\}.$$

There exist many classes of balanced games; see for example [4, 5].

Now, consider a sequence of games in coalitional form where the vector of coalitional values $\mathbf{v}(t) = [v(t, S)]_{S \subseteq N}$ is a function of time $t$ and fluctuates in a bounded polyhedron $\mathcal{V}$:

$$< N, \mathbf{v}(t) >, \; t = 1, 2, \ldots \text{ with } \mathbf{v}(t) \in \mathcal{V} \text{ for all } t. \tag{1}$$

This sequence of games has a *vector of average coalitions' values* $\bar{\mathbf{v}}$, being defined by

$$\bar{\mathbf{v}} = \lim_{T \longrightarrow \infty} \frac{1}{T} \sum_{k=0}^{T} \mathbf{v}(k). \tag{2}$$

Before recalling the next result we need to define the following notation. Let us first introduce the column vector of nonnegative surplus variables $s = [s_1, \ldots, s_{m-1}]' \geq 0$ where $\zeta'$ is the transposed of a given vector $\zeta$ and let $I$ be the $(m-1)$-dimensional identity matrix. Also, denote by $B \in \mathbb{R}^{m \times n}$ the matrix whose rows are the characteristic vectors $e^S \in \mathbb{R}^n$. Recall that the components of a characteristic vector are $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. Finally, let us introduce matrix $A \in \mathbb{R}^{m \times (n+m-1)}$ defined by

$$A = \left[ \begin{array}{c|c} B & \begin{array}{c} -I \\ ---- \\ 0 \ldots 0 \end{array} \end{array} \right]. \tag{3}$$

With the above notation in mind, we are ready to recall the following result.

If $< N, v >$ is a balanced game then finding an allocation vector $a$ in the core corresponds to finding an "augmented" *allocation vector* $u := \left[\begin{smallmatrix} a \\ s \end{smallmatrix}\right] \in \mathbb{R}^{n+m-1}$ in the set (4), where $a_{min} \in \mathbb{R}^n$ is an apriori given vector:

$$\mathcal{U}(v) = \left\{ u : Au = v, \; u \geq \begin{bmatrix} a_{min} \\ 0 \end{bmatrix} \right\}. \tag{4}$$

To elaborate more on the meaning of the surplus variables, note that each surplus variable $s_S$ corresponds to a coalition $S$ of players and describes the difference between the allocated value and the coalitional value, $s_S = \sum_{i\in S} a_i - v(S)$. Also, notice that there are only $m-1$ surplus variables because coalition $N$ has no surplus ($\sum_{i\in N} a_i - v(N) = 0$) due to the efficiency condition of the core.

### 2.2 Fast periodical allocations

Throughout the rest of the paper, we assume that allocations to players are made at an integer rate $1/\Theta$, $\Theta < 1$. This is larger than the rate of change of the coalitional values, which equals one by default. Then $\Theta$ is the time between two successive allocations. To facilitate our analysis, we stretch the time scale by the rate $1/\Theta$ and consider a new sequence of instantaneous games,

$$v(k) = \mathbf{v}(t)\Theta, \quad k = \frac{t-1}{\Theta} + 1, \ \ldots, \frac{t}{\Theta}, \quad t = 1, 2, \ldots. \tag{5}$$

This new sequence of games has the following interpretation. In the original time interval $(t-1, t]$ the vector of coalitional values equals $\mathbf{v}(t)$. We distribute these values equally over the $1/\Theta$ allocations that occur in this time period, so this results in values $\mathbf{v}(t)\Theta$ for each point in time where allocations are made. This ensures that the total amount allocated to the players in the new interval $((t-1)/\Theta, t/\Theta]$ does not exceed the available amount $v(t, N)$.

All the above reasoning leads us to the following crucial observation. Denote $\mathcal{V}^\Theta = \Theta \cdot \mathcal{V}$ and $\bar{v} = \Theta\bar{\mathbf{v}}$, the sequence of games (1)-(2) is equivalent to the sequence of games

$$\begin{array}{c} < N, v(k) >, \, k = 1, 2, \ldots \text{ with } v(k) \in \mathcal{V}^\Theta \text{ for each } k = 1, 2, \ldots \\ \bar{v} = \lim_{T\longrightarrow\infty} \frac{1}{T} \sum_{k=0}^{T} v(k). \end{array} \tag{6}$$

In the remainder of this paper, we will always refer to the sequence of instantaneous games expressed by (6).

### 2.3 Dynamics of coalitions' excesses

In this section we briefly comment on the nature of lower and upper bounds on allocations and introduce the dynamic model. More specifically, we assume that the allocation vector is bounded within a polyhedron

$$\mathcal{U} := \{u : u_{min} \leq u \leq u_{max}\}$$

with $u_{min} := \left[\begin{smallmatrix} a_{min} \\ 0 \end{smallmatrix}\right] \leq 0 \leq u_{max}$. Define $x(k+1) \in \mathbb{R}^m$ as the state variable of the system, with $x(0)$ the excess at time 0. This vector of variables describes the aggregate coalitions' excesses over the games $v(1), \ldots, v(k)$ up to time $k$:

$$x(k+1) = x(k) + Au(k) - v(k), \ v(k) \in \mathcal{V}^\Theta, \ u(k) \in \mathcal{U}, \, k = 1, 2, \ldots. \tag{7}$$

Here we write $u(k) = \left[\begin{smallmatrix} a(k) \\ s(k) \end{smallmatrix}\right]$, $a(k) = [a_i(k)]_{i \in N}$ with $a_i(k)$ the revenue allocated to player $i$, and $s(k) = [s_S(k)]_{S \subset N}$.

With the introduction of dynamics (7), we conclude this first part of the work regarding the problem statement and move to the presentation of results for the balanced case.

# 3 Balanced games

## 3.1 Core and $\epsilon$-core of the instantaneous game

In this section we analyze sequences of games (6) that are balanced. The lemma below provides necessary and sufficient conditions for this to be true in terms of the sets $\mathcal{V}^\Theta$ and $\mathcal{U}$.

