On Homogeneous Skewness of Unimodal Distributions

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Abstract

We introduce a new concept of skewness for unimodal continuous distributions which is built on the asymmetry of the density function around its mode. The asymmetry is captured through a skewness function. We call a distribution homogeneously skewed if this skewness function is consistently positive or negative throughout its domain, and partially homogeneously skewed if the skewness function changes its sign at most once. This type of skewness is shown to exist in many popular continuous distributions such as Triangular, Gamma, Beta, Lognormal and Weibull. Two alternative ways of partial ordering among the partially homogeneously skewed distributions are described. Extensions of the notion to broader classes of distributions including discrete distributions have also been discussed.
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1 Introduction

For any symmetric unimodal distribution, the three most popular measures of central tendency, viz., the mean μ, the median m, and the mode M coincide. The lack of symmetry for other distributions is expressed through measures of skewness. Although skewness has been studied extensively in the past, statisticians still differ in their approach for quantifying skewness. Possibly the earliest measure of skewness can be attributed to Pearson (1895) who advocated $\frac{\mu-M}{\sigma}$, with σ being the standard deviation of the distribution; this measure is usually referred to as the *Pearson's measure of skewness*. The *standardized third central moment*, $\frac{\mu_3}{\sigma^3}$, as proposed by Yule (1911) is possibly the most commonly used skewness measure till date. Another group of measures are based on percentiles, such as $\frac{(F^{-1}(0.75)-m)-(m-F^{-1}(0.25))}{F^{-1}(0.75)-F^{-1}(0.25)}$ given by Bowley (1901), where F is the cumulative distribution function. Later on many other measures have been proposed by modifying these measures (see David and Johnson, 1956; Doksum, 1975; Benjamini and Krieger, 1985; MacGillivray, 1986). Arnold and Groeneveld (1995) proposed a mode based measure $1 - 2F(M)$. More recent research in this area is focused on robust measures of skewness (see Brys et al., 2003, 2004; Aucremanne et al., 2004).

A single summary statistic is an inadequate tool to capture the skewness or lack of symmetry of a distribution as evidenced by the fact that no single measure could acquire universal acceptability even after prolonged research on this topic. Consequently we propose to look at the characteristics of the asymmetry more closely over the entire domain. It is to be noted that the asymmetry can be considered with respect to many different measures of "center". However, since by asymmetry of a probability distribution one commonly understands differential probability mass on either side of the center, for unimodal distributions we find it most appropriate to choose the center to be the point having the highest mass at/around it, i.e., the mode. We observe that many commonly used asymmetric distributions show certain specific patterns in their asymmetry. For example, for some distributions, the mass over any interval on one side of the mode is *always* higher (or lower) than that over the symmetric interval on the other side of the mode.

This leads us to define the asymmetry through a "skewness function" which captures the difference of densities at equidistant points from the mode. It turns out that for many popular families of distributions, this skewness function either does not change its sign (always positive or always negative) or changes its sign at most once. In this work, we limit our study of skewness to such probability distributions, referred to as (partially) homogeneously skewed distributions.

The rest of the article is organized as follows. In Section 2 the class of partially homogeneously skewed distributions is introduced. We show that several commonly occurring continuous unimodal distributions are partially homogeneously skewed. The associated measure of homogeneous skewness, its properties and two alternative ways of ordering the distributions in this class are discussed in Section 3. A comparison between the measure of homogeneous skewness and Pearson's measure of skewness and standardized third moment is also carried out in this section. The applicability of the concept of homogeneous skewness and its measure for distributions with flat modal regions, distributions with unique antimode and for discrete distributions are dealt in Section 4. Finally, Section 5 concludes the article with the outline of an application and a summary.

2 Concept of (partial) homogeneous skewness

In this section we define the concepts of homogeneous skewness and partial homogeneous skewness for unimodal continuous distributions, i.e., distributions with densities having unique maxima. This notion is further extended to broader classes of distributions in Section 4. Formally, for the purpose of the present work, a unimodal distribution may be defined as follows: A distribution with probability density function (p.d.f.) $f(x)$ is called *unimodal* if there exists a unique M such that $f(x)$ is non-decreasing on $(-\infty, M)$, and non-increasing on (M, ∞). The value M is called the *mode* of the distribution. Note that a distribution with non-increasing (resp. non-decreasing) p.d.f. also fall under unimodal distribution by taking M to be the left (resp. right) end point of the support of the density function. Also, in our convention, it is possible for the density function to be infinite, or even undefined at M. Note further that the support of the considered distributions need not be finite.

Since our proposed treatment of skewness is built around the lack of symmetry of the density function around the mode, we formally define the skewness function as follows:

Definition 1 *The* skewness function *of a unimodal distribution with mode M and p.d.f. $f(\cdot)$ is defined as*

$$\gamma_f(x) = f(M+x) - f(M-x), \quad x \in (0, \gamma_u], \tag{1}$$

where $\gamma_u = \max(M-L, U-M)$, and $[L, U]$ is the support of $f(\cdot)$, which need not be finite.

We now introduce the class of *homogeneously skewed* distributions.

Definition 2 *A unimodal distribution (equivalently, the random variable having a unimodal distribution) with mode M and density function $f(\cdot)$ is said to be* homogeneously right-skewed *if*

$$\gamma_f(x) \geq 0 \quad \textit{i.e.,} \quad f(M+x) \geq f(M-x), \quad \forall x > 0. \tag{2}$$

Homogeneously left-skewed distributions are defined analogously, i.e., with the inequality sign in (2) reversed.

Definition 3 *A unimodal distribution is said to be* homogeneously skewed *provided it is either homogeneously right-skewed or homogeneously left-skewed.*

We denote the class of homogeneously skewed distributions by $\mathcal{F}$. In order to broaden the coverage of this study, we now introduce the notion of partial homogeneous skewness.

Definition 4 *A unimodal distribution with skewness function $\gamma_f(\cdot)$ is said to be* partially homogeneously skewed *if $\gamma_f(\cdot)$ changes its sign at most once in its domain.*

The function $\gamma_f(\cdot)$ changing its sign only once implies that $\exists\ C > 0$ such that

$$\text{either} \quad \gamma_f(x) \begin{cases} \geq 0 & \text{for } 0 < x < C \\ \leq 0 & \text{for } C < x \leq \gamma_u \end{cases} \quad \text{or} \quad \gamma_f(x) \begin{cases} \leq 0 & \text{for } 0 < x < C \\ \geq 0 & \text{for } C < x \leq \gamma_u. \end{cases} \tag{3}$$

The "direction of skewness" for partially homogeneously skewed density is established by the aggregate skewness on either side of C and the "degree of partiality" is decided by the probability of the regions on which the skewness function is negative or positive.

