Filtering and Identification of Affine Term Structures from Yield Curve Data

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Abstract

We consider a slight perturbation of the Hull-White short rate model and the resulting modified forward rate equation. We identify the model coefficients by using the martingale property of the normalized bond price. The forward rate and the system parameters are then estimated by infinite dimensional Kalman filtering equations, coupled with the usual statistical techniques.
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1 Introduction

There is a vast literature ([8, 5, 6]) on estimating parameters of short rate models in finance. One popular approach is to take a short rate model that leads to an exponential-affine expression for the corresponding bond price. The yield is then easy to calculate. Artificial noises are then added to yields of different maturities and Kalman filtering is used to estimate the model parameters. As a byproduct one also gets the minimum variance estimate of the short rate, which is otherwise not observed in the market.

There are two fundamental drawbacks to this approach. The first is the artificial noises added to yields of different maturities, which is hard to justify. This is done only for the purpose of the filtering algorithm to work. The other difficulty is the robustness of the model. If a model is very close to one of the usual short rate models, but not exactly matches that model, there is no guarantee that the bond price will still be of the exponential-affine form.

We, therefore, approach the problem from a different perspective. We start with the usual Hull-White model of the short rate, and assume that a slightly different model will lead to a slightly perturbed bond price of the usual one derived from the Hull-White model. We also consider this perturbation to be generated by an infinite dimensional noise, as it should depend on all times to maturity. In this approach, we do not need to add artificial noises to bond yield of different maturities in order to use the Kalman filtering algorithm. We estimate forward rates as solutions of the resulting filtering problem, and estimate model parameters by usual statistical techniques.

2 A New Model for the Short Rate

Suppose that the short rate evolves according to the Hull-White model:

$$dr(t) = \{\Theta(t) - ar(t)\}dt + \sigma_r dW_r(t) \tag{2.1}$$

where $\{W_r(t), t \geq 0\}$ is a scalar standard Brownian motion, $\Theta(t)$ is a deterministic function of time, while a and σ_r are constants. It then follows that the bond price $P(t,T), 0 < T < \hat{T}$, is given by

$$P(t,T) = \exp\{-\int_0^{T-t} [A(t,x) + B(t,x)r(t)]dx\} \tag{2.2}$$

For exact expressions of A(t,x) and B(t,x), see [4].

Let us now assume that the short rate does not exactly follow (2.1), but is very close to this model. This will cause the bond price to be somewhat perturbed from the formula given in (2.2). Suppose that this perturbed bond price has the following expression:

$$P(t,T) = \exp\{-\int_0^{T-t} [A(t,x) + B(t,x)r(t) + \int_0^t \sigma dw(s, x+t-s)]dx\} \tag{2.3}$$

where $r(t)$ follows equation (2.1) with $\Theta(t)$ a random function of time and $w(t,x)$ is a two parameter Brownian motion process represented by

$$w(t,x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} e_k(x)\beta_k(t)$$

where e_k is a sequence of differentiable functions forming an orthonormal basis in $L(0,\hat{T})$ and $\{\beta_k(t)\}$ are mutually independent Brownian motion processes. In the sequel, we set $H = L^2(0,\hat{T})$ with the inner product $(\cdot,\cdot)$. Hence the Brownian motion process $w(t,\cdot)$ is regarded as the H-valued BMP with its incremental covariance operator Q;

$$E\{(\phi_1, w(t))(w(t), \phi_2)\} = (\phi_1, Q\phi_2)t, \text{ for } \phi_1, \phi_2 \in H$$

with

$$Tr\{Q\} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} < \infty.$$

We also represent this kernel by

$$Q = \int_0^{\hat{T}} q(x,y)(\cdot)dy. \tag{2.4}$$

Hence we have

$$\int_0^t \sigma dw(s,x+t-s) = \sum_{k=1}^{\infty} \int_0^t \sigma \frac{1}{\lambda_k} e_k(x+t-s) d\beta_k(s). \tag{2.5}$$

The above stochastic integral term denotes the modeling error between the Affine term structure constructed by A and Br and the true term structure.

Hence the usual forward process $f(t,x)$ which is defined by $P(t,T) = \exp\{-\int_0^{T-t} f(t,x)dx\}$ is given by

$$f(t,x) = A(t,x) + B(t,x)r(t) + \int_0^t \sigma dw(s,x+t-s). \tag{2.6}$$

See [1] for the general form of hyperbolic type formulation for $f(t,x)$ and the structure of $w(s,x+t-s)$.

Noting that the exact spot rate $r^e(t)$ is given by $f(t,0)$, we have

$$r^e(t) = A(t,0) + B(t,0)r(t) + \int_0^t \sigma dw(s,t-s), \tag{2.7}$$

with $r^e(0) = r(0)$. It should be noted that the $r^e(t)$-process belongs to $L^2(\Omega; C([0,t_f]; R^1)$ from the following estimate.

$$E\{\sup_{0 \le t \le t_f} |\int_0^t \sum_{k=1}^{\infty} \sigma \frac{1}{\lambda_k} e_k(t-s) d\beta_k(s)|^2\} = \int_0^{t_f} \sum_{k=1}^{\infty} (\sigma \frac{1}{\lambda_k} e_k(t_f - s))^2 ds \le \sigma^2 Tr\{Q\} t_f.$$

Now we identify A and B by using the fact that the discount process $\bar{P}(t,T) = P(t,T)/\exp\{\int_0^t r^e(s)ds\}$ must be a martingale.

Noting that

$$\begin{aligned}
& d[\int_0^{T-t} \{\int_0^t \sigma dw(s,x+t-s)\}dx \\
& = -\{\int_0^t \sigma dw(s,T-s)\}dt + \int_0^{T-t} d\{\sum_{k=1}^{\infty} \int_0^t \sigma \frac{1}{\lambda_k} e_k(x+t-s) d\beta_k(s)\}dx \\
& = -\{\int_0^t \sigma dw(s,T-s)\}dt + \{\int_0^{T-t} \{\sum_{k=1}^{\infty} \int_0^t \sigma \frac{1}{\lambda_k} \frac{\partial e_k(x+t-s)}{\partial t} d\beta_k(s)\}dx\}dt \\
& + \sum_{k=1}^{\infty} \{\int_0^{T-t} \sigma \frac{1}{\lambda_k} e_k(x)dx\} d\beta_k(t)
\end{aligned} \tag{2.8}$$

and

$$\int_0^{T-t}\{\sum_{k=1}^{\infty}\int_0^t\sigma\frac{1}{\lambda_k}\frac{\partial e_k(x+t-s)}{\partial t}d\beta_k(s)\}dx$$
$$=\int_0^{T-t}\{\sum_{k=1}^{\infty}\int_0^t\sigma\frac{1}{\lambda_k}\frac{\partial e_k(x+t-s)}{\partial x}d\beta_k(s)\}dx$$
$$=\sum_{k=1}^{\infty}\int_0^t\sigma\frac{1}{\lambda_k}\{e_k(T-s)-e_k(t-s)\}d\beta_k(s)$$
$$=\int_0^t\sigma dw(s,T-s)-\int_0^t\sigma dw(s,t-s), \tag{2.9}$$

we have

$$(2.8)=-\{\int_0^t\sigma dw(s,t-s)\}dt+\sum_{k=1}^{\infty}\{\int_0^{T-t}\sigma\frac{1}{\lambda_k}e_k(x)dx\}d\beta_k(t). \tag{2.10}$$