**Lemma 1** *All the games in the sequence (6) are balanced if and only if*

$$\mathcal{V}^\Theta \subseteq A\mathcal{U}. \tag{8}$$

*Then there exists an allocation rule $u(k) := \Phi(v(k))$ in the core of the game $< N, v(k) >$ for all $k$, i.e.,*

$$a(k) \in C(v(k)), \forall k. \tag{9}$$

*Proof* From the definitions it follows that balancedness is equivalent to (8). Next, we prove (9).

(Sufficiency) If (8) is true, then there exists a vector $u(k) \in \{u \in \mathcal{U} : Au = v(k)\} \subseteq \mathcal{U}(v(k))$. Thus $a(k) \in C(v(k))$ by (4).

(Necessity) If (8) is false, i.e., $\mathcal{V}^\Theta \not\subseteq A\mathcal{U}$, then there exists a vertex $v^{(r)}$ of $\mathcal{V}^\Theta$ such that $Au \neq v^{(r)}$ for all $u \in \mathcal{U}$; $U \cap U(v^{(r)}) = \emptyset$. On each time $k$ where $v(k) = v^{(r)}$ there exists no $a(k) \in C(v(k))$; the core $C(v(k))$ is empty. $\square$

Thus, if all the games are balanced then we can select allocations in the core $C(v(k))$ of the instantaneous game $< N, v(k) >$ at any time $k$. Allocations in the core $C(v(k))$ are not applicable if $v(k)$ is not known at time $k$. In this case, allocations outside the core may be approximately close to the core according to a certain tolerance $\epsilon$. This is captured by the definition of $\epsilon$-core as provided in [8].

**Definition 1** *($\epsilon$-core [8]) The $\epsilon$-core is the set of all allocations such that the total amount received by each coalition exceeds or is equal to the value of the coalition reduced by a given tolerance $\epsilon$:*

$$C_\epsilon(v) := \{a \in \mathbb{R}^n : \frac{a}{v(N)} \in \Delta^n, \sum_{i \in S} a_i \geq v(S) - \epsilon \textit{ for all } S \subseteq N\}.$$

We say that a game is $\epsilon$-balanced if and only if its $\epsilon$-core is nonempty.

Given a function $f(k)$, denote by $\bar{f}$ the long term average of a given function $f(k)$, i.e., $\lim_{k\longrightarrow\infty} \frac{1}{k}\sum_{j=0}^{k} f(j)$ and $\bar{f}^k$ the average up to time $k$. The game $< N, \bar{v}^k >$ is called the *average game* (up to time $k$).

Take $\epsilon = \max_{v\in\mathcal{V}^\Theta} \|v\|$, the infinity norm of the vector $v$, and consider a tolerance $\varepsilon(k) := \frac{\epsilon}{k}$. Average games and $\varepsilon$-balancedness are related as follows.

**Lemma 2** *Assume $v(k)$ is unknown at time $k$. There exists a time $\tilde{k}$ such that all average games $< N, \bar{v}^k >$ are $\varepsilon(k)$-balanced for all $k \geq \tilde{k}$ if and only if*

$$\mathcal{V}^\Theta \subseteq int\{A\mathcal{U}\}. \tag{10}$$

*Furthermore, there exists an allocation rule $u(k) := \Phi(x(k))$ such that the average allocations $\bar{a}^k$ are in the $\varepsilon(k)$-core of the average game $< N, \bar{v}^k >$ for all $k \geq \tilde{k}$, i.e.,*

$$\bar{a}^k \in C_{\varepsilon(k)}(\bar{v}^k), \ \forall k \geq \tilde{k}. \tag{11}$$

*Proof*

(Sufficiency) Assume first that $\tilde{k}$ exists (we prove its existence in the second part of the proof), and take for it the first time instant where $-x(\tilde{k}) \in \mathcal{V}^\Theta$. We show that $-x(\tilde{k}+1) \in \mathcal{V}^\Theta$ for some $u(\tilde{k}) \in \mathcal{U}$. By (10), there exists an allocation rule $u(\tilde{k}) \in \mathcal{U}$ such that $Au(\tilde{k}) = -x(\tilde{k})$. Then $x(\tilde{k}+1) = -v(\tilde{k})$ by (7) and also $-x(\tilde{k}+1) \in \mathcal{V}^\Theta$. We can repeat the same argument inductively for $\tilde{k}+2$ and so on. This proves that $-x(k) \in \mathcal{V}^\Theta$ for all $k \geq \tilde{k}$.

Now, we prove the existence of $\tilde{k}$. Consider any time instant $k$ such that $-x(k) \notin \mathcal{V}^\Theta$. Define a new variable $w(k) = x(k-1) + Au(k-1)$ so that $w(k) - x(k) = v(k-1) \in \mathcal{V}^\Theta$ (6). Now, choose $u(k) := u_1(k) + u_2(k) \in \mathcal{U}$ with $u_1(k)$ satisfying

$$Au_1(k) = w(k) - x(k) = v(k-1) \in \mathcal{V}^\Theta.$$

Using the above equality, the dynamics for $w(k)$ turns out to be

$$\begin{aligned} w(k+1) &= x(k) + Au(k) \\ &= x(k) + Au_1(k) + Au_2(k) \\ &= w(k) + Au_2(k). \end{aligned}$$

Now select $u_2(k)$ such that the norm of $w(k+1)$ is minimized, i.e.,

$$u_2(k) = \arg\min_{u\in\mathcal{U}} \|w(k) + Au\|. \tag{12}$$

If $u_2$ is chosen such that $Au_2$ is in the opposite direction of $w$ then the formulation of $w(k+1)$ implies that the norm of $w$ is reduced. Because of condition $0 \in int\{A\mathcal{U}\}$, which follows from (10) and $0 \in \mathcal{V}^\Theta$, we certainly have

$$\|w(k+1)\| - \|w(k)\| \leq \beta < 0$$

for some $\beta$, until $\|w(\tilde{k})\| = 0$ for a large enough $\tilde{k}$. This in turn implies that $-x(k)$, whose dynamics can be rewritten as $-x(k) = -w(k) + v(k-1)$, is ultimately bounded in $\mathcal{V}^{\Theta}$.