Definition 5 *A partially homogeneously skewed density is said to be skewed to the right if*

$$\int_{\{\gamma_f \geq 0\}} |\gamma_f(x)|\,dx \quad > \int_{\{\gamma_f \leq 0\}} |\gamma_f(x)|\,dx. \tag{4}$$

Definition 6 *The* degree *of partial homogeneous skewness of a partially homogeneously skewed distribution is defined as*

$$\delta \;=\; \max(\delta_1, 1-\delta_1), \quad \textit{where} \quad \delta_1 = \int_{M-C}^{M+C} f(x)dx. \tag{5}$$

Note that C in (3) is not necessarily unique as such; to achieve uniqueness, C would be taken as the minimum or maximum such value (satisfying (3)) that results in the highest possible value for δ.

Clearly, $0.5 \leq \delta \leq 1$ and any homogeneously skewed distribution can be considered as partially homogeneously skewed with degree 1. Let us denote the class of partially homogeneously skewed distributions of order δ by $\mathcal{F}_\delta$. The class of all partially homogeneously skewed distributions may be denoted by $\mathfrak{I} = \mathcal{F} \cup_\delta \mathcal{F}_\delta$.

Contrary to what may seem at the first look, the introduced concept of (partial) homogeneous skewness is not excessively restrictive. It can be observed that many commonly used distributions belong to $\mathfrak{I}$. For example, Triangular distribution, standard Gamma distribution, standard Beta distribution, Lognormal distribution are all homogeneously skewed. Typical forms of the skewness functions for these distributions are depicted in Figures 1 and 2. For Weibull distribution we have the following result.

Theorem 1 *Suppose X has a Weibull distribution with the density function*

$$f(x) = c\,x^{c-1}\exp(-x^c), \quad x > 0,$$

where the parameter $c > 0$. Then the following hold

(a.) X is homogeneously right-skewed if $c \leq 3$.

(b.) X is partially homogeneously skewed if $c > 3$. It is partially homogeneously skewed to the right if $c < \frac{1}{1-\ln(2)}$ and partially homogeneously skewed to the left if $c > \frac{1}{1-\ln(2)}$.

The proof of the theorem is given in the appendix. We note here that obtaining an expression for C, the point at which the skewness function changes its sign (viz. equation (3)), is quite tricky, thus making it difficult to express the degree of partial homogeneous skewness as an explicit function of c. However, if C is known then the δ_1 of equation (5) is given by $e^{-(M-C)^c} - e^{-(M+C)^c}$, where $M = \left(\frac{c-1}{c}\right)^{\frac{1}{c}}$ is the mode and subsequently the degree can be determined. Our computations show that for $c = 3.2$ and $c = 3.5$ the change of sign for the skewness function occurs at $C \approx 0.5478$ and $C \approx 0.7499$, respectively and the degrees are 0.9273 and 0.9956, respectively.