By using Ito's formula, the differential form of $P(t,T)$ becomes

$$\frac{dP(t,T)}{P(t,T)}=[A(t,T-t)-\int_0^{T-t}\frac{\partial A(t,T)}{\partial t}dx$$
$$-\int_0^{T-t}B(t,x)dx\Theta(t)+\frac{\sigma_r^2}{2}(\int_0^{T-t}B(t,x)dx)^2$$
$$+\frac{1}{2}\sum_{k=1}^{\infty}(\int_0^{T-t}\sigma\frac{1}{\lambda_k}e_k(x)dx)^2+\int_0^t\sigma dw(s,t-s)]dt$$
$$+[B(t,T-t)-\int_0^{T-t}\frac{\partial B(t,x)}{\partial t}dx+\int_0^{T-t}B(t,x)dxa]r(t)dt$$
$$-\int_0^{T-t}B(t,x)dx\sigma_r dW_r(t)-\sum_{k=1}^{\infty}(\int_0^{T-t}\sigma\frac{1}{\lambda_k}e_k(x)dx)d\beta_k(t). \tag{2.11}$$

The differential form of the discounted bond price $\bar{P}(t,T)=P(t,T)/\exp\{\int_0^t r^e(s)ds\}$ becomes

$$\frac{d\bar{P}(t,T)}{\bar{P}(t,T)}=[A(t,T-t)-A(t,0)-\int_0^{T-t}\frac{\partial A(t,T)}{\partial t}dx$$
$$-\int_0^{T-t}B(t,x)dx\Theta(t)+\frac{\sigma_r^2}{2}(\int_0^{T-t}B(t,x)dx)^2+\frac{1}{2}\sum_{k=1}^{\infty}(\int_0^{T-t}\sigma\frac{1}{\lambda_k}e_k(x)dx)^2]dt$$
$$+[B(t,T-t)-B(t,0)-\int_0^{T-t}\frac{\partial B(t,x)}{\partial t}dx+\int_0^{T-t}B(t,x)dxa]r(t)dt$$
$$-\int_0^{T-t}B(t,x)dx\sigma_r dW_r(t)-\sum_{k=1}^{\infty}(\int_0^{T-t}\sigma\frac{1}{\lambda_k}e_k(x)dx)d\beta_k(t).$$

Noting that

$$A(t,T-t)-A(t,0)=\int_0^{T-t}\frac{\partial A(t,x)}{\partial x}dx$$

and

$$B(t,T-t)-B(t,0)=\int_0^{T-t}\frac{\partial B(t,x)}{\partial x}dx,$$

we derive the following two equations for supporting the arbitrage free condition:

$$\begin{aligned}\frac{\partial A(t,x)}{\partial t} = \frac{\partial A(t,x)}{\partial x} - B(t,x)\Theta(t) + \sigma_r^2 B(t,x)\int_0^x B(t,y)dy \\ +\sigma^2\int_0^x q(x,y)dy \end{aligned} \tag{2.12}$$

and

$$\frac{\partial B(t,x)}{\partial t}=\frac{\partial B(t,x)}{\partial x}+aB(t,x)$$

with the boundary conditions:

$$A(t,0)=0 \text{ and } B(t,0)=1.$$

It is easy to show that

$$B(t,x)=e^{-ax},$$

and

$$\sum_{k=1}^{\infty}\int_0^{T-t}\sigma\frac{1}{\lambda_k}e_k(y)dy\sigma\frac{1}{\lambda_k}e_k(x)=\int_0^{T-t}q(y,x)dy.$$

Hence the forward process $f(t,x)$ satisfies the following stochastic partial differential equation:

$$\begin{aligned} df(t,x) &= \frac{\partial f(t,x)}{\partial x}dt+\{\sigma_r^2 e^{-ax}\int_0^x e^{-ay}dy+\sigma^2\int_0^x q(x,y)dy\}dt \\ &\quad + e^{-ax}\sigma_r dW_r(t)+\sigma dw(t,x), \qquad (2.13)\\ f(0,x) &= f_o(x). \qquad (2.14)\end{aligned}$$

It is also possible to show the relation between $\Theta(t)$ and $f(t_1,t)$ for $t_1<t$. From (2.6), we have

$$A(t_1,x)=f(t_1,x)-e^{-ax}r(t_1)-\int_0^{t_1}\sigma dw(s,x+t_1-s).$$

Hence it follows from (2.12) that

$$\begin{aligned} A(t,x) &= \{f(t_1,x+t-t_1)-e^{-a(x+t-t_1)}r(t_1)-\int_0^{t_1}\sigma dw(s,x+t-s)\} \\ &\quad +\int_{t_1}^t\left[-e^{-a(x+t-s)}\Theta(s)+\sigma_r^2e^{-a(x+t-s)}\int_0^{x+t-s}e^{-ay}dy+\sigma^2\int_0^{x+t-s}q(x+t-s,y)dy\right]ds. \end{aligned} \tag{2.15}$$

It follows from $A(t,0)=0$ that

$$e^{at_1}r(t_1)-e^{at}f(t_1,t-t_1)+e^{at}\int_0^{t_1}\sigma dw(s,t-s)$$
$$=\int_{t_1}^{t}\left[-e^{as}\Theta(s)+\sigma_r^2e^{as}\int_0^{t-s}e^{-ay}dy+\sigma^2e^{at}\int_0^{t-s}q(t-s,y)dy\right]ds. \tag{2.16}$$

Differentiating the above equation with respect to t, we have

$$-ae^{at}f(t_1,t-t_1)-e^{at}\frac{\partial f(t_1,t-t_1)}{\partial x}+ae^{at}\int_0^{t_1}\sigma dw(s,t-s)+e^{at}\int_0^{t_1}\sigma d\frac{\partial w(s,t-s)}{\partial x}$$
$$=-e^{at}\Theta(t)+\int_{t_1}^{t}\Bigg[\sigma_r^2e^{as}e^{-a(t-s)}+\sigma^2ae^{at}\int_0^{t-s}q(t-s,y)dy$$
$$+\sigma^2e^{at}q(t-s,t-s)+\sigma^2e^{at}\int_0^{t-s}\frac{\partial q(t-s,y)}{\partial x}dy\Bigg]ds$$

Taking a limit as $t_1\to t$, we get

$$\Theta(t)=af(t,0)+\frac{\partial f(t,0)}{\partial x}-a\int_0^t\sigma dw(s,t-s)-\int_0^t\sigma d\{\frac{\partial w(s,t-s)}{\partial x}\}. \tag{2.17}$$

We also have for $t_1\to 0$,

$$\Theta(t)=af(0,t)+\frac{\partial f(0,t)}{\partial x}+e^{-at}\int_0^t\Bigg[\sigma_r^2e^{as}e^{-a(t-s)}+\sigma^2ae^{at}\int_0^{t-s}q(t-s,y)dy$$
$$+\sigma^2e^{at}q(t-s,t-s)+\sigma^2e^{at}\int_0^{t-s}\frac{\partial q(t-s,y)}{\partial x}dy\Bigg]ds.$$

