# Abstractions of Stochastic Hybrid Systems


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**Abstract**

In this paper we define a stochastic bisimulation concept for a very general class of stochastic hybrid systems, which subsumes most classes of stochastic hybrid systems. The definition of this bisimulation builds on the concept of zigzag morphism defined for strong Markov processes. The main result is that this stochastic bisimulation is indeed an equivalence relation. The secondary result is that this bisimulation relation for the stochastic hybrid system models used in this paper implies the same kind of bisimulation for their continuous parts and respectively for their jumping structures.



Keywords: *stochastic hybrid systems, Markov processes, simulation morphism, zigzag morphism, bisimulation, category theory*


## 1 Introduction

Significant progress in verification of probabilistic systems has been done, however mostly for discrete distributions or Markov chains. Continuous stochastic processes are much more difficult to verify. It is notorious that theorem proving of stochastic properties (with the probability one) can be carried out on the unit circle only. Model checking and reachability analysis are strongly conditioned by abstraction techniques. When the state space is not only infinite but also continuous, abstraction techniques must be very strong. Hybrid systems add an extra level of complexity because of the hybrid nature of the state space (discrete and continuous states coexist) and stochastic hybrid systems push further this complexity by adding non-determinism and uncertainty. Therefore, it is necessary to have an abstraction theory for stochastic processes that can be used for verification and analysis of stochastic hybrid systems.

Reachability analysis and model checking are much easier when a concept of bisimulation is available. The state space can be drastically abstracted in some cases. In this paper, we focus on defining bisimulation relations for stochastic hybrid systems, as a first step towards creating a framework for verification.

Besides of different bisimulation concepts in concurrency theory, the notion of bisimulation is present

• in the 'deterministic world': continuous and dynamical systems [23] or hybrid systems [16];

• or in the 'probabilistic world': probabilistic discrete systems [19], labelled Markov processes [9], piecewise deterministic Markov processes [24].

In this paper we define different bisimulation concepts for a very large class of Markov processes. This work is motivated by the fact that the realizations of different models for stochastic hybrid systems make up, under mild assumptions, stochastic processes with the strong Markov property. Our interest is related with the so-called general stochastic hybrid systems, abbreviated GSHS, introduced in [2, 1, 12]. Mainly, we present two approaches to define stochastic bisimulation, both of them defined a categorical framework.

The first definition of bisimulation builds on the ideas of Edalat [9, 15] and of Larsen and Skou [19] and of Joyal, Nielsen and Winskel [18]. We extend the definition of bisimulation for labelled Markov processes to continuous time strong Markov processes defined on analytic spaces. We consider the category of continuous time strong Markov processes defined on analytic spaces, eqquiped with some arrows (called zigzag morphisms), which are morphisms defined on the state spaces that "preserve" the transition probabilities. The stochastic bisimulation is defined via a span of zigzag morphisms. The main result is that this bisimulation is indeed an equivalence relation. This turns out to be a rather hard mathematical result, which employs the whole stochastic analysis apparatus associated to this class of strong Markov processes.

The second definition of bisimulation uses the same class of Markov processes, but another definition of zigzag morphism. In this case, the zigzag morphism between two Markov processes is defined between the cones of excessive functions associated to those two processes, provided that it commutes with the kernel operators corresponding to the processes considered.

These two concepts of stochastic bisimulation are defined in a category theory framework. Therefore, these stochastic bisimulations, as notions of system equivalence, have some fundamental mathematical properties. Moreover, we give an algebraic characterization of these bisimulation through a measurable relation between the state spaces, which induces equivalent quotient processes. For the case of GSHS, we prove that this is a natural notion of bisimulation since the bisimilarity of two GSHS realizations implies that their diffusion / jumping components are bisimilar.

The rest of the paper is organized as follows. Next section gives some necessary background on Markov processes. Section 3 gives a short presentation of GSHS. Section 4 is the main body of the paper. It presents two categorical approaches to define bisimulation for Markov processes. It starts with a quick tour on stochastic bisimulation, and the main difficulties, which we have to overcome when we aim to define a concept of bisimulation for very general Markov processes. It is stressed the fact that the key point in the construction of bisimulation is the definition of morphism. Two concepts of categorical bisimulation are defined. The main result is that these bisimulations are equivalence relations. Algebraic characterizations and specific features in the case of GSHS are provided. The paper ends with some conclusions and further work.

# 2 Preliminaries

Stochastic processes, we consider here, are non-deterministic systems with a continuous state space, where "non-determinism" can be measured using transition probability measures. Markov processes form a subclass of stochastic systems for which, at any stage, future evolutions are conditioned only by the present state (in other words, they do not depend on the past).

A probability space $(\Omega, \mathcal{F}, P)$ is fixed and all $X-$valued random variables are defined on this probability space. The trajectories in the state space are modelled by a family of random variables $(x_t)$, where $t$ denotes the time. The reasoning about state change is carried out by a family of probabilities $P_x$ one for each state $x \in X$. The construction is similar to the coalgebraic reasoning in the semantics of specification languages: the system behavior is described by giving for each state the possible evolutions. For Markov processes, for each state $x$, the probability $P_x(x_t \in A)$ to reach a given set of state $A \subset X$ (provided that $A$ is measurable) starting at $x_0 = x$ describes the system evolution. We remark two ingredients that make the difference from the deterministic case: the evolutions are described from an initial state to a set of final set (nondeterminism) and all we know is a probability to have such trajectories (randomness).

For the purposes of this paper, we have to give some background about strong Markov processes.

## 2.1 Strong Markov Processes

Let $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ be a Markov process with the state space $X$. Suppose that $X$ is an analytic space. Then, we take the measurable space $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ or $\mathcal{B}$ is the Borel $\sigma$-algebra of $X$ (i.e. the $\sigma$-algebra generated by the open sets). An analytic space is the image of a Polish space under a continuous function from one Polish space to another. A Polish space is a topological space homeomorphic with a complete separable space. A Borel space is a topological space, which is homeomorphic to a Borel subset of a complete separable metric space. Any Borel space is an analytic space.

$(\Omega, \mathcal{F}, P_x)$ denotes the sample probability space for each process with initial start point $x$. The family of $\sigma$-algebras $\{\mathcal{F}_t^0\}$ denotes the *natural filtration*, i.e. $\mathcal{F}_t^0 = \sigma\{x_t, s \leq t\}$ and $\mathcal{F}_\infty^0 = \vee_t \mathcal{F}_t^0$. The trajectories of $M$ are modelled by a family of $X$-valued random variables $(x_t)$, which, as functions of time, might have some continuity properties (as the càdlàg property, i.e. right continuous with left limits). This means that, for each $t > 0$ the function $x_t : (\Omega, \mathcal{F}) \to (X, \mathcal{B})$ is a $\mathcal{F}^0/\mathcal{B}$-measurable function for all $t \geq 0$. The *(Wiener) probability* $P_x : (\Omega, \mathcal{F}) \to [0,1]$ is a probability measure such that $P_x(x_t \in A)$ is $\mathcal{B}$-measurable in $x \in X$, for each $t \in [0, \infty)$ and $A \in \mathcal{B}$, and $P_x(x_0 = x) = 1$.
If $\mu$ is a probability measure on $(X, \mathcal{B})$ then one can define

$$P_\mu(\Lambda) = \int_X P_x(\Lambda)\mu(dx), \Lambda \in \mathcal{F}^0.$$

We then denote by $\mathcal{F}$ (resp. $\mathcal{F}_t$) the completion of $\mathcal{F}_\infty^0$ (resp. $\mathcal{F}_t^0$) with respect to all $P_\mu$, probability measure on $(X, \mathcal{B})$. The family $\{\mathcal{F}_t\}_t$ denotes the *natural filtration* of $M$.
We say that a family $\{\mathcal{M}_t\}$ of sub-$\sigma$-algebras of $\mathcal{F}$ is an admissible filtration if $\mathcal{M}_t$ is increasing in $t$ and $x_t \in \mathcal{M}_t/\mathcal{B}$ for each $t \geq 0$. Then $\mathcal{F}_t^0$ is the *minimum admissible filtration*. An admissible filtration $\{\mathcal{M}_t\}$ is *right continuous* if $\mathcal{M}_t = \mathcal{M}_{t+} = \cap\{\mathcal{M}_{t'} | t' > t\}$.
Given an admissible filtration $\{\mathcal{M}_t\}$, a $[0, \infty]$-valued function $\tau$ on $\Omega$ is called an $\{\mathcal{M}_t\}$-*stopping time* if $\{\tau \leq t\} \in \mathcal{M}_t$, $\forall t \geq 0$.

For an admissible filtration $\{\mathcal{M}_t\}$, we say that $M$ is called *strong Markov process* with respect to $\{\mathcal{M}_t\}$ if $\{\mathcal{M}_t\}$ is *right continuous* and

$$P_\mu(x_{\tau+t} \in E | \mathcal{M}_\tau) = P_{x_\tau}(x_t \in E);\ P_\mu - a.s.$$

for all probability measures $\mu$ on $(X, \mathcal{B})$, $E \in \mathcal{B}$, $t \geq 0$, and for any $\{\mathcal{M}_t\}$-stopping time $\tau$.

The stochastic analysis identifies concepts (like infinitesimal generator, semigroup of operators, resolvent of operators) that characterize in an abstract sense the evolutions of a Markov process. Under standard assumptions, all these concepts are equivalent, in the sense that given one concept then all the others can be constructed from it. For a detailed presentation of these notions and the connections between them, the reader can consult, for example [20]. These tools can be further used to define a concept of stochastic bisimulation.

## 2.2 Operator Characterizations of Markov Processes

In this subsection, we shortly present some standard notions associated to a Markov process as operator semigroup, operator resolvent, infinitesimal generator. These will be used in the next subsection to define the simulation morphisms and the zigzag morphisms for our processes.

