A water wave model with horizontal circulation and accurate dispersion

Colin Cotter[1] and Onno Bokhove[2*]
1. Department of Aeronautics, Imperial College, London, U.K.
2. Department of Mathematics, University of Twente, Enschede, The Netherlands

March 26, 2009

Abstract

We describe a new water wave model which is variational, and combines a depth-averaged vertical (component of) vorticity with depth-dependent potential flow. The model facilitates the further restriction of the vertical profile of the velocity potential to n-th order polynomials or a finite element profile with a small number of elements (say), leading to a framework for efficient modelling of the interaction of steepening and breaking waves near the shore with a large-scale horizontal flow. The equations are derived from a constrained variational formulation which leads to conservation laws for energy, mass, momentum and vertical vorticity (or circulation). We show that the potential flow water wave equations and the shallow-water equations are recovered in the relevant limits, and provide approximate shock relations for the model which can be used in numerical schemes to model breaking waves.

1 Introduction

It always remains fascinating to watch waves near the shore line. They approach the shore, steepen as the water becomes more shallow, break upon further approach, and run up and down the beach or dike. In Fig. 1, wave trains are seen to approach the shore. We see that far from the shore no wave breaking occurs. Upon reaching the shore, part of the wave starts breaking while part of the same wave has only steepened significantly. Finally, waves very near to the shore will break along their entire crest, and subsequently enter the swash zone where the waves run up and down the beach or (rocky) shore.

In hydrodynamics and coastal engineering, depth-averaged shallow water modelling is and has been very successful. Due to the depth-averaging, the full three-dimensional fluid dynamical (multiphase, Navier-Stokes or Euler) equations of motions are reduced in complexity. Only dependence on horizontal coordinates and time is then retained. Wave breaking within such a shallow water model is commonly approximated by using line discontinuities in the horizontal, leading to the well-known bores and hydraulic jumps (see [Stoker 1957]).

Breaking waves emerge in various forms [Peregrine 1983]: plunging breakers are multivalued viewed along the direction of gravity before the free surface breaks apart, while spilling breakers remain single-valued before the free surface breaks apart. When the free surface breaks apart into spray and foam during wave breaking, the topology of the domain changes. Bores or hydraulic jumps in depth-averaged shallow water modelling are a straightforward approximation of these breaking waves as localized discontinuities, and when viewed from above these discontinuities form horizontal lines (of finite length). Mass and momentum are conserved across a bore or jump, while kinetic and potential energy are not. These bores and jumps are akin to shocks in gas dynamics where mass, momentum and total energy conservation is combined with entropy increase. In contrast we find energy loss in shallow water bores. After all the dependence on internal energy and entropy in compressible fluids is lost due to the assumed incompressibility of water. While the mathematical theory for gas shocks and hydraulic jumps is the same, the physics in both cases is different.

The simplicity of such a shallow water model is also its pitfall. Internal wave dispersion is lost as vertical velocity profiles across the depth have been ignored. Boussinesq models are therefore useful, in

*Corresponding author: Institute of Mechanics, Processes and Control, Twente; University of Twente, P.O. Box 217, 7500AE, Enschede The Netherlands; o.bokhove@math.utwente.nl

(a) (b)

Figure 1: (a) A steepening wave with partly broken and unbroken crest is approaching the shoreline. (b) Two different waves with broken and steepening crests approach the shore line. Courtesy: D. Howell Peregrine.

the coastal engineering community, because they include internal wave dispersion to a higher degree of accuracy. Generally, Boussinesq models are derived from the three-dimensional potential flow water wave equations, the latter which describe internal water wave dispersion fully (in the absence of vorticity). Disadvantages of (most) Boussinesq models are threefold. First, dispersion always seems to beat nonlinearity such that wave overturning as the precursor or first stage of wave breaking is always prevented. This is unphysical: waves are observed to break in nature and also do overturn within the potential flow water wave model. Second, Boussinesq models tend to be derived from the "parent" water wave model under the potential flow assumption, in which case (horizontal) circulation is eliminated from the outset. This contrasts with the derivation of the classical depth-averaged shallow water model, which contains horizontal circulation consistently from its derivation of the (hydrostatic) incompressible Euler equations. Moreover, in depth-averaged shallow water models wave breaking along non-uniform bores leads to the generation of potential vorticity anomaly [Pratt 1983, Peregrine 1998, Peregrine and Bokhove 1998, Peregrine 1999, Bühler 2000, Ambati and Bokhove 2007, Tassi *et al.* 2007]. Sometimes, horizontal circulation is added in a posteriori manner to Boussinesq models derived from parent potential flow water wave equations. It is unclear whether this is formally justified. Third, Boussinesq models do not always conserve the geometric structure of the parent equations. Hence, preservation of the original variational and Hamiltonian structure with its associated conservation of mass, momentum, and energy is then lost. The variational Boussinesq model of [Klopman et al. 2007] is a notable exception, but it is based on the potential flow Ansatz.

Finally, fully three-dimensional numerical models are in use but computational power is too limited for them to be wave resolving over large areas. Wave forecasting is therefore generally based on statistical wave modelling. Near the shore for tens of kilometers along the coast and one to two kilometers off the coast, deterministic wave and current modelling is required to forecast longshore currents and wave run-up and run-down for flood forecasting. Regions of such size appear feasible for numerical Boussinesq models and shallow water models, but not for fully three-dimensional ones.

Consequently, our aims are to overcome these various shortcomings. Our aims are to derive a simplified water wave model with accurate dispersion, horizontal circulation and vertical (component of) vorticity, and a bore model for wave breaking. Our derivation will be based on variational techniques and an immediate consequence is that the geometric structure of the parent incompressible Euler equations will be preserved. Conservation of mass, momentum and energy is therefore guaranteed. To illustrate our technique, we will show in passing that Luke's and Miles' variational principles [Luke 1967, Miles 1985] for potential flow water wave equations follow by a constraint approximation of the parent variational principle for the incompressible Euler equations. Our model with water wave dispersion and vertical vorticity is able to describe the wave steepening and breaking as seen in Fig. 1 and Fig. 2 in an advanced treatment of the bore approximation. The broken wave will be modelled as a discontinuity or shallow water bore as sketched in Fig. 2(a). Our model is then also of interest to model the propagation of undular bores on rivers, in which dispersion and nonlinearity are balanced in the deeper mid channel, while wave breaking occurs in shallowing water near the shore lines. This steepening and wave breaking behavior is clearly seen on the Severn bore in Fig. 3, and the Mascaret bore in Fig. 2(b).

The outline of our paper is as follows. In §2, our starting point, the variational principle of the parent incompressible Euler equations, is investigated. To illustrate our manipulations, Luke's and Miles' variational principles are rederived in a perhaps novel way from this starting principle in §3. Our new

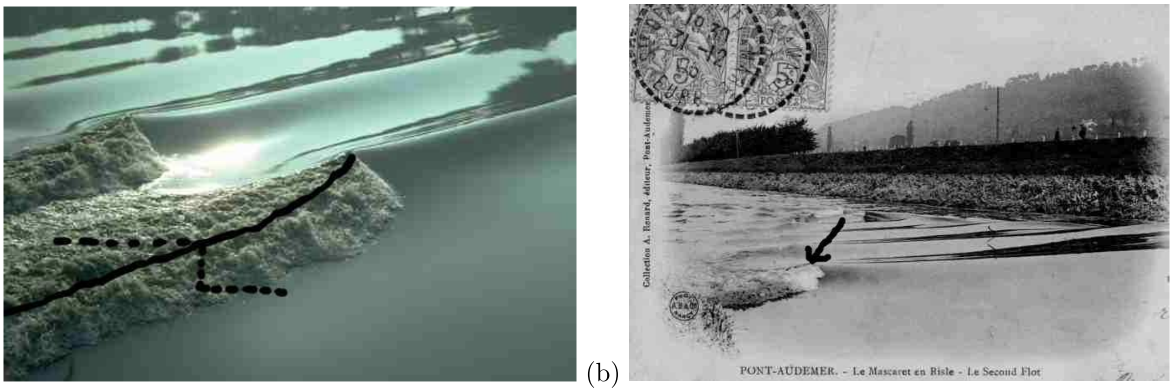
(a) (b)


Figure 2: (a) In the undular bore on the Severn River the competition between dispersion and nonlinearity is clearly seen. Part of the wave is broken, in the shallower water near the shore, on the left in the picture, and part is not. Photograph: D. Howell Peregrine. The black line added indicates where the bore discontinuity would be, and the dashed line a cross section of the free surface approximation of such a bore. (b) The Mascaret on the Risle, a small tributary of the Seine River at Pont-Audemer, shows the same breaking and non-breaking features on one wave crest. Courtesy: J.J. Malandain [Malandain 1988]. The arrow indicates where wave breaking starts.

model is derived and analyzed in §4. We conclude with a summary and discussion in §5.

