Optimized Variational Boussinesq Modelling; part 1: Broad-band waves over flat bottom

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Abstract

The Variational Boussinesq Model (VBM) for waves above a layer of ideal fluid conserves mass, momentum, energy, and has decreased dimensionality compared to the full problem. It is derived from the Hamiltonian formulation via an approximation of the kinetic energy, and can provide approximate dispersion characteristics. Having in mind a signalling problem, we search for optimal dispersive properties of the 1-D linear model over flat bottom and, using finite element and (pseudo-) spectral numerical codes, investigate its quality. For the optimization we restrict to the class of potentials with hyperbolic vertical profiles that are parametrized by the wavenumber. The optimal wavenumber is obtained by minimizing the kinetic energy for the given signal and produces good results for two realistic test cases. Besides this kinetic energy principle we also consider various ad-hoc least square type of minimization problems for the error of the phase or group velocity. The test cases are two examples of focussing wave groups with broad spectra for which accurate experimental data are available from MARIN hydrodynamic laboratory. To determine the quality of an ‘optimized’ wavenumber for the governing dynamics, we use accurate numerical simulations with the AB-equation to compare with VBM calculations for the whole range of possible wavenumbers. The comparison includes the errors in the signal at the focussing position, as well as the integrated errors of maximal and minimal wave heights along a spatial and temporal interval that is symmetric around the focussing event.

\textit{Keywords:} Variational Boussinesq Model, Surface waves, Optimized dispersion, AB-equation.

1 Introduction

The Variational Boussinesq Model (VBM) is based on the fact that the surface wave evolution can be described as an infinite dimensional Hamiltonian system.
The canonical variables are the surface elevation $\eta$ and the fluid potential $\phi$ at the free surface. The Hamiltonian is the total energy, of which the kinetic energy is given by

$$K(\phi, \eta) = \int \int \frac{1}{2} |\nabla \Phi|^2 d\eta d\phi,$$  \hspace{1cm} (1)

where the fluid potential $\Phi$ satisfies the Laplace equation in the interior, the impermeability condition at the bottom $z = -h$ and the prescribed value $\Phi = \phi$ at the free surface $z = \eta$. In this paper we consider the bottom to be flat; the case of varying bottom will be dealt with in a forthcoming paper. This potential $\Phi$ has the extremal property that it minimizes the kinetic energy over all potentials that satisfy the prescribed surface value $\phi$. This minimization property of the kinetic energy will be exploited further on in an essential way to obtain the best dispersive properties.

Since the Laplace problem cannot be solved explicitly for nontrivial $\eta$, the kinetic energy has to be approximated to make the model useful for numerical simulations.

In this paper, just as in [5, 6, 7], we choose to take the approximation that follows by writing $\Phi$ as a one-term perturbation of the surface potential:

$$\Phi(x, z) = \phi(x) + F(z) \psi(x),$$ \hspace{1cm} (2)

requiring $F(\eta) = 0$ along with the bottom impermeability condition. The vertical profile function $F$ has to be chosen in advance, and the function $\psi$ at the free surface becomes an additional variable for which an additional elliptic equation has to be solved together with $\eta$ and $\phi$.

The dispersion relation of the resulting dynamical system depends (strongly) on the choice of the function $F$. The choice $F = 0$ leads to the shallow water equations (SWE) with no dispersion. From linear theory, the fluid potential of a small amplitude harmonic wave with wave number $\kappa$ can be exactly represented by the choice

$$F(z) = \frac{\cosh \kappa(z + h)}{\cosh \kappa(\eta + h)} - 1.$$ \hspace{1cm} (3)

But for non-harmonic waves, or when nonlinearity is essential, the form (3) can at best be approximative.

In this paper we address the question for which choice of $F$ one gets the best dispersive properties. We restrict to the linearized equations, and consider the signaling problem for wave fields in 1D with broad-band spectra. We will keep the form (3) as Ansatz, but allow the value of $\kappa$ to be chosen in an optimal way. Intuitively, the optimal $\kappa$ will be some averaged wave number, the value of which will depend on properties of the wave field, in particular on the initial profile or the initial signal.

Since the dispersion relation related to the choice of $F$ in (3) will depend on $\kappa$, it is natural to choose $\kappa$ in such a way that the difference with the exact
dispersion relation, given by

\[ \omega = \Omega_{ex}(k) = \text{sign}(k) \cdot \sqrt{gk \tanh kh}, \tag{4} \]

is as small as possible for relevant wave numbers. But it is not obvious which norm to choose for measuring the difference. Moreover, it is not simple to know what the effect is of differences in the dispersion relation on the behaviour of the wavefields.

In the following we will show that a natural choice for \( \kappa \) is obtained by minimizing the kinetic energy for the given initial time signal. This optimization is well-founded by the minimality property of the kinetic energy. Nevertheless we will also consider some ad-hoc least square formulations. These are of the form of minimizing the phase speed error or the group speed error in the \( L_2 \)-norm, weighted with the initial spectrum. It will be shown that these optimal values give almost the same optimization result for the two test cases considered here.

The two test cases are focusing wave groups that have been generated and measured at MARIN hydrodynamic laboratory, Wageningen, the Netherlands. Instead of the MARIN numbering 109001 and 101013, we will refer to these cases as the mild and the strong focusing group, denoted by mFG and sFG respectively.