Let $\mathcal{B}^b(X)$ be the lattice of bounded non-negative measurable functions on $X$. This is a Banach space under the norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

Let $\mathcal{P} = (P_t)_{t>0}$ denote the *operator semigroup* associated to $M$, which maps $\mathcal{B}^b(X)$ into itself given by

$$P_t f(x) = E_x f(x_t), \forall x \in X \tag{1}$$

where $E_x$ is the expectation w.r.t. $P_x$. The semigroup $\mathcal{P} = (P_t)_{t>0}$ can be thought of as an *abstraction* of $M$, since that from $\mathcal{P}$ one can recuperate the initial process [8]. This kind of abstraction can be related with the concept of abstract control system from [25], but in our case due to the stochastic features of the model, the domain of the abstraction is not longer the state space $X$, but $\mathcal{B}^b(X)$. The transition probabilities associated to $M$ are defined as follows

$$p_t(x, A) = P_t(I_A)(x) = P_x(x_t \in A). \tag{2}$$

A function $f$ is *excessive* (w.r.t. the semigroup $(P_t)$ or the resolvent $(V^\alpha)$) if it is measurable, non-negative and $P_t f \leq f$ for all $t \geq 0$ and $P_t f \nearrow f$ as $t \searrow 0$. Let denote by $\mathcal{E}_M$ the set of all excessive functions associated to $M$. The strong Markov property can be characterized in terms of excessive functions [**?**].
The *resolvent of operators* $\mathcal{V} = (V_\alpha)_{\alpha\geq 0}$ associated with the semigroup $\mathcal{P}$ are given by formula

$$V_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt, \alpha > 0.$$

Let us denote by $V$ the initial operator $V_0$ of $\mathcal{V}$, which is known as the *kernel operator* of Markov process $M$. The operator resolvent $(V_\alpha)_{\alpha\geq 0}$ is the Laplace transform of the semigroup.
The *strong generator* $L$ is the derivative of $P_t$ at $t = 0$. Let $D(L) \subset \mathcal{B}^b(X)$ be the set of functions $f$ for which the following limit exists (denoted by $Lf$)

$$\lim_{t\searrow 0} \frac{1}{t}(P_t f - f) \tag{3}$$

## 2.3 Fine Topology and Order on Trajectories

Suppose that $M$ is a transient Markov process, i.e. there exists a strict positive Borel measurable function $q$ such that $Vq$ is a bounded function. The transience means that for any Borel set $E$ in $X$ for almost all trajectories there exists a finite stopping time $t^*$such that $x_t \notin E$ for all $t > t^*$ (for more explanations about the transience property see [**?**]).

On the state space $X$ we define a preorder relation $\prec_M$ given

$$x \prec_M y \Longleftrightarrow Vf(y) \leq Vf(x), \forall f \in \mathcal{B}^b(X)$$

Intuitively, $\prec_M$ is the order on the trajectories of $M$. In particular, if $M$ degenerates in a semi-dynamical system, $\prec_M$ is exactly the order relation on the trajectories.

One can define on $X$ the *fine topology*, which consists of the sets $G \subseteq X$ with the following property: for each $x \in G$ there exists a measurable set $A \supset X \backslash G$ and $P_x(T_A > 0) = 1$, where

$$T_A(\omega) = \inf\{t > 0 | x_t(\omega) \in A\} \tag{4}$$

is the first *hitting time* of $A$. Intuitively, this means that each trajectory starting from $x$ remains for a while in $G$. The fine topology is separated and is finer than the initial topology.

# 3 Stochastic Hybrid Systems

Stochastic hybrid systems are 'traditional' hybrid systems with some stochastic features. These systems typically contain variables or signals that take values from a continuous set and also variables that take values from a discrete (finite or countable) set. Differential equations or stochastic differential equations tipically give the continuous dynamics of such systems. Usually, a Markov chain governs the discrete-variable dynamics. The stochastic features might be present in the continuous dynamics or in the discrete dynamics, or in both. The continuous and discrete dynamics coexist and interact with each other and because of this it is important to use models that accurately describe the dynamic behaviour of such hybrid systems. The realizations of the different models of stochastic hybrid systems (see [21] for an overview) can be thought of as particular classes of strong Markov processes with the continuous evolution disturbed by forced or spontaneous transitions.

## 3.1 Syntax

In this section we give a short presentation of the general model for stochastic hybrid systems, introduced in [2], which is used in the following sections. In [7], a quite general model of stochastic hybrid systems that can be related to GSHS as a particular case, has been implemented in Charon [4]).

General Stochastic Hybrid Systems (GSHS) are a class of non-linear stochastic continuous-time hybrid dynamical systems. GSHS are characterized by a hybrid state defined by two components: the continuous state and the discrete state. The continuous and the discrete parts of the state variable have their own natural dynamics, but the main point is to capture the interaction between them.

The time $t$ is measured continuously. The state of the system is represented by a continuous variable $z$ and a discrete variable $i$. The continuous variable evolves in some "cells" $X^i$ (open sets in the Euclidean space) and the discrete variable belongs to a countable set $Q$.

As usual, we define the hybrid state space of the GSHS as $X = \bigcup_{i\in Q}\{i\}\times X^i$ and $x = (i, x^i) \in X$ the hybrid state. The closure of the hybrid state space is

$$\overline{X} = X \cup \partial X,$$

where

$$\partial X = \bigcup_{i\in Q}\{i\} \times \partial X^i.$$

It is known that $X$ can be endowed with a metric $\rho$ whose restriction to any component $X^i$ is equivalent to the usual component metric [14]. Then $(X, \mathcal{B}(X))$ is a Borel space, where $\mathcal{B}(X)$ is the Borel $\sigma$-algebra of $X$.

The intrinsic difference between the discrete and continuous variables, consists of the way that they evolve through time. The continuous state evolves according to an SDE whose vector field and drift factor depend on the hybrid state. The discrete dynamics produces transitions in both (continuous and discrete) state variables $z, i$. Switching between two discrete states is governed by a probability law or occurs when the continuous state hits the boundary of its state space. Whenever a switching occurs, the hybrid state is reset instantly to a new state according to a probability law which depends itself on the past hybrid state. Transitions, which occur when the continuous state hits the boundary of the state space are called forced transitions, and those which occur probabilistically according to a state dependent rate are called spontaneous transitions.

Formally, a GSHS is defined as follows.

**Definition 1** *A General Stochastic Hybrid System (GSHS) is a collection* $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$ *where*

- $Q$ *is a countable set of discrete variables;*
- $d : Q \to \mathbb{N}$ *is a map giving the dimensions of the continuous state spaces (for each location);*
- $m : Q \to \mathbb{N}$ *is a map giving the dimensions of the Wiener processes*
- $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ *maps each* $q \in Q$ *into an open subset* $X^q$ *of* $\mathbb{R}^{d(q)}$*;*
- $b : X \to \mathbb{R}^{d(\cdot)}$ *is a vector field;*
- $\sigma : X \to \mathbb{R}^{d(\cdot)\times m(\cdot)}$ *is a* $X^{(\cdot)}$*-valued matrix;*
- $Init : \mathcal{B}(X) \to [0, 1]$ *is an initial probability measure on* $(X, \mathcal{B}(S))$*;*
- $\lambda : \overline{X} \to \mathbb{R}^+$ *is a transition rate function;*
- $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0, 1]$ *is a transition measure.*

### 3.2 Semantics

A probability space $(\Omega, \mathcal{F}, P)$ is fixed and all $X-$valued random variables are defined on this probability space. The realization of a GSHS $H$ is built as a *Markov string* [3, 12] obtained by the concatenation of some diffusion processes $(z_t^i)$, $i \in Q$ together with a jumping mechanism given by a family of stopping times $(S^i)$. Let $\omega_i$ be a diffusion trajectory, which starts in $(i, z^i) \in X$. Let $t_*(\omega_i)$ be the first hitting time of $\partial X^i$ of the process $(z_t^i)$. Define the function

$$F(t, \omega_i) = I_{(t<t_*(\omega_i))} \exp(-\int_0^t \lambda(i, x_s^i(\omega_i)))ds. \tag{5}$$

This function will be the survivor function for the stopping time $S^i$ associated to the diffusions $(z_t^i)$.

**Definition 2 (GSHS Execution)** *A stochastic process $x_t = (q_t, z_t)$ is called a GSHS execution if there exists a sequence of stopping times $T_0 = 0 < T_1 < T_2 \leq \ldots$ such that for each $k \in \mathbb{N}$,*

- *$x_0 = (q_0, x_0^{q_0})$ is a $Q \times X$-valued random variable extracted according to the probability measure $Init$;*
- *For $t \in [T_k, T_{k+1})$, the discrete state $q_t$ remains constant*

$$q_t = q_{T_k}$$

*and the continuous state $z_t$ is a solution of the SDE:*

$$dz_t = b(q_{T_k}, z_t)dt + \sigma(q_{T_k}, z_t)dW_t \tag{6}$$

*where $W_t$ is a the $m(q_{T_k})$-dimensional standard Wiener;*

- *$T_{k+1} = T_k + S^{i_k}$ where $S^{i_k}$ is a stopping time chosen according with survivor function $F$ given by (5).*
- *The probability distribution of $z_{T_{k+1}}$ is governed by the law $R\left((q_{T_k}, z_{T_{k+1}^-}), \cdot\right)$.*

It is known, from [1], that the realization of any GSHS, $H$, under standard assumptions (see below) is a strong Markov process.