2 Parent —three-dimensional incompressible Euler— equations

2.1 Variational principle

We construct a Lagrangian density by taking the difference between the kinetic and potential energy density

$$\frac{1}{2}D|\boldsymbol{u}|^2 - gDz, \tag{1}$$

in which g is the acceleration of gravity, $D = D(x, y, z, t)$ the weighted density, and $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t) = (u, v, w)^T$ the velocity vector with horizontal coordinates (x, y) and vertical coordinate z, and time t. Density D is the Jacobian between Eulerian coordinates (x, y, z) and Lagrangian label coordinates $\boldsymbol{a} = (a_1, a_2, a_3)^T$, such that $D\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}a_1\,\mathrm{d}a_2\,\mathrm{d}a_3$. The fluid resides in a domain Ω with solid domain walls $\partial\Omega_w$ and a free surface at $\partial\Omega_s$, such that $\partial\Omega = \partial\Omega_w \cup \partial\Omega_s$. The solid walls consist of bottom topography at $z = b(x, y)$ and possibly vertical walls. The free surface resides at $z = b(x, y) + h(x, y, t)$; we restrict attention to this single-valued free surface.

We introduce a Lagrange multiplier p to enforce incompressibility as the constraint

$$D - 1 = 0. \tag{2}$$

The constrained variational principle then reads

$$0 = \int_0^T \delta \int_\Omega \frac{1}{2}D|\boldsymbol{u}|^2 - g\,D\,z + p\,(1 - D)\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t. \tag{3}$$

Variations in D and $\boldsymbol{u}$ are related to the Eulerian variation of the label displacements $\mathbf{w}(x, y, z, t)$ [Holm et al. 1998, for example]. Define the flow map $\boldsymbol{\chi}$ such that $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{a}, t)$. Hence, the time derivative and variation of $\boldsymbol{\chi}$, i.e.,

$$\dot{\boldsymbol{\chi}} = \boldsymbol{u} \circ \boldsymbol{\chi} \equiv \boldsymbol{u}(\boldsymbol{\chi}, t) \quad \text{and} \quad \delta\boldsymbol{\chi} = \mathbf{w} \circ \boldsymbol{\chi} \equiv \mathbf{w}(\boldsymbol{\chi}, t), \tag{4}$$

are the Eulerian velocity and displacement variation mapped back into the Lagrangian label framework. Manipulation of (4) and use of the chain rule yields the variation of the velocity to be

$$\delta\boldsymbol{u} = \mathbf{w}_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\mathbf{w} - (\mathbf{w} \cdot \boldsymbol{\nabla})\boldsymbol{u} \tag{5}$$

Density D obeys the conservation law

$$D_t + \nabla \cdot (\boldsymbol{u}D) = 0, \tag{6}$$

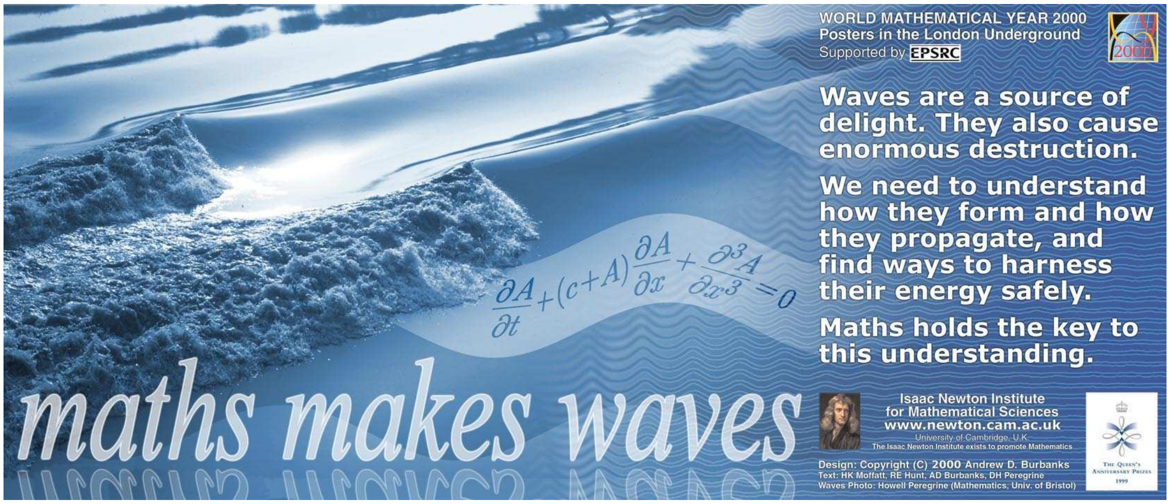


Figure 3: This "maths makes waves" poster features an undular bore on the Severn River; the competition between dispersion and nonlinearity is clearly seen. Part of the wave is broken, in the shallower water near the shore, and part is not. Photograph: Howell Peregrine. Poster design copyright: Andrew Burbanks.

and by analogy its variation relates to the displacement flow $\mathbf{w}$

$$\delta D + \nabla \cdot (\mathbf{w} D) = 0. \tag{7}$$

The kinematic condition at the free surface $z - h - b = 0$ is

$$h_t + (\boldsymbol{v}_s \cdot \boldsymbol{\nabla})(h+b) - w_s = 0 \tag{8}$$

with the suffix indicating evaluation at the free surface, e.g., $\boldsymbol{u}_s = \boldsymbol{u}(x, y, z = h+b, t)$; and, $\boldsymbol{v} = (u, v, 0)^T$. Likewise we find the variation

$$\delta h + (\mathbf{w}_s \cdot \boldsymbol{\nabla})(h+b) - \mathsf{w}_s = 0 \tag{9}$$

with $\mathbf{w}_s = (\mathsf{w}_{s1}, \mathsf{w}_{s2}, 0)^T$ is the horizontal components of $\mathbf{w}$ evaluated at the surface and $\mathsf{w}_s = \mathsf{w}_3$ is the vertical component of $\mathbf{w}$ evaluated at the surface. Relations (7), (9), (5) are used to evaluate variations in (3) as follows:

$$0 = \delta \int_0^T \int_\Omega \frac{1}{2} D|\boldsymbol{u}|^2 - g\,D\,z + p\,(1-D)\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t \tag{10a}$$

$$\begin{aligned}&= \int_0^T \int_\Omega D\boldsymbol{u}\cdot\delta\boldsymbol{u} + \big(\tfrac{1}{2}|\boldsymbol{u}|^2 - g\,z - p\big)\,\delta D + (1-D)\,\delta p\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t + \\ &\qquad \int_0^T \int_{\partial\Omega_s} D_s\,B_{-s}\,\delta h_s\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\end{aligned} \tag{10b}$$

$$\begin{aligned}&= \int_\Omega D\,\boldsymbol{u}\cdot\mathbf{w}\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\Big|_0^T - \\ &\quad \int_0^T \int_\Omega \mathbf{w}\cdot\partial_t(D\boldsymbol{u}) + D\boldsymbol{u}(\mathbf{w}\cdot\boldsymbol{\nabla})\boldsymbol{u} + (D-1)\delta p + \mathbf{w}\boldsymbol{\nabla}\cdot(D\boldsymbol{u}\boldsymbol{u}) - D\mathbf{w}\cdot\boldsymbol{\nabla}B_-\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t + \\ &\qquad \int_0^T \int_{\partial\Omega_s} D_s(\hat{\mathbf{n}}_s\cdot\boldsymbol{u}_s\,\boldsymbol{u}_s\cdot\mathbf{w}_s - \hat{\mathbf{n}}_s\cdot\mathbf{w}B_-)\,\mathrm{d}s\;\mathrm{d}t + \\ &\qquad \int_0^T \int_{\partial\Omega_s} D_s\,B_{-s}\,\delta h_s - D_s\,\boldsymbol{u}_s\cdot\mathbf{w}_s\,\partial_t h\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\end{aligned} \tag{10c}$$