Hence we have shown that there exists a time $\tilde{k}$ such that $-x(k) \in \mathcal{V}^{\Theta}$, and $\|x(k)\| \le \epsilon$ as well, for all $k \ge \tilde{k}$. Now, using $\varepsilon(k) = \frac{\epsilon}{k}$, the latter inequality implies the inequalities $-\varepsilon(k) \le \sum_{i\in S} \bar{a}_i^k - \bar{s}_S^k - \bar{v}_S^k \le \varepsilon(k)$ for all $S \subset N$ from which we have $\sum_{i\in S} \bar{a}_i^k \ge \bar{v}_S^k + \bar{s}_S^k - \varepsilon(k) \ge \bar{v}_S^k - \varepsilon(k)$. This proves that all average games $< N, \bar{v}^k >$ for all $k \ge \tilde{k}$ are $\varepsilon(k)$-balanced and also that (11) holds true.

(Necessity) Assume that (10) is false. Then there certainly exists a vertex $v^{(r)}$ of the polytopic set $\mathcal{V}^{\Theta}$ such that either there no exists $u_1(k) \in \mathcal{U}$ for which $Au_1(k) = v^{(r)}$ or, if it exists, it lies on the frontier, i.e., $u_1(k) \in \partial\mathcal{U}$. In both cases $u_2(k) = 0$, which means that we cannot select $u_2(k)$ as in (12). Thus if $v(k) = v^{(r)}$ for all $k > 0$ then we cannot bound $x(k)$ in $\mathcal{V}^{\Theta}$ for any $k$ large enough. Then the average game $< N, \bar{v}^k >$ is not $\varepsilon(k)$-balanced for some $k > 0$, and condition (11) is not true. $\square$

Under certain conditions we can design allocation rules such that averaging the allocations over the long run results in a desired value, called the *nominal allocation*. We analyze such conditions in the next section.

### 3.2 Nominal allocations

Let $a_{nom} \in \mathbb{R}_+^N$ be an apriori given allocation vector, referred to as the *nominal allocation*, and denote by $v^{(r)}$ with $r \in Ext\{\mathcal{V}^{\Theta}\}$ a vertex of the polytopic set $\mathcal{V}^{\Theta}$. To be more precise, $Ext\{\mathcal{V}^{\Theta}\}$ is the set of indices of all vertices of $\mathcal{V}^{\Theta}$. Assume that $\bar{v} = v_{nom}$, the vector of known *nominal coalitions' values*. Consider a matrix $D \in \mathbb{R}^{(n+m-1)\times m}$ subject to the conditions (see also [3, Lemma 1]):

$$AD = I \in \mathbb{R}^{m\times m} \tag{13}$$

$$u_{min} \le D(v^{(r)} - v_{nom}) + u_{nom} \le \quad u_{max} \quad \forall r \in Ext\{\mathcal{V}^{\Theta}\}, \tag{14}$$

where $u_{nom} = \begin{bmatrix} a_{nom} \\ s_{nom} \end{bmatrix} \in U$ such that $Au_{nom} = v_{nom}$. We investigate under which conditions

$$\bar{a}(k) = a_{nom} \in C(v_{nom}) \quad \text{for some } v_{nom} \in \mathcal{V}^{\Theta}. \tag{15}$$

That is, when does the long run average allocation converge to the nominal allocation?

**Lemma 3** *Assume that $v(k)$ is known at time $k$, condition (10) is satisfied, and the vector of average coalitions' values $\bar{v}$ is equal to a given $v_{nom} \in \mathcal{V}^{\Theta}$. Furthermore, consider $u_{nom} = \begin{bmatrix} a_{nom} \\ s_{nom} \end{bmatrix} \in \mathcal{U}$ such that $Au_{nom} = v_{nom}$. There exists an allocation rule $u(k) := \Phi(v(k))$ such that (9) and (15) hold if and only if there exists a matrix $D \in \mathbb{R}^{(n+m-1)\times m}$ that satisfies (13) and (14). The allocation rule is linear on $v(k)$, that is,*

$$u(k) = u_{nom} + D(v(k) - v_{nom}). \tag{16}$$

*Proof* (Sufficiency) Assume that (13) and (14) are true. We show that the allocation rule (16) satisfies (9) and (15). To prove (9), it suffices to show that $Au(k) = v(k)$ with $u(k)$ as in (16). This is evident as by substitution we have

$$Au(k) = A(u_{nom} + D(v(k) - v_{nom})) = Au_{nom} + v(k) - v_{nom} = v(k).$$

To prove (15), observe that $\bar{u} = u_{nom} + D(\bar{v} - v_{nom}) = u_{nom}$ which implies (15).

(Necessity) Assume now that (9) and (15) are true. We show that (13) and (14) are satisfied, and that the allocation rule is of type (16).