- Assumption about the diffusion coefficients ensures that for any $i \in Q$, the existence and uniqueness of the solution of the equations (6):

**Assumption 1** *Suppose that $b : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)}$, $\sigma : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot) \times m(\cdot)}$ are bounded and Lipschitz continuous in $z$.*

- Assumption about non-Zeno executions: We denote

$$N_t(\omega) = \sum I_{(t \geq T_k)}$$

**Assumption 2** *For every starting point $x \in X$, $\mathbf{E}^x N_t < \infty$, for all $t \in \mathbb{R}_+$.*

- Assumption about the transition measure the transition rate function is:

**Assumption 3** *(A) $\lambda : X \to \mathbb{R}_+$ is a measurable function such that $t \to \lambda(x_t^i(\omega_i))$ is integrable on $[0, \varepsilon(x^i))$, for some $\varepsilon(x^i) > 0$, for each $z^i \in X^i$ and each $\omega_i$ starting at $z^i$.*
*(B) (i) for all $A \in \mathcal{B}(X)$, $R(\cdot, A)$ is measurable; (ii) for all $x \in \overline{X}$ the function $R(x, \cdot)$ is a probability measure; (iii) $R(x, \{x\}) = 0$ for $x \in X$.*

Let $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ be a strong Markov process. In the following sections, usually $M$ will represent the realization of a stochastic hybrid system model, $H$.
For a GSHS, $H$, the expression of the infinitesimal generator $L$ of its Markov process semantics is given in [2]. For $f \in \mathcal{D}(L)$ (the domain of generator) $Lf$ is given by

$$Lf(x) = L_{cont}f(x) + \lambda(x) \int_{\overline{X}} (f(y) - f(x))R(x, dy) \tag{7}$$

where:

$$L_{cont}f(x) = \mathcal{L}_b f(x) + \frac{1}{2} Tr(\sigma(x)\sigma(x)^T \mathbb{H}^f(x)). \tag{8}$$

It should be noticed that $\mathcal{D}(L)$ contains at least those measurable functions $f$ on $X \cup \partial X$, which are twice differentiable and satisfy the following boundary condition

$$f(x) = \int_{\mathbb{X}} f(y)R(x, dy),\ x \in \partial X.$$

Given a function $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and a vector field $b : \mathbb{R}^n \to \mathbb{R}^n$, we use $\mathcal{L}_b f$ to denote the Lie derivative of $f$ along $b$ given by $\mathcal{L}_b f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) b_i(x)$. Given a function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, we use $\mathbb{H}^f$ to denote the Hamiltonian operator applied to $f$, i.e. $\mathbb{H}^f(x) = (h_{ij}(x))_{i,j=1\ldots n} \in \mathbb{R}^{n \times n}$, where $h_{ij}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$. $A^T$ denotes the transpose matrix of a matrix $A = (a_{ij})_{i,j=1\ldots n} \in \mathbb{R}^{n \times m}$ and $Tr(A)$ denotes its trace.

For a strong Markov process defined on an analytic space (which is the case for the GSHS realization), the opus of the kernel operator is the inverse operator of the infinitesimal generator of the process [**?**].

A stochastic differential equation generates a much richer structure than just a family of stochastic processes, each solving the stochastic differential equation for a given value. In fact, it gives a flow of random diffeomorphism, i.e. it generates a random dynamical system (RDS) [5]. Therefore, the construction of a GSHS as a Markov string (see [3]) of diffusions does not only generate a Markov process, but it also generates an RDS (which is a 'string' of the RDS components). The theory of random dynamical systems is relatively new and we refer to [5], as the first systematic presentation of this theory. We present only the necessary definitions that we need in this paper.

Let $\theta_t : \Omega \to \Omega$ for all $t \in [0, \infty)$. $(\Omega, \mathcal{F}, P, \theta_t)$ (abbreviated $\theta$) is called a *metric dynamical system*, if:

1. The map $\theta : \Omega \times [0, \infty) \to \Omega$, $(\omega, t) \mapsto \theta_t(\omega)$ is measurable from $(\Omega \times [0, \infty)$, $\mathcal{F} \otimes \mathcal{B}([0, \infty))$ to $(\Omega, \mathcal{F})$;
2. $\theta$ satisfies the flow properties: (i) $\theta_0 = id_\Omega$ and (ii) $\theta(t + s) = \theta_t \circ \theta_s$ $\forall s, t \in [0, \infty)$;
3. $\theta$ is measure preserving, i.e. $\theta_t P = P$ $\forall t \in [0, \infty)$ (where $fP := P \circ f^{-1}$). The metric dynamical system is necessary to model the random perturbations of an RDS.

A measurable *random dynamical system* on the measurable space $(X, \mathcal{B})$ over the metric dynamical system $\theta$ with time $[0, \infty)$ is a map $\varphi : [0, \infty) \times \Omega \times X \to X$, $(t, \omega, x) \mapsto \varphi(t, \omega, x)$ with the following properties:

1. $\varphi$ is $\mathcal{B}([0, \infty)) \otimes \mathcal{F} \otimes \mathcal{B}/\mathcal{B}$ - measurable;
2. If $\varphi(t, \omega) = \varphi(t, \omega, \cdot)$ then $\varphi$ forms a perfect cocycle over $\theta$, i.e. (i) $\varphi(0, \omega) = id_X$ and (ii) $\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)$ $\forall \omega \in \Omega$ $\forall s, t \in [0, \infty)$.

The RDS associated to a GSHS realization arises from its construction as a Markov string: the shift operator $(\theta_t)$ of the corresponding Markov string is exactly the metric dynamical system for the RDS and for each $x \in X$, $\omega \in \Omega$, $t \geq 0$ the value of the RDS cocycle $\varphi(t, \omega, x)$ is exactly $x_t(\omega)$ with $x$ as the starting point (or $\varphi(t, \omega, x)$ is the execution of GSHS with $x$ as the starting point). In other words, the cocycle $\varphi$ is a replacement of the flow from the determinist case.

# 4 Categorical Approaches to Bisimulation

In the first subsection, we discuss a general methodology to define bisimulation for strong Markov processes on analytic spaces. Then, in the next two subsections we give two possible methods to define the category of strong Markov processes with analytic state spaces. The difference consists in the way to define the arrows in such categories. In each case we define the concept of bisimulation and we show that the respective category has semi-pullback. The later result implies that the bisimulation is an equivalence relation. The resulting concept of bisimulation will be compared with a concept of bisimulation via open maps (as introduced by Winskel et.al. and applied to continuous dynamical system by Tabuada et.al.) for semi-dynamical system. After that we introduce the concept of bisimulation for GSHS and study some specific properties.

## 4.1 Methodology

The classical paper of Joyal, Nielsen and Winskel [18] presents a general categorical view of what bisimulation is for deterministic systems. The bisimulation concept is given in terms of open maps and simulation morphisms.

Then, based on the work of Larsen and Skou [19] and of Joyal, Nielsen and Winskel [18], the concept of bisimulation has been extended to a specific class of Markov processes (labelled Markov processes) by Edalat et.al. [15].

For the continuous-time continuous-space Markov processes, this definition can not be adapted in a direct manner. The main problem is how to define the simulation morphisms and the open maps. In this case, we say that a Markov process $M^1$ *simulates* another Markov $M^2$ if there exists a surjective continuous morphism $\psi$ between their state spaces such that each transition probability of $M^2$ 'is matched' by a transition probability on $M^1$, in sense that for each measurable set $A \subset X^1$ and for each $u \in X^2$ we have

$$p_t^2(u, \psi^{-1}(A)) \leq p_t^1(\psi(u), A) \qquad \forall t \geq 0 \qquad (*)$$

where $(p_t^2)$ and $(p_t^1)$ are the transition functions corresponding to $M^2$, respectively to $M^1$. A such morphism $\psi$ is called *simulation morphism*.
The open maps are replaced by the so-called *zigzag morphisms*, which are simulation morphism for which the above condition holds with equality.

Practically, a simulation condition as before is hard to be checked because the time $t$ runs in a 'continuous' set. Then, it is necessary to require supplementary assumptions about the transition probabilities of the processes we are talking about. This kind of simulation morphisms and zigzag morphisms have been defined for labelled Markov processes and for stationary Markov processes with discrete time defined on Polish or analytic spaces (see [15] and the references therein). The categories considered there have the above Markov processes as objects and the zigzag morphisms as morphisms. Then the bisimulation notion for these processes is given in a classical way. Two labelled Markov processes, for example, are probabilistically bisimilar if there exists a *span of zigzag morphisms* between them. In this context, we point out also another reason why only some special kind of Markov processes are considered, as follows. This bisimulation relation is always reflexive and symmetric. But, the transitivity of a such relation (the bisimulation should be an equivalence relation) is usually implied by the existence of *semi-pullbacks* in the Markov process category considered [18, 15]. That means, in the respective category, for any pair of morphisms

$$\varphi^1 : M^1 \to M \text{ and } \varphi^2 : M^2 \to M$$

where $M^1, M^2, M$ are objects in that category, there exist an object $M^0$ and morphisms $\pi^i : M^0 \to M^i$ $(i = 1, 2)$ such that

$$\varphi^1 \circ \pi^1 = \varphi^2 \circ \pi^2.$$

The construction of the semi-pullback in the above categories of Markov processes is strongly based on the stationarity property of the Markov processes considered [15]. In this case the transition probabilities do not depend on time. Then the construction mechanism of the semi-pullback in a such categories of Markov processes is reduced to the construction of the semi-pullback in the category of transition probability functions and surjective transition probability preserving Borel maps (as morphisms in the respective category).

We develop a novel concept of *stochastic bisimulation* for strong Markov processes defined on analytic spaces. The novelty consists of the way to define the simulation morphisms and the zigzag morphisms. Specifically, we replace the condition ($*$) by a 'global condition' given in terms of kernel operators. We present two approaches to define these morphisms:

1. In the first approach these are defined on the state spaces and commute with the kernel operators of the processes considered.

2. In the second approach these are defined on the cones of excessive functions and have a similar commutativity property with the kernel operators. These kind of functions can be thought of as general solutions associated to the processes generator. In this case the zigzag morphisms change the directions of arrows. The simulator process has a larger cone of excessive functions. Then the zigzag morphism spans between Markov processes used to define the bisimulation relation become co-spans of morphims between the excessive function cones.

Then the bisimulation relation is naturally given via zigzag morphism spans between Markov processes. Moreover, the category of strong Markov processes defined on analytic spaces with these kinds of zigzag morphisms as arrows has semi-pullback.

Therefore, the bisimulation relation is an equivalence relation.

## 4.2 First Approach

### 4.2.1 The Category of Markov Processes

Let **GMP** be the category of the strong Markov processes, defined on analytic spaces, with continuous time, as objects. In this category, the arrows are the zigzag morphisms defined on the state spaces, which will be defined in the following. The aim of this subsection is to give an appropriate definition of these *zigzag morphisms (and of simulation morphisms)* between such processes, which will be used to define the concept of stochastic bisimulation in this category.