$$\begin{aligned}&= -\int_0^T \int_\Omega \big(\partial_t(D\boldsymbol{u}) + \boldsymbol{\nabla}\cdot(D\boldsymbol{u}\boldsymbol{u}) + \boldsymbol{\nabla}(p+gz)\big)\cdot\mathbf{w} + (D-1)\,\delta p\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t + \\ &\qquad \int_0^T \int_{\partial\Omega_s} D_s\big(B_{-s}\delta h - \mathsf{w}_s + \mathbf{w}\cdot\boldsymbol{\nabla}(h+b)\big) - \\ &\qquad\qquad D_s\,\boldsymbol{u}_s\cdot\mathbf{w}_s\,\big(\partial_t h + \boldsymbol{v}_s\cdot\boldsymbol{\nabla}(h+b) - w_s\big)\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,\end{aligned} \tag{10d}$$

where we have used the endpoint condition $\mathbf{w}|_{t=0}^{t=T} = 0$ and no normal flow conditions at solid walls. In addition, the function

$$B_- = \frac{1}{2}|\boldsymbol{u}|^2 - g\,z - p$$

was used, the outward normal $\hat{\mathbf{n}}_s$ at the free surface with surface element $\mathrm{d}s$, and $\hat{\mathbf{n}}_s \cdot \boldsymbol{u}_s \,\mathrm{d}s = (w_s - \boldsymbol{v}_s \cdot \boldsymbol{\nabla}(h+b))\,\mathrm{d}x\,\mathrm{d}y$. Condition (9) yields the vanishing of the first boundary term in (10d). The arbitrariness of the variation δp yields $D = 1$, which together with (6) yields the incompressibility condition. In turn, the arbitrariness of $\mathbf{w}$ yields via $D = 1$ the momentum equations. Finally, via the arbitrariness of $\mathbf{w}_s$ we derive the kinematic condition (8). Hence, we find

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}(p+gz) = 0 \quad \text{and} \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{11}$$

as well as kinematic condition (8). We note that the multiplier p can be interpreted as the pressure field.

In this paper, our aim is to develop reduced models which impose a specified vertical profile for the velocity $\boldsymbol{u}$, following the methodology of [Klopman et al. 2007, Ambati et al. 2008/9], whilst preserving vertical vorticity and the variational structure. In general, the above methodology does not permit the restriction of the vertical profile, unless one assumes that the horizontal components of velocity are depth-independent (columnar motion), in which case one obtains the Green-Naghdi equations [Green et al. 1974] (for a description of the above variational process restricted to columnar velocity profiles, see [Percival et al, 2008, for example]). The reason that the Green-Naghdi equations cannot be extended to more general polynomial profiles is that it is not possible to find a subgroup of the group of flow maps (diffeomorphisms) χ which satisfies (4). In this paper we pursue an alternative direction which is to start from an equivalent formulation in which we remove the implicit constraints (5), and replace them with the explicit constraint (enforced by Lagrange multipliers) that particle labels (the initial conditions for Lagrangian particles, for example) are advected by the velocity $\boldsymbol{u}$; an additional constraint that the density D satisfies (6) is also required. In [Cotter et al. 2007] it was shown that formulations of this type are equivalent to formulations of the type described above. For the case of the three-dimensional incompressible Euler equations discussed above, the variational principle becomes

$$0 = \delta\int_0^T\int_\Omega \frac{1}{2}D|\boldsymbol{u}|^2 - g\,D\,z + p\,(1-D) + \underbrace{\boldsymbol{\pi}\cdot(\boldsymbol{l}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{l})}_{\text{labels term}} + \phi\,(D_t + \nabla\cdot(\boldsymbol{u}D))\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t, \tag{12}$$

where $\boldsymbol{l}(\boldsymbol{x},t) \in \mathbb{R}^3$ are the particle labels, $\boldsymbol{\pi}(\boldsymbol{x},t) \in \mathbb{R}^3$ is the vector of Lagrange multipliers that constrain the labels to be advected by the velocity field $\boldsymbol{u}$, and ϕ is the Lagrange multiplier which constrains the density D to satisfy equation (6). The kinematic boundary condition (8) can either by explicitly enforced *via* another Lagrange multiplier, or be retained implicitly (it turns out not to be necessary to compute an equivalent formula to (9)); in this paper we choose the latter. As shown in [Cotter et al. 2007], this variational principle results in the same equations (11). We note that removing the indicated "labels term" from the action principle still results in the same equations but restricts the velocity $\boldsymbol{u}$ to potential (vorticity-free) flow $\boldsymbol{u} = \nabla\phi$: the labels are required for velocity fields with non-zero vorticity, and a total of three dynamical constraints are required for the most general flows with non-zero helicity. We actually have four constraints in (12), the three components of the labels $\boldsymbol{l} = (l_1, l_2, l_3)$ and the density equation constraint, which is more than is strictly necessary, but this keeps the formulation tidy.

In section 3, we shall see that this leads to Luke's variational principle for the case of water waves. In section 4 we shall reintroduce vertical vorticity into this formulation by replacing the labels term by a modified constraint that the horizontal components of the particle labels should be advected by the vertically-averaged velocity. This allows a model with dispersive non-hydrostatic waves and vertical vorticity.

3 Luke's variational principle as constrained formulation

Our aim is to derive water wave equations with horizontal circulation from the parent variational principle (3). To gain more insight, we first (re)derive Luke's variational principle as a constrained formulation from our parent equations. It appears to be a novel derivation; more importantly it defines the methodology we use. Luke's principle, however, is only valid for water waves under potential flow without any horizontal circulation.

We remove the labels term from the variational principle (12) to give

$$0 \;=\; \delta\int_0^T\int_\Omega \frac{1}{2}D|\boldsymbol{u}|^2 - gDz + p(1-D) + \phi\,(D_t + \nabla\cdot(\boldsymbol{u}D))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t. \tag{13}$$

We will obtain Luke's variational principle by a Legendre transform as follows:

1. use the variational principle to obtain an equation for $\boldsymbol{u}$, and then
2. eliminate $\boldsymbol{u}$ from the action; finally, take variations to obtain the water wave dynamics.

Taking (unconstrained) variations of (13) in $\boldsymbol{u}$ gives

$$\begin{aligned} 0 &= \delta \int_\Omega \frac{1}{2} D\,|\boldsymbol{u}|^2 - D\,\boldsymbol{u}\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \delta \int_{\partial\Omega_s} D_s\,\phi_s\,\partial_t h\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega D\,\delta\boldsymbol{u}\cdot(\boldsymbol{u}-\nabla\phi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \end{aligned}$$

and we obtain

$$\boldsymbol{u} = \nabla\phi.$$

Using this expression we eliminate $\boldsymbol{u}$ from the action principle (13) and use Gauss' theorem in space on the term $\phi\nabla\cdot(\boldsymbol{u}D)$ to obtain

$$0 =\delta \int_0^T \int_\Omega \frac{1}{2} D|\nabla\phi|^2 - gDz + p(1-D) + \phi\,(D_t + \nabla\cdot((\nabla\phi)D)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{14a}$$

$$=\delta \int_0^T \int_\Omega -\frac{1}{2} D|\nabla\phi|^2 - gDz + p(1-D) + \phi D_t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\partial\Omega_s} D_s\phi_s h_t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \tag{14b}$$

Making use of

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega D\phi\,\mathrm{d}V = \int_\Omega \frac{\partial}{\partial t}(D\phi)\,\mathrm{d}V + \int\int h_t D_S\phi_S\,\mathrm{d}x\,\mathrm{d}y, \tag{15}$$

variational principle (14) becomes

$$0 =\delta \int_0^T \int_\Omega \frac{1}{2} D|\nabla\phi|^2 + gDz + p(D-1) + D\phi_t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t, \tag{16}$$

provided we consider $\int_\Omega D\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ as a global constant in time. The latter integral constraint determines ϕ, which would otherwise be fixed up to a constant. Variation of (16) with respect to density D yields the Bernoulli condition

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\,z + p = 0 \tag{17}$$

throughout the fluid.