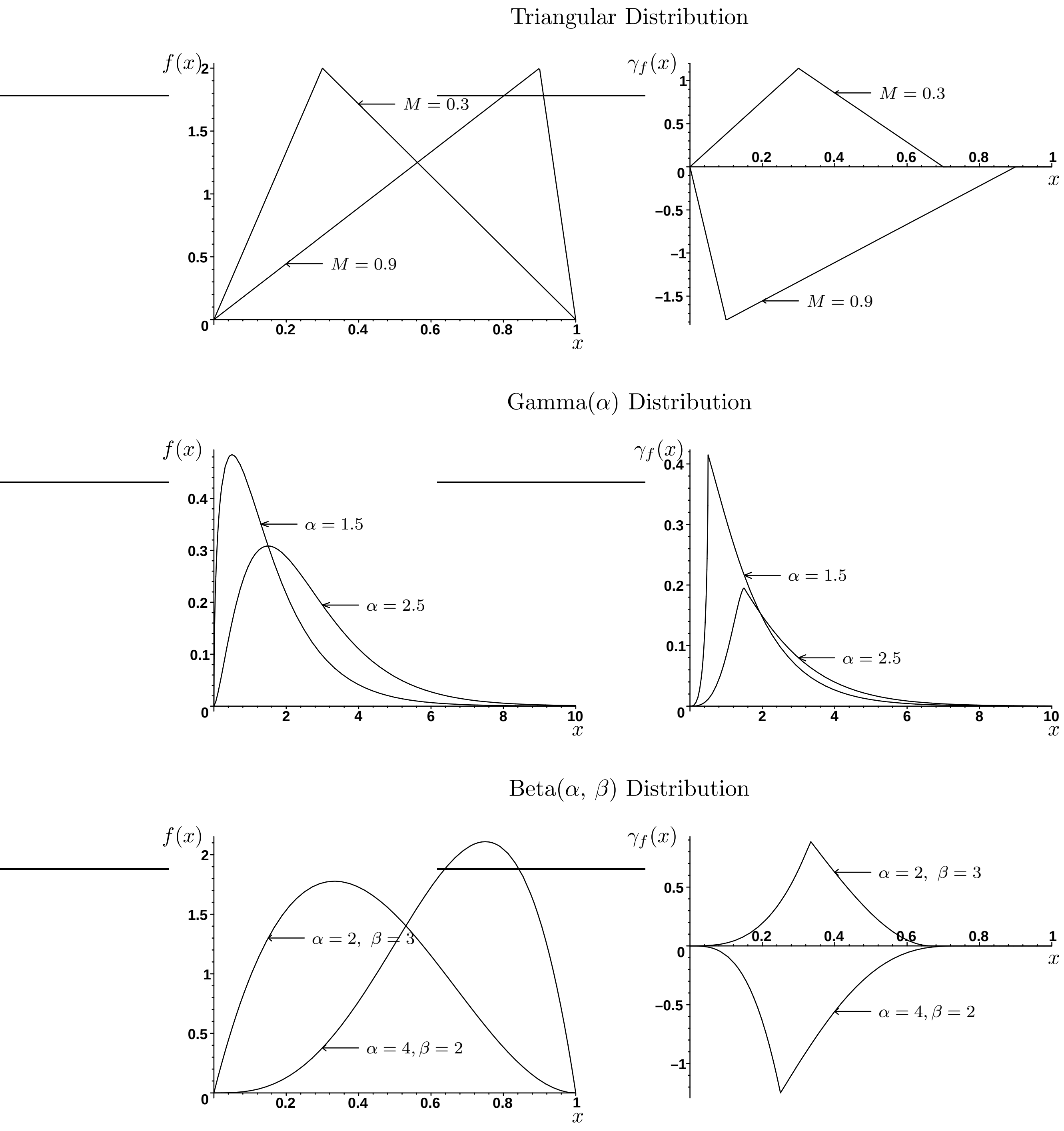

Figure 1: Density and skewness functions of some continuous distributions

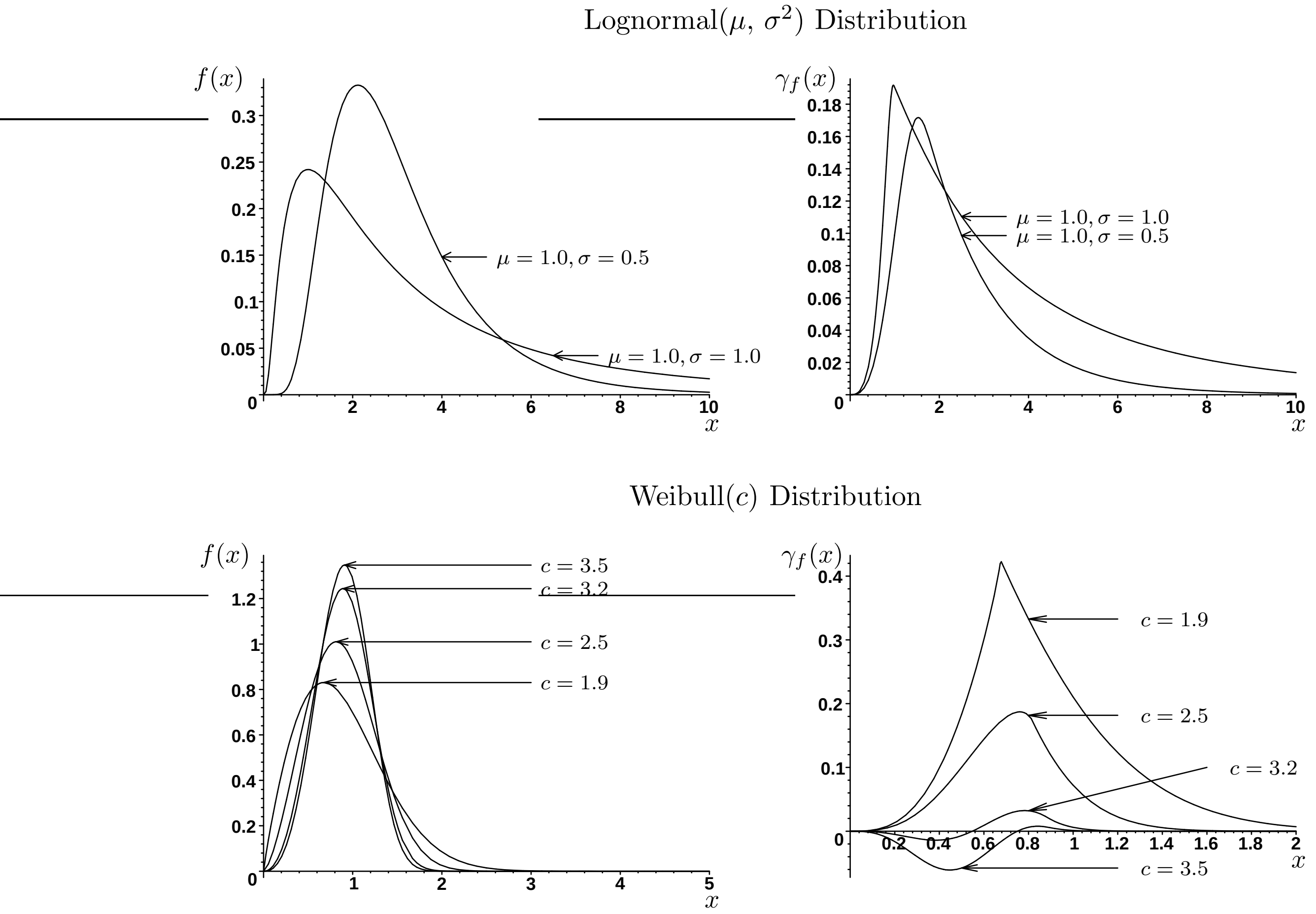

Figure 2: Density and skewness functions of some continuous distributions

3 Measure of homogeneous skewness, its properties and ordering of distributions

The notion of homogeneous skewness leads to the following natural definition of a measure of skewness.

Definition 7 *The measure of (partial) homogeneous skewness of a distribution in* $(\beth)$ F*, with Mode* M *and p.d.f.* $f(\cdot)$*, is defined as*

$$\tau_f = \int_0^\infty \gamma_f(x)dx. \tag{6}$$

Remark: Clearly, a homogeneously skewed density is skewed to the right if and only if $\tau_f > 0$. It also follows from equation (4) that the same holds for a partially homogeneously skewed density.
Remark: It is easy to see that the proposed measure reduces to $1 - 2F(M)$, the measure of skewness proposed by Arnold and Groeneveld (1995).

Arnold and Groeneveld (1995) have shown that the skewness measure τ_f defined by (6) has the following properties.

(i) $\tau_f = 0$ if the distribution is symmetric.

(ii) $-1 \leq \tau_f \leq 1$.

(iii) If $g(x) = \frac{1}{a} f((x-b)/a)$, then $\tau_g = \text{Sign}(a)\tau_f$, i.e., the measure is scale and location invariant.

Furthermore, we note the following additional properties of the measure. For any distribution $F \in \mathcal{F}$ with density f,

(i) $\tau_f = 0$ only if the distribution is symmetric.

(ii) $\tau_f > 0$ (< 0) if and only if the distribution is homogeneously right-skewed (left-skewed).

(iii) $\tau_f = +1(-1)$ if and only if the density is nonincreasing (nondecreasing) on its support.

The following two results show that the measure given by (6) is more stringent than some existing measures.

Theorem 2 *If a random variable or its distribution is homogeneously right-skewed (left-skewed), then its Pearson's measure of skewness is necessarily positive (respectively, negative).*

Proof:

$$\begin{aligned}
\mu - M &= \int_{-\infty}^{\infty} (x-M) f(x) dx \qquad (7) \\
&= \int_{-\infty}^{M} (x-M) f(x) dx + \int_{M}^{\infty} (x-M) f(x) dx \\
&= -\int_{0}^{\infty} y f(M-y) dy + \int_{0}^{\infty} y f(M+y) dx \\
&= \int_{0}^{\infty} y \gamma_f(y) dy. \qquad (8)
\end{aligned}$$

Now for homogeneously right-skewed (left-skewed) distributions, $\gamma_f(x) \geq$ (respectively, $\leq$) 0, $\forall x$, and consequently $\mu \geq$ (respectively, $\leq$)M, implying that the distribution is positively (respectively, negatively) skewed as per Pearson's definition. □

It is easy to visualize distributions that are not partially homogeneously skewed but have positive or negative Pearson's measure of skewness. This shows that the concept of homogeneous skewness is stronger than that of Pearson's skewness.