Let $W = [v^{(1)}\ v^{(2)}\ \ldots v^{(s)}]$ be a matrix whose columns are the vertices of $\mathcal{V}^{\Theta}$. Also, because of (9), we can consider a matrix $U = [u^{(1)}\ u^{(2)}\ \ldots u^{(s)}]$ whose columns are the allocations $u^{(j)} \in \mathcal{U}$ that satisfy $Au^{(j)} = v^{(j)}$ for all $j = 1, \ldots, s$. Take, without loss of generality, $v_{nom} = 0$ and $u_{nom} = 0$. This is possible after translating the origin of the $v$-space and $u$-space in $v_{nom}$ and $u_{nom}$ respectively. Now, condition (15) means that $\bar{u} = 0$ if and only if $\bar{v} = 0$. Note that $\bar{w}$ and $\bar{u}$ can always be rewritten as convex combinations of the columns of $W$ and $U$ (vertices of $\mathcal{V}^{\Theta}$ and $\mathcal{U}$). Then, $\bar{u} = 0 \Leftrightarrow \bar{v} = 0$ implies that the kernels of $W$ and $U$ coincide, i.e., $Ker(W) = Ker(U)$. Thus there exists a matrix $\hat{D} \in \mathbb{R}^{(m \times n)}$ such that $U = \hat{D}W$. Since $W = AU = A\hat{D}W$, $\hat{D}$ is a right inverse of $A$. This proves (13). To prove (14), we use convex combinations with coefficients $\sum_{j=1}^{s} \alpha_j = 1$, and $\alpha_j \geq 0$ for all $j = 1, \ldots, s$. Then $u(k) = \hat{D}v(k) = \hat{D}\sum_{j=1}^{s} \alpha_j(v^{(j)}) = \sum_{j=1}^{s} \alpha_j \hat{D}(v^{(j)}) = \sum_{j=1}^{s} \alpha_j u^{(s)} \in \mathcal{U}$ and so (14) is proved as well. Finally note that the resulting allocation rule $u(k) = \hat{D}v(k)$ corresponds to (16) if we translate the origins of the $u$-space and $v$-space back to the zero-vector. □

We proceed by analysing what happens if $v(k)$ is not known at time $k$.

**Lemma 4** *Assume that $v(k)$ is unknown at time $k$, condition (10) is satisfied, and the average coalitions values $\bar{v}$ are equal to a given $v_{nom} \in \mathcal{V}^{\Theta}$. Furthermore, consider $u_{nom} = \begin{bmatrix} a_{nom} \\ s_{nom} \end{bmatrix} \in \mathcal{U}$ such that $Au_{nom} = v_{nom}$. There exists an allocation rule $u(k) := \Phi(x(k), .)$ such that (11) with $\epsilon = \max_{v \in \mathcal{V}^{\Theta}} \|v\|$ and (15) hold if and only if there exists a matrix $D \in \mathbb{R}^{(n+m-1)\times m}$ that satisfies (13) and (14).*

*Proof* Consider the matrices $A$, as in (3), and $D$ satisfying (13) and (14). Using a standard property of linear algebra, we can find two matrices $C$ and $F$ that "square" $A$ and $D$, and satisfy

$$\begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} D & F \end{bmatrix} = I. \tag{17}$$

The introduction of matrix $C$ is useful to construct the augmented system

$$\begin{aligned} x(k+1) &= x(k) + Au(k) - v(k) \\ y(k+1) &= y(k) + Cu(k). \end{aligned} \tag{18}$$

On the other hand, we can use matrix $F$ to define a new variable $z(k) \in \mathbb{R}^{(n+m-1)}$ as expressed below

$$z(k) = \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} z(k).$$

This variable evolves according to the dynamic equation (19). Now, because of its "simple form", such dynamics will soon lead us to the construction of an opportune allocation rule:

$$\begin{aligned} z(k+1) &= \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k+1) \\ y(k+1) \end{bmatrix} \\ &= \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} u(k) - \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} v(k) \\ 0 \end{bmatrix} \\ &= z(k) + u(k) - Dv(k). \end{aligned} \tag{19}$$

It turns useful to rewrite the above dynamics componentwise. So, we have

$$z_i(k+1) = z_i(k) + u_i(k) - D_i v(k), \tag{20}$$

where $D_i$ is the $i$th row of $D$ and $u_{i,\min} \leq u_i \leq u_{i,\max}$. With reference to (20), a possible allocation rule is

$$u_i(k) = sat_{[u_{i,\min}, u_{i,\max}]}[-z_i(k)] := \begin{cases} u_{i,min} & \text{if} \quad z_i(k) > -u_{i,\min} \\ u_{i,max} & \text{if} \quad z_i(k) < -u_{i,\max} \\ -z_i(k) & \text{if} \quad -u_{i,\max} \leq z_i(k) \leq -u_{i,\min} \end{cases}. \tag{21}$$

Let the time $\tilde{k}$ be such that $-u_{i,\max} \leq z_i(\tilde{k}) \leq -u_{i,\min}$ (hereafter we prove the existence of such a $\tilde{k}$). Take without loss of generality $u_{nom} = v_{nom} = 0$. Because of (14), note that the dynamics (20) and (21) imply $-u_{i,\max} \leq z_i(\tilde{k}+1) = -D_i v(\tilde{k}) \leq -u_{i,\min}$. Repeating the same argument forward in time results in $-u_{i,\max} \leq z_i(k+1) = -D_i v(k) \leq -u_{i,\min}$ for all $k \geq \tilde{k}$.

Now, we show that the time $\tilde{k}$ exists. Take $z_i(k) < -u_{i,\max}$. Using (21) in the dynamics (20) and again because of (14), we obtain $z_i(k+1) - z_i(k) = u_{i,\max} - D_i v(k) > 0$ until $-u_{i,max} \leq z_i(\tilde{k})$ for a large enough $\tilde{k}$. The proof for $z_i(k) > -u_{i,\min}$ is similar.

We have proved that for a large enough $\tilde{k}$ we may reach the condition

$$\|z(k)\| \leq \max_{\tau \geq \tilde{k}} \|Dv(\tau)\| \quad \text{for all } k \geq \tilde{k}. \tag{22}$$

This last condition also implies that for all $k \geq \tilde{k}$

$$\|x(k)\| \leq \max_{k \geq \tilde{k}} \|Az(k)\| \leq \max_{v \in \mathcal{V}^\Theta} \|ADv\| = \max_{v \in \mathcal{V}^\Theta} \|v\| = \max_{j=1,\ldots,s} \|v^{(j)}\| = \epsilon,$$

which proves condition (11).