Finally, we notice that the incompressibility constraint $\nabla\cdot\boldsymbol{u} = 0$ drops out of the formulation if we eliminate the terms $p(1-D)+\phi D_t$ by substituting $D=1$ in the variational principle. Hence, we obtain Luke's variational principle

$$0 =\delta \int_0^T \int_\Omega \frac{1}{2}|\nabla\phi|^2 + gz + \phi_t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t, \tag{18}$$

for water waves [Luke 1967]. Variation of (18) yields

$$\begin{aligned} 0 = &\int_0^T \int_\Omega \nabla^2\phi\,\delta\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t + \\ &\int_0^T \int_{\partial\Omega_s} \big(\frac{1}{2}|\nabla\phi|^2 + g(h+b) + \partial_t\phi\big)_s \delta h + \big(w_s - \partial_t h - (\boldsymbol{\nabla}\phi)_s\cdot\boldsymbol{\nabla}(h+b)\big)\,(\delta\phi)_s\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \end{aligned} \tag{19}$$

These lead to the "classical" water wave equations under the assumption of potential flow:

$$\nabla^2\phi = 0 \tag{20a}$$

$$(\partial_t\phi)_s + \frac{1}{2}|\nabla\phi|_s^2 + g(h+b) = 0 \tag{20b}$$

$$\partial_t h + (\boldsymbol{\nabla}\phi)_s\cdot\boldsymbol{\nabla}(h+b) - (\partial_z\phi)_s = 0. \tag{20c}$$

Luke's variational principle can be rewritten in another form by noting that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega \phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_\Omega \partial_t\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int\int \phi_s h_t\,\mathrm{d}x\,\mathrm{d}y,$$

leading to Miles' variational principle [Miles 1985]:

$$0 = \delta \int_0^T \int_\Omega \frac{1}{2}|\nabla\phi|^2 + gz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \int\int \phi_s h_t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

4 Constrained variational principle: water waves with vorticity

4.1 Variational principles

In this extension, we introduce horizontal labels $\boldsymbol{l}(x,y,t) \in \mathbb{R}^2$ which are from the onset constrained to depend on the horizontal coordinates only, in contrast to the general case considered in §2. These labels are in essence advected by a depth and density weighted horizontal velocity, that is

$$\frac{1}{h}\int_b^{h+b} D\,\mathrm{d}z\boldsymbol{l}_t + \frac{1}{h}\int_b^{h+b} D\,\boldsymbol{u}\,\mathrm{d}z \cdot \boldsymbol{\nabla}\boldsymbol{l} = 0. \tag{21}$$

After multiplication by depth h, these advected constrained labels are enforced by additional Lagrange multipliers $\boldsymbol{\pi} = \boldsymbol{\pi}(x,y,t)$, which are also independent of z. Again, we are already anticipating that D will be set to unity later. Hence, the variational principle, either (12) or (13), is changed further to

$$0 =\delta \int_0^T \int_\Omega \frac{1}{2}D|\boldsymbol{u}|^2 - gDz + p(1-D) + \phi\left(D_t + \nabla\cdot(\boldsymbol{u}D)\right) - D\,\boldsymbol{\pi}\,\cdot\big(\partial_t\boldsymbol{l} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{l}\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t. \tag{22}$$

A similar calculation as before for the $\boldsymbol{u}$-variation yields

$$\boldsymbol{u} \equiv \nabla\phi + \mathbf{v} \equiv \nabla\phi + (\boldsymbol{\nabla}\boldsymbol{l})^T\boldsymbol{\pi},$$

adopting the notation

$$\big[(\nabla\boldsymbol{l})^T\boldsymbol{\pi}\big]_{ij} = \frac{\partial l_j}{\partial x_i}\pi_j,$$

which we substitute into action principle (22) to get:

$$\begin{aligned} 0 =&\delta \int_0^T \int_\Omega \frac{1}{2}D|\nabla\phi + \mathbf{v}|^2 - gDz + p(1-D) + \phi\big(D_t + \nabla\cdot(D(\nabla\phi+\mathbf{v}))\big) - \\ &\qquad D\,\boldsymbol{\pi}\cdot(\boldsymbol{l}_t + (\boldsymbol{\nabla}\phi + \mathbf{v})\cdot\boldsymbol{\nabla}\boldsymbol{l})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t \quad \text{(23a)} \\ =& -\delta\int_0^T \int_\Omega \frac{1}{2}D|\nabla\phi+\mathbf{v}|^2 + gDz + p(D-1) + D\,\boldsymbol{\pi}\cdot\partial_t\boldsymbol{l} - \phi D_t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int D_s\phi_S h_t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \quad \text{(23b)} \end{aligned}$$

Making use of (15) and the corresponding integral constraint, and then changing the sign of the variational principle yields

$$0 =\delta \int_0^T \int_\Omega \frac{1}{2}D|\nabla\phi + \mathrm{v}|^2 + gDz + p(D-1) + D\,\boldsymbol{\pi}\cdot\partial_t\boldsymbol{l} + D\partial_t\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t. \tag{24}$$

Variation with respect to D provides an extended Bernoulli's relation

$$\partial_t\phi + \frac{1}{2}|\nabla\phi + \mathbf{v}|^2 + gz + p + \boldsymbol{\pi}\cdot\partial_t\boldsymbol{l} = 0. \tag{25}$$

Finally, we include incompressibility by setting $D=1$ in (24) to obtain the extended variational principle

$$0 =\delta \int_0^T \int_\Omega \frac{1}{2}|\nabla\phi + \mathbf{v}|^2 + \partial_t\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t + \delta\int_0^T\int_{\partial\Omega_s} h\,\boldsymbol{\pi}\cdot\partial_t\boldsymbol{l} + \frac{1}{2}g\big((h+b)^2 - b^2\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \tag{26}$$

Variation of (26) gives

$$
\begin{aligned}
0 =& \int_0^T \int_\Omega (\nabla\phi + \mathbf{v})\cdot(\nabla\delta\phi + \delta\mathbf{v}) + \partial_t\delta\phi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \\
& \int_0^T \int_{\partial\Omega_s} \Big(\partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi + \mathbf{v}|^2 + g\,(h+b) + \boldsymbol{\pi}\cdot\partial_t\boldsymbol{l}\Big)_s \delta h + h\,\delta\boldsymbol{\pi}\cdot\partial_t\boldsymbol{l} + h\,\boldsymbol{\pi}\cdot\partial_t\delta\boldsymbol{l}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \qquad (27a)\\
=& -\int_0^T \int_\Omega (\nabla^2\phi + \boldsymbol{\nabla}\cdot\mathbf{v})\delta\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_\Omega \delta\phi|_{t=0}^{t=T}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \\
& \int_0^T \int_{\partial\Omega_s} \Big(\partial_t\phi + \frac{1}{2}|\nabla\phi + \mathbf{v}|^2 + g\,(h+b) + \boldsymbol{\pi}\cdot\partial_t\boldsymbol{l}\Big)_s \delta h + h\,\delta\boldsymbol{\pi}\cdot\partial_t\boldsymbol{l} + h\,\boldsymbol{\pi}\cdot\partial_t\delta\boldsymbol{l} + \\
& \qquad (\delta\phi)_s(\boldsymbol{u}_s\cdot(-\boldsymbol{\nabla}_H(h+b),1)^T - \partial_t h\Big) + h\,\bar{\boldsymbol{u}}\cdot\delta\mathbf{v}\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \qquad (27b)\\
=& -\int_0^T \int_\Omega (\nabla^2\phi + \boldsymbol{\nabla}\cdot\mathbf{v})\delta\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \\
& \int_0^T \int_{\partial\Omega_s} \Big(\partial_t\phi + \frac{1}{2}|\nabla\phi + \mathbf{v}|^2 + g\,(h+b) + \boldsymbol{\pi}\cdot\partial_t\boldsymbol{l}\Big)_s \delta h + \\
& \qquad h\,\delta\boldsymbol{\pi}\cdot\big(\partial_t\boldsymbol{l} + \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{l}\big) - \big(\partial_t(h\,\boldsymbol{\pi}) + \boldsymbol{\nabla}\cdot(h\bar{\boldsymbol{u}}\boldsymbol{\pi})\big)\cdot\delta\boldsymbol{l} + \\
& \qquad \big(\delta\phi\big)_s(w_s - \boldsymbol{v}_s\cdot\boldsymbol{\nabla}(h+b) - \partial_t h\big)\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int_{\Omega_s} h\boldsymbol{\pi}\cdot\delta\boldsymbol{l}|_{t=0}^{t=T}\;\mathrm{d}x\,\mathrm{d}y, \qquad (27c)
\end{aligned}
$$

where

$$
\mathbf{v} = (\nabla\boldsymbol{l})^T\boldsymbol{\pi} \qquad \text{and} \qquad h\bar{\boldsymbol{u}} = \int_b^{b+h} \boldsymbol{u}_H\,\mathrm{d}z. \qquad (28)
$$