It is not possible to draw a direct comparison with other measures of skewness, which are typically based on asymmetry around mean or median. In order to draw a general comparison we need to consider suitable modifications of the standard measures. We therefore modify the standardized third central moment by considering the the third moment around the mode. The following theorem shows that the notion of homogeneous skewness is also more stringent than this measure.

Theorem 3 *For distributions homogeneously right-skewed (left-skewed), the skewness measure based on third moment around the mode is also positive (respectively, negative).*

Proof: Following steps similar to (7) – (8), one can show that

$$E(X - M)^3] = \int_0^\infty y^3 \gamma_f(y) dy;$$

this proves the theorem on similar lines to Theorem 2. $\square$

Ordering of distributions

All the standard measures of skewness impose an ordering of distributions. This has been elaborately explored by, among others, MacGillivray (1986), Oja (1981), and van Zwet (1979). In order to achieve the same with the newly introduced homogeneous skewness we have two alternative ways of ordering partially homogeneously skewed distributions.

One can define an ordering of the distributions in $\gimel$ primarily through its degree of partial homogeneous skewness, and then through the skewness measure. Thus, if $F_1 \in \digamma_{\delta_1}$, and $F_2 \in \digamma_{\delta_2}$ with $\delta_1 < \delta_2$, then the distribution F_2 is said to be more homogeneously skewed than F_1. However, if $F_1, F_2 \in \digamma_\delta$ for some $\delta \in [0.5, 1]$, then F_1 is said to be less homogeneously skewed than F_2 if and only if

$$|\tau_{F_1}| \leq |\tau_{F_2}|. \tag{9}$$

This defines a total ordering of distributions in $\gimel$.

Alternatively, a partial ordering may be defined in $\gimel$ by replacing (9) by

$$|\gamma_{f_1}(x)| \leq |\gamma_{f_2}(x)| \quad \forall x. \tag{10}$$

In some sense, this latter ordering is more intuitive and in line with the notion of homogeneous skewness. It is, however, very much restrictive. For example, it presents problems in cases where the skewness functions have different domains.

4 Homogeneous skewness for other types of distributions

In this section, we investigate the applicability of the concept of homogeneous skewness to distribution functions other than continuous unimodal distributions.

Distributions with one flat modal region

It is easy to extend the concept of homogeneous skewness to distributions with a flat modal region in the density function (see Figure 3) instead of a unique mode. For these distributions, the density function $f(\cdot)$ is nondecreasing on $(-\infty, M_1)$, non-increasing on (M_2, ∞) and constant on (M_1, M_2) with $f(x) < f(M_1)$, $\forall x \notin (M_1, M_2)$. Then the skewness function (1) may be modified to:

$$\gamma_f(x) = f(M_2 + x) - f(M_1 - x),$$

and subsequently the measure of homogeneous skewness may be defined as before.

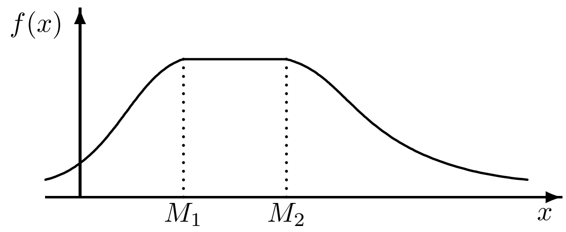

Figure 3: Density function of a distribution with flat modal region

Discrete unimodal distributions

The concept of homogeneous skewness carries over naturally to discrete distributions *per se.*, whose density function with respect to an appropriate counting measure, $f(\cdot)$, also known as the probability mass function, is assumed to have a peak at M. For such distributions, the skewness function $\gamma(\cdot)$ remains unaltered. In principle, the definition of the skewness measure τ can also remain same. Then, however, it is no longer possible to achieve the extreme values ± 1, because the probability mass at the mode would never be accounted for. However, the interpretation of the extreme values can be maintained by defining the measure for discrete unimodal distributions as follows:

Definition 8 *For a discrete distribution in $\mathcal{F}$ with unique mode M and mass function $f(\cdot)$, the measure of homogeneous skewness may be defined as*

$$\tau_f = \sum_{x>0}\{f(M+x) - f(M-x)\} + f(M) \times \text{Sign}\Big(\sum_{x>0}\{f(M+x) - f(M-x)\}\Big). \qquad (11)$$

Note that the second term in (11) has no impact on the sign of the skewness measure and is brought in only to ensure that the skewness of a decreasing/increasing density is 1 or -1. However, the measure will satisfy the other properties mentioned in Section 3 with or without this additional term.

With similar ideology, to make sure that the probability mass at mode M has no unwanted influence on the degree δ, of partial homogeneous skewness, the definition of δ for discrete distributions should be changed to

$$\delta = \max\left(\sum_{0<|x-M|<C} f(x), \sum_{|x-M|\geq C} f(x)\right) + f(M),$$

where C is the point where $\gamma_f(x)$ changes its sign.

We again observe that many commonly used discrete distributions such as Geometric, Poisson, Binomial belong to the class of (partially) homogeneously skewed distributions. Since the Geometric probabilities are decreasing, by definition it is homogeneously right-skewed. For Poisson and Binomial we have the following results.

Theorem 4 *Suppose X is a Poisson random variable with parameter μ.*

(a.) If μ is an integer or $\mu < 1$ then X is homogeneously right-skewed.

(b.) If $\mu > 1$ is not an integer then there exists a $\theta_0 \equiv \theta_0(\mu)$ with $0 < \theta_0 \leq 1/2$ and $\lim_{\mu\to\infty} \theta_0(\mu) = 1/2$ such that X is

(i) partially homogeneously skewed if $\mu - \lfloor \mu \rfloor < \theta_0$ and

(ii) homogeneously right-skewed if $\mu - \lfloor \mu \rfloor \geq \theta_0$.

We were not able to make a complete characterisation of the direction of partial homogeneous skewness. However, based on numerical experimentation, our conjecture is that there is a $\theta_1 \in (.1, .2)$ such that if $\mu - \lfloor \mu \rfloor < \theta_1$ then it is partially homogeneously skewed to the left whereas for $\theta_1 < \mu - \lfloor \mu \rfloor < \theta_0$ it is partially homogeneously skewed to the right.