To prove (15), we sum (19) for $k = 1, 2, \ldots$ to obtain

$$\frac{1}{T} \sum_{k=0}^{T-1} u(k) - \frac{1}{T} \sum_{k=0}^{T-1} Dv(k) = \frac{z(T) - z(0)}{T} \to 0$$

as $T \to \infty$, since the numerator is a finite quantity whereas the denominator tends to infinity. Therefore $\bar{u} = D\bar{v}$, which concludes the proof. $\square$

# 4 Unbalanced games

In this section, we move to consider sequences of games that are, in general, not balanced. This is the case, for instance, when condition (8) in Lemma 1 does not hold. The following assumption applies.

**Assumption 1** *Assume that the following condition is satisfied:*

$$\mathcal{V}^{\Theta} \not\subseteq AU. \tag{23}$$

It is reasonable to assume that the expected coalitions' values are not correlated with the state and coincide at each time with the long term average. We establish this in the following assumption.

**Assumption 2** *The vector of coalitions' values $v(k)$ is a mean ergodic stochastic process, i.e., $E[v(k)] = \bar{v}$, that is not correlated with $x(k)$, i.e., $E[x(k)^T v(k)] = 0$.*

We can always translate the origin of the $u$ and $v$ spaces without loss of generality. So, let us move the origins in $u_{nom}$ and $v_{nom}$ as established next.

**Assumption 3** *For ease of calculations set $u_{nom} = v_{nom} = 0$ and assume that $\bar{v} = v_{nom} = 0 \in int\{A\mathcal{U}\}$.*

Note that $0 \in int\{A\mathcal{U}\}$ means that the interior of $A\mathcal{U}$ includes at least the nominal coalitions' values. So, this is a relaxation of condition (10) where the interior of $A\mathcal{U}$ includes all coalitions' values. Further, it means that the average game is balanced.

Before introducing the next result we define the *distance* between a point $x \in \mathbb{R}^n$ and a set $S$ in $\mathbb{R}^n$ as $d(x, S) = \min_{y \in S} \|x - y\|$, and define the function $V(x) = x^T x/2$.

**Theorem 1** *Under Assumptions 2 and 3, there exists an allocation rule $u(k) := \Phi(.)$ such that $d(\bar{a}^k, C(\bar{v}^k)) \to 0$, for $k \to \infty$ with probability one. Furthermore, such an allocation rule satisfies (15). A possible control is*

$$u(k) = \arg\min_{u \in \mathcal{U}} V(x(k) + Au). \tag{24}$$

*Proof* The first part of the theorem, which establishes $d(\bar{a}^k, C(\bar{v}^k)) \to 0$ for $k \to \infty$, is proved if we show that $x(k)$ tends to zero with probability one. Let $u(k)$ be defined as in (24), and recall that $0 \in int\{A\mathcal{U}\}$. For $x \neq 0$ the new variable $w(k) = x(k-1) + Au(k-1)$ satisfies the condition

$$V(w(k+1)) = V(x(k) + Au(k)) < V(x(k) + A0) = V(x(k))$$

by definition of $u(k)$, and because $0 \in \mathcal{U}$. Further, $w(k+1) = x(k) + Au(k) = x(k+1) + v(k)$ by definition of $w(k)$ and (7). This implies $x(k+1) = w(k+1) - v(k)$. Applying the triangle inequality results in

$$V(x(k+1)) < V(w(k+1)) + V(v(k)) < V(x(k)) + V(v(k)).$$

If we take expectations we obtain $E(V(x(k+1))) < E(V(x(k))) + E(V(v(k)))$. Finally, recall that $\lim_{\Theta \to 0} E(V(v(k))) = 0$ as $v(k) \in \mathcal{V}^{\Theta}$ and $\mathcal{V}^{\Theta} = \Theta \mathcal{V}$. Then, taking the limit we have $\lim_{\Theta \to 0} E(V(x(k+1))) < \lim_{\Theta \to 0}(E(V(x(k))) + E(V(v(k)))) = \lim_{\Theta \to 0} E(V(x(k)))$. This last inequality implies that $x(k)$ tends to zero with probability one (and $x(k)$ is said to be stochastically stable).

The proposed strategy does imply stochastic stability but it does not necessarily satisfy (15). To enforce (15) we use (20) which we rewrite below

$$z_i(k+1) = z_i(k) + u_i(k) - \delta_i(k),$$

where $\delta_i(k) = D_i v(k)$, and $D_i$ is the $i$th row of any matrix $D$ satisfying (13) but not necessarily (14). Note that $E[\delta_i] = D_i E[v(k)] = 0$. If we consider the function $V(z_i) = z_i(k)^2/2$, and slightly modify (24) to

$$u_i(k) = \arg \min_{u_{i,\min} \leq \mu \leq u_{i,\max}} V(z_i(k) + \mu),$$

then we see that $E(V(z_i(k+1))) \leq E(V(z_i(k))) + E(V(\delta_i(k)))$. Taking limits results in $\lim_{\Theta \to 0} E(V(z_i(k+1))) \leq \lim_{\Theta \to 0} E(V(z_i(k))) + \underbrace{\lim_{\Theta \to 0} E(V(\delta_i(k)))}_{=0} \leq \lim_{\Theta \to 0} E(V(z_i(k)))$ which means that the $z_i(k)$ subsystem is stable with probability one. Then, $\frac{1}{T}[z_i(T) - z_i(0)] \to 0$, which implies $\frac{1}{T}\sum_{k=0}^{T} [u_i(k) - \delta_i(k)] \to 0$ with probability one and (15) is proved. $\square$

# 5 Numerical example.

Consider three players and the following intervals for the coalitions' values.

$$\begin{array}{cccc} v(\{1\}) = 0 & v(\{2\}) = 0 & v(\{3\}) = 0 & \\ v(\{1,2\}) \in [0,5] & v(\{1,3\}) \in [0,5] & v(\{2,3\}) \in [0,7] & v(N) \in [0,12]. \end{array}$$