End-point conditions used are $\delta\boldsymbol{l}|_{t=0,T} = 0$, whence the last term in (27) vanishes. The dynamics arising from (27) are therefore

$$
\begin{aligned}
\nabla^2\phi + \boldsymbol{\nabla}\cdot\mathbf{v} =& 0 \qquad (29a)\\
(\partial_t\phi)_s + \frac{1}{2}|\nabla\phi + \mathbf{v}|_s^2 + g\,(h+b) + \boldsymbol{\pi}\cdot\partial_t\boldsymbol{l} =& 0 \qquad (29b)\\
\partial_t\boldsymbol{l} + \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{l} =& 0 \qquad (29c)\\
w_s - \boldsymbol{v}_s\cdot\boldsymbol{\nabla}(h+b) - \partial_t h =& 0 \qquad (29d)\\
\partial_t(h\,\boldsymbol{\pi}) + \boldsymbol{\nabla}\cdot(h\bar{\boldsymbol{u}}\boldsymbol{\pi}) =& 0 \qquad (29e)
\end{aligned}
$$

together with (28). The above system can be expressed in terms of $\mathbf{v}$ and $\bar{\boldsymbol{u}}$, instead of $\boldsymbol{\pi}$ and $\boldsymbol{l}$, by multiplication of (29c) by $\boldsymbol{\pi}$ and elimination of $\boldsymbol{\pi}\cdot\partial_t\boldsymbol{l}$, and by evaluation of $\partial_t(h\bar{\mathbf{v}})$. In addition, we can rewrite (29d). The reformulated system becomes more familiar and reads

$$
\begin{aligned}
\nabla^2\phi + \boldsymbol{\nabla}\cdot\mathbf{v} =& 0 \qquad (30a)\\
(\partial_t\phi)_s + \frac{1}{2}|(\nabla\phi)_s + \mathbf{v}|^2 + g\,(h+b) - \mathbf{v}\cdot\bar{\boldsymbol{u}} =& 0 \qquad (30b)\\
\partial_t h + \boldsymbol{\nabla}\cdot(h\,\bar{\boldsymbol{u}}) =& 0 \qquad (30c)\\
\partial_t(h\,\mathbf{v}) + \boldsymbol{\nabla}\cdot(h\bar{\boldsymbol{u}}\mathbf{v}) + h\mathbf{v}\boldsymbol{\nabla}\bar{\boldsymbol{u}} =& 0 \qquad (30d)\\
h\bar{\boldsymbol{u}} =& \int_b^{b+h} \boldsymbol{\nabla}_H\phi\,\mathrm{d}z + h\mathbf{v}. \qquad (30e)
\end{aligned}
$$

It is seen to be a modification of the water wave equations under the potential flow condition; it includes a component of the horizontal momentum $h\mathbf{v}$ which has a layer-averaged vertical component of the vorticity.

The extended variational principle can be transformed partially back towards the original one for the parent Euler equations. We obtain:

$$
0 = \delta\int_0^T \int_\Omega \frac{1}{2}|\nabla\phi + \mathbf{v}|^2 + \partial_t\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t + \delta\int_0^T \int_{\partial\Omega_s} -\frac{1}{2}\,h\,\bar{\boldsymbol{u}}\cdot\mathbf{v} + \frac{1}{2}\,g\big((h+b)^2 - b^2\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \qquad (31)
$$

in which $\mathbf{v}$ is a shorthand notation for $h\mathbf{v} = h\bar{\boldsymbol{u}} - \int_b^{b+h}\boldsymbol{\nabla}_H\phi\,\mathrm{d}z$. We subtracted the constraint on the horizontal labels, which implies the use of the following variations for the depth and depth-averaged velocity:

$$
\delta h + \boldsymbol{\nabla}\cdot(h\,\bar{\mathbf{w}}) = 0 \quad \text{and} \quad \delta\bar{\boldsymbol{u}} = \bar{\mathbf{w}}_t + (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\bar{\mathbf{w}} - (\bar{\mathbf{w}}\cdot\boldsymbol{\nabla})\bar{\boldsymbol{u}}. \qquad (32)
$$

Also, $\boldsymbol{u} = \boldsymbol{\nabla}\phi + \mathbf{v}$. Variation of (31) yields

$$
\begin{aligned}
0 = &-\int_0^T \int_\Omega (\nabla^2\phi + \boldsymbol{\nabla}\cdot\mathbf{v})\delta\phi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \\
&\int_0^T \int_{\partial\Omega_s} \big(\partial_t\phi + \frac{1}{2}|\boldsymbol{u}|^2 + g\,(h+b) - \bar{\boldsymbol{u}}\cdot\mathbf{v}\big)_s \delta h - h\mathbf{v}\cdot\delta\bar{\boldsymbol{u}} + \\
&\qquad \big(\delta\phi\big)_s (w_s - \boldsymbol{v}_s\cdot\boldsymbol{\nabla}(h+b) - \partial_t h\big)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\text{(33a)}\\
= &-\int_0^T \int_\Omega (\nabla^2\phi + \boldsymbol{\nabla}\cdot\mathbf{v})\delta\phi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \\
&\int_0^T \int_{\partial\Omega_s} \bar{\mathbf{w}}\cdot\Big(h\,\boldsymbol{\nabla}_H\big((\partial_t\phi)_s + \frac{1}{2}|\boldsymbol{u}_s|^2 + g\,(h+b) - \bar{\boldsymbol{u}}\cdot\mathbf{v}\big) + \partial_t(h\mathbf{v}) + \boldsymbol{\nabla}\cdot(h\bar{\boldsymbol{u}}\mathbf{v}) + h\mathbf{v}\boldsymbol{\nabla}\bar{\boldsymbol{u}}\Big) - \\
&\qquad \big(\delta\phi\big)_s(\partial_t h + \boldsymbol{\nabla}\cdot(h\,\bar{\boldsymbol{u}})\big)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\text{(33b)}
\end{aligned}
$$

where we used $\hat{\mathbf{n}}\cdot\bar{\mathbf{w}} = \hat{\mathbf{n}}\cdot\bar{\boldsymbol{u}} = 0$ and $\bar{\mathbf{w}}|_{t=0,T} = 0$. Given the arbitrariness of the variations $\delta\phi, (\delta\phi)_s$ and $\bar{\mathbf{w}}$, the resulting equations are

$$\nabla^2\phi + \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \tag{34a}$$

$$\partial_t h + \boldsymbol{\nabla}\cdot(h\,\bar{\boldsymbol{u}}) = 0 \tag{34b}$$

$$h\boldsymbol{\nabla}_H\big((\partial_t\phi)_s + \frac{1}{2}|\boldsymbol{u}_s|^2 + g\,(h+b) - \mathbf{v}\cdot\bar{\boldsymbol{u}}\big) + \partial_t(h\,\mathbf{v}) + \boldsymbol{\nabla}\cdot(h\bar{\boldsymbol{u}}\mathbf{v}) + h\mathbf{v}\boldsymbol{\nabla}\bar{\boldsymbol{u}} = 0 \tag{34c}$$

$$h\mathbf{v} = h\bar{\boldsymbol{u}} - \int_b^{b+h} \boldsymbol{\nabla}_H\phi\,\mathrm{d}z. \tag{34d}$$

We can rewrite (34c) with the aid of (34b) to

$$\partial_t\mathbf{v} + \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\boldsymbol{\nabla}\bar{\boldsymbol{u}} + \boldsymbol{\nabla}_H\big((\partial_t\phi)_s + \frac{1}{2}|\boldsymbol{u}_s|^2 + g\,(h+b) - \mathbf{v}\cdot\bar{\boldsymbol{u}}\big) = 0. \tag{35}$$