Note that the process $f_w(t,x)=\int_0^t\sigma dw(s,x+t-s)$ is a solution of

$$df_w(t,x)=\frac{\partial f_w(t,x)}{\partial x}dt+\sigma dw(t,x),\ f_w(0,x)=0 \tag{2.18}$$

Let

$$q_a(x)=\sigma_r^2e^{-ax}\int_0^x e^{-ay}dy+\sigma^2\int_0^x q(x,y)dy.$$

Defining $f_r(t,x)$ and $R(t)$ by

$$\frac{\partial f_r(t,x)}{\partial t}=\frac{\partial f_r(t,x)}{\partial x}+q_a(x),\ f_r(0,x)=f_o(x), \tag{2.19}$$

and

$$dR(t)=-aR(t)dt+\sigma_r dW_r(t),R(0)=0. \tag{2.20}$$

we readily see that $f(t,x)$ may be decomposed as

$$f(t,x)=f_r(t,x)+e^{-ax}R(t)+f_w(t,x), \tag{2.21}$$

3 The Observation Mechanism

In practice, we only observe the yield curve data from the market, and do not precisely know the forward rate process. The observed yield is given by

$$Y(t,T-t) = -\frac{1}{T-t}\log P(t,T).$$

Setting the time-to-maturity $\tau = T-t$ as constant, we have

$$\begin{aligned} Y(t,\tau) &= -\frac{1}{\tau}\log P(t,t+\tau) \\ &= \frac{1}{\tau}\int_0^\tau f(t,x)dx \\ &= \frac{1}{\tau}\int_0^\tau \{f_r(t,x)+e^{-ax}R(t)+f_w(t,x)\}dx. \end{aligned} \tag{3.1}$$

Hence for $i = 1,2,\cdots,m$

$$\begin{aligned} dY(t,\tau_i) &= \frac{1}{\tau_i}\int_0^{\tau_i}(df_r(t,x)+e^{-ax}dR(t)+df_w(t,x))dx, \\ &= \frac{1}{\tau_i}\{f_r(t,\tau_i)-f_r(t,0)\}dt \\ &+ \frac{1}{\tau_i}\int_0^{\tau_i} q_a(x)dxdt + \frac{1-e^{-a\tau_i}}{a\tau_i}[-aR(t)dt+\sigma_r dW_r(t)] + \frac{1}{\tau_i}\{f_w(t,\tau_i)-f_w(t,0)\}dt \\ &+ \frac{1}{\tau_i}\sigma\sum_{j=1}^{\infty}\frac{1}{\lambda_i}\int_0^{\tau_i} e_j(x)dxd\beta_j(t) \\ &= [H_i\{f_r(t,\cdot)+f_w(t,\cdot)\} - aG_i(a)R(t)]dt + F_i(a)dt + G_i(a)\sigma_r dW_r(t) + \sigma K_i dw(t) \end{aligned} \tag{3.2}$$

where $\forall\phi(x) \in H^1$

$$\begin{aligned} H_i\phi &= (\phi(\tau_i)-\phi(0))\frac{1}{\tau_i} \\ G_i(a) &= \frac{1-e^{-a\tau_i}}{a\tau_i} \\ F_i(a) &= \frac{1}{\tau_i}\int_0^{\tau_i} q_a(x)dx \\ &= \frac{1}{\tau_i}\int_0^{\tau_i}[\sigma_r^2 e^{-ax}\int_0^x e^{-ay}dy + \sigma^2\int_0^x q(x,y)dy]dx \end{aligned}$$

and

$$K_i dw(t) = \frac{1}{\tau_i}\sum_{j=1}^{\infty}\frac{1}{\lambda_i}\int_0^{\tau_i} e_j(x)dxd\beta_j(t). \tag{3.3}$$

Now we set the m-dimensional observation;

$$\vec{Y}_{(m)}(t) = [Y(t,\tau_1)\ Y(t,\tau_2)\ \cdots Y(t,\tau_m)]'. \tag{3.4}$$

The differential form of $\vec{Y}_{(m)}$ becomes

$$d\vec{Y}_{(m)}(t) = [H(f_r(t,\cdot) + f_w(t,\cdot)) - aG(a)R(t)]dt + F(a)dt + \sigma_r G(a)dW_r(t) + \sigma K dw(t), \quad (3.5)$$

where

$$Kdw(t) = \frac{1}{\tau_i}[\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_0^{\tau_i} e_k(x)dxd\beta_k(t)]_{m\times 1}$$

with

$$\begin{aligned} KQK^* &= [\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2 \tau_i \tau_j} \int_0^{\tau_i} e_k(x)dx \int_0^{\tau_j} e_k(x)dx]_{m\times m} \\ &= [\frac{1}{\tau_i \tau_j} \int_0^{\tau_i} \int_0^{\tau_j} q(x,y)dxdy]_{m\times m}. \end{aligned}$$

4 The Filtering Problem

Now we summarize the system and observation mechanism in the usual vector notation;

$$d \begin{bmatrix} f_r(t,x) \\ R(t) \\ f_w(t,x) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_r(t,x)}{\partial x} \\ -aR(t) \\ \frac{\partial f_w(t,x)}{\partial x} \end{bmatrix} dt + \begin{bmatrix} q_a(x) \\ 0 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma_r dW_r(t) \\ \sigma dw(t,x) \end{bmatrix},$$

with

$$d\vec{Y}_{(m)}(t) = \mathscr{H}(a) \begin{bmatrix} f_r(t,\cdot) \\ R(t) \\ f_w(t,\cdot) \end{bmatrix} dt + F(a)dt + \sigma_r G(a)dW_r(t) + \sigma K dw(t,\cdot),$$

where

$$\mathscr{H}(a) = [H,\ -aG(a),\ H].$$

Under the following assumption (see [3]):

$$\sigma_r^2 G(a)G^*(a) + \sigma^2 KQK^* > 0, \quad (4.1)$$

we can derive the optimal filtering equations. Before writing down the optimal filtering equations, we first show the feasibility of the above assumption by using the US-treasury bond data.

Example 4.1 *We will check the assumption (4.1) numerically by using the US-treasury bond data. As shown in Fig.1, we used the 7-dimensional yield curve data, i.e., $\vec{Y}_7(t)$. It is well known that [7],[2],[3]*

$$\lim_{n\to\infty} \frac{1}{n\Delta t} \sum_{i=1}^{n} (\vec{Y}_7(t_{i+1}) - \vec{Y}_7(t_i))(\vec{Y}_7(t_{i+1}) - \vec{Y}_7(t_i))^* = \sigma_r^2 GG^* + \sigma^2 KQK^* \ \ a.s. \quad (4.2)$$

In this numerical example, we set $\Delta t = t_{i+1} - t_i = 0.0027 year = 1 day$.