These intervals may be interpreted as the intervals of cost-savings of the coalitions in a single-period one-warehouse multi-retailer inventory system, cf. [3]. Let the nominal coalitions' values and the nominal average allocation vector be

$$v_{nom} = [0,0,0,2,3,4,10]^T, \quad u_{nom} = [3,5,2,3,5,2,6,2,3]^T.$$

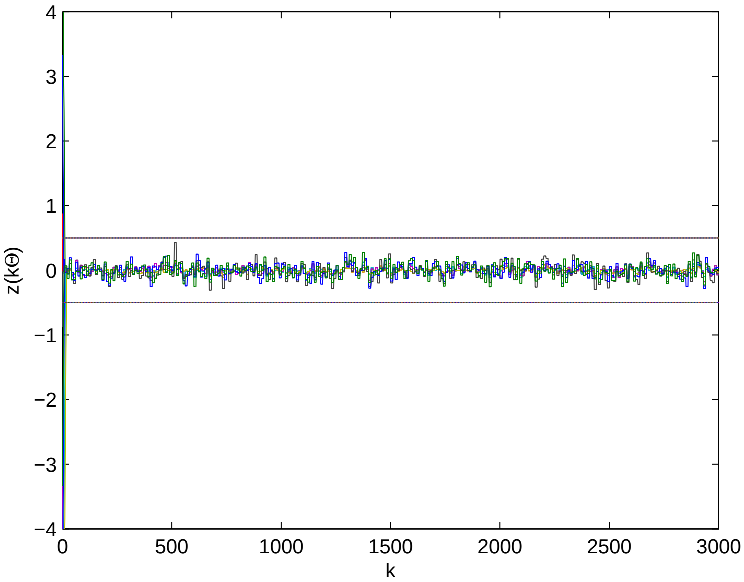


Figure 1: Time plot of $z(.)$. The variable is $\epsilon$-stabilized with $\epsilon = 0.5$.

Note that $Au_{nom} = v_{nom}$. We translate the origin of the $u$-$v$ space to $u_{nom}$-$v_{nom}$.

First, we calculate $D$ as illustrated in the appendix. Then we compute the matrices $C$ and $F$ that square $B$ and $D$ using the method explained in detail in the appendix of [3]. For the maximum sample time we get $\Theta^* > 0.1$ and choose $\Theta = 0.1$.

We divide this simulation section in two parts: balanced and unbalanced games. In a first set of simulations, we consider the bounding polyhedron $\mathcal{U} := \{u \in \mathbb{R}^{10} : -0.5 \cdot \mathbf{1} \leq u \leq 0.5 \cdot \mathbf{1}\}$, where $\mathbf{1}$ is the 10-dimensional vector of ones. Condition (10) holds true and the resulting games in the sequence are balanced. Now, by using the results of Lemma 4, we implement the dynamic allocation rule (21) to simulate the evolution of the system as displayed in Figures 1-4. More specifically, Fig. 1 shows the time plot of the variable $z(.)$. This variable was defined in the proof of Lemma 4 and its evolution follows (19). The variable is $\epsilon$-stabilized with $\epsilon = 0.5$. We recall that $\epsilon$-stabilizing $z(.)$ is a key part in the proof of Lemma 4 that shows the validity of (11) and (15). For the same simulation scenario, Fig. 2 shows the time plot of $\bar{u}^k - u_{nom}$, where $\bar{u}^k$ is the average of $u(k)$ up to time $k$. All plots tend to zero for increasing time which means that the average $\bar{u}^k$ tends to $u_{nom}$ according to the condition $\bar{u} = u_{nom}$. Let us turn to Fig. 3, which depicts the time plot of $\bar{v}^k - v_{nom}$, where $\bar{v}^k$ is the average of $v(k)$ up to time $k$. All plots tend to zero for increasing time which means that the average $\bar{v}^k$ tends to $v_{nom}$ according to the hypothesis of Lemma 4, $\bar{v} = v_{nom}$. We can conclude our comments on the first set of simulations with Fig. 4 which illustrates the time plot of $x(.)$. The variable is $\epsilon$-stabilized with $\epsilon < 0.5$.

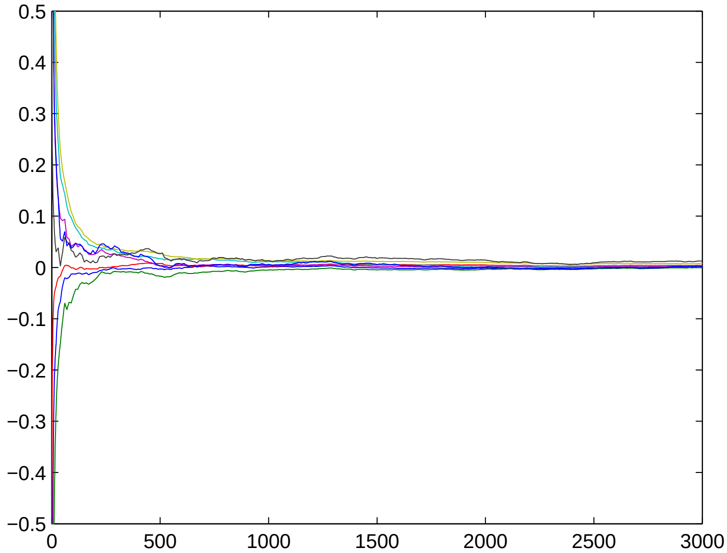


Figure 2: Time plot of $\bar{u}^k - u_{nom}$. The average tends to $u_{nom}$ for increasing time according to the condition $\bar{u} = u_{nom}$.