In order to qualify the VBM results, we compare the evolutions with simulations using the linear version of the AB-equation [2, 3]. These AB-simulations have exact dispersive effects, and turn out to be very accurate when compared to the point measurements of MARIN. With these accurate simulations, we compare the VBM calculations. We compare the time signal at the focusing point, but also the whole spatial evolution of the focusing and defocusing behaviour as represented by the maximal and minimal temporal amplitudes. It will turn out that in all cases the optimal \( \kappa \) is far away from the peak-wave number, which could have been a first guess for an optimum.

The outline of the paper is as follows. In section 2 we present the dynamic equations, and investigate the dispersion relation in its dependence on \( \kappa \). In particular, we show that for each \( \kappa \), all sufficiently short waves have the same, finite propagation speed; this erroneous behaviour is an inevitable consequence of the fact that we represent the fluid potential as in (2).

In section 3 we derive the kinetic energy optimization principle to calculate the optimal \( \kappa \)-value. In addition, we propose various optimization criteria which could be used as well. In section 4 we describe the two test cases of focusing wave groups and show that the simulations with the linear and non-linear AB-equation are very close to the MARIN experiments. In section 5 we calculate the optimal \( \kappa \)-values according to the previously proposed optimization criteria, and we present the comparison between the optimal VBM calculations and the AB-simulations. In section 6 we provide some remarks and conclusions.

It should be remarked that the analysis in this paper is concentrated on the errors caused by the modelling process, and not on numerical accuracy errors. The numerical simulations are performed for the exact dispersion with a spectral
for the VBM simulation we used a FE-implementation with sufficiently
fine grid.

2 The VBM dispersion relations

As written in the introduction, every Variational Boussinesq model is obtained
via approximation of the kinetic energy. We accomplish this by approximating
the fluid potential $\Phi$ by the expression (2). Since we are studying the dispersive
properties in this paper, we will restrict to linearized equations. This is obtained
by replacing in (1) the vertical integration interval till the still water level 0
instead of till the surface elevation $\eta$, and putting $\eta = 0$ in (3).

The kinetic energy is then given by

$$K \approx \frac{1}{2} \int [h(\partial_x \phi)^2 + \alpha(\partial_x \psi)^2 + \gamma \psi^2 + 2\beta \partial_x \phi \partial_x \psi] dx$$  \hspace{1cm} (5)

with integral coefficients $\alpha$, $\beta$ and $\gamma$ that are given by

$$\alpha = \int_0^h F^2 dz, \hspace{0.5cm} \beta = \int_0^h F dz, \hspace{0.5cm} \gamma = \int_0^h (F')^2 dz.$$  \hspace{1cm} (6)

Based on Luke’s variational principle [4, 8] (see also [1, 11, 12]), variations
of the following Lagrangian should be equal to zero.

$$\delta L = \delta \int (\int \phi \partial_t \eta dx - K - P) dt = 0,$$

where $P = \frac{1}{2} \int g \eta^2 dx$ is the potential energy.

Variations of the Lagrangian with respect to $\phi$, $\eta$ and $\psi$ give the following
system of PDEs, which we call the Linear Variational Boussinesq Model,

$$\begin{align*}
\hat{\partial}_t \eta &= -h \hat{\partial}_x^2 \phi - \beta \hat{\partial}_x^2 \psi \\
\hat{\partial}_t \phi &= -g \eta \\
-\alpha \hat{\partial}_x^2 \psi + \gamma \psi &= \beta \hat{\partial}_x^2 \phi.
\end{align*}$$  \hspace{1cm} (7)

The coefficients $\alpha$, $\beta$ and $\gamma$ depend on the approximation of the vertical po-
tential profile $F$ of the model. For a parabolic approximation of the profile $F$,
taken in [5] and [6], they depend only on depth. For the cosine hyperbolic ap-
proximation, on which we concentrate in this paper, the coefficients depend also
on the wave number $\kappa$; the value of this parameter will shortly be determined
in an optimal way.

2.1 VBM dispersion relation

We obtain an analytic expression for the dispersion of the system (7): we look
for harmonic profiles with frequency $\omega(k)$ depending on the wave number $k$:

$$\eta = ae^{i(kx \cdot \omega t)}, \hspace{0.5cm} \phi = be^{i(kx \cdot \omega t)}, \hspace{0.5cm} \psi = ce^{i(kx \cdot \omega t)}.$$
Substituting these profiles into the system (7), we obtain a matrix equation in a form $L \cdot (a, b, c)^T = 0$, for which non-trivial solutions exist only when $\det L = 0$. This gives the dispersion relation is given through the phase velocity $C_{VBM} = \omega/k$, by

$$C_{VBM} = c_0 \sqrt{1 - \frac{\beta^2}{h} \cdot \frac{k^2}{\gamma + \alpha k^2}},$$

where $c_0 = \sqrt{gh}$. Unlike the exact phase speed, this approximation has the nonzero limit for short waves

$$\lim_{k \to 0} C_{VBM}(k) = c_0 \sqrt{1 - \frac{\beta^2}{\alpha h}}.$$  \hspace{1cm} (9)

Indeed, this limit is real and nonzero as a consequence of the fact that $\beta^2 \leq \alpha h$ because of Cauchy–Schwarz inequality:

$$\left( \int_{h}^{0} F \cdot 1dz \right)^2 \leq \int_{h}^{0} F^2dz \cdot \int_{h}^{0} 1dz,$$

while equality in this expression is only possible for trivial functions $F$.