Let $M^1$ and $M^2$ be two objects of **GMP**, defined on $X^{(1)}$, respectively $X^{(2)}$.

**Definition 3** *A simulation morphism between the processes $M^2$ and $M^1$ (the process $M^1$ simulates the process $M^2$) is a measurable, monotone, finely continuous map $\psi : X^{(2)} \to X^{(1)}$ such that*

$$V^2(f \circ \psi) \leq V^1 f \circ \psi, \forall f \in \mathcal{B}^b(X^{(1)})$$

*where $V^1$ (resp. $V^2$) is the kernel operator associated to $M^1$ (resp. $M^2$).*

Def. 3 illustrates, in terms of kernel operators, that the simulating process can make all the transitions of the simulated process with greater probability than in the process being simulated.

A surjective simulation morphism $\psi : X^{(2)} \to X^{(1)}$ induces an equivalence relation $\sim_\psi$ on $X^{(2)}$

$$u \sim_\psi v \Leftrightarrow \psi(u) = \psi(v). \tag{9}$$

In this way, to each $x \in X^{(1)}$ we can associate an equivalence class $[u]_\psi$ w.r.t. $\sim_\psi$ such that $[u]_\psi = \psi^{-1}(x)$. We call $\sim_\psi$ the *simulation relation* induced by $\psi$.

**Definition 4** *A surjective simulation morphism $\psi$ between the processes $M^2$ and $M^1$ is called zigzag morphism if the condition from Def. 3 holds with equality, i.e.*

$$V^2(f \circ \psi) = V^1 f \circ \psi, \forall f \in \mathcal{B}^b(X^{(1)}). \tag{10}$$

It is easy to show that a surjective simulation morphism $\psi$ between the processes $M^2$ and $M^1$ is a zigzag morphism if and only if for almost all[1] $t \geq 0$ the following equality holds

$$P_t^2(f \circ \psi)(u) = (P_t^1 f)(\psi(u)), \forall f \in \mathcal{B}^b(X^{(1)}),\ u \in X^{(2)}, \tag{11}$$

where $(P_t^1)$ (resp. $(P_t^2)$) is the semigroup of operators associated to $M^1$ (resp. $M^2$).

**Remark 1** *The monotony of a zigzag morphism $\psi$ can be derived from condition satisfied by a zigzag morphism. Roughly speaking, this means that whilst the process $M^2$ evolves from $u$ to $\psi^{-1}(A)$ ($A \in \mathcal{B}(X^{(1)})$) on a trajectory with a given probability, the process $M^1$ evolves from $\psi(u)$ to $A$ with the same probability.*

**Remark 2** *A zigzag morphism $\psi : X^{(2)} \to X^{(1)}$ induces a morphism between the lattices of measurable functions associated with the two processes: $\Psi : \mathcal{B}^b(X^{(1)}) \to \mathcal{B}^b(X^{(2)})$ such that*

$$\Psi(f) = f \circ \psi \tag{12}$$

*for all $f \in \mathcal{B}^b(X^{(1)})$. Then the condition (11) can be written as follows*

$$\Psi(P_t^1 f) = P_t^2(\Psi(f)) \tag{13}$$

*for all $f \in \mathcal{B}^b(X^{(1)})$, or equivalently the following diagram commutes*

$$\begin{array}{ccc} \mathcal{B}^b(X^{(2)}) & \xrightarrow{P_t^2} & \mathcal{B}^b(X^{(2)}) \\ \Psi \uparrow & & \uparrow \Psi \\ \mathcal{B}^b(X^{(1)}) & \xrightarrow{P_t^1} & \mathcal{B}^b(X^{(1)}) \end{array}$$

The Remark 2 shows that a zigzag morphism between two Markov processes can be thought of as a generalization for the stochastic case of the simulation concept for abstract control systems defined in [25].

[1] i.e. except with a zero Lebesgue measure set of times

More explicitly, the operator semigroups $(P_t^1)$ and $(P_t^2)$ define two dynamical systems (the *abstractions* of $M^1$, resp. $M^2$) on the Banach spaces $\mathcal{B}^b(X^{(1)})$ and $\mathcal{B}^b(X^{(2)})$, respectively

$$\begin{array}{rcl}\phi^1 & : & \mathbb{R}_+ \times \mathcal{B}^b(X^{(1)}) \to \mathcal{B}^b(X^{(1)});\ \phi^1(t,f) = P_t^1 f \\ \phi^2 & : & \mathbb{R}_+ \times \mathcal{B}^b(X^{(2)}) \to \mathcal{B}^b(X^{(2)});\ \phi^2(t,f) = P_t^2 f.\end{array}$$

A zigzag morphim between $M^2$ and $M^1$ implies that $\phi^1$ is simulated by $\phi^2$. The condition (13) is, in fact, the characterization of an open map between these dynamical systems [16]: when $\phi^1$ evolves from $f$ to $P_t^1$ then $\phi^2$ evolves from $\Psi f$ to $\Psi(P_t^1 f) = P_t^2 \Psi f$.

#### 4.2.2 Stochastic Bisimulation

We consider the category **GMP** of strong Markov processes, defined on analytic spaces, with continuous time as objects and zigzag morphisms as arrows.

Then, we define the *stochastic bisimulation* between two processes in this category as the existence of a span of zigzag morphisms between them.

**Definition 5** *Let $M^1$ and $M^2$ be two objects in* **GMP**. *$M^1$ is stochastic bisimilar to $M^2$ (written $M^1 \sim M^2$) if there exists a span of zigzag morphisms between them, i.e. there exists a Markov process $M^{12}$ (object in* **GMP***) and the zigzag morphisms $\psi^1$ (where $\psi^1 : X^{12} \to X^{(1)}$) and $\psi^2$ (where $\psi^2 : X^{12} \to X^{(2)}$) such that*

$$\begin{array}{ccccc} & & M^{12} & & \\ & \overset{\psi^1}{\swarrow} & & \overset{\psi^2}{\searrow} & \\ M^1 & & & & M^2 \end{array}$$

**Proposition 6** *$M^1$ and $M^2$ are stochastic bisimilar if and only if there exists a co-span between their lattices of measurable functions, i.e. there exists a Markov process $M^{12}$ and the zigzag morphisms $\psi^1$ (where $\psi^1 : X^{12} \to X^{(1)}$) and $\psi^2$ (where $\psi^2 : X^{12} \to X^{(2)}$) such that*

$$\begin{array}{ccccc} & & \mathcal{B}^b(X^{12}) & & \\ & \overset{\Psi^1}{\nearrow} & & \overset{\Psi^2}{\nwarrow} & \\ \mathcal{B}^b(X^1) & & & & \mathcal{B}^b(X^2) \end{array}$$

*where $\Psi^1$ and $\Psi^2$ are induced from $\psi^1$ and $\psi^2$ by the formula (12).*

**Lemma 7** *The category* **GMP** *has semi-pullbacks.*

**Proof.** Let $M^1, M^2, M$ be three strong Markov processes defined on the analytic spaces $X^{(1)}, X^{(2)}, X$, respectively. Suppose that there exist two zigzag morphisms $\psi^1 : X^{(1)} \to X$, $\psi^2 : X^{(2)} \to X$. We have to prove that there exist another object $M^0$ (a strong Markov process defined on a analytic space $X^{(0)}$) and two zigzag morphisms $\pi^1 : X^{(0)} \to X^{(1)}$ and $\pi^2 : X^{(0)} \to X^{(2)}$ such that the following diagram commutes

$$
\begin{array}{ccccc}
 & & X^{(0)} & & \\
 & \pi^1 \swarrow & & \searrow \pi^2 & \\
X^{(1)} & & & & X^{(2)} \\
 & \psi^1 \searrow & & \swarrow \psi^2 & \\
 & & X & &
\end{array}
$$

Let $X^{(0)} = \{(x^1, x^2) | \psi^1(x^1) = \psi^2(x^2)\}$ equipped with the subspace topology of the product topology on $X^{(1)} \times X^{(2)}$. Note that $X^{(0)}$ is nonempty since $\psi^1$ and $\psi^2$ are surjective (in fact, for all $x^1 \in X^{(1)}$ there exists some $x^2 \in X^{(2)}$ such that $(x^1, x^2) \in X^{(0)}$). Clearly, $X^{(0)}$ is an analytic space since it is a closed subset of the analytic space $X^{(1)} \times X^{(2)}$. We take $M^0$ as the part of the product of the Markov processes $M^1, M^2$ restricted to $X^{(0)}$, i.e. $M^0$ is the product process $M^1 \otimes M^2$ "killed" outside of $X^{(0)}$. More explicitly, $M^0$ is the subprocess of $M^1 \otimes M^2$ with respect to the multiplicative functional $N_t = I_{[0,T)}(t)$, where $T$ is the first exit time of $X^{(0)}$ and $I_{[0,T)}$ is the indicator function of $[0, T)$ (see [8], Ch.3 for background on multiplicative functionals and subprocesses).
Let $(P_t^1), (P_t^2)$ be the operator semigroups associated with $M^1$ and $M^2$. Let $\widehat{P_t^1}$ and $\widehat{P_t^2}$ the semigroups defined on $\mathcal{B}^b(X^{(1)} \times X^{(2)})$ by

$$\widehat{P_t^1} f(x^1, x^2) = P_t^1(f(\cdot, x^2))(x^1), \widehat{P_t^2} f(x^1, x^2) = P_t^2(f(x^1, \cdot))(x^2).$$

The semigroup associated with $M^1 \otimes M^2$ is $P_t = \widehat{P_t^1}\widehat{P_t^2} = \widehat{P_t^2}\widehat{P_t^1}$ [13]. Then according with the Th. 3.3 [8], the process $M^0$ is a Markov process and the semigroup associated with it is $Q_t f(x^1, x^2) = E_x[f(x_t^1 \otimes x_t^2) N_t]$ for any $f \in \mathcal{B}^b(X^{(1)} \times X^{(2)})$ or, equivalently, $Q_t f(x^1, x^2) = P_t f(x^1, x^2)$ for any $f \in \mathcal{B}^b(X^{(0)})$.