Theorem 5 *Suppose $X \sim$ Binomial(n, p), where $0 < p < \frac{1}{2}$.*

(a.) X is homogeneously right-skewed if $(n + 1)p$ is an integer or $(n + 1)p < 1$.

(b.) If $(n+1)p > 1$ and is not an integer, then there exists a $\theta_0 \equiv \theta_0(n, p)$ with $0 < \theta_0 \leq 1/2$ and $\lim_{n\to\infty} \theta_0 = 1/2$ such that X is

(i) partially homogeneously skewed if $(n + 1)p - \lfloor (n + 1)p \rfloor < \theta_0$ and

(ii) homogeneously right-skewed if $(n + 1)p - \lfloor (n + 1)p \rfloor \geq \theta_0$.

Remark: Since $n - X \sim$ Binomial$(n, 1 - p)$ when $X \sim$ Binomial(n, p), the skewness property of Binomial(n, p) when $\frac{1}{2} < p < 1$, would be the same as that of Binomial$(n, 1-p)$ but on the opposite direction.

Proofs of these theorems are given in the appendix.

Distributions with Unique Antimodes

A distribution with p.d.f. $f(\cdot)$ is said to have a unique antimode $\bar{M}$ if $f(\cdot)$ is decreasing (non-increasing) on $(a, \bar{M})$ and increasing (non-decreasing) on $(\bar{M}, b)$. Our notion of homogeneous skewness could be applied to such distributions as well, with only $\bar{M}$ replacing M. Beta distributions with both $\alpha, \beta < 1$ are examples of distribution with unique antimodes. As seen in Figure 4 these are partially homogeneously skewed distributions.

Another distribution family having a unique antimode is depicted in Figure 5 in its standardized form. This family may be referred to as inverted Triangular distributions, parameterized by slopes S_1 and S_2 and the antimode $\bar{M}$. In Figure 5a we have a homogeneously skewed distribution to the left and in Figure 5b we have a homogeneously skewed distribution to the right. For $S_1 > S_2, \bar{M} < 0.5$, or $S_1 < S_2, \bar{M} > 0.5$, the distributions are partially homogeneously skewed.

5 Summary and an application

In this paper, we examine a concept of skewness for unimodal distributions, which is based on the asymmetry of the density function around the mode. We introduce the class of partially homogeneously skewed distributions (in Section 2) for which this concept is meaningful. Although the class of partially homogeneously skewed distribution initially seems

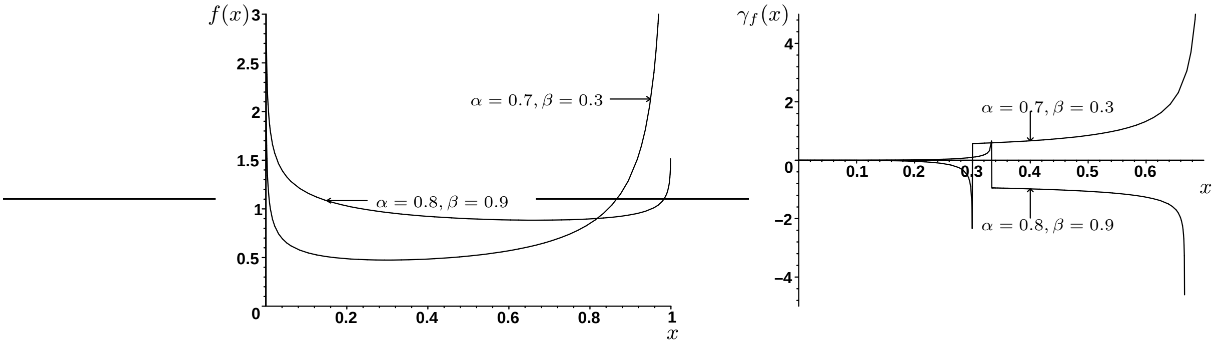

Figure 4: Density and skewness functions of Beta(α, β) distributions with antimode

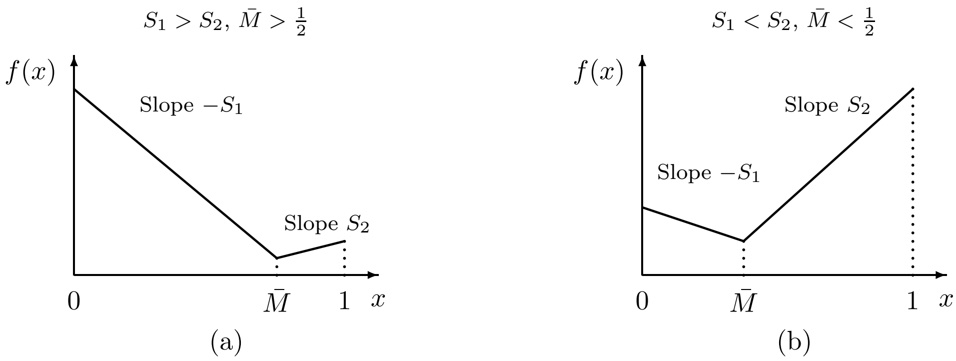

Figure 5: Inverted triangular distributions

to be restrictive, we show that many commonly encountered continuous distributions like the Triangular, Beta, Gamma, Lognormal, and Weibull distributions are partially homogeneously skewed. The natural skewness measure that arises out of the skewness function leads to partial/total ordering of distributions in terms of skewness within the class of partially homogeneously skewed distributions. We finally extend this class to distributions having one flat modal region, to discrete distributions and to distributions with unique antimode. In particular, we notice that the Binomial and Poisson distributions are also partially homogeneously skewed.

Let us conclude with an outline of a direct application of this notion of homogeneous skewness in discrete optimization problems (DOP). Traditional notion of optimality used to solve a DOP such as a traveling salesman problem needs to be revised if some elements are stochastic in nature, as opposed to all being deterministic. One approach is to look for a solution with minimum expected regret, where the regret for a solution S is defined based on the difference between S and the best (deterministic) solution given the realizations of the random elements. If the regret function is linear, such optimal solutions can be found by replacing the random elements by their mean (expected) values. Under general regret function, the search for optimal solution is more complicated. In many situations, however, it is possible to show existence of deterministic surrogate values, based on the probability distributions of the random elements, which may be used to replace the stochastic elements. For instance, in a DOP with one random element, this surrogate value is the mean/median of the distribution, if it is symmetric. In general, however, this surrogate need not always be the mean value and neither is it easy to obtain analytically. In this context, Ghosh et al. (2005) obtain useful bound for the surrogate if the random element has a homogeneously skewed distribution. Some limiting results, e.g., with change in degree of homogenous skewness, have also been obtained. For details, see Ghosh et al. (2005).