Using (34d) and again (34b), we find

$$
\begin{aligned}
&\partial_t(h\bar{\boldsymbol{u}}) + \boldsymbol{\nabla}\cdot(h\,\bar{\boldsymbol{u}}\,\bar{\boldsymbol{u}} + \frac{1}{2}\,g\,h^2) + g\,h\boldsymbol{\nabla}b + \\
&\underline{h\boldsymbol{\nabla}_H(\partial_t\phi)_s - h\partial_t\big(\frac{1}{h}\int_b^{b+h}\boldsymbol{\nabla}_H\phi\,\mathrm{d}z\big) + \frac{1}{2}\,h\boldsymbol{\nabla}_H(|\boldsymbol{u}_s|^2 - |\bar{\boldsymbol{u}}|^2) + h\,(\bar{\boldsymbol{u}}\,\boldsymbol{\nabla}\overline{\boldsymbol{\nabla}_H\phi} - \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\overline{\boldsymbol{\nabla}_H\phi})} = 0
\end{aligned}
\tag{36}
$$

with $\overline{\boldsymbol{\nabla}_H\phi} \equiv (1/h)\int_b^{b+h}\boldsymbol{\nabla}_H\phi\ \mathrm{d}z$. The last two terms concern a rotational component, which is nonzero due to the averaging. The underline indicates the terms which are extra compared to the shallow-water equations.

4.2 Conservation laws

As described in [Cotter et al. 2007], variational principles of the form (26) have relabelling symmetries of the type:

$$\delta\boldsymbol{l} = \boldsymbol{\xi}(\boldsymbol{l}), \quad \delta\boldsymbol{\pi} = -(\nabla\boldsymbol{\xi}(\boldsymbol{l}))^T\boldsymbol{\pi},$$

for any vector field $\boldsymbol{\xi}$, which gives rise to the conservation law

$$\frac{\partial}{\partial t}(h\boldsymbol{\pi}\cdot\boldsymbol{\xi}(\boldsymbol{l})) + \nabla\cdot(\bar{\boldsymbol{u}}\boldsymbol{\pi}\cdot\boldsymbol{\xi}(\boldsymbol{l})h) = 0.$$

When combined with the continuity equation (30c), we obtain

$$\big(\frac{\partial}{\partial t} + \bar{\boldsymbol{u}}\cdot\nabla\big)\,(\boldsymbol{\pi}\cdot\boldsymbol{\xi}(\boldsymbol{l})) = 0.$$

Following [Cotter et al. 2007], if we choose a closed loop $C(t)$ advected in the horizontal by $\bar{\boldsymbol{u}}$, and take $\boldsymbol{\xi}$ to be tangent to the loop at $t = 0$ with $|\boldsymbol{\xi}| = 1$, then

$$\boldsymbol{\xi}\cdot\mathrm{d}\boldsymbol{x} = (\nabla\boldsymbol{l})\cdot\mathrm{d}\boldsymbol{x}$$

for all times. Hence, we obtain the Kelvin theorem

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\oint_{C(t)}\big((\boldsymbol{\pi}\cdot\nabla)\boldsymbol{l}\big)\cdot\mathrm{d}\boldsymbol{x}=0.$$

Recalling that $\mathbf{v}=\boldsymbol{\pi}\cdot\nabla\boldsymbol{l}$, Stokes' theorem gives

$$\left(\frac{\mathrm{d}}{\mathrm{d}\,t}+\bar{\boldsymbol{u}}\cdot\nabla\right)\nabla\times\mathbf{v}=0,$$

which is the horizontal potential vorticity equation for this system. Thus we have a horizontal vorticity which is advected by the vertically-averaged velocity inside the fluid, coupled to a vector potential which keeps the flow incompressible and interacts with the free surface. The variational principle is also invariant under translations in time, and so it has a locally-conserved energy $|\boldsymbol{u}|^2/2+gz$. Similarly, the variational principle (for the case of flat topography with constant b) is invariant under translations in space, leading to the locally-conserved momentum $\boldsymbol{u}$.

4.3 Water wave and shallow water limit

The system (30) reduces to the water wave equations under potential flow (20) when we either take $\mathbf{v}=0$ or $\mathbf{v}=\boldsymbol{\nabla}_H\varphi$. In the latter case, the equation for the time evolution of $\mathbf{v}$ can be written as the horizontal gradient of a Bernoulli-type function. This then recombines with the dynamic condition at the free surface such that the total potential becomes $\phi+\varphi$. Replacement of $\phi+\varphi$ by ϕ notationally then yields the desired result, and similarly for Laplace's equation.

The shallow water limit is reached when we restrict ϕ to ϕ_s in the variational principle (26). The result is:

$$\partial_t\phi_s+\frac{1}{2}|\bar{\boldsymbol{u}}|^2+g\,(h+b)+\boldsymbol{\pi}\cdot\partial_t\boldsymbol{l}=0 \tag{37a}$$

$$\partial_t\boldsymbol{l}+\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{l}=0 \tag{37b}$$

$$-\boldsymbol{\nabla}_H\cdot(h\,\bar{\boldsymbol{u}})-\partial_t h=0 \tag{37c}$$

$$\partial_t(h\,\boldsymbol{\pi})+\boldsymbol{\nabla}\cdot(h\bar{\boldsymbol{u}}\boldsymbol{\pi})=0. \tag{37d}$$

This shallow water system can be rewritten as

$$\partial_t h+\boldsymbol{\nabla}\cdot(h\,\bar{\boldsymbol{u}})=0 \tag{38a}$$

$$\partial_t\phi_s+\frac{1}{2}|\bar{\boldsymbol{u}}|^2+g\,(h+b)-\mathbf{v}\cdot\bar{\boldsymbol{u}}=0 \tag{38b}$$

$$\partial_t\mathbf{v}+\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\mathbf{v}+\mathbf{v}\boldsymbol{\nabla}\bar{\boldsymbol{u}}=0 \tag{38c}$$

$$\bar{\boldsymbol{u}}=\boldsymbol{\nabla}_H\phi_s+\mathbf{v}. \tag{38d}$$

The last three equations can be reduced to the regular shallow water momentum equation

$$\partial_t\bar{\boldsymbol{u}}+\bar{\boldsymbol{u}}^{\perp}\,\zeta+\boldsymbol{\nabla}\big(g\,(h+b)+\frac{1}{2}|\bar{\boldsymbol{u}}|^2\big)=0 \tag{39}$$

with vertical component of vorticity $\zeta=\partial_x\bar{u}_2-\partial_y\bar{u}_1$ and $\bar{\boldsymbol{u}}^{\perp}=(-\bar{u}_2,\bar{u}_1)^T$. Alternatively, note that the last (underlined) terms in (36) vanish in the shallow water limit, when $\boldsymbol{u}$ and ϕ evaluated at the surface equal their depth-averaged counterparts, whence (36) becomes the regular shallow water momentum equation in nearly conservative form.

4.4 Weak solutions with a single-valued free surface

A weak formulation is sought such that overturning of the free surface is approximated by a local discontinuity. In the shallow water limit the resulting hydraulic jump or bore relations across the discontinuity should reduce to those for the shallow water equations. Consequently, the depth h and the depth-averaged velocity will be single-valued except for local line discontinuities in the horizontal. In addition, a new boundary condition for ϕ is required for the elliptical equation (30a) along the jump. A sketch of a domain with an overturning wave and the approximation with one jump is given in Fig. 4.

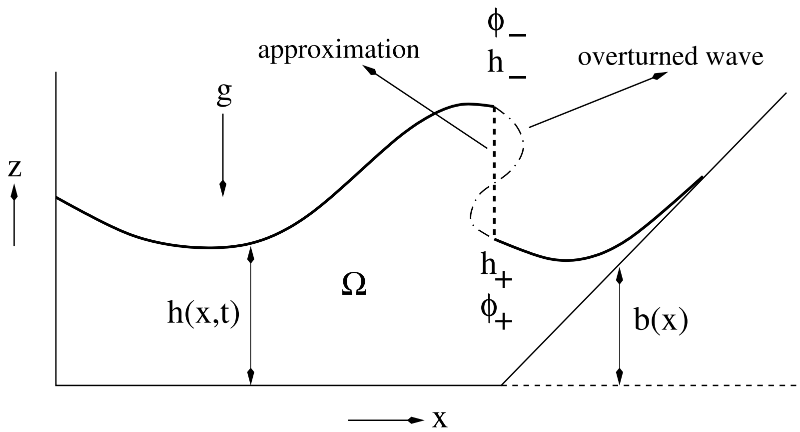


Figure 4: A sketch is given of a vertical cross section of the domain with an overturned wave and its bore approximation.