Moreover, $M^0$ is a strong Markov process since $N_t$ is a strong functional multiplicative (see Prop. 3.12 [8]).
Then $\pi^1$ and $\pi^2$ can be taken as the projection maps. Using product semigroup and the Prop.4.2.1, it follows that these projection maps are indeed zigzag morphisms. For example, for $\pi^1$, we have $f \circ \pi^1$ (for $f \in \mathcal{B}^b(X^{(1)})$) depends only on $x^1$ and $P_t^2$ does not change it. Then $Q_t(f \circ \pi^1)(x^1, x^2) = (P_t^1 f)(\pi^1(x^1, x^2))$ for all $(x^1, x^2) \in X^{(0)}$. The surjectivity of $\pi^1$ or $\pi^2$ can be easily derived using the sujectivity of $\psi^1$ and $\psi^2$ and the definition of $X^{(0)}$. The equality $\psi^1 \circ \pi^1 = \psi^2 \circ \pi^2$ trivially holds.□

**Proposition 8** *If a category has semi-pullbacks then the bisimulation relation is an equivalence relation.*

Then, an immediate consequence of the existence of semi-pullbacks in the category **GMP** is the following result:

**Theorem 9** *The stochastic bisimulation in the category* **GMP** *is an equivalence relation.*

## 4.3 Second Approach

### 4.3.1 The Category of Markov Processes

Let $\widetilde{\mathbf{GMP}}$ be the category of the strong Markov processes defined on analytic spaces as the objects and the $\mathcal{E}$-zigzag morphisms (which will be defined in the following) as the arrows.

The zigzag morphisms between Markov processes can be defined also as morphisms between their cones of excessive functions. Let $M^1, M^2$ be two strong Markov processes defined on analytic spaces $X^{(1)}$, respectively $X^{(2)}$. Let $\mathcal{E}_{M^1}, \mathcal{E}_{M^2}$ the associated cones of excessive functions.

**Definition 10** *An* $\mathcal{E}$-morphism *(between these two cones) can be defined as an application*

$$\Psi : \mathcal{E}_{M^1} \to \mathcal{E}_{M^2} \tag{14}$$

*such that the following properties hold: (i)* $\Psi(f+g) = \Psi(f) + \Psi(g), \forall f, g \in \mathcal{E}_{M^1}$*; (ii)* $f \leq g \Rightarrow \Psi(f) \leq \Psi(g)$*;* $f_k \nearrow f \Rightarrow \Psi(f_k) \nearrow \Psi(f)$*; (iv)* $\Psi(f \cdot g) = \Psi(f) \cdot \Psi(g)$, $\forall f, g \in \mathcal{E}_{M^1}$*; (v)* $\Psi(1) = 1$*. An $\mathcal{E}$-morphism $\Psi$ is called* finite *if* $f < +\infty \Rightarrow \Psi(f) < +\infty$*.*

**Proposition 11** *If* $\psi : X^{(2)} \to X^{(1)}$ *is a $H$-*map *then* $\Psi : \mathcal{E}_{M^1} \to \mathcal{E}_{M^2}$ *given by*

$$\Psi(f) = f \circ \psi \tag{15}$$

*for all $f \in \mathcal{E}_{M^1}$, is a finite $\mathcal{E}$-morphism.*

Intuitively, in the formula (15) the $H$-map $\psi$ can be thought of as a *variable change*, i.e. for all $f \in \mathcal{E}_{M^1}$

$$\Psi(f)(u) = f(\psi(u)), \forall u \in X^{(2)}. \tag{16}$$

**Remark 3** *(i) The map $\Psi$ defined by (15) can be extended as a map between the two cones of measurable positive functions defined on $X^{(1)}$, respectively $X^{(2)}$, loosing the property of finely continuity. Prop.11 shows how a function between the state spaces of $M^1, M^2$ can provide an $\mathcal{E}$-morphism.*
*(ii) Conversely, if $\Psi$ is an $\mathcal{E}$-morphism as in (14) then there exists a unique measurable monotone and finely continuous application $\overline{\psi}$ from $X^{(2)}$ to an extension of $X^{(1)}$ such that: $\Psi(f) = f \circ \overline{\psi}, \forall f \in \mathcal{E}_{M^1}$. To obtain this result one can use results from [22].*

Using (16), each function $g$ belonging to the range of $\Psi$ can be extended to $X^{(2)}/_{\sim\psi}$, i.e. $g([u]_\psi) = f(x)$ provided that $[u]_\psi = \psi^{-1}(x)$ and $g = \Psi(f)$.

**Proposition 12** *If $\psi : X^{(2)} \to X^{(1)}$ is a surjective and finely open $H$-map such that each excessive function $g \in \mathcal{E}_{M^2}$ has the property*

$$u \sim_\psi v \Rightarrow g(u) = g(v) \tag{17}$$

*then the $\mathcal{E}$-morphism $\Psi : \mathcal{E}_{M^1} \to \mathcal{E}_{M^2}$ given by formula (15) is surjective.*

**Proof.** For each $g \in \mathcal{E}_{M^2}$ we have to define $f \in \mathcal{E}_{M^1}$ such that $\Psi(f) = g$. Let $f : X^{(1)} \to [0, \infty)$ defined by $f(x) = g(u)$ for each $x \in X^{(1)}$, where $u \in X^{(2)}$ is such that $\psi(u) = x$ (there exists a such $u$ since $\psi$ is surjective). The function $f$ is well defined because of (17). Then $f$ can be written as $f = g \circ \psi^{-1}$ and for any open set $D \subset [0, \infty)$ we have $f^{-1}(D) = \psi(g^{-1}(D))$. Since $\psi$ is a finely open map we obtain that $f^{-1}(D)$ is finely open in $X^{(1)}$. Then $f \in \mathcal{E}_{M^1}$.□

**Remark 4** *It is easy to check that if in the Prop. 11 both $\psi$ and $\Psi$ are surjective then $\Psi$ must be bijective. Therefore the two excessive function cones can be identified and the two processes are equivalent.*

**Definition 13** *A simulation $\mathcal{E}$-morphism between $M^1, M^2$ is an $\mathcal{E}$-morphism such that*

$$V^2 \circ \Psi \leq \Psi \circ V^1. \tag{18}$$

A surjective $\mathcal{E}$-morphism $\Psi$ is called zigzag $\mathcal{E}$-morphism if

$$V^2 \circ \Psi = \Psi \circ V^1 \tag{19}$$

i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_{M^1} & \stackrel{\Psi}{\to} & \mathcal{E}_{M^2} \\ V^1 \uparrow & & \uparrow V^2 \\ \mathcal{E}_{M^1} & \stackrel{\Psi}{\to} & \mathcal{E}_{M^2} \end{array}$$

**Remark 5** *It is clear that if $\psi$ is a $H$-map which is a zigzag morphism in the sense of the first approach, i.e. it satisfies the condition (10) then the $\mathcal{E}$-morphism generated by (15) is a zigzag $\mathcal{E}$-morphism.*

#### 4.3.2 Stochastic Bisimulation

We can define another version of the stochastic bisimulation via $\mathcal{E}$-morphisms:

**Definition 14** *Let $M^1$ and $M^2$ be two objects in $\widetilde{\mathbf{GMP}}$. $M^1$ is stochastic bisimilar to $M^2$ (written $M^1 \sim M^2$) if there exists a cospan of $\mathcal{E}$-zigzag morphisms between them, i.e. there exists a Markov process $M^{12}$ (object in $\widetilde{\mathbf{GMP}}$)*

and the $\mathcal{E}$-morphisms $\Psi^1$ and $\Psi^2$ between their excessive function cones

$$\begin{array}{ccccc} & & \mathcal{E}_{H^{12}} & & \\ & \stackrel{\Psi^1}{\nearrow} & & \stackrel{\Psi^2}{\nwarrow} & \\ \mathcal{E}_{H^1} & & & & \mathcal{E}_{H^2} \end{array}$$

**Proposition 15** *The category $\widetilde{\mathbf{GMP}}$ has semi-pullbacks.*

**Proof.** If we define the stochastic bisimulation defined via zigzag $\mathcal{E}$-morphisms, then the semi-pullback existence for the category of Markov processes (with morphisms given by zigzag $\mathcal{E}$-morphisms) is equivalent with the *pushout existence* in the category of their excessive function cones (with the morphisms given by zigzag $\mathcal{E}$-morphisms). Let us take the following span of morphims between the excessive function cones

$$\begin{array}{ccccc} & & \mathcal{E}_M & & \\ & \overset{\Psi^1}{\swarrow} & & \overset{\Psi^2}{\searrow} & \\ \mathcal{E}_{M^1} & & & & \mathcal{E}_{M^2} \end{array}$$

Naturally, we consider $\mathcal{E}$ as the tensor product $\mathcal{E}_{M^1} \otimes \mathcal{E}_{M^1}$ of the cones $\mathcal{E}_{M^1}, \mathcal{E}_{M^1}$ (which correspond to the product of operator semigroups or to Markov process product defined on $X^{(1)} \times X^{(2)}$). Then the 'inclusions' $\mathcal{E}_{M^1} \overset{\Gamma^1}{\hookrightarrow} \mathcal{E}$, $\Gamma^1(f^1) = \Psi^1(f) \otimes \Psi^2(f)$ if $f^1 = \Psi^1(f)$ and $\mathcal{E}_{M^2} \overset{\Gamma^2}{\hookrightarrow} \mathcal{E}$, $\Gamma^2(f^2) = \Psi^1(f) \otimes \Psi^2(f)$ if $f^2 = \Psi^2(f)$ (essentially, $\Psi^1$ and $\Psi^2$ are surjective) gives the desired pushout construction, i.e. the following diagram commutes

$$\begin{array}{ccccc} & & \mathcal{E}_M & & \\ & \overset{\Psi^1}{\swarrow} & & \overset{\Psi^2}{\searrow} & \\ \mathcal{E}_{M^1} & & & & \mathcal{E}_{M^2} \\ & \underset{\Gamma^1}{\searrow} & & \underset{\Gamma^2}{\swarrow} & \\ & & \mathcal{E} & & \end{array}$$