The limit for long waves is, as it should be, $c_0 = \sqrt{gh}$. The Taylor expansion around $k = 0$ yields

$$\Omega_{VBM} \approx c_0 k \left[ 1 - \frac{\beta^2}{2h\gamma} k^2 + \left( \frac{\alpha \beta^2}{2h\gamma^2} - \frac{\beta^4}{8h^2\gamma^2} \right) k^4 - \right.$$

$$\left. \frac{\alpha^2 \beta^2}{2h^3\gamma^3} - \frac{\alpha \beta^4}{4h^2\gamma^3} + \frac{\beta^6}{16h^3\gamma^3} \right] k^6 + O(k^8).$$  \hspace{1cm} (10)

It should be noticed that these expressions are valid for a Variational Boussinesq Model with any vertical potential approximation $F(z)$ in (2), although the integral coefficients (6) will depend on model parameters, e.g. the wave number $\kappa$ as appears in the cosh-approximation below.

Fig. 1 shows normalized plots of the phase and group speed for the parabolic and the cosine hyperbolic models; for comparison, also the plot of the exact phase and group speed is given, where the group speed is expressed by $V = \frac{d\omega}{dk}$.

### 2.2 Parabolic Approximation

The parabolic approximation for the function $F$ has been extensively discussed by Klopmann e.a. [5, 6]; we will briefly recall the results. The function $F$ in (2) is taken to be

$$F(z) = (A + 1)\frac{z}{h} + A\frac{z^2}{h^2}.$$  \hspace{1cm} (11)

We set $A = 1$ because of the bottom impermeability condition $F'(-h) = 0$. The surface condition $F(0) = 0$ is satisfied as well, and additionally, we can normalize the function so that $F(-h) = -1$. Then the coefficients are given by

$$\alpha = \frac{8}{15}h, \quad \beta = -\frac{2}{3}h, \quad \gamma = \frac{4}{3} \cdot \frac{1}{h}.$$  \hspace{1cm} (12)
Figure 1: The phase speed (upper curves) and the group speed (lower curves) for different models: the shallow water equation (dotted black), the exact dispersion (dashed black), the parabolic approximation (red solid), the cosine hyperbolic approximation with $\kappa = 5.73$ (blue solid).
After substitution of $\alpha$, $\beta$ and $\gamma$ from these formulae in expression (8), one gets the expression by Klopman e.a. [5, 6] (compare to the appropriate expressions in [9, 10]):

$$\frac{\omega^2 h}{g} = (kh)^2 \cdot \frac{1 + \frac{1}{3}(kh)^2}{1 + \frac{2}{3}(kh)^2}.$$  

The phase speed $\omega/k$ by this formula has the limit $\sqrt{gh/6}$ for $k \to \infty$.

Comparing the Taylor expansion around $k = 0$

$$\Omega_{VBM}^{par} \approx c_0 k \left[ 1 - \frac{1}{6}(kh)^2 + \frac{19}{360}(kh)^4 - \frac{193}{10800}(kh)^6 + O((kh)^8) \right]$$  

(13)

to the Taylor expansion of the exact dispersion relation (4)

$$\Omega_{ex} \approx c_0 k \left[ 1 - \frac{1}{6}(kh)^2 + \frac{19}{360}(kh)^4 - \frac{55}{3024}(kh)^6 + O((kh)^8) \right],$$  

(14)

one can conclude that the dispersion of the VBM equations is correct up to and including the 5-th order for long waves.

### 2.3 Hyperbolic Cosine Approximation

According to the linear theory for small-amplitude gravity driven waves on a layer of ideal fluid, the expression (3) leads to the correct fluid potential for harmonic waves with wave number $\kappa$.

The surface condition $F(0) = 0$ is satisfied along with the bottom impermeability condition $F'(-h) = 0$. The integral coefficients $\alpha$, $\beta$ and $\gamma$ in (6) are:

$$\alpha(\kappa) = -\frac{h}{2\kappa} \tanh \kappa h + \frac{h}{2 \cosh^2 \kappa h} + h,$$

$$\beta(\kappa) = \frac{1}{h} \tanh \kappa h - h,$$

$$\gamma(\kappa) = \frac{h}{2} \tanh \kappa h - \frac{h^2}{2 \cosh^2 \kappa h}.$$  

(15)

The Taylor expansion of these coefficients around $\kappa = 0$ gives

$$\alpha(\kappa) = h \left[ \frac{1}{15}(kh)^4 - \frac{4}{105}(kh)^6 + O((kh)^8) \right],$$

$$\beta(\kappa) = h \left[ -\frac{1}{3}(kh)^2 + \frac{2}{15}(kh)^4 - \frac{17}{315}(kh)^6 + O((kh)^8) \right],$$

$$\gamma(\kappa) = \frac{h}{6} \left[ (kh)^2 - \frac{4}{15}(kh)^4 + O((kh)^6) \right],$$

which is the same expression as in the parabolic case (13).