Appendix: Proofs of partial homogeneous skewness for commonly used distributions

In this appendix, we show mathematically that many commonly used distributions are (partially) homogeneously skewed. The proofs of Theorems 1, 4, 5 are also given. First note that a random variable X is (partially) homogeneously skewed if and only if any linear transformation of it $aX + b$ is (partially) homogeneously skewed.

Triangular distribution

In view of the previous observation, it is sufficient to work with the standardized form of the density function:

$$f(x) = \begin{cases} \frac{2x}{M} & \text{for } 0 \le x \le M, \\ \frac{2(1-x)}{(1-M)} & \text{for } M \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where M is the mode of the distribution. First assume that $M \le 0.5$. Then it is easy to see that, the skewness function is given by

$$\gamma(y) = \frac{2y(1-2M)}{M(1-M)}, \quad \text{for} \quad 0 < y < M, \quad \text{and} \quad \gamma(y) = f(M+y), \quad \text{for} \quad y \ge M.$$

Hence, the distribution is homogeneously right-skewed.
Similarly one can see that, it is homogeneously left-skewed if $M \ge 0.5$.

Gamma distribution

Note that the standard gamma density $f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \exp(-x)$ (for $x > 0$) is decreasing when $\alpha \le 1$ and consequently is homogeneously right-skewed. For $\alpha > 1$, observe that the mode of the distribution is at $M = (\alpha - 1)$ and $f(M+y) \ge 0 = f(M-y)$ for $y \ge M$. Now, defining the function $h(y) = f(M+y)/f(M-y)$, $(0 \le y < M)$ we have

$$h'(y) = 2y^2 \, (M+y)^{(M-1)} \, (M-y)^{-(M+1)} \, \exp(-2y) \ge 0, \quad \text{for } 0 \le y < M.$$

This implies that $h(y) \ge h(0) = 1$, i.e., $f(M+y) \ge f(M-y)$ for $0 < y < M$. Hence the distribution is homogeneously right-skewed.

Beta distribution

Recall that the standard beta density is given by $f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ (for $0 < x < 1$) where α, $\beta > 0$. We consider the case when at least one of α, $\beta > 1$, because otherwise the density is not unimodal. Furthermore, for $\alpha \le 1, \beta > 1$ and $\alpha > 1, \beta \le 1$, the density functions are respectively decreasing and increasing and hence homogeneously skewed. When $\alpha > 1$, and $\beta > 1$, according to the existing measures, the distribution is positively (negatively) skewed for $\beta >$ (respectively, $<$) α. In the following, we show that the distribution is homogeneously right-skewed if $\beta > \alpha$. That the distribution is homogeneously left-skewed for $\beta < \alpha$ can be shown similarly.

First, note that if $\beta > \alpha > 1$, the mode of the beta distribution is at $M = (\alpha - 1)/(\alpha + \beta - 2) < 1/2$ and hence $f(M + y) \ge 0 = f(M - y)$ for $y \ge M$. Further, defining $h(\cdot) = f(M + y)/f(M - y)$ for $0 \le y < M$, we have

$$h'(y) = 2y^2(\beta - \alpha)(M - y)^{-\alpha}(M + y)^{\alpha-2}(1 - M - y)^{\beta-2}(1 - M + y)^{-\beta} \ge 0,$$

for $0 \le y < M < \frac{1}{2}$ and $\beta > \alpha$. This implies that the distribution is homogeneously right-skewed.

Lognormal distribution

To see that the standard lognormal distribution is homogeneously right-skewed, recall that the density is given by $f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right)$ (for $x \ge 0$). The mode of the distribution is $M = \exp(\mu - \sigma^2)$. Since $f(M + x) \ge 0 = f(M - x)$ for $x \ge M$, we need to show that $f(M + x) \ge f(M - x)$ for $0 < x < M$. But for $0 < x < M$, using the concavity of $\ln(\cdot)$, we have

$$\begin{aligned}
& \ln(M) - \ln(M - x) \ge \ln(M + x) - \ln(M), \\
\iff\ & 2(\mu - \sigma^2) \ge \ln(M + x) + \ln(M - x), \\
\iff\ & -\frac{1}{2\sigma^2}(\ln(M + x) + \ln(M - x) - 2\mu) \ge 1 \\
\iff\ & -\frac{1}{2\sigma^2}[(\ln(M + x) + \ln(M - x) - 2\mu)(\ln(M + x) - \ln(M - x))] \\
& \qquad \ge (\ln(M + x) - \ln(M - x)), \\
\iff\ & -\frac{1}{2\sigma^2}[(\ln(M + x) - \mu)^2 - (\ln(M - x) - \mu)^2] \ge (\ln(M + x) - \ln(M - x)) \\
\iff\ & \exp\{-\frac{1}{2\sigma^2}[(\ln(M + x) - \mu)^2 - (\ln(M - x) - \mu)^2]\} \ge \frac{M + x}{M - x} \\
\iff\ & \frac{f(M + x)}{f(M - x)} \ge 1.
\end{aligned}$$

Weibull distribution

We now prove Theorem 1.

Consider the Weibull density function, parametrized by $c > 0$, and given by

$$f(x) = c\, x^{c-1} \exp(-x^c), \quad x > 0.$$

Clearly, if $c \le 1$ then the density function is decreasing and hence homogeneously right-skewed. So assume $c > 1$. The mode of this distribution is at $M = \left(\frac{c-1}{c}\right)^{\frac{1}{c}}$. Since the skewness function

$$\gamma(x) = f(M + x) - f(M - x) = f(M + x) \ge 0, \quad \text{for } x \ge M,$$

it suffices to consider $\gamma(x)$ for $0 \le x < M$. Note that for $0 \le x < M$