**Proposition 16** *The stochastic bisimulation defined by Def. 14 in* $\widetilde{\mathbf{GMP}}$ *is an equivalence relation.*

## 4.4 Characterization of stochastic bisimulation

Let us consider two bisimilar processes $M^1$ and $M^2$ and $\psi^1$ and $\psi^2$ are zigzag morphisms as in the Def.5. Then we can define a relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$, called *bisimulation relation*, given by

$$x^1 \mathcal{R} x^2 \Leftrightarrow (\psi^1)^{-1}(x^1) \cap (\psi^2)^{-1}(x^2) \neq \emptyset \tag{20}$$

**Proposition 17** $x^1 \mathcal{R} x^2$ *if and only if*

$$x^1 \in \psi^1[(\psi^2)^{-1}(x^2)].$$

For $A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$, we define

$$\begin{array}{rcl} \mathcal{R}^{-1}(A^1) & = & \{(x^1, x^2) | x^1 \in A^1, x^1 \mathcal{R} x^2\} \\ \mathcal{R}^{-1}(A^2) & = & \{(x^1, x^2) | x^2 \in A^2, x^1 \mathcal{R} x^2\} \end{array}$$

$\mathcal{R}$ is called *measurable* if for all $A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$ the sets $\mathcal{R}^{-1}(A^1)$, $\mathcal{R}^{-1}(A^2)$ are measurable w.r.t. $\sigma$-algebra produs $\mathcal{B}(X^{(1)}) \otimes \mathcal{B}(X^{(2)})$.

Then we can extend the bisimulation relation (20) to the measurable sets

$$A^1 \mathcal{R} A^2 \Leftrightarrow (\mathcal{R}^1)^{-1}(A^1) = (\mathcal{R}^2)^{-1}(A^2).$$

or, equivalently,

$$A^1 \mathcal{R} A^2 \text{ iff } \forall x^1 \in A^1 \exists x^2 \in A^2 \text{ s.t. } x^1 \mathcal{R} x^2 \text{ and viceversa.}$$

$\mathcal{R}$ is called *weak measurable* if for all $A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$ with $A^1 \mathcal{R} A^2$ then $\mathcal{R}^{-1}(A^1)$, $\mathcal{R}^{-1}(A^2)$ are measurable w.r.t. $\sigma$-algebra produs $\mathcal{B}(X^{(1)}) \otimes \mathcal{B}(X^{(2)})$.

**Lemma 18** *If $\psi^1, \psi^2$ are finely open $H$-maps then*
*(i) $A^1 \mathcal{R} A^2$ iff*

$$\begin{aligned} A^1 &= \psi^1[(\psi^2)^{-1} A^2] \\ A^2 &= \psi^2[(\psi^1)^{-1} A^1] \end{aligned}$$

*(ii) $\mathcal{R}$ is a weak measurable relation.*

**Proof.** (ii) The conclusion is clear since for $A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$ we have

$$\begin{aligned} \mathcal{R}^{-1}(A^1) &= \{(x^1, x^2) | x^1 \in A^1 \text{ and } x^2 \in \psi^2[(\psi^1)^{-1} x^1]\} \\ \mathcal{R}^{-1}(A^2) &= \{(x^1, x^2) | x^2 \in A^2 \text{ and } x^1 \in \psi^1[(\psi^2)^{-1} x^2]\}.\square \end{aligned}$$

Let $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ be the equivalence relation given by (20). This relation will induce other two relations $\mathcal{R}^1$ and $\mathcal{R}^2$ define on $X^{(1)}$ and $X^{(2)}$, respectively, as follows.

$$x^1 \mathcal{R}^1 y^1 \Leftrightarrow \exists x^2 \in X^{(2)} \text{ s.t. } x^1 \mathcal{R} x^2 \text{ and } y^1 \mathcal{R} x^2 \tag{21}$$

and a similar definition for $\mathcal{R}^2$.

**Lemma 19** *$x^1 \mathcal{R}^1 y^1$ iff there exist $u, v \in X^{12}$ such that $x^1 = \psi^1 u$ and $y^1 = \psi^1 v$ provided that $u \sim_{\psi^2} v$. Similarly, for $\mathcal{R}^2$.*

**Proposition 20** *$\mathcal{R}^1$ and $\mathcal{R}^2$ are equivalence relations.*

**Proof**. The previous lemma ensures the transitivity property and the surjectivity of $\psi^1, \psi^2$ gives the reflectivity property. The symmetry is clear.□

**Remark 6** *In a similar way, for a relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ with $\Pi^1(\mathcal{R}) = X^{(1)}$ and $\Pi^2(\mathcal{R}) = X^{(2)}$ we can define the induced equivalence relations $\mathcal{R}^1, \mathcal{R}^2$ (If it is necessary we have to take the transitive closure of these relations).*

Let

$$\mathcal{B}^*(X^{(1)}) = \mathcal{B}(X^{(1)}) \cap \{A^1 \subset X^{(1)} | \text{ if } x^1 \in A^1 \text{ and } [x^1] = [y^1] \text{ then } y^1 \in A^1\}$$

be the collection of all Borel sets in which any equivalence class of $X^{(1)}$ is either totally contained or totally not contained. Here, for $x^1 \in X^{(1)}$ (resp. $x^2 \in X^{(2)}$) we denote its class of equivalence by $[x^1]$ (resp. $[x^2]$) w.r.t. $\mathcal{R}^1$ (resp. $\mathcal{R}^2$). It can be checked that $\mathcal{B}^*(X^{(1)})$ is a $\sigma$-algebra. Let $X^{(1)}/_{\mathcal{R}^1}$ be the set of equivalence classes of $X^{(1)}$, let $\pi_{X^{(1)}} : X^{(1)} \to X^{(1)}/_{\mathcal{R}^1}$ be the mapping that maps each $x^1 \in X^{(1)}$ to its equivalence class and let

$$\mathcal{B}(X^{(1)}/_{\mathcal{R}^1}) = \{A^1 \subset X^{(1)}/_{\mathcal{R}^1} | \pi^{-1}_{X^{(1)}}(A^1) \in \mathcal{B}^*(X^{(1)})\}.$$

Then $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}(X^{(1)}/_{\mathcal{R}^1}))$, which is a measurable space, is called the quotient space of $X^{(1)}$ with respect to $\mathcal{R}^1$. The quotient space of $X^{(2)}$ with respect to $\mathcal{R}^2$ is defined in a similar way. Clear, $\mathcal{B}(X^{(i)}/_{\mathcal{R}^i})$ can be identified with $\mathcal{B}^*(X^{(i)})$. Then, for $i = 1, 2$, the space $X^{(i)}/_{\mathcal{R}^i}$ can be endowed with the $\sigma$-algebra $\mathcal{B}^*(X^{(i)})$, which is the "saturation" of the Borel $\sigma$-algebra of $X^{(i)}$ w.r.t. $\mathcal{R}^i$.

The following proposition shows that only the saturated sets can be bisimilar.

**Proposition 21** *If $A^1 \in \mathcal{B}(X^{(1)})$ is such that there exists $A^2 \in \mathcal{B}(X^{(2)})$ with $A^1 \mathcal{R} A^2$ then $A^1$ is saturated, i.e. $A^1 \in \mathcal{B}^*(X^{(1)})$.*

**Proof**. If $y^1 \mathcal{R}^1 x^1 \in A^1$ then there exist $u, v \in X^{12}$ such that $y^1 = \psi^1 u$ and $x^1 = \psi^1 v$ with $\psi^2 u = \psi^2 v$. Since $\psi^2 v \in \psi^2(\psi^1)^{-1}(A^1) = A^2$ there exists $x^2 \in A^2$ such that $\psi^2 v = x^2$. Therefore, $\psi^2 u = x^2 \in A^2$ and $\psi^1 u \in \psi^1(\psi^2)^{-1}(A^2) = A^1$, i.e. $y^1 \in A^1$. That means $A^1$ is saturated.□

**Proposition 22** *If $\psi^1, \psi^2$ are finely open $H$-maps then the quotient spaces $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)}))$, $(X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$ are homeomorph.*

**Proof**. We can define an application $\varphi : (X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)})) \to (X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$ such that, for all $[x^1] \in X^{(1)}/_{\mathcal{R}^1}$ we have

$$\varphi([x^1]) = [x^2] \tag{22}$$

provided that $x^1 \mathcal{R} x^2$. Definition of $\mathcal{R}^1$ and $\mathcal{R}^2$ ensure that $\varphi$ is well-defined and bijective. For measurability, let us consider an arbitrary $A^2 \in \mathcal{B}^*(X^{(2)})$ then

$$\varphi^{-1}(A^2) = \psi^1[(\psi^2)^{-1} A^2]$$

is a measurable set in $X^{(1)}$, where $A^2$ is considered as a measurable set in $X^{(2)}$. The Prop.17 and the fact that $A^2$ is saturated w.r.t. $\mathcal{R}^2$ ensure that $\varphi^{-1}(A^2)$ is saturated w.r.t. $\mathcal{R}^1$. Then $\varphi$ is measurable. Similarly, $\varphi^{-1}$ is measurable. □

**Remark 7** *The map (22) from the previous proposition shows that an equivalence class $[x^2] \in X^{(2)}/_{\mathcal{R}^2}$ is identified with an equivalence class $[x^1] \in X^{(1)}/_{\mathcal{R}^1}$ where $x^1$ corresponds to $[u]_{\psi^1}$ given by $(\psi^2)^{-1}([u]_{\psi^1}) = [x^2]$.*