Whatever value is taken for $\kappa$, the dispersion relation gives the exact value for $k = \kappa$ for the phase and the group speed:

$$C_\kappa(k) = C_{ex}(k) \mid_{k=\kappa},$$

$$V_\kappa(k) = V_{ex}(k) \mid_{k=\kappa}.$$

**Remark.** We observe from Fig. 1 that it seems that $C_\kappa \geq C_{ex}$: the approximate phase velocity is for each wavenumber larger than or equal to
the exact phase speed. This is actually true for any approximate VBM and is a direct consequence of the minimization property of the (quadratic) kinetic energy, mentioned in the introduction. Indeed, the kinetic energy can be written for any linear dispersive wave equation as

$$K(\phi) = \frac{1}{2g} \int u C^2 u \, dx$$

with $u = \delta z \phi$, where $C$ is the phase velocity. For the exact dispersion with $C_{\text{ex}}$ and an approximate VBM model, such as $C_\kappa$ in the hyperbolic approximation, we have

$$K(\phi) = \min_{\phi \to 0 \text{ at } z \to 0} \frac{1}{2} \int |\nabla \Phi|^2 \, dz \, dx = \frac{1}{2g} \int u C_{\text{ex}}^2 u \, dx$$

Since this holds for each $\phi$ (each $u$) we conclude that $C_{\text{ex}}^2 \leq C_\kappa^2$ for each $\kappa$. For the hyperbolic profile we have

$$C_{\text{ex}}(\kappa) = C_\kappa(\kappa), \text{ and } C_{\text{ex}}(k) < C_\kappa(k) \text{ for } k \neq 0, \kappa.$$  

### 3 Optimization criteria

As stated above, it is not clear in advance how to choose an optimal value of $\kappa$. To illustrate the problem, in Fig. 2 we present the power spectrum $S(\omega)$ of a time signal of the wave at one position for the two test cases that we will consider in the next section.

In the same figure we plotted the graph of the phase velocity $C_t(\omega)$; here we used the subscript $t$ to indicate that we consider the phase velocity as a function of frequency. Hence, for given dispersion relation $\omega = \Omega(k)$, we consider the inverse $k = K(\omega)$ and define $C_t$ as

$$C_t(\omega) = C(K(\omega)) = \frac{\omega}{K(\omega)}.$$  

When using the exact dispersion relation $\omega = \Omega_{\text{ex}}(k)$, we will specify this by writing $C_{t,\text{ex}}$. The same figure shows a plot of the group velocity $V_t(\omega)$, where again the subscript $t$ indicates that we take the group velocity as a function of frequency. It is defined as $V_t(\omega) = V(K(\omega))$.

When using the VBM-hyperbolic profile, we have to transfer a choice for $\kappa$ to the frequency domain. Since the exact and the VBM-hyperbolic dispersion relation coincide at $\kappa$, $\Omega_{\text{ex}}(\kappa) = \Omega_\kappa(\kappa)$, the transformation is independent of this choice, and we find a unique value $\nu = \Omega_{\text{ex}}(\kappa) = \Omega_\kappa(\kappa)$ corresponding to $\kappa$. Therefore we will write $C_{t,\nu}$ to denote the VBM-phase speed as function of the frequency:

$$C_{t,\nu}(\omega) = \frac{\omega}{K_\kappa(\omega)} \text{ with } \nu = \Omega_\kappa(\kappa).$$
Figure 2: The broad-band signal spectrum (with convenient normalization, green, dashed) is shown as function of frequency for the two MARIN test cases to be studied below: at the left for the mFG and at the right for the sFG. In the upper plots the exact phase velocity and in the lower plots the exact group speed are given (blue, solid). For both the error between the exact dispersion and the VBM for some (optimal) value of $\nu$ is given (red, solid).

The difference between the two phase speeds is denoted by $\Delta C_{t,\nu}$

$$\Delta C_{t,\nu} (\omega) = C_{t,\nu} (\omega) - C_{t,ex} (\omega).$$

Since the derivatives of the exact dispersion relation $\Omega_{ex} (k)$ and the hyperbolic one coincide in the point $k$, the group speed error

$$\Delta V_{t,\nu} (\omega) = V_{t,\nu} (\omega) - V_{t,ex} (\omega)$$

is also zero for $\omega = \nu$. It turns out that there is another zero value, and the frequency for which the group speeds are the same is quite close to the peak frequency of the shown spectra (for the chosen value of $\nu$).

In the figure we also plot the error for the phase and group velocity as function of $\nu$. Actually, the specific value of $\nu$ is not relevant for the present reasoning; the chosen values are actually the optimal choices according to the kinetic energy minimization to be defined below.

This figure illustrates the problem how to choose an optimal value $\nu$. Intuitively, we would like the approximate velocity to be accurate, i.e. small $\Delta C$ or small $\Delta V$, where the spectrum is large, but for applications with a broad
spectrum as in this example, the best value is not obvious. Besides that, the problem is made even more intricate because we have only very limited intuition what the effect of changes in phase or group speed is on the actual evolution of the waves.

The error $\Delta V$ is much larger than $\Delta C$ in the tail of the spectrum close to $\omega = 10$: the error $\Delta V$ is comparable to the actual value, while the error $\Delta C$ is approximately 20% near $\omega = 10$. Since $\Delta V$ vanishes also in a point close to the peak frequency, we observe that an optimal $\nu$-value gives a group speed curve that has minimal error over a rather large frequency range, but the error increases much faster than the error in phase speed for higher frequencies.