The candidate equations for the weak formulation are (30c) and (36), i.e.:

$$\partial_t h + \boldsymbol{\nabla} \cdot (h\, \bar{\boldsymbol{u}}) = 0 \tag{40a}$$

$$\partial_t(h\bar{\boldsymbol{u}}) + \boldsymbol{\nabla}_H \cdot (h\, \bar{\boldsymbol{u}}\, \bar{\boldsymbol{u}} + \frac{1}{2}\, g\, h^2) + g\, h \boldsymbol{\nabla} b + \underline{h \boldsymbol{\nabla}_H (\partial_t \phi)_s - h \partial_t \big(\frac{1}{h} \int_b^{b+h} \boldsymbol{\nabla}_H \phi \, \mathrm{d}z\big) +}$$
$$\underline{\frac{1}{2}\, h \boldsymbol{\nabla}_H (|\boldsymbol{u}_s|^2 - |\bar{\boldsymbol{u}}|^2) + h\, (\bar{\boldsymbol{u}}\, \boldsymbol{\nabla} \overline{\boldsymbol{\nabla}_H \phi} - \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \overline{\boldsymbol{\nabla}_H \phi})} = 0. \tag{40b}$$

There is a theory for hyperbolic systems with nonconservative products [Dal Maso 1995] and numerical applications thereof [Parés 2006, Rhebergen et al. 2008]. The term $h\boldsymbol{\nabla} b$ is an example of an intrinsic nonconservative product, also when we consider the shallow water limit. It is intrinsic because the term cannot be brought in conservative form without destroying the conservative structure in the remaining shallow water momentum equations. The underlined terms are also nonconservative terms but the theory of Dal Maso [Dal Maso 1995] does not directly apply, because our extended water wave system is a coupled elliptic-hyperbolic system. The theory of hyperbolic systems with nonconservative products shows that the extended Rankine-Hugonot or shock relations depend on the path chosen in the calculation. Besides the strict nonapplicability of this theory for our coupled elliptic-hyperbolic system, the choice of path is open. In many cases, Rhebergen et al. [Rhebergen et al. 2008] show numerically that a linear path suffices as a reasonable choice.

Without a derivation of an extended theory for elliptic-hyperbolic systems and non-conservative products, two approximations may be reasonable. First, the additional underlined terms in (40b) are ignored at leading order, and the jump relations for the shallow water equations can be used as a crude approximation. Second, this theory of nonconservative products can be applied in a heuristic manner while using a linear path. In simplified terms, generalized Rankine-Hugoniot relations emerge by consideration of (40) in space time. Upon using Gauss' theorem in space time, the weak formulation across the shock surface in space time amounts to [Rhebergen et al. 2008]:

$$-S_n \int_0^1 \partial_\tau \tilde{h}(\tau; U^-, U^+)\, \mathrm{d}\tau + \int_0^1 \partial_\tau \widetilde{(h\bar{\boldsymbol{u}})}(\tau; U^-, U^+) \cdot \hat{\mathbf{n}}\, \mathrm{d}\tau \tag{41}$$

for the continuity equation where we use a vector $(-S_n, n_x, n_y)^T$ parallel to the space time normal vector, space normal $\hat{\mathbf{n}} = (n_x, n_y)^T$, and the shock speed S_n. The variables with a tilde depend on the states of the variables, denoted by U^-, U^+, across the shock and the path τ, with $U = (h, h\bar{\boldsymbol{u}}, b, \phi)$. Furthermore, $\tilde{h}(0; U^-, U^+) = h^-$ and $\tilde{h}(1; U^-, U^+) = h^+$, and so forth. However, due to its conservative form expression (41) is immediately integrable and leads to the classic jump relation, expressing mass conservation across the shock:

$$[h\, (\bar{\boldsymbol{u}} \cdot \hat{\mathbf{n}} - S_n)] = 0 \tag{42}$$

with $[U]=U^+-U^-$ denoting the jump across. Similarly for the momentum equations we find:

$$
\begin{aligned}
&- S_n\int_0^1 \partial_\tau(\widetilde{h\bar{\boldsymbol{u}}})(\tau;U^-,U^+)\;\mathrm{d}\tau + \int_0^1 \partial_\tau(\widetilde{h\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}+\frac{g}{h}^2})(\tau;U^-,U^+)\,\hat{\mathbf{n}}\;\mathrm{d}\tau +\\
&\int_0^1 \tilde{h}(\tau;U^-,U^+)\,\partial_\tau\tilde{b}(\tau;U^-,U^+)\,\hat{\mathbf{n}}\,\mathrm{d}\tau + S_n\int_0^1 \tilde{h}(\tau;U^-,U^+)\,\partial_\tau\big(\widetilde{\frac{1}{h}\int_b^{b+h}\boldsymbol{\nabla}_H\phi\,\mathrm{d}z}\big)(\tau;U^-,U^+)\;\mathrm{d}\tau +\\
&\int_0^1 \tilde{h}(\tau;U^-,U^+)\,\partial_\tau\widetilde{(\partial_t\phi)_s}(\tau;U^-,U^+)\,\hat{\mathbf{n}}\;\mathrm{d}\tau + \frac{1}{2}\int_0^1 \tilde{h}(\tau;U^-,U^+)\,\partial_\tau(\widetilde{|\boldsymbol{u}_s|^2-|\bar{\boldsymbol{u}}|^2})(\tau;U^-,U^+)\,\hat{\mathbf{n}}\;\mathrm{d}\tau +\\
&\qquad\int_0^1 \big(\widetilde{h\,\bar{\boldsymbol{u}}}(\tau;U^-,U^+)\,\partial_\tau\widetilde{\overline{\boldsymbol{\nabla}_H\phi}}(\tau;U^-,U^+) - \widetilde{h\,\bar{\boldsymbol{u}}}(\tau;U^-,U^+)\cdot\partial_\tau\widetilde{\overline{\boldsymbol{\nabla}_H\phi}}\big)\hat{\mathbf{n}}\;\mathrm{d}\tau \qquad (43a)\\
&= [h\bar{\boldsymbol{u}}(\bar{\boldsymbol{u}}\cdot\hat{\mathbf{n}}-S_n)] + [\frac{1}{2}\,g\,h^2]\hat{\mathbf{n}} +\\
&\int_0^1 \tilde{h}(\tau;U^-,U^+)\,\partial_\tau\tilde{b}(\tau;U^-,U^+)\,\hat{\mathbf{n}}\,\mathrm{d}\tau + S_n\int_0^1 \tilde{h}(\tau;U^-,U^+)\,\partial_\tau\big(\widetilde{\frac{1}{h}\int_b^{b+h}\boldsymbol{\nabla}_H\phi\,\mathrm{d}z}\big)(\tau;U^-,U^+)\;\mathrm{d}\tau +\\
&\int_0^1 \tilde{h}(\tau;U^-,U^+)\,\partial_\tau\widetilde{(\partial_t\phi)_s}(\tau;U^-,U^+)\,\hat{\mathbf{n}}\;\mathrm{d}\tau + \frac{1}{2}\int_0^1 \tilde{h}(\tau;U^-,U^+)\,\partial_\tau(\widetilde{|\boldsymbol{u}_s|^2-|\bar{\boldsymbol{u}}|^2})(\tau;U^-,U^+)\,\hat{\mathbf{n}}\;\mathrm{d}\tau +\\
&\quad\int_0^1 \big(\widetilde{h\,\bar{\boldsymbol{u}}}(\tau;U^-,U^+)\,\partial_\tau\widetilde{\overline{\boldsymbol{\nabla}_H\phi}}(\tau;U^-,U^+) - \widetilde{h\,\bar{\boldsymbol{u}}}(\tau;U^-,U^+)\cdot\partial_\tau\widetilde{\overline{\boldsymbol{\nabla}_H\phi}}(\tau;U^-,U^+)\big)\hat{\mathbf{n}}\;\mathrm{d}\tau = 0. \qquad (43b)
\end{aligned}
$$