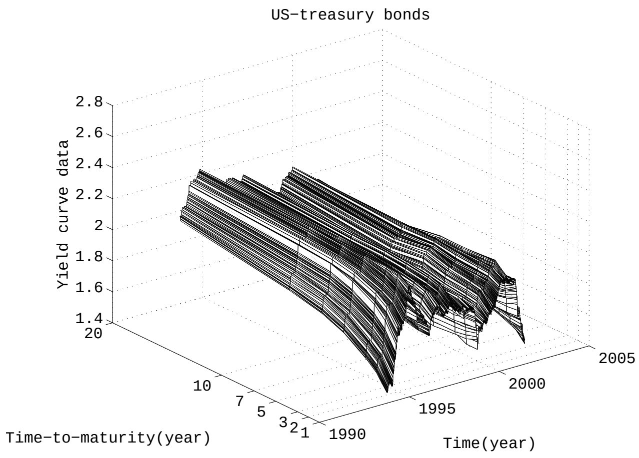

Figure 1: US-treasury bond (yield curve data)

From (4.2), we get $\sigma_r^2 GG^* + \sigma^2 KQK^*$ *numerically as shown in Fig.2.*

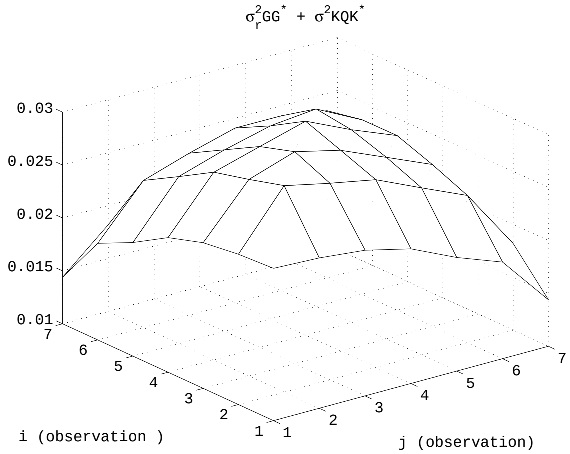

Figure 2: $\sigma_r^2 GG^* + \sigma^2 KQK^*$

As the determinant of the obtained $\sigma_r^2 GG^* + \sigma^2 KQK^*$ *matrix becomes positive, it is also possible to calculate the inverse of* $\sigma_r^2 GG^* + \sigma^2 KQK^*$ *matrix as also shown in Fig.3.*

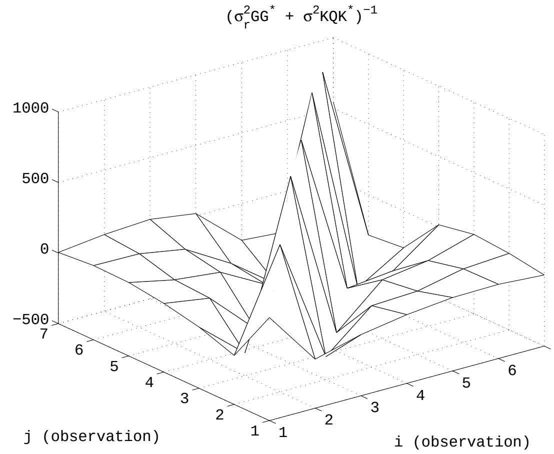

Figure 3: $(\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1}$

Usually we would construct the optimal filter for $f_r(t,x), R(t)$ and $f_w(t,x)$. The process $f_r(t,x)$ does not contain the random additive noise but the initial $f_o(x)$ is random and unknown. However the filter gain equation then consists of 6 equations, which are not easy to apply in practical situations. (We will list this exact algorithm in the Appendix-A, because the estimate $f_r(t,x)$ form $\vec{Y}_m(t)$ is more accurate than the estimate $\tilde{f}_r(t,x)$((4.3) below) only from $\vec{Y}_m(0)$.)

In this section, we present an empirical method for estimating the $f_r(t,x)$-process and separate out this process for the filtering algorithm.

4.1 Empirical estimation for $f_r(t,x)$

In practice, from the initial yield curve $\vec{Y}_m(0)$, one can construct the initial factor curve $f_o(x)$ [2];

- Construct the whole yield curve $\hat{Y}(0,x)$ from $\vec{Y}_m(0)$ by using the spline interpolation method.
- The initial estimate for $f_o(x)$ is given by

$$\tilde{f}_o(x) = x\frac{d\hat{Y}(0,x)}{dx} + \hat{Y}(0,x).$$

Hence the estimate $\tilde{f}_r(t,x)$ is a solution of

$$\frac{\partial \tilde{f}_r(t,x)}{\partial t} = \frac{\partial \tilde{f}_r(t,x)}{\partial x} + q_a(x),\ \tilde{f}_r(0,x) = \tilde{f}_o(x). \tag{4.3}$$

4.2 Optimal estimates for $R(t)$ and $f_w(t,x)$

The optimal estimates for $R(t)$ and $f_w(t,x)$ are given by

$$
\begin{aligned}
&d\begin{bmatrix} \hat{R}(t) \\ \hat{f}_w(t,x) \end{bmatrix} = \begin{bmatrix} -a\hat{R}(t) \\ \frac{\partial \hat{f}_w(t,x)}{\partial x} \end{bmatrix} dt \\
&+ \left(\mathscr{P}(t) \begin{bmatrix} -aG^*(a) \\ H^* \end{bmatrix} + \begin{bmatrix} \sigma_r^2 G(a) \\ \sigma^2 QK^* \end{bmatrix} \right) (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1} \\
&\times (d\vec{Y}_{(m)}(t) - \mathscr{H}(a) \begin{bmatrix} \tilde{f}_r(t,\cdot) \\ \hat{R}(t) \\ \hat{f}_w(t,\cdot) \end{bmatrix} dt - F(a)dt),
\end{aligned} \tag{4.4}
$$

where $\tilde{f}_r(t,x)$ is a solution of (4.3)

$$
QK^* = [\frac{1}{\tau_1}\int_0^{\tau_1} q(x,y)dy, \cdots, \frac{1}{\tau_m}\int_0^{\tau_m} q(x,y)dy], \tag{4.5}
$$

$$
\mathscr{P}(t) = \begin{pmatrix} \mathbf{P}_R(t) & \mathbf{P}_{Rw}(t) \\ \mathbf{P}_{wR}(t) & \mathbf{P}_w(t) \end{pmatrix}, \tag{4.6}
$$

$\mathbf{P}_{Rw} = \mathbf{P}_{wR}^*$ and

$$
\mathbf{P}_R(t) = p_R(t), \;\; \mathbf{P}_{Rw}(t) = p_{Rw}(t,x), \;\; \mathbf{P}_W(t) = \int_0^{\hat{T}} p_w(t,x,y)(\cdot)dy.
$$

