Generating All Permutations by Context-Free Grammars in Greibach Normal Form

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Abstract

We consider context-free grammars G_n in Greibach normal form and, particularly, in Greibach m-form ($m = 1, 2$) which generates the finite language L_n of all $n!$ strings that are permutations of n different symbols ($n \geq 1$). These grammars are investigated with respect to their descriptional complexity, i.e., we determine the number of nonterminal symbols and the number of production rules of G_n as functions of n. As in the case of Chomsky normal form these descriptional complexity measures grow faster than any polynomial function.

Keywords: context-free grammar, Greibach normal form, permutation, descriptional complexity, unambiguous grammar.

1 Introduction

A finite set, coded in some way as a finite language, can be generated in a trivial way by a context-free grammar with a single nonterminal symbol and as many rules as there are elements present in that finite language. This straightforward approach is no longer possible when we require that the context-free grammar possesses a special form such as Chomsky normal form (CNF) or Greibach normal form (GNF). If that finite language X_n belongs to an indexed family $\{X_n\}_{n\geq 1}$ of similar languages, then for each number $n \geq 1$ we have to construct a grammar G_n such that $L(G_n) = X_n$. The descriptional complexity of the resulting family of grammars $\{G_n\}_{n\geq 1}$ is usually expressed by a few descriptive complexity measures such as the number $\nu(n)$ of nonterminal symbols of G_n and the number $\pi(n)$ of productions of G_n; cf. e.g. [15, 17, 18, 8, 6, 1, 7]. An additional complexity measure has been introduced in [2, 3], viz. the number $\delta(n)$ of all possible leftmost derivations according to G_n, which makes sense particularly when dealing with finite languages. Clearly, the grammar G_n is unambiguous if and only if $\delta(n)$ equals the number of words in X_n.

In order to provide some concrete examples of the rather abstract setting sketched above, a few historical remarks are in order. So consider an alphabet of n symbols $\Sigma_n = \{a_1, a_2, \ldots, a_n\}$ and the language L_n consisting of all $n!$ permutations of these n symbols. In 2002 G. Satta [21] conjectured that "any context-free grammar G_n in CNF that generates L_n must have a number of nonterminal symbols that is not bounded by any polynomial function in n". This statement has been proved in [10], but without showing how to generate the languages $\{L_n\}_{n\geq 1}$ by context-free grammars $\{G_n\}_{n\geq 1}$ in CNF. In [2] we provided some approaches to obtain such grammar families for $\{L_n\}_{n\geq 1}$ together with the corresponding measures $\nu(n)$ and $\pi(n)$. The relative descriptional complexity of these grammar families is anything but straightforward and the quest for a family of minimal grammars (with respect to any of these complexity measures) remains a challenging problem.

Then in [3] we restricted our attention to some specific permutations over Σ_n, viz. to the so-called circular or cyclic shifts. When we provide Σ_n with a linear order, e.g., $a_1 < a_2 < \cdots < a_n$, then the set C_n of *circular* or *cyclic shifts* over Σ_n is defined by

$$C_n = \{a_1 a_2 \cdots a_{n-1} a_n, a_2 a_3 \cdots a_n a_1, a_3 a_4 \cdots a_1 a_2, \ldots, a_n a_1 \cdots a_{n-2} a_{n-1}\}.$$

Since C_n can be obtained from the word $a_1 a_2 \cdots a_n$ by moving the symbol from one end to the other end of the string iteratively, the number of elements in C_n equals n. This also follows from an alternative definition of C_n in terms of the so-called *circular closure operator* c on languages which is defined by $c(L) = \{vu \mid uv \in L\}$ for each language L [9]. Then the language C_n can be defined by $C_n = c(\{a_1 a_2 \cdots a_n\})$.

In [3] we defined some families $\{G_n\}_{n\geq 1}$ in CNF that generate $\{C_n\}_{n\geq 1}$ such that both $\nu(n)$ and $\pi(n)$ are bounded by polynomial functions of low degree, culminating in a "minimal" family of which ν and π are linear functions with very small coefficients. In case of GNF [4] there is still an open problem. Although ν and π can be bounded by polynomial functions of low degree, the quest for a minimal family remains open in this case. We conjectured in [4] that "any context-free grammar G_n in GNF that generates C_n must have a number of nonterminals that is not bounded by any linear function in n" and that for such a minimal family $\nu(n)$ and $\pi(n)$ are in $\Theta(n \cdot \log_2 n)$ rather than in $\Theta(n)$.