**Proposition 23 (reachability equivalence)** *If $M^1$ and $M^2$ are stochastic bisimilar via finely open zigzag morphisms then for all pairs $(x^1, x^2) \in X^{(1)} \times X^{(2)}$ and $(A^1, A^2) \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$ such that $x^1 \mathcal{R} x^2$ and $A^1 \mathcal{R} A^2$ the equality between the transition probabilities*

$$p_t^1(x^1, A^1) = p_t^2(x^2, A^2) \tag{23}$$

*is fulfilled for almost all $t > 0$.*

**Proof**. Since $A^1 \mathcal{R} A^2$ then, from the Prop.21, we get that $(A^1, A^2) \in \mathcal{B}^*(X^{(1)}) \times \mathcal{B}^*(X^{(2)})$. Formula (11) can be written for the sets $A^1$ and $A^2$ as follows

$$\begin{aligned} p_t^1(x^1, A^1) &= p_t^{12}[u, (\psi^1)^{-1}(A^1)],\ u \in (\psi^1)^{-1}(x^1) \\ p_t^2(x^2, A^2) &= p_t^{12}[v, (\psi^2)^{-1}(A^2)],\ v \in (\psi^2)^{-1}(x^2) \end{aligned}$$

But, from Lemma 18

$$\begin{aligned} A^1 &= \psi^1[(\psi^2)^{-1} A^2] \\ A^2 &= \psi^2[(\psi^1)^{-1} A^1] \end{aligned}$$

which implies that $(\psi^1)^{-1}(A^1) = (\psi^2)^{-1}(A^2)$ since $A^1, A^2$ are saturated. In fact, the equality (23) results from the definition of zigzag morphism and the existence of the homeomorphism $\varphi$ between the quotient spaces (see Prop.22).□

Now with these two results (Prop.22 and Prop.21) in hand we can introduce the quotient stochastic processes $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ with

- the quotient spaces $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)}))$, $(X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$, respectively, as state spaces;
- transition probabilities given by

  $$\begin{aligned} &\widetilde{p}_t^1([x^1], A^1) = p_t^1(x^1, A^1), \text{ for all } A^1 \in \mathcal{B}^*(X^{(1)}); x^1 \in X^1 \\ &\widetilde{p}_t^2([x^2], A^2) = p_t^1(x^2, A^2), \text{ for all } A^2 \in \mathcal{B}^*(X^{(2)}); x^2 \in X^2 \end{aligned}$$

  defined for all $t > 0$. The way to define the induced equivalence relations $\mathcal{R}^1$, $\mathcal{R}^2$ ensures that these transition probabilities are well-defined, i.e. they do not depend on the representants of equivalence classes $[x^1]$ or $[x^2]$.

**Proposition 24** *The quotient stochastic processes $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ are Markov processes.*

From Prop.22, we are able now to make the connection between stochastic bisimulation and equivalence of stochastic processes as follows.

**Proposition 25** *If $\psi^1, \psi^2$ are finely open H-maps then the quotient stochastic processes $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ are equivalent.*

**Proof**. According to the Prop.22 the quotient spaces $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)}))$, $(X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$ are homeomorph. Then the equality (23) becomes

$$\widetilde{p}_t^1([x^1], A^1) = \widetilde{p}_t^2([x^2], A^2)$$

for all $A^1 \in \mathcal{B}^*(X^{(1)})$; $x^1 \in X^1$; $A^2 \in \mathcal{B}^*(X^{(2)})$; $x^2 \in X^2$ and for almost all $t > 0$ provided that $\varphi([x^1]) = [x^2]$ and $\varphi(A^1) = A^2$ with $\varphi$ defined as in the Prop.22. This means that $\mathcal{R}$ preserves the transition probabilities, i.e. $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ are equivalent.□

The properties of the bisimulation relation $\mathcal{R}$ induced by the existence of a span of zigzag morphisms between $M^1$ and $M^2$ give the idea to introduce a general concept of bisimulation relation, which will not depend on a given span.

**Definition 26** *A relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ is called a bisimulation relation between $M^1$ and $M^2$ if the following conditions are satisfied:*

1. $\Pi^1(\mathcal{R}) = X^{(1)}$ *and* $\Pi^2(\mathcal{R}) = X^{(2)}$*;*
2. $\mathcal{R}$ *is measurable;*
3. *the quotient stochastic processes $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ are equivalent.*

For a bisimulation relation $\mathcal{R}$, let us define

$$X^{12} = \{(x^1, x^2) \in X^{(1)} \times X^{(2)} | x^1 \mathcal{R} x^2\} \tag{24}$$

The $\sigma$-algebra of $X^{12}$ is defined as the product $\sigma$-algebra

$$\mathcal{B}(X^{12}) = \sigma\{\mathcal{R}^{-1}(A^1) \text{ and } \mathcal{R}^{-1}(A^2) | A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \otimes \mathcal{B}(X^{(2)})\}. \tag{25}$$

**Assumption 4 (Analiticity of $\mathcal{R}$)** *We suppose that if $X^{(1)}$ and $X^{(2)}$ are analytic spaces then $X^{12}$ is analytic.*

**Theorem 27** *Under the Ass.4 the following assertions hold:*
*(A) $M^1$ is stochastic bisimilar with $M^2$ via finely open zigzag morphisms then there exists a weak measurable bisimulation relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ between them.*
*(B) If there exists a measurable bisimulation relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ between $M^1$ and $M^2$ then they are stochastic bisimilar.*

**Proof**. Given two bisimilar processes $M^1$ and $M^2$ via finely open zigzag morphisms, the construction of the bisimulation relation $\mathcal{R}$ is given by (20) and the assertion (A) follows from Prop.22, Prop.20, and Prop.25.
Suppose now there exists a bisimulation relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ which satisfies the conditions of Def.26. In fact, the direct sum of the quotient spaces $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)}))$, $(X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$ is embedded in $(X^{12}, \mathcal{B}(X^{12}))$.
We construct the Markov process $M^{12}$ with the following transition probabilities

$$\begin{aligned} p_t^{12}[(x^1, x^2), \mathcal{R}^{-1}(A^1)] &= p_t^1(x^1, A^1) \\ p_t^{12}[(x^1, x^2), \mathcal{R}^{-1}(A^2)] &= p_t^2(x^2, A^2). \end{aligned}$$

Then we define for $i = 1, 2$ two maps $\psi^i$ from $X^{12}$ to $X^{(i)}$ as the canonical projections

$$\begin{aligned} \psi^i \quad &: \quad X^{12} \to X^{(i)} \\ \psi^i(x^1, x^2) \quad &= \quad x^i \end{aligned}$$

Clear, $\psi^i$ are surjective since $\Pi^i(\mathcal{R}) = X^{(i)}$, $i = 1, 2$. If $A^{(i)}$ is a measurable set of $X^{(i)}$ then

$$(\psi^i)^{-1}(A^{(i)}) = \mathcal{R}^{-1}(A^i) \in \mathcal{B}(X^{12}),$$

i.e. $\psi^i$ is measurable. On the other hand we have

$$\psi^i[\mathcal{R}^{-1}(A^i)] = A^i$$

but for $i \neq j$ we do not have the measurability of $\psi^i[\mathcal{R}^{-1}(A^j)]$. If $A^i \mathcal{R} A^j$ then $\mathcal{R}^{-1}(A^i) = \mathcal{R}^{-1}(A^j)$ and

$$\psi^i[\mathcal{R}^{-1}(A^j)] = \psi^i[\mathcal{R}^{-1}(A^i)] = A^i.$$

Then $\psi^i$ are open maps only w.r.t. the $\sigma$-algebras generated by bisimilar sets.□

## 4.5 Specific Features of Bisimulation for GSHS

Let $H^1$ and $H^2$ be two GSHS with the realizations $M^1$ and $M^2$, respectively.

**Definition 28** *$H^1$ and $H^2$ are stochastic bisimilar if their realizations $M^1$ and $M^2$ are stochastic bisimilar.*

### 4.5.1 Properties of zigzag morphims

A zigzag morphism $\psi : X^{(2)} \to X^{(1)}$ between $M^1$ and $M^2$, induces a relation $\mathcal{R}_\psi \subset X^{(2)} \times X^{(1)}$ as follows: $u\mathcal{R}_\psi x \Leftrightarrow \psi(u) = x$. Then the equivalence relation $\sim_\psi$on $X^{(2)}$ can be thought of as the equivalence relation induced by $\mathcal{R}_\psi$ in sense of [24], i.e. $u \sim_\psi v$ iff there exists $x \in X^{(1)}$ such that $u\mathcal{R}_\psi x$ and $v\mathcal{R}_\psi x$ (which is exact the meaning of (9)). The equivalence relation induced by $\mathcal{R}_\psi$ on $X^{(2)}$ is the trivial one ($x$ can be equivalent only with itself).

The space $X^{(2)}/_{\sim_\psi}$ can be endowed with the $\sigma$-algebra $\mathcal{B}^*(X^{(2)})$, which is the "saturation" of the Borel $\sigma$-algebra of $X^{(2)}$ w.r.t. $\sim_\psi$ (i.e. the collection of all Borel sets of $X^{(2)}$ in which any equivalence class of $X^{(2)}$ is either totally contained or totally not contained). A function on $g : X^{(2)} \to \mathbb{R}$, which is measurable w.r.t. $\mathcal{B}^*(X^{(2)})$ will be called *saturated measurable function.* It is clear that a function measurable $g$ is saturated measurable iff (17) holds. Each function $f : X^{(1)} \to \mathbb{R}$ measurable w.r.t. $\mathcal{B}(X^{(1)})$ can be identified with a saturated measurable function $g$ such that $g = f \circ \psi$.

The morphism $\psi$ can be viewed as a bijective mapping $\psi : X^{(2)}/_{\sim_\psi} \to X^{(1)}$. It is clear that $\psi$ is a measurable application. To identify the two above measurable spaces $\psi^{-1}$ must be measurable. The main idea, which results from this reasoning, is that the measurable space $(X^{(1)}, \mathcal{B}(X^{(1)}))$ can be embedded in the measurable space $(X^{(2)}, \mathcal{B}(X^{(2)}))$ and the measurable function on $X^{(1)}$ can be identified with the saturated measurable functions on $X^{(2)}$.