The kernel equations are given by

$$
\begin{aligned}
&\frac{dp_R(t)}{dt} = -2ap_R(t) + \sigma_r^2 - \Big[-p_R(t)\frac{1-e^{-a\tau_i}}{\tau_i} \\
&+ (p_{Rw}(t,\tau_i) - p_{Rw}(t,0))\frac{1}{\tau_i} + \sigma_r^2 \frac{1-e^{-a\tau_i}}{a\tau_i} \Big]_{1\times m} (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1} \\
&\times \Big[-p_R(t)\frac{1-e^{-a\tau_j}}{\tau_j} + (p_{Rw}(t,\tau_j) - p_{Rw}(t,0))\frac{1}{\tau_j} + \sigma_r^2\frac{1-e^{-a\tau_j}}{a\tau_j} \Big]_{m\times 1}, \; p_R(0) = 0,
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
&\frac{\partial p_w(t,x,y)}{\partial t} = \frac{\partial p_w(t,x,y)}{\partial x} + \frac{\partial p_w(t,x,y)}{\partial y} + \sigma^2 q(x,y) - \Big[-p_{wR}(t,x)\frac{1-e^{-a\tau_i}}{\tau_i} + (p_w(t,x,\tau_i) \\
&- p_w(t,x,0))\frac{1}{\tau_i} + \frac{\sigma^2}{\tau_i}\int_0^{\tau_i} q(x,y)dy \Big]_{1\times m} (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1} \Big[-p_{wR}(t,y)\frac{1-e^{-a\tau_j}}{\tau_j} \\
&+ (p_w(t,\tau_j,y) - p_w(t,0,y))\frac{1}{\tau_j} + \frac{\sigma^2}{\tau_j}\int_0^{\tau_j} q(x,y)dx \Big]_{m\times 1}, \; p_w(t,x,y) = 0,
\end{aligned} \tag{4.8}
$$

$$\frac{\partial p_{Rw}(t,x)}{\partial t} = -ap_{Rw}(t,x) + \frac{\partial p_{Rw}(t,x)}{\partial x} - \left[-p_R(t)\frac{1-e^{-a\tau_i}}{\tau_i} + (p_{Rw}(t,\tau_i) - p_{Rw}(t,0))\frac{1}{\tau_i} \right.$$
$$\left. +\sigma_r^2 \frac{1-e^{-a\tau_i}}{a\tau_i} \right]_{1\times m} (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1} \left[-p_{Rw}(t,x)\frac{1-e^{-a\tau_j}}{\tau_j} \right.$$
$$\left. +(p_w(t,\tau_j,x) - p_w(t,0,x))\frac{1}{\tau_j} + \frac{\sigma^2}{\tau_j}\int_0^{\tau_j} q(y,x)dy \right]_{m\times 1}, \; p_{Rw}(0,x) = 0. \tag{4.9}$$

Remark 4.1 *From above results, we obtain the estimate of* $\Theta(t)$ *from (2.17)*

$$\hat{\Theta}(t) = a\tilde{f}_r(t,0) + \frac{\partial \tilde{f}_r(t,0)}{\partial x}. \tag{4.10}$$

5 Identification

5.1 Identification of the Noise Kernel

As stated in the previous example, we can identify the noise kernel from the yield curve data. The mathematical expression of (4.2) becomes

$$\frac{1}{t}\{\vec{Y}_{(m)}(t)\vec{Y}^*_{(m)}(t) - \vec{Y}_{(m)}(0)Y^*_{(m)}(0) - \int_0^t \vec{Y}_{(m)}(s)d\vec{Y}^*_{(m)}(s) - \int_0^t d\vec{Y}_{(m)}(s)\vec{Y}^*_{(m)}(s)\}$$
$$= \sigma_r^2 GG^* + \sigma^2 KQK^* \;\; a.s. \tag{5.1}$$

Hence we get from (5.1)

$$[F(a)]_i = \tau_i[\sigma_r^2 GG^* + \sigma^2 KQK^*]_{ii}. \tag{5.2}$$

For the US-bond data, we have

$$F(a) = [0.0122, 0.0295, 0.0450, 0.0755, 0.1003, 0.1290, 0.1835],$$

as shown in Fig.4.

In practice, we realize the above formula (5.1) in the discrete-time version (4.2) as used in the previous example. Now we will identify a, σ_r and σ from the value of (5.1).

- We set a number $0 < \rho < 1$ and identify $\hat{a}$ and $\hat{\sigma}_r$ such that

$$\hat{\sigma}_r G_1(\hat{a})G_1(\hat{a}) = \hat{\sigma}_r^2 \frac{1-e^{-\hat{a}\tau_1}}{\hat{a}\tau_1}$$
$$= \rho\left\{\frac{1}{t}\{\vec{Y}_{(1)}(t)\vec{Y}_{(1)}(t) - \vec{Y}_{(1)}(0)\vec{Y}_{(1)}(0) - 2\int_0^t \vec{Y}_{(1)}(s)d\vec{Y}_{(1)}(s)\}\right\}. \tag{5.3}$$

- Check the positivity of

$$\sigma_r^2 GG^* + \sigma^2 KQK^* - \hat{\sigma}_r^2 G(\hat{a})G(\hat{a})^*. \tag{5.4}$$

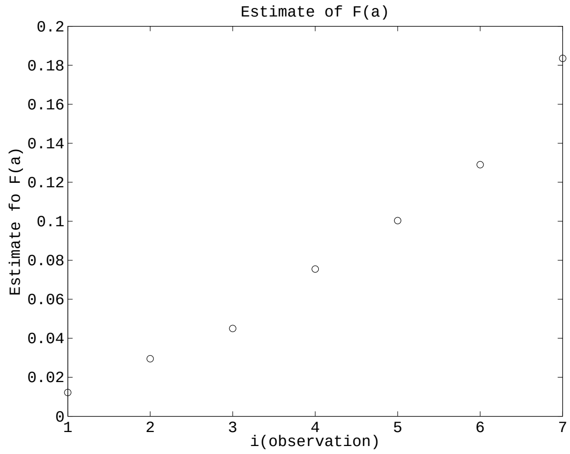

Figure 4: $F(a)$ for US-bond

- We adjust the value of ρ to be as large as possible supporting the positivity of (5.4).

The above identification procedure works due to the fact that a is positive and $r(t)$ has the fundamental property of the spot rate. The yield curve $\vec{Y}_{(1)}(t)$ contains the most crucial information for the spot rate rather than $\vec{Y}_{(2)}(t)\cdots$. The parameter ρ implies that $\sigma_r^2 G_1 G_1$ is assumed to be $100\rho\%$ of $\sigma_r^2 G_1 G_1 + [\sigma^2 KQK^*]_{11}$.

In the US-bond case, we set

$$\rho = 0.3$$

and the estimated results are

$$\hat{\sigma}_r = 0.2940,\ \hat{a} = 3.3114$$

The exact shape of $\hat{\sigma}_r^2 GG^*$ is presented in Fig.5.

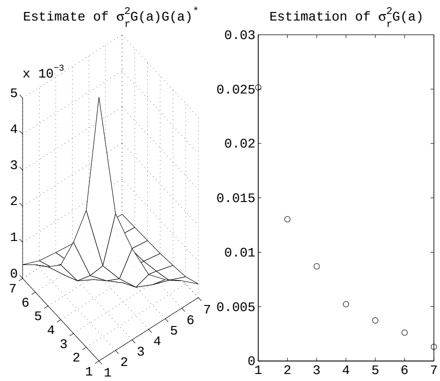

Figure 5: $\sigma_r^2 GG^*$ and $\sigma_r^2 G$ for US-bond

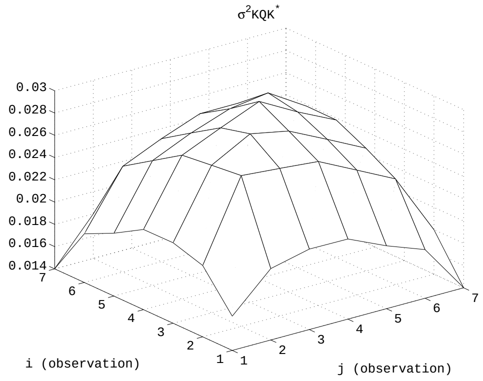

Figure 6: $\overline{\sigma^2 KQK^*}$ for US-bond

Hence we obtain the value of

$$\overline{\sigma^2 KQK^*} = \sigma_r^2 GG^* + \sigma^2 KQK^* - \hat{\sigma}_r^2 G(\hat{a})G^*(\hat{a}). \tag{5.5}$$

For the US-bond case, the obtained $\overline{\sigma^2 KQK^*}$ from (5.5) is shown in Fig.6.