$$\begin{aligned}
\gamma(x) &= c\,(M + x)^{c-1} \exp(-(M + x)^c) - c\,(M - x)^{c-1} \exp(-(M - x)^c) \\
&= c\,M^{c-1} \left[\left(1 + \frac{x}{M}\right)^{c-1} e^{-M^c(1+\frac{x}{M})^c} - \left(1 - \frac{x}{M}\right)^{c-1} e^{-M^c(1-\frac{x}{M})^c}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma(x) \gtreqless 0 &\Leftrightarrow \left(1 + \frac{x}{M}\right)^{c-1} e^{-M^c(1+\frac{x}{M})^c} \gtreqless \left(1 - \frac{x}{M}\right)^{c-1} e^{-M^c(1-\frac{x}{M})^c} \\
&\Leftrightarrow \ln\left(\frac{1 + \frac{x}{M}}{1 - \frac{x}{M}}\right) \gtreqless \frac{M^c}{c-1}\left[\left(1 + \frac{x}{M}\right)^c - \left(1 - \frac{x}{M}\right)^c\right] \\
&\Leftrightarrow \ln\left(\frac{1 + \frac{x}{M}}{1 - \frac{x}{M}}\right) \gtreqless \frac{1}{c}\left[\left(1 + \frac{x}{M}\right)^c - \left(1 - \frac{x}{M}\right)^c\right] \\
&\Leftrightarrow h\left(\frac{x}{M}\right) \gtreqless g\left(\frac{x}{M}\right), \quad \text{where} \qquad \text{(A.1)}
\end{aligned}$$

$$h(y) = \ln\left(\frac{1 + y}{1 - y}\right) \quad \text{and} \quad g(y) = \frac{1}{c}\left[(1 + y)^c - (1 - y)^c\right], \text{ for } 0 \le y < 1.$$

Also note that for $0 \le y < 1$

$$h'(y) = \tfrac{2}{1-y^2} > 0, \qquad h''(y) = \tfrac{4y}{(1-y^2)^2} > 0, \qquad g'(y) = (1 + y)^{c-1} + (1 - y)^{c-1} > 0,$$

$$\text{and } g''(y) = (c - 1)\left[(1 + y)^{c-2} - (1 - y)^{c-2}\right] \begin{cases} \ge 0 & \text{for } c \ge 2, \\ < 0 & \text{for } c < 2. \end{cases}$$

Hence h is an increasing convex function in $(0, 1)$ while g is either the same or an increasing concave function. In either case, the two functions can intersect each other at the most at one point in $(0, 1)$. Now, note that $\lim_{x\to1} h(x) = \infty$ and $g(1) = 2^c/c$. Hence $h(y) > g(y)$ in the neighborhood of $y = 1$. Furthermore, since $h(0) = g(0) = 0$, there are two possibilities:

(i) $h(y) - g(y) \ge 0$ for all $0 \le y \le 1$ or

(ii) there exists $y_0 \in (0, 1)$ such that $h(y) - g(y) < 0$ for $0 < y < y_0$ and $h(y) - g(y) \ge 0$ for $y_0 \le y \le 1$.

Hence from (A.1), the skewness function is either always nonnegative or it starts with negative values and at some point becomes nonnegative and remains nonnegative. That is to say that the density is either homogeneously right-skewed or partially homogeneously right-skewed.

To complete the proof we need to show that if $1 < c \le 3$ then $h(y) \ge g(y)$ for $0 < y < 1$ and for $c > 3$ there exists y_0 such that $h(y_0) < g(y_0)$.

Considering the Taylor series expansions around zero we have for $|y| < 1$,

$$\begin{aligned} g(y) &= \frac{1}{c}\left[(1+y)^c - (1-y)^c\right] = 2y + \sum_{n=1}^{\infty} \frac{2}{2n+1} \frac{(c-1)(c-2)\cdots(c-2n)}{(2n)!} y^{2n+1}, \\ h(y) &= \ln\left(\frac{1+y}{1-y}\right) = 2y + \sum_{n=1}^{\infty} \frac{2}{2n+1} y^{2n+1}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} h(y) - g(y) &= \sum_{n=1}^{\infty} \frac{2}{2n+1} \left(1 - \frac{(c-1)(c-2)\cdots(c-2n)}{(2n)!}\right) y^{2n+1} \\ &= \sum_{n=1}^{\infty} \frac{2}{2n+1} (1-a_n)\, y^{2n+1}, \text{ where } a_n = \frac{(c-1)(c-2)\cdots(c-2n)}{(2n)!}. \end{aligned} \tag{A.2}$$

Now note that for $1 < c \le 3$

$$|a_1| = \left|\frac{(c-1)(c-2)}{2!}\right| = \frac{(c-1)}{2} \cdot |c-2| \le |c-2| \le 1$$

and for $n \ge 2$

$$\begin{aligned} |a_n| &= \left|\frac{(c-1)(c-2)\cdots(c-2n)}{(2n)!}\right| = \frac{2n-c}{2n} \cdot \frac{(2n-1)-c}{2n-1} \cdots \frac{4-c}{4} \cdot \frac{3-c}{3} \cdot |a_1| \\ &\le |a_1| \le 1. \end{aligned}$$

Hence when $1 < c \le 3$, $h(y) - g(y) \ge 0$ for $0 < y < 1$. When $c > 3$, it follows from (A.2) that

$$h(y) - g(y) = \frac{2}{3}(1-a_1)\, y^3 + o(y^3) \quad \text{where} \quad a_1 = \frac{(c-1)(c-2)}{2} > 1.$$

Hence one can choose a $y_0 > 0$ small enough so that $h(y_0) < g(y_0)$. To complete the proof of Theorem 1, note that in this case the cdf is given by $F(x) = 1 - \exp(x^c)$. Hence, $\tau = 1 - 2F(M) = 2\exp\left(-1 + \frac{1}{c}\right) - 1$. Subsequently, $\tau > 0$ if $c < \frac{1}{1-\ln(2)}$ and $\tau < 0$ if $c > \frac{1}{1-\ln(2)}$.

Poisson distribution

We give a proof of Theorem 4. Recall that the mode of a Poisson(μ) distribution is given by $M = \lfloor \mu \rfloor$ if μ is not an integer, otherwise both μ and $\mu - 1$ are modes.