In this report we first investigate in subsection 3.1 some ad-hoc, but reasonable optimization criteria. Then we will use the criterion of kinetic energy minimization in subsection 3.2. These optimization criteria will be used in the next section to determine the quality of the resulting dynamics, and to verify that the kinetic energy minimizer is the best choice.

### 3.1 Weighted least square formulations

In this subsection we will consider several ad-hoc least square formulations, each of which aims to reduce the velocity error over the whole relevant frequency interval. The methods can be formulated as a temporal optimization problem

$$ \text{Err}_\nu := \int |\Delta W_\nu(\omega)|^2 \rho_t(\omega) d\omega \to \min_\nu $$

or a spatial optimization problem

$$ \text{Err}_\kappa := \int |\Delta W_\kappa(k)|^2 \rho_s(k) dk \to \min_\kappa $$

where $\rho_t$ and $\rho_s$ denote temporal and spatial weight functions to be chosen. $\Delta W$ is the difference of the velocity $W$ in the VBM model and the exact velocity: $\Delta W = W_{VBM} - W_{ex}$, where for $W$ we will consider the phase or the group velocity. Be warned that we use rather sloppy, but efficient, notation: $\Delta W_\nu(\omega) = \Delta W_\kappa(k)$ for $\omega = \Omega_\kappa(k)$ and $\nu = \Omega_\kappa(k)$. Also note that $\rho_t(\omega) d\omega = \rho_t(\Omega_\kappa(k))V_\kappa(k) dk$ so that the formulations are closely related, but that if $\rho_t = \rho_s$ formulations differ by the group velocity. Different cases arise by making different specific choices for the velocity and for the weight functions.

In principle also the integration boundaries can be chosen, but for the confined — yet broad-band — examples we will consider, it is most appropriate to take the integration over the total real line, which we will do in the following.

We will consider two choices for the speed: the phase velocity $C$ and the group velocity $V$. As weight function we will take the power spectrum of (the influx of) the wave field under consideration. This leads us to four possible criteria, for $W = C$ (the phase velocity) or $W = V$ (the group velocity), in the expressions $\text{Err}_\nu = \int |\Delta W_\nu(\omega)|^2 S(\omega) d\omega$ or $\text{Err}_\kappa = \int |\Delta W_\kappa(k)|^2 S(k) dk$. Observe that for given $S(\omega)$ we have $S(k) = S(\Omega_{VBM}(k))$. Hence, the difference
between the formulations is the additional group velocity from the transformation \(d\omega = V dk\). For these four ad-hoc optimization criteria we will determine the minimizer for the two test cases in the next section.

### 3.2 Kinetic energy optimization criteria

The exact kinetic energy for linear equations with dispersion relation \(\omega = \Omega(k)\) can be written like

\[
K = \frac{1}{4g\pi} \int \Omega \hat{\phi}^2 dk.
\]

A basic ingredient of the VBM is that the kinetic energy (1) is minimized for all fluid potentials \(\Phi\) that satisfy \(\Phi = \phi\) at the surface. We will look for the restricted minimization on the set of potentials given by (2) with \(F\) given by (3) where we minimize with respect to the parameter \(\kappa\). To make this operational for the case of a signalling problem, we have to translate the uni-directional influx of a given initial signal \(\eta_0(t)\) to the corresponding kinetic energy. This is achieved in two steps.

First, we recall the dynamic equation \(\partial_t \phi = -gn\); besides that we realize that a uni-directional influx will lead to an initial evolution given by \(\partial_t \phi = -i\Omega \phi\). Combining these two expressions, we get for the spatial Fourier transform of the initial surface potential \(\hat{\phi}_0\):

\[
d\Omega(k) \hat{\phi}_0(k) = g \hat{\eta}_0(k)
\]

with \(\hat{\eta}_0\) the spatial Fourier transform of the initial profile \(\eta_0(x)\). The kinetic energy now becomes

\[
K = \frac{g}{4\pi} \int |\hat{\eta}_0(k)|^2 dk.
\]

In a second step we relate the spatial Fourier transform of \(\eta\) to the temporal Fourier transformation \(\hat{\eta}_0(\omega)\) of the wave elevation \(\eta_0(t)\) at \(x = 0\). Realizing that for uni-directional propagation it holds that

\[
\eta(x, t) = \int \hat{\eta}_0(k) e^{i(kx + \Omega(k)t)} dk = \int \hat{\eta}_0(\omega) e^{i(K(\omega)x - \omega t)} d\omega,
\]

and that \(d\omega = V(k) dk\), we get from \(\hat{\eta}_0(\omega) d\omega = \hat{\eta}_0(k) dk\) that

\[
\hat{\eta}_0(\omega)V(K(\omega)) = \hat{\eta}_0(K(\omega)).
\]

Substituted in the last expression for the kinetic energy we get

\[
K = \frac{g}{4\pi} \int \hat{\eta}_0(\omega)^2 V(\omega) d\omega = \frac{g}{2} \int S(\omega) V(\omega) d\omega,
\]

where we simply write \(V(\omega) = V(K(\omega))\) and \(S(\omega) = \frac{\tilde{\eta}_0(\omega) \tilde{\eta}_0^* (\omega)}{2n}\) for the power spectrum.
In the case of the VBM the dispersion relation depends on $\kappa$, $\Omega = \Omega_\kappa$, and correspondingly $V(\omega) = V_\nu(\omega)$ with $\nu = \Omega_\kappa(\kappa)$. Hence for a given power spectrum the minimization problem becomes

$$K_\nu = \frac{g}{2} \int S(\omega)V_\nu(\omega)d\omega \to \min_\nu$$ (18)

This is the minimization problem we will consider as the 'natural' way to find the optimal parameter $\nu$ and the related $\kappa$.