In a second set of simulations, we consider the bounding polyhedron $\mathcal{U} := \{u : -0.2 \cdot \mathbf{1} \leq u \leq 0.2 \cdot \mathbf{1}\}$. Condition (10) holds no longer (the bounds in $\mathcal{U}$ are too tight) and the resulting games in the sequence are not balanced. We can start by illustrating the time plot of $z(.)$ in Fig. 5. Peaks show that the variable is not $\epsilon$-stabilized with $\epsilon = 0.5$. Next Fig. 6 shows the time plot of $\bar{u}^k - u_{nom}$, where $\bar{u}^k$ is the average of $u(k)$ up to time $k$. All plots tend to zero for increasing time which means that the average $\bar{u}(k)$ tends to $u_{nom}$ according to the condition $\bar{u} = u_{nom}$. Further, Fig. 7 illustrates the time plot of $\bar{v}^k - v_{nom}$, where $\bar{v}^k$ is the average of $v(k)$ up to time $k$. All plots tend to zero for increasing time which means that the average $\bar{v}(k)$ tends to $v_{nom}$ according to the hypothesis of Theorem 1, $\bar{v} = v_{nom}$. Finally, Fig. 8 displays the time plot of $x(.)$. Peaks illustrate that the variable is not $\epsilon$-stabilized with $\epsilon < 0.5$.

## 6 Conclusions and future directions

In this work, we investigate what is necessary to capture the dynamic and uncertain nature of the coalitions' values in most real contexts. We highlight how $\epsilon$-stabilizability of a control strategy collapses into $\epsilon$-balancedness and $\epsilon$-core allocations inclusions. We believe that the robust dynamic approach proposed here is likely to become an alternative perspective on the study of sequences of coalitional games and interval valued games.

Future research will include contributions to distributed allocation algorithms. This line of

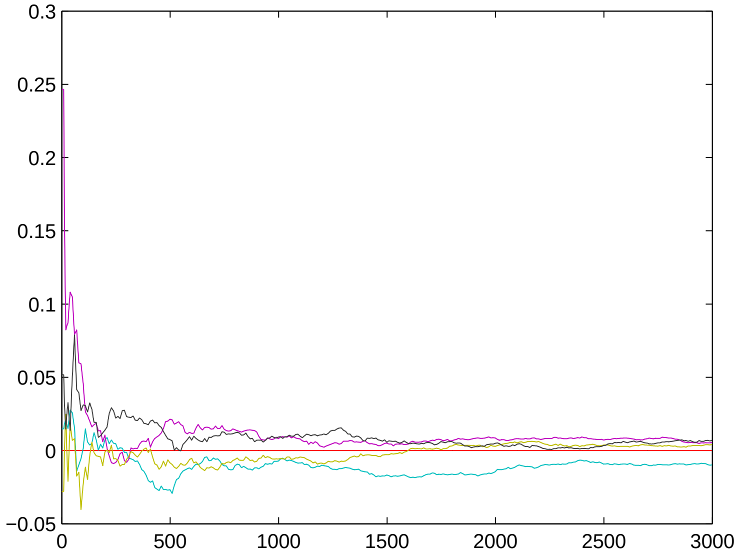


Figure 3: Time plot of $\bar{v}^k - v_{nom}$.

research is inspired by so-called consensus algorithms already present in the control literature. The underlying idea is that people exchange money only with neighbors (this motivates the meaning of the word distributed) until an allocation in the core is reached. So, no centralized thinking, no supervisor, but a decentralized allocation procedure that behaves like a centralized one.

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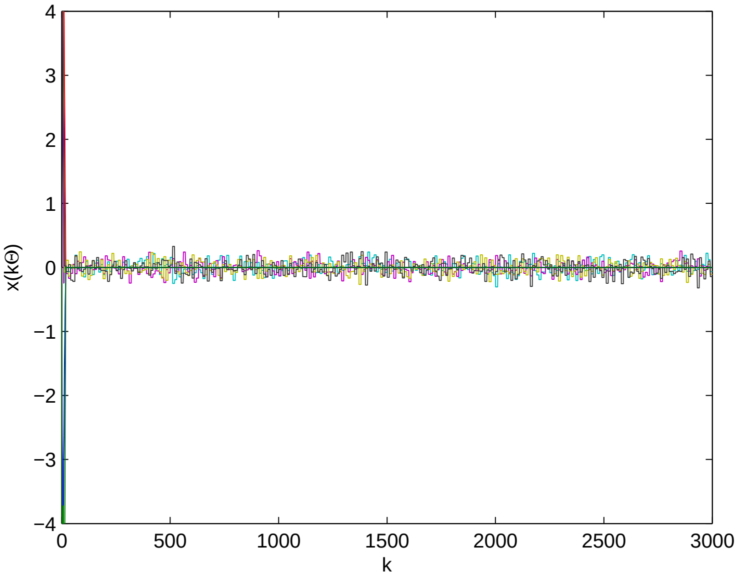


Figure 4: Time plot of $x(k)$. The variable is $\epsilon$-stabilized with $\epsilon = 0.5$.

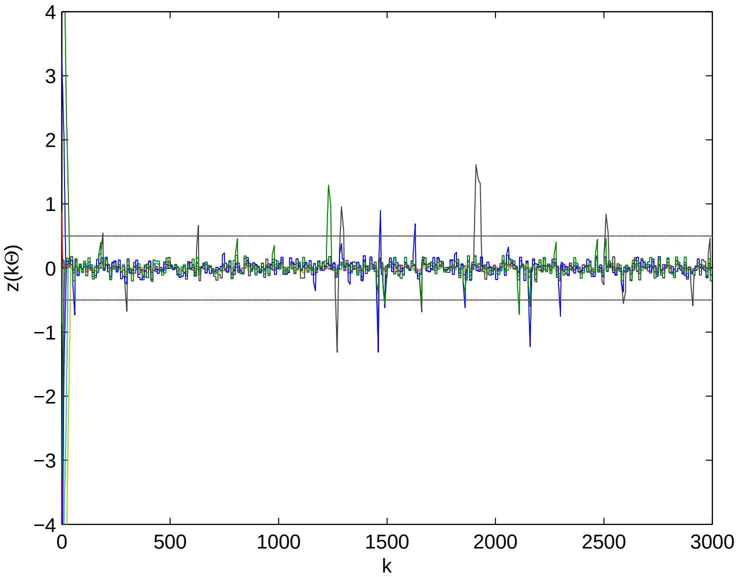


Figure 5: Time plot of $z(.)$. Peaks show that the variable is not $\epsilon$-stabilizable with $\epsilon = 0.5$.