Clearly, if $\mu < 1$ then the probabilities are decreasing. To see that if μ is an integer the distribution is homogeneously right-skewed, we modify, in consistence with section 4, the skewness function as:

$$\gamma_f(x) = f(M+x) - f(M-1-x), \quad x = 1, 2, 3, \ldots$$

Then note that for $x \geq M$, $\gamma_f(x) = f(M+x) > 0$, and for $1 \leq x \leq M-1$

$$\frac{f(M+x)}{f(M-1-x)} = \prod_{k=1}^{x} \frac{\mu^2}{(\mu^2 - k^2)} \;\geq\; 1.$$

For non-integer μ (> 1), we define $\theta \equiv \theta(\mu) = \mu - \lfloor \mu \rfloor$. Note that for $1 \leq x \leq M$,

$$\begin{aligned} g(x) \;&:=\; \frac{f(M+x)}{f(M-x)} \;=\; \prod_{k=1}^{x} \frac{\mu^2}{(M+k)(M-k+1)} \;=\; \prod_{k=1}^{x} \frac{(M+\theta)^2}{M^2+M-k(k-1)} \\ &=\; \prod_{k=1}^{x} \psi(k;\theta,M), \quad \text{where } \; \psi(k;\theta,M) = \frac{(M+\theta)^2}{M^2+M-k(k-1)}. \end{aligned}$$

Clearly, $\psi(k;\theta,M)$ is increasing in k, $\;(1 \leq k \leq M)$. Hence if $\psi(x;\theta,M) < 1$ for some $1 \leq x \leq M$, then $g(x) < 1$. Equivalently, if $g(x) \geq 1$ for some $1 \leq x \leq M$, then $\psi(x;\theta,M) \geq 1$ and subsequently,

$$g(y) = g(x) \cdot \prod_{k=x+1}^{y} \psi(k;\theta,M) \geq 1 \quad \text{ for all } x \leq y \leq M.$$

This, together with the fact that $\gamma_f(x) = f(M+x) - f(M-x) = f(M+x) \geq 0$ for $x > M$, implies that once the skewness function $\gamma_f(x)$ becomes nonnegative it remains so. Hence if $\psi(1;\theta,M) \geq 1$, i.e., $f(M+1) \geq f(M-1)$, the distribution is homogeneously right-skewed, otherwise, it is partially homogeneously skewed since the skewness function would be negative up to some point and changing into positive values subsequently.

Thus the distribution will be *partially* homogeneously skewed if and only if

$$1 > \psi(1;\theta,M) = \frac{(M+\theta)^2}{M(M+1)} \quad \Leftrightarrow \quad \theta < \sqrt{M(M+1)} - M \;=:\; \theta_0, \quad \text{say.}$$

Finally, note that θ_0 can be rewritten as $\theta_0 \;=\; \frac{1}{2}\left(1 - (\sqrt{M+1} - \sqrt{M})^2\right)$ and hence

$$\theta_0 \leq \frac{1}{2} \quad \text{with} \quad \lim_{\mu \to \infty} \theta_0 \;=\; \lim_{M \to \infty} \frac{1}{2}\left(1 - (\sqrt{M+1} - \sqrt{M})^2\right) \;=\; \frac{1}{2}.$$

This completes the proof of Theorem 4.

Binomial distribution

To prove Theorem 5 we recall that the mode of the Binomial(n,p) distribution is given by $M = \lfloor (n+1)p \rfloor$ if $(n+1)p$ is not an integer, otherwise, both M and $M-1$ are modes.

First note that if $(n+1)p < 1$, then the probabilities are decreasing. Next, consider the case when $(n+1)p$ is an integer. We modify the skewness function in consistence with section 4, i.e., $\gamma_f(x) = f(M+x) - f(M-1-x)$, $x \geq 1$. Since for $x \geq M$, $\gamma_f(x) = f(M+x) \geq 0$, it is sufficient to prove that $g(x) := f(M+x)/f(M-1-x) \geq 1$, for

$1 \leq x \leq M-1$. But for $1 \leq x \leq M-1$, $g(x)$ can be rewritten as $g(x) = \prod_{k=1}^{x} \psi(k; n, M)$, where

$$\begin{aligned}
\psi(k; n, M) &= \frac{(n-M+1-k)(n-M+1+k)}{(M+k)(M-k)} \cdot \left(\frac{p}{1-p}\right)^2 \\
&= \frac{(n-M+1)^2 - k^2}{M^2 - k^2} \cdot \left(\frac{M}{n-M+1}\right)^2 = \frac{1 - \frac{k^2}{(n-M+1)^2}}{1 - \frac{k^2}{M^2}} \geq 1.
\end{aligned}$$

The last inequality follows from the fact that $p < \frac{1}{2}$ and hence $M = (n+1)p \leq \frac{n+1}{2}$, i.e., $n - M + 1 \geq M$.

Finally consider the case when $(n+1)p$ (> 1) is not an integer. Note that the skewness function $\gamma_f(x) = f(M+x) - f(M-x) \geq 0$ for $x \geq M+1$. So we just need to check $\gamma_f(x)$ for $1 \leq x \leq M$. Define $\theta \equiv \theta(n,p) = (n+1)p - \lfloor (n+1)p \rfloor$. Then noting that $p = (M+\theta)/(n+1)$, we can rewrite the function $g(x) := f(M+x)/f(M-x)$, $1 \leq x \leq M$, as $g(x) = \prod_{k=1}^{x} \psi(k; n, \theta, M)$, where

$$\psi(k; n, \theta, M) = \frac{(n-M-k+1)(n-M+k)}{(M-k+1)(M+k)} \cdot \frac{(M+\theta)^2}{(n-M+1-\theta)^2}.$$

Furthermore, since $(n+1)p$ is not an integer and $p < \frac{1}{2}$, it follows that $M = \lfloor (n+1)p \rfloor \leq n/2$, or equivalently, $n - M \geq M$. Using this, one can check that $\psi(k; n, \theta, M)$ is increasing in k, $(1 \leq k \leq M)$. Arguing exactly similarly as in the case of Poisson(μ) with non-integer μ, we see that the Binomial(n, p) distribution with $p < \frac{1}{2}$ is either partially homogeneously skewed or homogeneously right-skewed. It is partially homogeneously skewed if and only if

$$\begin{aligned}
\psi(1; n, \theta, M) < 1 \quad &\Leftrightarrow \quad \frac{(n-M)(n-M+1)}{M(M+1)} \cdot \frac{(M+\theta)^2}{(n-M+1-\theta)^2} < 1 \\
&\Leftrightarrow \quad \theta^2(n-2M) \; + \; 2\theta M(n-M+1) \; - \; M(n-M+1) < 0 \\
&\Leftrightarrow \quad \theta < \frac{\sqrt{M(M+1)(n-M)(n-M+1)} \; - \; M(n-M+1)}{n-2M} \quad =: \; \theta_0 \equiv \theta_0(n).
\end{aligned}$$

Finally, note that θ_0 can be rewritten as

$$\theta_0(n) \;=\; 1 \bigg/ \left(1 + \sqrt{\frac{M+1}{M + \frac{M}{n-M}}}\right) \;\leq\; \frac{1}{2} \qquad (\text{since } \; n - M \geq M.)$$

Also, $\lim_{n\to\infty} \theta_0(n) = 1/2$ because $\lim_{n\to\infty} M/n = p$.

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