Observe that for the exact dispersion relation we would obtain the lowest minimal value, so that (18) provides the same optimal value as

$$K_\nu - K_{ex} = \frac{g}{2} \int S(\omega)[V_\nu(\omega) - V_{ex}(\omega)]d\omega \to \min_\nu$$ (19)

In section 5 we will conclude that this optimization criterion leads to acceptable results for the two test cases to be considered.

4 Test cases

We consider two cases of focusing wave groups. We describe in this section the main characteristics of the initial signal and spectrum, consider the evolution with the accurate AB-equation for non-linear and linear evolutions and compare these with the measurement at the focusing point. We will use these numerical simulations in section 5 to be able to quantify the quality of the optimized VBM-model.

4.1 Focusing wave groups

Both cases are examples of constructed waves that were designed for use at MARIN, the Maritime Hydrodynamic Laboratory Netherlands, to generate high waves in a long wave tank. The design is to exploit dispersive focusing: short period, small amplitude waves are generated at a waveflap, followed by successively larger and longer period waves. The design is such that the longer, i.e. faster, waves catch up with the slower shorter waves at a predetermined position in the tank. This dispersive focusing requires a broad spectrum, and the different speeds of the different frequency components determine the focusing process in a critical way. Hence, the behaviour will be most sensitive for perturbation in the phase speeds, which is why we choose these examples.

For these cases real laboratory measurements in a wavetank with depth of 1 m are available: the time signal on a waveflap $x = X_0 = 0$ m and near the focusing point at $x = X_1 = 20.8$ m for one case (MARIN test case #109001). And for another case (MARIN test case #101013) the surface elevation at $x = X_0' = 10$ m from the waveflap and at the (designed) focusing point $x = 50$ m. The first case will be called the Mildly Focusing Group (mFG) in the following, since the maximal waveheight at the focusing position is rather mild. The other case, the Strong Focusing Group (sFG), is more extreme: just downstream of
the focussing point some breaking (white capping) was observed. For the latter case of sFG we shift in our numerical model the starting position to the position \( x = x_0 = 0 \ m \) and accordingly the measurement position to \( x = X_1 = 40 \ m \).

In Fig. 3, we present for each case the time signal at \( X_0 \) together with the power spectrum. Observe that the spectra (which were used in the preceding Fig. 1), are rather broad, and that sFG contains many more waves.

4.2 Accurate simulation of the focussing process

The MARIN measurement of the elevation at \( X_1 \) is the only available information downstream of \( X_0 \). This gives only little information, which is why we performed additional numerical calculations. Another reason is that the experimental data include nonlinear effects, while we are here especially interested in the linear dispersive properties.

Therefore we performed calculations with a very accurate and efficient model for uni-directional waves, i.e. the AB-equation derived by Van Groesen & Andonowati ([2], see [3] for numerical results). The results with the linear version of this code — which uses the exact dispersion — will be used to compare with VBM calculations in the next section for various \( \nu \)-values. The remainder of this section is to justify that we can take the linear AB-simulations as sufficiently
Figure 4: At the left for mFG and at the right for sFG are shown in the upper pane the time signals at the focussing point (respectively \( X_1 = 20.8 \) and \( X_1 = 40 \) m) of the measurement and of the nonlinear AB simulation. In the lower pane MCH, MTD and the maximal wave elevation are shown as calculated by the nonlinear AB-equation.

accurate results to be valid as 'exact' results for the comparison with the VBM calculations.

First, we show the result of a nonlinear evolution with a pseudo-spectral implementation of the AB-equation, downstream from the measured elevation signal at \( X_0 \).

In Fig. 4 we show at \( X_1 \) the calculated signals and the measured signal. These results show that the simulations are remarkably accurate, even for the extreme case of sFG. In measurements and simulations the spectrum changes during the evolution due to nonlinear effects. In fact, detailed analysis of the simulations show that especially for sFG, long- and short-wave generation takes place very close to the focussing point. Since this paper deals with the linear VBM, we also consider linear evolutions of the initial signals which is simply the evolution according to the exact dispersion, and can be done with the linearized AB-equation.

Remarkably, also for these linear simulations the wave signals at \( X_1 \) are quite similar to measurements, albeit the amplitude is somewhat less, despite the fact that nonlinear effects do play a role in these test cases.

The numerical simulations — different from the available measurements — also provide information at any point in between \( X_0 \) and \( X_1 \). To get condensed information of that evolution, we will consider the maximal temporal
Figure 5: Similar to Fig. 4, now the AB-linear model is used. For these cases of focusing wave groups we observe that the linear simulations give results quite close to the real measurements.

amplitudes: the Maximal Crest Height (MCH) and the Minimal Trough Depth (MTD) at each position; the results are shown in Fig. 4 and Fig. 5, and we will use these to compare the MTA calculated with VBM in the next section.