In the present paper we investigate several families of context-free grammars $\{G_n\}_{n\geq 1}$ in Greibach normal form that generate the family of languages $\{L_n\}_{n\geq 1}$ where L_n is the set of all permutations of the word $a_1 a_2 \cdots a_n$. And for each of these families we determine the descriptive complexity measures $\nu(n)$ and $\pi(n)$. As in [2] we start with some preliminaries (Section 2) and elementary properties of context-free grammars G_n in GNF that generate L_n (Section 3). In Section 4 we establish a lower bound on the number of nonterminal symbols for each context-free grammar in Greibach m-form ($m = 1, 2$) generating L_n; the argument is similar to the one in [10]. This lower bound implies that any context-free grammar G_n in Greibach m-form ($m = 1, 2$) that generates L_n must have a number of nonterminals that is not bounded by any polynomial function in n; cf. Satta's conjecture [21] on the CNF. We introduce families of grammars based on the power set of Σ_n in Section 5. Then in Section 6 we study grammatical transformations to define grammar families for $\{L_n\}_{n\geq 1}$ inductively. Section 7 is devoted to a divide-and-conquer approach, and Section 8 consists of concluding remarks.

2 Preliminaries

For each finite set X, $\#X$ denotes the cardinality (i.e., the number of elements) of X and $\mathcal{P}(X)$ the power set of X, and $\mathcal{P}_+(X)$ the set of nonempty subsets of X, i.e., $\mathcal{P}_+(X) = \mathcal{P}(X) - \{\varnothing\}$.

For rudiments of discrete mathematics, particularly of combinatorics (counting, recurrence relations and difference equations), we refer to standard texts such as [14, 19, 20]. Often we use $C(n,k)$ to denote the binomial coefficient $C(n,k) = n!/(k!(n-k)!)$; in displayed formulas we apply the usual notation.

The reader is assumed to be familiar with basic terminology and notation from formal language theory; cf. e.g. [16]. We denote the empty word by λ and the length of a word w by $|w|$. For each word w over an alphabet Σ, $\mathcal{A}(w)$ is the set of all symbols from Σ that do occur in w, i.e., $\mathcal{A}(\lambda) = \varnothing$, and $\mathcal{A}(ax) = \{a\} \cup \mathcal{A}(x)$ for each $a \in \Sigma$ and $x \in \Sigma^\star$. This mapping is extended to languages L over Σ by $\mathcal{A}(L) = \bigcup\{\mathcal{A}(w) \mid w \in L\}$.

Recall that a λ-free context-free grammar $G = (V, \Sigma, P, S)$ is in *Chomsky normal form* (CNF) if $P \subseteq N \times (N - \{S\})^2 \cup N \times \Sigma$ where $N = V - \Sigma$. And such a G is in *Greibach normal form* (GNF) if $P \subseteq N \times \Sigma(N - \{S\})^\star$. Particularly, G is in *Greibach m-form* or in *m-standard form* [16] if $P \subseteq N \times \Sigma(\bigcup_{i=0}^{m}(N - \{S\})^i)$.

For each context-free grammar $G = (V, \Sigma, P, S)$ and each $A \in V$, let $L(G, A)$ be the language over Σ defined by $L(G, A) = \{w \in \Sigma^\star \mid A \Rightarrow^\star w\}$. Then the language $L(G)$ generated by G equals $L(G, S)$. Note that, if G is in CNF or in GNF, then G has no useless symbols, $L(G, \alpha)$ is a nonempty language for each α in V, and $L(G, a) = \{a\}$ for each a in Σ.

In the sequel $\Sigma_n = \{a_1, a_2, \ldots, a_n\}$ denotes an alphabet of n symbols ($n \geq 1$) and L_n is the finite language over Σ_n that consists of the $n!$ permutations of $a_1 a_2 \cdots a_n$. The finiteness of L_n implies that each context-free grammar G_n in CNF or in GNF for L_n does not possess any recursive nonterminal.

For each family of grammars $\{G_n\}_{n\geq 1}$ generating $\{L_n\}_{n\geq 1}$ to be considered in this paper, we always assume that the first two elements G_1 and G_2 are

- $G_1 = (V_1, \Sigma_1, P_1, S_1)$, $N_1 = \{S_1\}$, $P_1 = \{S_1 \to a_1\}$, and
- $G_2 = (V_2, \Sigma_2, P_2, S_2)$, $N_2 = \{S_2, A_1, A_2\}$, $P_2 = \{S_2 \to a_1 A_2 \mid a_2 A_1,\ A_1 \to a_1,\ A_2 \to a_2\}$,

respectively. This implies that specifying a family $\{G_n\}_{n\geq 1}$ for $\{L_n\}_{n\geq 1}$ reduces to defining the family $\{G_n\}_{n\geq 3}$.

3 Elementary Properties

This section is devoted to some straightforward properties of context-free grammars in GNF form that generate L_n. Following the convention made at the end of the previous section we restrict our attention to the case $n \geq 3$.

Proposition 3.1. *For $n \geq 3$, let $G_n = (V_n, \Sigma_n, P_n, S_n)$ be a context-free grammar in Greibach normal form that generates L_n, and let N_n be defined by $N_n = V_n - \Sigma_n$.*

(1) For each A in N_n, the language $L(G_n, A)$ is a nonempty subset of an isomorphic copy M_k of the language L_k for some k $(1 \leq k \leq n)$. Consequently, each string z in $L(G_n, A)$ has length k, z consists of k different symbols, and $\mathcal{A}(z) = \mathcal{A}(L(G_n, A))$.