The first terms are conservative and integrable for any path function, and lead to the well-known jump condition for the shallow water momentum equations, while we need to choose or determine a path for the remaining terms. We choose a linear path $U=U^-+\tau\,(U^+-U^-)$ based on the generally weak path dependence observed in [Rhebergen et al. 2008]. Hence, we obtain the approximated jump relation for our extended system:

$$
\begin{aligned}
&[h\bar{\boldsymbol{u}}(\bar{\boldsymbol{u}}\cdot\hat{\mathbf{n}}-S_n)] +\\
&[\frac{1}{2}\,g\,h^2+\{h\}b]\,\hat{\mathbf{n}} + \Big[S_n\int_b^{b+h}\boldsymbol{\nabla}_H\phi\,\mathrm{d}z + h\,(\partial_t\phi)_s + \frac{1}{2}\,h\,(|\boldsymbol{u}_s|^2-|\bar{\boldsymbol{u}}|^2) + h\,\big(\bar{\boldsymbol{u}}\overline{\boldsymbol{\nabla}_H\phi} - \bar{\boldsymbol{u}}\cdot\overline{\boldsymbol{\nabla}_H\phi}\big)\Big]\hat{\mathbf{n}}\\
&\qquad - \int_0^1\Big\{S_n\,[h]\,\big(\widetilde{\frac{1}{h}\int_b^{b+h}\boldsymbol{\nabla}_H\phi\,\mathrm{d}z}\big)(\tau;U^-,U^+) + [h]\,\widetilde{(\partial_t\phi)_s}(\tau;U^-,U^+) +\\
&\frac{1}{2}\,[h](\widetilde{|\boldsymbol{u}_s|^2-|\bar{\boldsymbol{u}}|^2})(\tau;U^-,U^+) + \big([h\,\bar{\boldsymbol{u}}]\widetilde{\overline{\boldsymbol{\nabla}_H\phi}}(\tau;U^-,U^+) - [h\,\bar{\boldsymbol{u}}]\cdot\widetilde{\overline{\boldsymbol{\nabla}_H\phi}}(\tau;U^-,U^+)\big)\Big\}\hat{\mathbf{n}}\;\mathrm{d}\tau = 0 \qquad (44)
\end{aligned}
$$

with mean $\{U\}=(U^-+U^+)/2$ and $[\{U\}b]=\{U\}[b]$. The last term with $|\bar{\boldsymbol{u}}|^2$ can be integrated directly, but the others need to be determined numerically or approximated further, e.g., by using a linear path in τ for $\boldsymbol{u}_s, (\partial_t\phi)_s,\ (1/h)\int_b^{b+h}\boldsymbol{\nabla}_H\phi\,\mathrm{d}z$ et cetera. When we approximate $[h]=[H]$ for some mean constant depth H when the topography is flat, then the last terms disappear; the momentum equations are conservative in this (inconsistent) small amplitude limit; it merely illustrates that the remaining terms are as expected.

Once the domain has been approximated an extra boundary condition is required to solve the elliptic equation. The normal velocity $\boldsymbol{u}\cdot\hat{\mathbf{n}}$, a component in the horizontal direction, then equals the horizontal shock speed $S_n=S_n(x,y,t)$, which is valid in the vertical plane of the shock and independent of z, see Fig. 2. Finally, we re-emphasize that these results are heuristic and may be based on severe approximations.

4.5 Linear waves

Linearization of (30) around a state of rest and over a flat bottom yields $\mathbf{v}=0$. Hence, the dispersion relation for the classical water wave equations holds again for our extended water wave equations. For linear flows over non-uniform topography the dispersion relation will be an extension of both the water wave results under potential flow and the shallow water equations.

5 Summary and discussion

The aims laid out in the introduction have been reached nearly completely. We have derived a reduced model with accurate surface water wave dispersion and depth-averaged horizontal circulation. It is derived systematically via simplification of the variational principle of the three-dimensional incompressible Euler equations with free surface dynamics. Associated conservation laws follow from the symmetries of our new variational principle. Mass, momentum, energy and material conservation of depth-averaged potential vorticity is thus guaranteed. The model is reduced, or simplified, because it contains only the depth-averaged horizontal circulation combined with the full three-dimensional velocity potential. On the one hand our new model reduces to the classical depth-averaged shallow water equations in the shallow water limit, and on the other hand it reduces to the classical potential flow water wave equations in the vorticity-free limit. Breaking waves are included in our model by extension of the classical bore relations in the usual depth-averaged shallow water equations. The multivalued free surface of an overturning wave is therefore limited locally to a vertical face. The advanced jump relations are based on the continuity equation and an analysis of the depth-averaged part of the momentum equations. It is an heuristic extension of the theory of shocks in hyperbolic equations with nonconservative products. The new shock relations clearly reduce to the classical bore relations for depth-averaged shallow water equations in the shallow water limit. Further analysis needs to sort out what the associated loss of energy and possibly some momentum is. For flow over a flat bottom, linear waves in our new model are completely the same as those for the potential flow water wave equations. Wave dispersion is therefore completely accurate.

It is important to stress that the preservation of the geometric structure in an Eulerian model is nontrivial. Our usage of the combination potential flow plus the depth-averaged vortical velocity component is special because it allows the preservation of the relabelling symmetry with its associated potential vorticity conservation material or integral laws [Cotter et al. 2007]. The preservation of the geometric (Hamiltonian) structure in Eulerian variables for higher order discretizations or (polynomials) truncations of the vortical component of the flow in the vertical will be nontrivial. The latter remains an outstanding research question. It goes without saying that conservation of energy and perhaps momentum is locally broken through a hydraulic jump or bore.

Two issues we aimed at have not yet been resolved. First, the new model is still quite complex as it contains the full three-dimensional potential flow component. The derivation of our new model with this dependence was necessary to pinpoint the coupling between the potential flow part and the vortical part. The further approximation of the velocity potential by a quadratic polynomial in the vertical z-direction and subsequent substitution thereof in the variational principle immediately leads to a Boussinesq-type model, with high order accurate dispersion, and horizontal circulation. Any such substitution and evaluation within the variational principle is permissible and straightforward; it contains no intrinsic challenges. In essence, it constitutes a discretization in the vertical direction only; the three-dimensional Laplace equation emerging in the water wave part of our model is then discretized in the vertical and replaced by an elliptical equation in the horizontal coordinates with non-constant depth-dependent coefficients. Such a simplification to Boussinesq-type models follows immediately from some work of [Klopman et al. 2007] which, however, only applies consistently to potential flows. Furthermore, it may be easier to recombine the potential and vortical part in a formulation with a depth-averaged velocity after such an additional Boussinesq-type simplification has been made. Second, for depth-averaged shallow water flows the generation of potential vorticity emerges for bores in which energy is dissipated in a non-uniform manner along the breaking wave crest. While we have derived advanced bore relations for our new model, the subsequent analysis of potential vorticity generation is left open.

Finally, future work will also involve discontinuous Galerkin finite element discretizations (DGFEM) of our new model, or Boussinesq versions thereof. It will combine our research on shallow water ocean models [Ambati and Bokhove 2007, Tassi *et al.* 2007, Cotter *et al.* 2009], including flooding and drying [Bokhove 2005], with our DGFEM discretization based on a Luke's variational principle [Ambati 2008, Ambati et al. 2008/9]. In addition, we aim to explore exact solutions, inspired by [Constantin 2001].

Acknowledgments
The photograph "sun-on-bore", as the late Professor Howell D. Peregrine (DHP) called it, in Fig. 2(a) featured most prominently on the poster "maths makes waves" (Fig. 3) in the campaign "World Mathematical Year 2000, Posters in the London Underground". Howell Peregrine was always keen that his photographs were used for good, whether it be in the sciences or (fine) arts. O.B. was a postdoctoral research associate with DHP (and Professor Andrew Woods) in 1998 and 1999 at the School of Mathematics in Bristol, U.K. Thereafter, our scientific, geological and walking tours continued, till March

2007.

It is a pleasure to thank the organizers of two workshops, Geometric and Stochastic Methods in Geophysical Fluid Dynamics in Bremen 2008 (a.o., Marcel Oliver) and Numerical Modelling of Complex Dynamical System at the Lorentz Center in 2008 (a.o., Jason Frank), for providing the seeding opportunities of the research presented

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