5 Optimized VBM simulations

In this section we will first determine the optimal parameter values according to the five optimization methods discussed in section 3 for both focusing wave groups. We performed numerical simulations using these optimal values in a Finite Element implementation of VBM. In subsection 5.2 we present the results of the VBM calculated time signals at the focusing points for the optimal values. In section 5.3 we provide the results when looking at the downstream evolution as measured by the MTA’s. For both test cases the kinetic energy (KE) optimized value performed best.

5.1 Calculation of optimal values

Using the initial power spectra of the two wave groups we show in Fig. 6 the plots of the four least square errors defined in section 3.1 and the values of the kinetic energy error as a function of the parameter $\nu$. The lowest point of each of these curves provides the optimal value for the corresponding optimization criterion. These optimal values are assembled in Table 1. Observe that the
KE-optimal value for mFG ($\nu = 5.73$) is best approximated by the ad-hoc optimization for spatial phase speed norm (5.68), while for sFG the KE-optimal value (5.65) is closest to the temporal group speed norm (5.70).

<table>
<thead>
<tr>
<th>$W$</th>
<th>weight</th>
<th>$\nu$ ($\kappa$) optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S\omega d\omega$</td>
<td>C</td>
<td>5.39 (2.98)</td>
</tr>
<tr>
<td>$S\kappa dk$</td>
<td>C</td>
<td>5.68 (3.30)</td>
</tr>
<tr>
<td>$S\omega d\omega$</td>
<td>V</td>
<td>6.04 (3.73)</td>
</tr>
<tr>
<td>$S\kappa dk$</td>
<td>V</td>
<td>6.40 (4.17)</td>
</tr>
</tbody>
</table>

Table 1: The first row provides for mFG and sFG the optimal values $\nu$ (and $\kappa$ in parentheses) according to the kinetic energy optimization. The other rows give the optimal values for for the four ad-hoc error-minimization norms of subsection 3.1

5.2 Signal at focussing point

We used a Finite Element implementation of the optimized VBM to calculate the time signal at the focussing point, with the optimal values obtained from the kinetic energy optimization, as given in Table 1. The result is shown for both test cases in Fig. 7, upper row. To compare the results, we also depicted in the same plot the signal as calculated with the linear exact dispersion code mentioned in section 4. It can be observed that the main wave is relatively well represented for both cases, but that certain frequency components disturb the signal before and after the time of focussing.
Figure 7: The mFG test case is shown to the left, the sFG test case at the right. Above are the temporal signals at the focussing point $X_1$, and below are the MCH, MTD and maximal amplitude wave elevations, simulated with the VBM-code for the optimal $\nu$-value according to the KE-optimization, and compared to the exact dispersive linear simulation at the focussing time.
In order to compare the sensitivity of the result on the value of \( \nu \), we calculated for each \( \nu \) in the interval from \( \nu = 0 \) till \( \nu = 10 \) the \( L_2 \)-norm of the difference of the VBM signal with the 'exact' signal, over the time interval that includes the dispersive focussing and defocussing, i.e. from \( t_0 = 40 \) s till \( t_1 = 50 \) s for mFG and from \( t_0 = 89 \) s till \( t_1 = 99 \) s for sFG:

\[
||\Delta S_1||^2 = \int_{t_0}^{t_1} [\eta_v(X_1, t) - \eta_{ex}(X_1, t)]^2 dt.
\]

The results are shown in Fig. 8 and Fig. 9; it should be remarked that enlarging the integration span does not affect much the calculated results. The minimal value of this curve is at \( \nu = 5.09 \) and \( \nu = 4.97 \) for mFG and sFG respectively. Referring to Table 1, these values are somewhat smaller than the optimal \( \nu \)-value, obtained from the KE-optimization criterion. The use of this norm to measure the error is, however, somewhat dubious, since a small phase error, as is clearly visible in Fig. 7, contributes largely to this error.

### 5.3 Maximal wave heights comparisons

In Fig. 7, second row, we show the spatial wave profile at the time of focussing as calculated with the KE-optimal VBM and for comparison, the 'exact' fully dispersive wave profile. Understandably, the additional oscillations in the VBM time signal in the first row, also have effects on this spatial profile. In the spatial plots we also show the curves of Maximal Crest Height (MCH) and Minimal Trough Depth (MTD) for each simulation. These maximal temporal amplitudes give a condensed indication of the downstream running wave process. Therefore, we considered the difference of the exact calculation with the VBM simulations for all values of the parameter \( \nu \). The results are shown in Fig. 10 for mFG and in Fig. 11 for sFG. The density plots do not give much interpretable information, but the plots on the lowest row provide a precise value for \( \nu \) for which the maximal error over the whole running down area is as small as possible. These values are given in Table 2 along with other optimal values of the parameter \( \nu \) (or corresponding \( \kappa \)), calculated as minima of \( L_2 \)-norms of the appropriate errors. The integration for the MTA’s differences is done in the symmetric interval around \( X_1 \): for mFG \( x \in [11, 31] \) and for SFG \( x \in [30, 50] \); enlarging the integration span does not affect much the results.