- We set the functional form of $\sigma^2 Q$ for $\hat{T} = 30$ as

$$\begin{aligned}\bar{\sigma}^2 Q(c) &= \bar{\sigma}^2 \sum_{i=1}^{20} \frac{1}{i^2} \exp(-cx)\sin(\frac{i\pi x}{30}) \int_0^{30} \exp(-cy)\sin(\frac{i\pi y}{30})(\cdot)dy \\ &= \int_0^{30} \bar{\sigma}^2 q_c(x,y)(\cdot)dy,\end{aligned} \tag{5.6}$$

  where $\bar{\sigma}$ and c are unknown parameters to be estimated.

- Find the values of $\bar{\sigma}$ and c such that

$$[\hat{\sigma},\ \hat{c}] = \mathrm{argmin}_{\bar{\sigma}c} ||\sigma^2 KQK^*(5.5) - \bar{\sigma}^2 KQ(c)K^*(5.6)||^2_{R^m}. \tag{5.7}$$

  For the US-bond data, we obtain

$$\hat{\sigma}^2 = 0.6269, \hat{c} = 0.1627$$

  The shapes of $\hat{\sigma}^2 q(x,y)$ and $\sigma^2 QK^*$ from (5.7) are shown in Figs.7 and 8, respectively.

- Finally we also obtain the value $q_a(x)$ as shown in Fig.9.

5.2 Filtering and Identification in the Real World

Our data used in the filtering for the factor process is a yield curve. As stated in the previous section, we got all parameters for the filtering algorithm. However from [6],[5] and our experimental studies for the US-bond data in [2], we have to include the market price of risk term in the factor

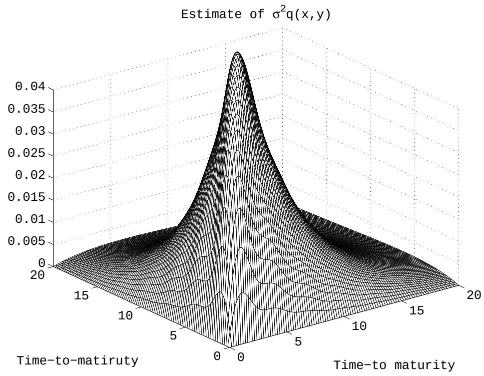

Figure 7: $\sigma^2 q(x,y)$ for US-bond

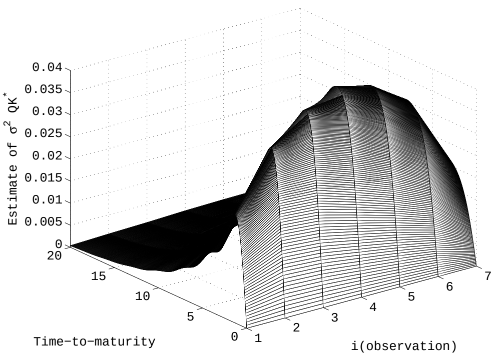

Figure 8: $\sigma^2 QK^*$ for US-bond

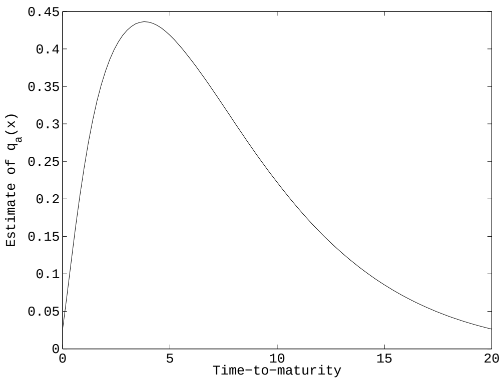

Figure 9: $q_a(x)$ for US-bond

model. Here we only consider the market price of risk term corresponding to $W_r(t)$. Now we reset $q_a(x)$ as $q_{a\lambda_r}(x)$;

$$\begin{aligned} q_{a\lambda_r}(x) &= \sigma_r^2 e^{-ax}\int_0^x e^{-ay}dy + \sigma^2\int_0^x q(x,y)dy - \lambda_r\sigma_r e^{-ax} \\ &= q_a(x) - \lambda_r\sigma_r g_a(x), \end{aligned} \tag{5.8}$$

where

$$g_a(x) = e^{-ax}.$$

$F(a)$ is also reset as $F_{a\lambda_r}$

$$F_{a\lambda_r} = F(a) - \lambda_r\sigma_r K g_a. \tag{5.9}$$

We denote the filtering outputs with $g_{a\lambda_r}, F_{a\lambda_r}$ terms by $\tilde{f}_{r\lambda_r}(t,x), \hat{R}_{\lambda_r}(t), \hat{f}_{w\lambda_r}(t,x)$. Hence λ_r should be estimated to maximize the likelihood functional, i.e.,

$$\begin{aligned} \hat{\lambda}_r = \mathrm{argmax}_{\lambda_r\in R^1}\{\int_0^{t_f}(\mathscr{H}(a)\begin{bmatrix}\tilde{f}_{r\lambda_r}(t)\\ \hat{R}_{\lambda_r}(t)\\ \hat{f}_{w\lambda_r}(t)\end{bmatrix} + F(a) - \lambda_r\sigma_r K g_a)^*(\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1}d\vec{Y}_{(m)}(t) \\ -\frac{1}{2}\int_0^{t_f}||(\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1/2}(\mathscr{H}(a)\begin{bmatrix}\tilde{f}_{r\lambda_r}(t)\\ \hat{R}_{\lambda_r}(t)\\ \hat{f}_{w\lambda_r}(t)\end{bmatrix} + F(a) - \lambda_r\sigma_r K g_a)||^2_{R^m}dt\}. \end{aligned} \tag{5.10}$$