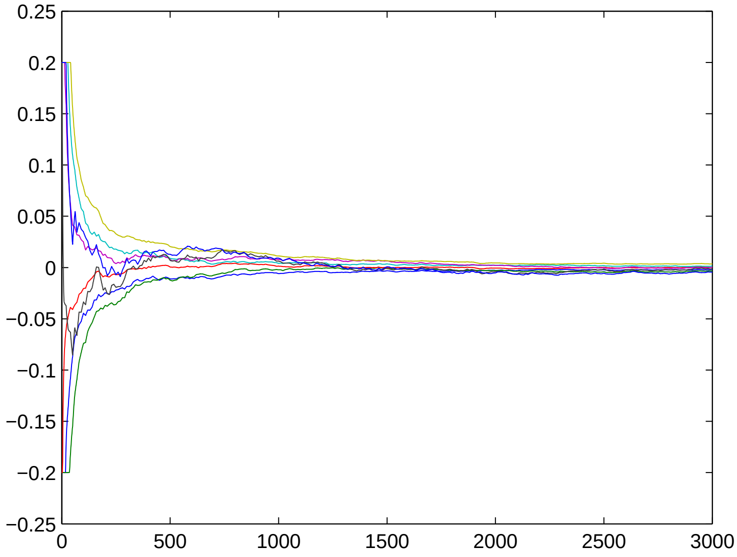


Figure 6: Time plot of $\bar{u}^k - u_{nom}$. The average tends to $u_{nom}$ for increasing time according to the condition $\bar{u} = u_{nom}$.

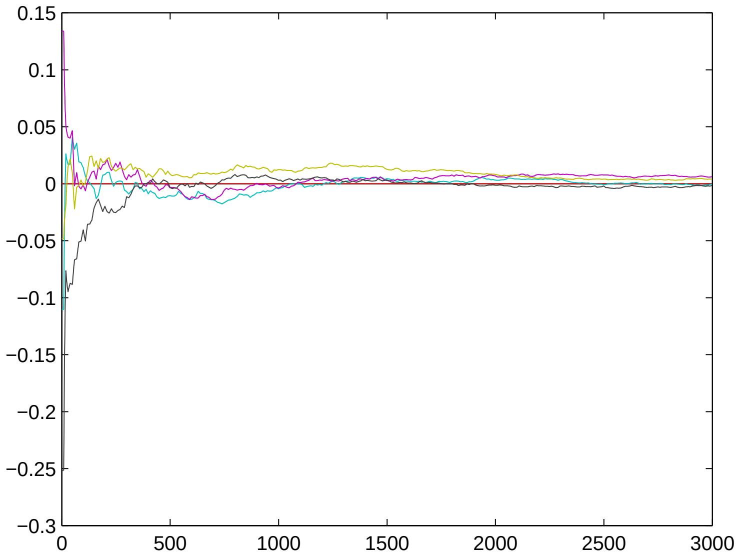


Figure 7: Time plot of $\bar{v}^k - v_{nom}$.

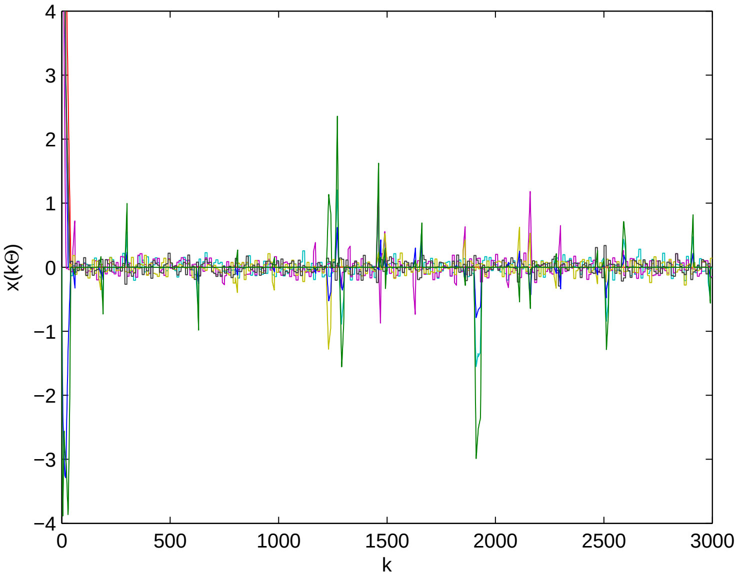


Figure 8: Time plot of $x(k)$. Peaks show that the variable is not $\epsilon$-stabilized with $\epsilon = 0.5$.

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# Appendix

To calculate matrix $D$ we formulate a linear programming problem as illustrated in details in Bauso and Timmer (2009) [3]) and obtain

$$D = \begin{bmatrix} 0.22222 & -0.11111 & -0.11111 & 0.11111 & 0.11111 & -0.22222 & 0.33333 \\ -0.11111 & 0.22222 & -0.11111 & 0.13695 & -0.18346 & 0.16279 & 0.46253 \\ -0.11111 & -0.11111 & 0.22222 & -0.24806 & 0.072351 & 0.059432 & 0.20413 \\ -0.77778 & -0.11111 & -0.11111 & 0.11111 & 0.11111 & -0.22222 & 0.33333 \\ -0.11111 & -0.77778 & -0.11111 & 0.13695 & -0.18346 & 0.16279 & 0.46253 \\ -0.11111 & -0.11111 & -0.77778 & -0.24806 & 0.072351 & 0.059432 & 0.20413 \\ 0.11111 & 0.11111 & -0.22222 & -0.75194 & -0.072351 & -0.059432 & 0.79587 \\ 0.11111 & -0.22222 & 0.11111 & -0.13695 & -0.81654 & -0.16279 & 0.53747 \\ -0.22222 & 0.11111 & 0.11111 & -0.11111 & -0.11111 & -0.77778 & 0.66667 \end{bmatrix}.$$