We see that the best value for MCH for mFG is given by 5.83, which is close to the KE-optimum 5.73; the best value for MTD-error is almost the same as this KE-optimum as well. For sFG the best values of \( \nu \) are now more pronounced, given by \( \nu = 5.41 \) for MCH and by \( \nu = 5.42 \) for MTD, both quite close to the KE-optimum value.

In Fig. 12 we plot the curves of the \( L_2 \)-norm of the MTA-differences. The same figure represents the plots of the maximal crest height 'positioning error’, i.e. \( |X_{\nu}^{max} - X_{ex}^{max}| \), where these \( x \)-values correspond to places in a domain,
Figure 8: Density plots of the signal error $\Delta S_1 = |\eta_\nu(X_1, t) - \eta_{ex}(X_1, t)|$ at $X_1$ for the mFG test case are shown in the top pane (at the right a zoom-in): the difference between signals of the exact dispersive simulations and VBM-hyperbolic simulations, as functions of $\nu$, $t$. The solid horizontal line correspond to the the KE-optimal $\nu$-value, the dashed lines show the ad-hoc optimal $\nu$-values. A side view of the surface is shown below at the left and the $L_2$-error $|\Delta S_1|^2$ is shown below at the right.
Figure 9: Similar to Fig. 10, but now for the sFG test case.
where the wave of maximal amplitude is obtained; similar for the minimal trough depth. Observe that the proposed KE-optimal choice of the parameter \( \nu \), the first in Table 1, is very close to the errors’ minima.

### 6 Conclusions and remarks

The freedom in the Variational Boussinesq Model (VBM) to choose the vertical profile of the fluid potential was exploited in this paper by determining the optimal parameter value in a parameterized class of profiles. This parameter is an effective wave number, the potential profile of which is taken as an approximation of the potential profile of all other waves with different wave numbers. The optimal parameter was found from an interesting minimum kinetic energy principle that depends on properties of the influxed signal (or of an initial wave profile). Hence, in contrast with most other wave models, optimal dispersive properties are determined before the model is used for simulating the evolution.

The quality of this optimal dispersion can be seen in Fig. 2 from the errors in phase and group velocity at all frequencies of the spectra. A good simulation of the focusing process requires all participating waves to evolve accurately. The result of the optimal performance is shown in Fig. 7. In both considered cases the spatial and temporal positioning of the maximal (focused) wave is rather accurate. The amplitudes of the maximal waves are too small, around 20% and 15% for the mild and strong case respectively, but the wave shape is well simulated. Some additional oscillations are noticeable which result from errors in the higher frequencies.

To make such visual observations more quantitative, we showed for both cases that the optimal parameter choice is close to optimal values, where error measures are minimal. The measures we used to judge the quality of the simulation are pointwise and integrated errors over long spatial and temporal intervals.

<table>
<thead>
<tr>
<th>difference</th>
<th>optimal ( \nu (\kappa) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mFG</td>
</tr>
<tr>
<td>max, ( \Delta MCH_{\nu}(x) )</td>
<td>5.90 (3.55)</td>
</tr>
<tr>
<td>max, ( \Delta MTD_{\nu}(x) )</td>
<td>6.18 (3.89)</td>
</tr>
<tr>
<td>(</td>
<td></td>
</tr>
<tr>
<td>(</td>
<td></td>
</tr>
<tr>
<td>(</td>
<td></td>
</tr>
<tr>
<td>KE-optimization</td>
<td>5.73 (3.36)</td>
</tr>
</tbody>
</table>

Table 2: Optimal values for mFG and sFG cases, calculated according to the difference between exact dispersion and VBM simulations. The first two rows show the values \( \nu \) for which the maximal error over the whole running area is as small as possible. The next two rows represent \( L_2 \)-errors of the difference of the MCH and MTD. Then the \( L_2 \)-error of a signal is provided (see the previous subsection), and the last row shows the optimal values according to the kinetic energy optimization.
Figure 10: Density plots of $\Delta$-MCH (first row) and $\Delta$-MTD (second row) for the mFG case: the differences between the exact dispersive simulations and VBM-hyperbolic simulations, as functions of $\nu$, $x$. The solid horizontal line correspond to the KE-optimal $\nu$-value, the dashed lines show the ad-hoc optimal $\nu$-values. Two side views of the surfaces are shown in the third row: $\Delta$-MCH to the left and $\Delta$-MTD to the right.
Figure 11: Similar to Fig. 10, but for the sFG test case.
Figure 12: Calculated errors for mFG at the left and sFG at the right. The difference is given between the VBM-hyperbolic simulations with varying parameter $\nu$ and the exact dispersive code. The $L^2$-errors $|MTA_\nu - MTA_{ex}|$ are shown above, and the maximum’s positioning errors $|X_{\nu}^{max} - X_{ex}^{max}|$ are shown below. The solid vertical line correspond to the the KE-optimal $\nu$-value.
that include the essential deformations before and after focusing. The combined results for these different errors give support to the conclusion that the optimal wave number from the kinetic energy principle produces good results. This implies that if better approximations are desired, the choice of the parameterized family, provided here by (2) and (3), has to be improved, for instance by taking a superposition of various (parameterized) profiles. This opens up new opportunities for extended optimized VBMs, where the optimization should use the kinetic energy principle introduced here.

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References