It is also possible to decompose the filtering outputs as the solutions in the risk neutral world and the λ_r term;

$$\begin{pmatrix}\tilde{f}_{r\lambda_r}(t,x)\\ \hat{R}_{\lambda_r}(t)\\ \hat{f}_{w\lambda_r}(t,x)\end{pmatrix} = \begin{pmatrix}\tilde{f}_r(t,x)\\ \hat{R}(t)\\ \hat{f}_w(t,x)\end{pmatrix} + \lambda_r\begin{pmatrix}e(t,x)\\ h(t)\\ u(t,x)\end{pmatrix}, \tag{5.11}$$

where $\tilde{f}_r(t,x), \hat{R}(t)$ and $\hat{f}_w(t,x)$ are solutions of (4.3) and (4.4), respectively and

$$\frac{\partial e(t,x)}{\partial t} = \frac{\partial e(t,x)}{\partial x} - \sigma_r g_a(x) \tag{5.12}$$

$$\begin{aligned} \begin{bmatrix}\frac{dh(t)}{dt}\\ \frac{\partial u(t,x)}{\partial t}\end{bmatrix} &= \begin{bmatrix}-ah(t)\\ \frac{\partial u(t,x)}{\partial x}\end{bmatrix} - \left(\mathscr{P}(t)\begin{bmatrix}-aG^*(a)\\ H^*\end{bmatrix} + \begin{bmatrix}\sigma_r^2 G(a)\\ \sigma^2 QK^*\end{bmatrix}\right) \\ &\times(\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1}(\mathscr{H}(a)\begin{bmatrix}e(t,x)\\ h(t)\\ u(t,x)\end{bmatrix} - \sigma_r K g_a), \end{aligned} \tag{5.13}$$

with

$$\begin{pmatrix}e(0,x)\\ h(0)\\ u(0,x)\end{pmatrix} = 0. \tag{5.14}$$

Noting that

$$\frac{\partial}{\partial\lambda_r}\begin{pmatrix}\tilde{f}_{r\lambda}(t,x)\\ \hat{R}_{\lambda}(t)\\ \hat{f}_{w\lambda}(t,x)\end{pmatrix} = \begin{pmatrix}e(t,x)\\ h(t)\\ u(t,x)\end{pmatrix}, \tag{5.15}$$

we get the necessary condition for MLE;

$$
\begin{aligned}
&\int_0^t (\mathscr{H}(a)\begin{bmatrix} e(s,x) \\ h(s) \\ u(s,x) \end{bmatrix} - \sigma_r K g_a)^*(\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1} d\vec{Y}_{(m)}(s) \\
&- \int_0^t (\mathscr{H}(a)\begin{bmatrix} e(s,x) \\ h(s) \\ u(s,x) \end{bmatrix} - \sigma_r K g_a)^*(\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1} \\
&\times (\mathscr{H}(a)\begin{bmatrix} \tilde{f}_r(s,x) + e(s,x)\hat{\lambda}_r \\ \hat{R}(s) + h(s)\hat{\lambda}_r \\ \hat{f}_w(s,x) + u(s,x)\hat{\lambda}_e \end{bmatrix} + F(a) - \hat{\lambda}_r \sigma_r K g_a) ds = 0.
\end{aligned} \tag{5.16}
$$

Consequently, the MLE $\hat{\lambda}_r$ can be obtained in the recursive form;

$$
\hat{\lambda}_r(t) = \frac{\text{num}(t)}{\text{den}(t)}, \tag{5.17}
$$

where

$$
\begin{aligned}
\text{num}(t) = &\int_0^t (\mathscr{H}(a)\begin{bmatrix} e(s,x) \\ h(s) \\ u(s,x) \end{bmatrix} - \sigma_r K g_a)^*(\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1} \\
&\times (d\vec{Y}_{(m)}(s) - (\mathscr{H}(a)\begin{bmatrix} \tilde{f}_r(s,x) \\ \hat{R}(s) \\ \hat{f}_w(s,x) \end{bmatrix} + F(a)) ds).
\end{aligned} \tag{5.18}
$$

$$
\begin{aligned}
\text{den}(t) = &\int_0^t (\mathscr{H}(a)\begin{bmatrix} e(s,x) \\ h(s) \\ u(s,x) \end{bmatrix} - \sigma_r K g_a)^*(\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1} \\
&\times (\mathscr{H}(a)\begin{bmatrix} e(s,x) \\ h(s) \\ u(s,x) \end{bmatrix} - \sigma_r K g_a) ds.
\end{aligned} \tag{5.19}
$$

Remark 5.1 *To realize the MLE for $\hat{\lambda}_r$, we can use the filter outputs in the risk neutral world. Hence the estimate for the factor process can be obtained as*

$$
\begin{pmatrix} \tilde{f}_{r\lambda}(t,x) \\ \hat{R}_\lambda(t) \\ \hat{f}_{w\lambda}(t,x) \end{pmatrix} = \begin{pmatrix} \tilde{f}_r(t,x) \\ \hat{R}(t) \\ \hat{f}_w(t,x) \end{pmatrix} + \hat{\lambda}_r(t) \begin{pmatrix} e(t,x) \\ h(t) \\ u(t,x) \end{pmatrix}. \tag{5.20}
$$

6 Conclusion

We start with a short rate model that deviates slightly from the traditional Hull-White model. We propose that this should lead to a new bond price that is a random perturbation of the exponential-affine form obtained in the usual approach. This leads to an infinite-dimensional HJM model for the forward rate. Using now observed yields of various maturities, we can estimate the forward rate and the unknown parameters by the infinite dimensional Kalman filter. To reduce the computational load, we propose estimating all the system parameters using standard statistical techniques. The approach proposed in this paper may be applied to any short rate model that leads to the bond price of the exponential affine form. The problem is that other models would lead to nonlinear filtering problems and approximate filtering techniques (like extended Kalman filter) would have to be used in such situations.

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A Appendix-A

Our optimal filtering equations for $f_r(t,x), R(t)$ and $f_w(t,x)$ are given by

$$d\begin{bmatrix}\hat{f}_r(t,x)\\ \hat{R}(t)\\ \hat{f}_w(t,x)\end{bmatrix} = \begin{bmatrix}\frac{\partial \hat{f}_r(t,x)}{\partial x}\\ -a\hat{R}(t)\\ \frac{\partial \hat{f}_w(t,x)}{\partial x}\end{bmatrix} dt + \begin{bmatrix} q_a(x)\\ 0\\ 0\end{bmatrix} dt$$
$$+\left(\mathscr{P}(t)\mathscr{H}^*(a) + \begin{bmatrix}0\\ \sigma_r^2 G(a)\\ \sigma^2 QK^*\end{bmatrix}\right)(\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1}$$
$$\times (d\vec{Y}_{(m)}(t) - \mathscr{H}(a)\begin{bmatrix}\hat{f}_r(t,\cdot)\\ \hat{R}(t)\\ \hat{f}_w(t,\cdot)\end{bmatrix} dt - F(a)dt). \tag{A.1}$$

The initial condition of $\mathscr{P}$ becomes a simple form as

$$\mathscr{P}(0) = \begin{pmatrix}\mathrm{cov}(f_o) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{A.2}$$

We list up the gain equation of $\mathscr{P}(t)$:

$$\mathscr{P}(t) = \begin{pmatrix}\mathbf{P}_{11}(t) & \mathbf{P}_{12}(t) & \mathbf{P}_{13}(t)\\ \mathbf{P}_{21}(t) & \mathbf{P}_{22}(t) & \mathbf{P}_{23}(t)\\ \mathbf{P}_{31}(t) & \mathbf{P}_{32}(t) & \mathbf{P}_{33}(t)\end{pmatrix}, \tag{A.3}$$

where $\mathbf{P}_{12} = \mathbf{P}_{21}^*, \mathbf{P}_{23} = \mathbf{P}_{32}^*, \mathbf{P}_{13} = \mathbf{P}_{31}^*$ and

$$\mathbf{P}_{11}(t) = \int_0^{\hat{T}} p_{11}(t,x,y)(\cdot)dy,\ \ \mathbf{P}_{12}(t) = p_{12}(t,x),\ \ \mathbf{P}_{13}(t) = \int_0^{\hat{T}} p_{13}(t,x,y)(\cdot)dy$$
$$\mathbf{P}_{22}(t) = p_{22}(t),\ \ \mathbf{P}_{23}(t) = p_{23}(t,x),\ \ \mathbf{P}_{33}(t) = \int_0^{\hat{T}} p_{33}(t,x,y)(\cdot)dy.$$