Based on the theory of semigroups of Markov processes, one can obtain from the zigzag condition (10): for almost all $t \geq 0$ (i.e. except with a zero Lebesgue measure set of times) the following equalities (versions of (**??**)) hold

$$\begin{aligned} p_t^2(u, \psi^{-1}(A)) &= p_t^1(x, A), \forall x \in X^{(1)}, \forall u \in [u]_\psi = \psi^{-1}(x), \forall A \in \mathcal{B}(X^{(1)}) \quad (26) \\ P_t^2(f \circ \psi)(u) &= P_t^1 f(x), \forall x \in X^{(1)}, \forall u \in [u]_\psi = \psi^{-1}(x), \forall f \in \mathcal{B}^b(X^{(1)}) \end{aligned}$$

Note that $\psi^{-1}(A) \in \mathcal{B}^*(X^{(2)})$. Therefore the transition probabilities of $M^1$ simulates 'equivalence classes' of transition probabilities of $M^2$.

**Remark 8** *The connection between the kernel operator and the infinitesimal generator of the strong process Markov process allows us transform the conditions (19) and (10) as follows*

$$\begin{gathered} L^{(2)} \circ \Psi = \Psi \circ L^{(1)} \\ L^{(2)}(f \circ \psi) = L^{(1)} f \circ \psi, \forall f \in \mathcal{D}(L^{(1)}) \end{gathered} \quad (27)$$

*where $L^{(1)}$ (resp. $L^{(2)}$) is the infinitesimal generator of $M^1$ (resp. $M^2$). The equality (27) holds provided that for each $f \in \mathcal{D}(L^{(1)})$ (the domain of $L^1$) the function $f \circ \psi$ belongs to $\mathcal{D}(L^{(2)})$ (the domain of $L^{(2)}$).*

Since for a GSHS realization the expression of the infinitesimal generator is known, to check if the formula (27) is true for two given GSHS is only a computation exercice.
Recall that the realization of an GSHS has been constructed as a Markov string, i.e. a sequence of diffusion processes with a jumping structure. Then the cone of excessive functions associated to a GSHS can be characterized as a 'sum' of the excessive function cones associated to the diffusion components. This characterization 'explains' the following result.

**Proposition 29** *A zigzag morphism $\psi$ between the realizations of two GSHS $H^1$ and $H^2$ defined as in Def. 4 preserves the continuous parts of the two models.*

**Proof**. Suppose that the two GSHS state spaces are $X^{(1)} = \bigcup_{i \in Q^1} \{i\} \times X^{i(1)}$ and $X^{(2)} = \bigcup_{q \in Q^2} \{q\} \times X^{q(2)}$. We can suppose without loosing the generality that each two modes have empty intersection and therefore $X^{(1)} = \bigcup_{i \in Q^1} X^{i(1)}$ and $X^{(2)} = \bigcup_{q \in Q^2} X^{q(2)}$. The function $\psi$ maps $X^{(2)}$ into $X^{(1)}$. From the construction of $H^1$, as Markov string, we have $V^1 f = \sum_{i \in Q^1} V^{i1} f^i, \forall f \in \mathcal{B}^b(X^{(1)})$,where, for each $i \in Q^1$, $V^{i1}$ is the kernel operators of the component diffusion of $H^1$ which operates on $X^{i(1)}$ and $f^i = f|_{X^{i(1)}} \in \mathcal{B}^b(X^{i(1)})$. A similar expression can be written for $V^2$ (i.e. $V^2 g = \sum\limits_{q \in Q^2} V^{q2} g^q$, $g \in \mathcal{B}^b(X^{(2)})$).

Let $f$ be an arbitrary positive bounded measurable function on $X^{(1)}$. Then for each $i \in Q^1$ consider $f^i$ as before. Let $Y^{i(2)} = \psi^{-1}(X^{i(1)})$ (note that $Y^{i(2)}$ is an open set) and $\psi^i$ be the restriction of $\psi$, which maps $Y^{i(2)}$ into $X^{i(1)}$. Denote $g^i = f^i \circ \psi^i \in \mathcal{B}^b(Y^{i(2)})$ and $g^{iq} = g^i|_{Y^{i(2)} \cap X^{q(2)}}$. The zigzag condition (10) becomes $W^{i2}(f^i \circ \psi^i) = V^{i1} f^i \circ \psi^i$, where $W^{i2}$ is the 'restriction' of $V^2$ to $Y^{i(2)}$, i.e. $W^{i2} g^i = \sum\limits_{q \in Q^2} V^{q2} g^{iq}$ (more intuitively, $W^{i2}$ is the sum of kernels associated to the component diffusions of $H^2$, which operate on $Y^{i(2)}$). Then, for all $x \in X^{i(1)}$ we have

$$W^{i2} g^i(u) = V^{i1} f^i(x), \tag{28}$$

provided that $\psi^i(u) = x$. Because $V^{i1}$ corresponds to a diffusion process, it must be the case that in the left hand side of (28) the 'jumping part' to not longer exist (at least for the saturated measurable functions). Then the kernel $W^{i2}$ corresponds to a continuous process (which might be a diffusion or a switching diffusion process).□

Any zigzag morphism $\psi$ can be extended by (finely) continuity to the boundary of the state spaces. Or, we can suppose from the beginning that the zigzag morphims operate on the closures of the state spaces. We have to assume that the zigzag morphims 'keep' the boundary points, or, in other words, $\psi : \partial X^{(2)} \to \partial X^{(1)}$ is surjective.

**Remark 9** *The finely continuity of a zigzag morphism between the realizations of two GSHS is important only when we use the connection with the associated excessive function cones. Otherwise, we can replace this continuity with the continuity w.r.t. to the initial topologies of the state spaces.*

**Proposition 30** *A zigzag morphism $\psi$ between the realizations of two GSHS $H^1$ and $H^2$ defined as in Def. 4 preserves the jumping structure of the two models.*

**Proof.** For each $x \in X^{(1)}$ there exist, by surjectivity of $\psi$, some elements $u \in X^{(2)}$ such that $\psi(u) = x$. Then, for each $f \in \mathcal{D}(L^{(1)})$, a simple computation of the right hand side of (27) gives

$$L^{(1)} f(x) = L^{(1)}_{cont} f(x) + \lambda^1(x) \int_{\overline{X}^{(1)}} (f(y) - f(x)) R^1(x, dy) \tag{29}$$

and after, the left hand side of (27) is

$$L^{(2)}(f \circ \psi)(u) = L^{(2)}_{cont}(f \circ \psi)(u) + \lambda^2(u) \int_{\overline{X}^{(2)}} [(f \circ \psi)(v) - (f \circ \psi)(u)] R^2(u, dv). \tag{30}$$

From the Prop. 29 we have the equality of the continuous parts of (29) and (30). Then the jumping parts (29) and (30) must coincide. Then

$$\lambda^1(x)\int_{\overline{X}^{(1)}}(f(y)-f(x))R^1(x,dy) = \lambda^2(u)\int_{\overline{X}^{(2)}}[(f\circ\psi)(v)-(f\circ\psi)(u)]R^2(u,dv).$$

The construction of GSHS $H^1$ and $H^2$, as Markov strings, shows that the transition measures $R^1$ and $R^2$ play the role of the transition probabilities when the processes jump from one diffusion to another (see Def.2). Then they satisfy (26), i.e.

$$R^2(u,\psi^{-1}(A)) = R^1(x,A), \forall A \in \mathcal{B}(X^{(1)}).$$

It easily follows that $\lambda^1(x) = \lambda^2(u), \forall u \in [u]_\psi = \psi^{-1}(x).\square$

### 4.5.2 Properties of bisimulation

Consider now two bisimilar GSHS, $H^1$ and $H^2$, with the realizations $M^1$ and $M^2$, respectively. Let $M^{12}$ and $\psi^1, \psi^2$ as in the Def.5. Define the bisimulation relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ by formula (20).

Continuity property of the operator semigroups of the quotient processes $M^1/_\mathcal{R}$ and $M^2/_\mathcal{R}$.

Therefore, the stochastic bisimulation between two GSHS reduces to the bisimulations between their continuous components and between their jump structures. In this way our concept of bisimulation can be related with the bisimulation for piecewise deterministic Markov processes (which are particular class of GSHS) defined in terms of an equivalence relation between the deterministic flows and the probabilistic jumps [24].

# 5 Conclusions

In this paper we develop a notion of stochastic bisimulation for a category of general models for stochastic hybrid systems (which are Markov processes) or, more generally, for the category of strong Markov processes defined on analytic spaces. The morphisms in this category are the zigzag morphims. A zigzag morphism between two Markov processes is a surjective (finely) continuous measurable functions between their state spaces which 'commutes' with the kernel operators of the processes considered. The fundamental technical contribution is the proof that this stochastic bisimulation is indeed an equivalence relation.

The second result of the paper is that this bisimulation relation for GSHS (the stochastic hybrid system models we are dealing in this paper) implies the same kind of bisimulation for their continuous parts and respectively for their jumping structures.

# 6 Further Work

From stochastic analysis viewpoint, most of the models of stochastic hybrid systems are strong Markov processes. Then, many tools available for the Markov process studying can be used to characterize their main features. On the other hand, some of them can be included in the class of random dynamical systems (stochastic extensions of the dynamical systems). Therefore the whole ergodic theory or stability results available for random dynamical systems might be applied to them. As well, stability results of random dynamical systems [6] can be lifted to these models of stochastic hybrid systems. Moreover, because in the deterministic case there are characterizations of the Lyapunov functions in terms of excessive function [17], it might be possible to investigate similar connections in the stochastic case.

From the verification and analysis of stochastic hybrid systems perspective, a concept of stochastic bisimulation can facilitate the way towards a model checking of stochastic hybrid systems.

The work presented in this paper and the above discussion allow us to point out some possible research directions in the stochastic hybrid system framework:

- Use the stochastic bisimulation to get manageable sized system abstractions;
- Use the stochastic bisimulation to investigate the reachability problem;
- Make a comparative study of the different approaches on reachability analysis for stochastic hybrid systems: 1. the approach based on the hitting times and hitting probabilities for a target set [11]; 2. the approach based on the so-called Dirichlet forms and excessive functions [10]; 3. the approach based on Lyapunov function (for the switching diffusion processes, see [26]).

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