The kernel equations are given by

$$\frac{\partial p_{11}(t,x,y)}{\partial t} = \frac{\partial p_{11}(t,x,y)}{\partial x} + \frac{\partial p_{11}(t,x,y)}{\partial y} - \Bigg[(p_{11}(t,x,\tau_i) - p_{11}(t,x,0))\frac{1}{\tau_i}$$
$$-p_{12}(t,x)\frac{1-e^{-a\tau_i}}{\tau_i} + (p_{13}(t,x,\tau_i) - p_{13}(t,x,0))\frac{1}{\tau_i}\Bigg]_{1\times m} (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1}$$
$$\times \Bigg[(p_{11}(t,\tau_j,y) - p_{11}(t,0,y))\frac{1}{\tau_j} - p_{12}(t,y)\frac{1-e^{-a\tau_j}}{\tau_j} + (p_{13}(t,\tau_j,y) - p_{13}(t,0,y))\frac{1}{\tau_j}\Bigg]_{m\times 1} \tag{A.4}$$

$$\frac{dp_{22}(t)}{dt} = -2ap_{22}(t) + \sigma_r^2 - \left[(p_{21}(t,\tau_i) - p_{21}(t,0))\frac{1}{\tau_i} \right.$$
$$\left. - p_{22}(t)\frac{1-e^{-a\tau_i}}{\tau_i} + (p_{23}(t,\tau_i) - p_{23}(t,0))\frac{1}{\tau_i} + \sigma_r^2 \frac{1-e^{-a\tau_i}}{a\tau_i} \right]_{1\times m} (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1}$$
$$\times \left[(p_{21}(t,\tau_j) - p_{21}(t,0))\frac{1}{\tau_j} - p_{22}(t)\frac{1-e^{-a\tau_j}}{\tau_j} + (p_{23}(t,\tau_j) - p_{23}(t,0))\frac{1}{\tau_j} + \sigma_r^2 \frac{1-e^{-a\tau_j}}{a\tau_j} \right]_{m\times 1} \tag{A.5}$$

$$\frac{\partial p_{33}(t,x,y)}{\partial t} = \frac{\partial p_{33}(t,x,y)}{\partial x} + \frac{\partial p_{33}(t,x,y)}{\partial y} + \sigma^2 q(x,y) - \left[(p_{31}(t,x,\tau_i) - p_{11}(t,x,0))\frac{1}{\tau_i} \right.$$
$$\left. - p_{32}(t,x)\frac{1-e^{-a\tau_i}}{\tau_i} + (p_{33}(t,x,\tau_i) - p_{33}(t,x,0))\frac{1}{\tau_i} + \frac{\sigma^2}{\tau_i}\int_0^{\tau_i} q(x,y)dy \right]_{1\times m} (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1}$$
$$\times \left[(p_{31}(t,\tau_j,y) - p_{31}(t,0,y))\frac{1}{\tau_j} - p_{32}(t,y)\frac{1-e^{-a\tau_j}}{\tau_j} + (p_{33}(t,\tau_j,y) - p_{33}(t,0,y))\frac{1}{\tau_j} \right.$$
$$\left. + \frac{\sigma^2}{\tau_j}\int_0^{\tau_j} q(x,y)dx \right]_{m\times 1} \tag{A.6}$$

$$\frac{\partial p_{12}(t,x)}{\partial t} = \frac{\partial p_{12}(t,x)}{\partial x} - ap_{12}(t,x) - \left[(p_{11}(t,x,\tau_i) - p_{11}(t,x,0))\frac{1}{\tau_i} \right.$$
$$\left. - p_{12}(t,x)\frac{1-e^{-a\tau_i}}{\tau_i} + (p_{13}(t,x,\tau_i) - p_{13}(t,x,0))\frac{1}{\tau_i} \right]_{1\times m} (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1}$$
$$\times \left[(p_{21}(t,\tau_j) - p_{21}(t,0))\frac{1}{\tau_j} - p_{22}(t)\frac{1-e^{-a\tau_j}}{\tau_j} + (p_{23}(t,\tau_j) - p_{23}(t,0))\frac{1}{\tau_j} + \sigma_r^2 \frac{1-e^{-a\tau_j}}{a\tau_j} \right]_{m\times 1} \tag{A.7}$$

$$\frac{\partial p_{13}(t,x,y)}{\partial t} = \frac{\partial p_{13}(t,x,y)}{\partial x} + \frac{\partial p_{13}(t,x,y)}{\partial y} - \left[(p_{11}(t,x,\tau_i) - p_{11}(t,x,0))\frac{1}{\tau_i} \right.$$
$$\left. - p_{12}(t,x)\frac{1-e^{-a\tau_i}}{\tau_i} + (p_{13}(t,x,\tau_i) - p_{13}(t,x,0))\frac{1}{\tau_i} \right]_{1\times m} (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1}$$
$$\times \left[(p_{31}(t,\tau_j,y) - p_{31}(t,0,y))\frac{1}{\tau_j} \right.$$
$$\left. - p_{32}(t,y)\frac{1-e^{-a\tau_j}}{\tau_j} + (p_{33}(t,\tau_j,y) - p_{33}(t,0,y))\frac{1}{\tau_j} + \frac{\sigma^2}{\tau_j}\int_0^{\tau_j} q(x,y)dx \right]_{m\times 1} \tag{A.8}$$

$$
\begin{aligned}
\frac{\partial p_{23}(t,x)}{\partial t} &= -ap_{23}(t,x) + \frac{\partial p_{23}(t,x)}{\partial x} - \Bigg[(p_{21}(t,\tau_i) - p_{21}(t,0))\frac{1}{\tau_i} - p_{22}(t)\frac{1-e^{-a\tau_i}}{\tau_i} \\
&+ (p_{23}(t,\tau_i) - p_{23}(t,0))\frac{1}{\tau_i} + \sigma_r^2 \frac{1-e^{-a\tau_i}}{a\tau_i} \Bigg]_{1\times m} (\sigma_r^2 GG^* + \sigma^2 KQK^*)^{-1} \\
&\times \Bigg[(p_{31}(t,\tau_j,y) - p_{31}(t,0,y))\frac{1}{\tau_j} \\
&- p_{32}(t,y)\frac{1-e^{-a\tau_j}}{\tau_j} + (p_{33}(t,\tau_j,y) - p_{33}(t,0,y))\frac{1}{\tau_j} + \frac{\sigma^2}{\tau_j}\int_0^{\tau_j} q(x,y)dx \Bigg]_{m\times 1} \qquad \text{(A.9)}
\end{aligned}
$$