(2) Let A and B be nonterminal symbols in N_n. If $L(G_n, A) \cap L(G_n, B) \neq \varnothing$, then $\mathcal{A}(L(G_n, A)) = \mathcal{A}(L(G_n, B))$.

(3) If $A \to aA_1A_2 \cdots A_m$ is a rule in G_n, then for each pair (i, j) with $1 \leq i < j \leq m$, $\mathcal{A}(L(G_n, A_i)) \cap \mathcal{A}(L(G_n, A_j)) = \varnothing$, $a \notin \mathcal{A}(L(G_n, A_k))$ with $1 \leq k \leq m$, and

$$\mathcal{A}(L(G_n, A)) = \{a\} \cup \mathcal{A}(L(G_n, A_1)) \cup \mathcal{A}(L(G_n, A_2)) \cup \cdots \cup \mathcal{A}(L(G_n, A_m)).$$

Proof. The proofs of (1) and (2) are as the ones for Proposition 3.1 in [2]; they rely on the facts that for each A in N_n, $L(G, A)$ is a nonempty subset of Σ_n^+, and that each word in $L(G, A)$ is a nonempty substring of a permutation, i.e., of a word in L_n.

(3) Suppose that for some pair (i, j) the intersection is nonempty: if it contains a symbol b, then we have a subderivation $A \Rightarrow aA_1A_2 \cdots A_m \Rightarrow^\star ax_1bx_2bx_3$ which cannot be a subderivation of a derivation that yields a permutation.

Now the inclusion$\{a\} \cup \bigcup_{i=1}^m \mathcal{A}(L(G_n, A_i)) \subseteq \mathcal{A}(L(G_n, A))$ is trivial. Suppose that it is proper: there exists a symbol b with $b \neq a$ and $b \in \mathcal{A}(L(G_n, A)) - \bigcup_{i=1}^m \mathcal{A}(L(G_n, A_i))$. Then there is a rule $A \to dB_1B_2 \cdots B_k$ with $b \in \{d\} \cup \bigcup_{i=1}^k \mathcal{A}(L(G_n, B_i))$. Consider the derivation $S_n \Rightarrow^\star uAv \Rightarrow uaA_1A_2 \cdots A_mv \Rightarrow^\star uxv$ with $b \in \mathcal{A}(uv)$ and $b \notin \mathcal{A}(x)$, yielding the permutation uxv. Using this alternative rule $A \to dB_1B_2 \cdots B_k$ for A we obtain the derivation $S_n \Rightarrow^\star uAv \Rightarrow udB_1B_2 \cdots B_kv \Rightarrow^\star uyv$ with $b \in \mathcal{A}(y)$; consequently, uyv contains at least two b's and therefore it is not a permutation. Hence, the inclusion cannot be proper, and so we have equality. □

Proposition 3.1(2) gives rise to the following equivalence relation on N_n.

Definition 3.2. Two nonterminal symbols A and B from N_n are called *equivalent* if $|x| = |y|$ for some $x \in L(G_n, A)$ and some $y \in L(G_n, B)$. The corresponding equivalence classes are denoted by $\{E_{n,k}\}_{k=1}^n$. The number of elements $\#E_{n,k}$ of the equivalence class $E_{n,k}$ will be denoted by $D(n, k)$ $(1 \leq k \leq n)$. □

From this definition and Proposition 3.1(3) we obtain the following property: if $A \to aA_1A_2 \cdots A_m$ is a rule in G_n and for each i $(1 \leq i \leq m)$ A_i belongs to $E_{n,k(i)}$, then we have that A is in $E_{n,p}$ with $p = 1 + \sum_{i=1}^m k(i)$.

Proposition 3.1 suggests a partial order relation on N_n which is induced by the inclusion relation on $\mathcal{P}(\Sigma_n)$ and which is a more general notion than the linear order present in the concept of *sequential grammar*; cf. [11, 5].

Definition 3.3. Let A and B be nonterminal symbols from N_n. Then the partial order $\sqsubseteq$ on N_n and the corresponding strict order $\sqsubset$ are given by:

$A \sqsubseteq B$ if and only if $\mathcal{A}(L(G_n, A)) \subseteq \mathcal{A}(L(G_n, B))$,

$A \sqsubset B$ if and only if $\mathcal{A}(L(G_n, A)) \subset \mathcal{A}(L(G_n, B))$. □

For the descriptional complexity of a context-free grammar G_n from a family $\{G_n\}_{n\geq 1}$, we use well-known measures like the number $\nu(n)$ of nonterminal symbols and the number $\pi(n)$ of production rules of G_n; so $\nu(n) = \#N_n$ and $\pi(n) = \#P_n$. As in [2, 3, 4] we will consider ν and π as functions of n. These measures are anything but original, since they have been studied frequently in the literature concerning context-free grammars [15, 17, 18, 8, 6, 1, 7]. A somewhat less-known descriptional complexity measure has been introduced recently in [2, 3, 4]; viz. the number of left-most derivations $\delta(n)$ according to a context-free grammar, i.e., $\delta(n) = \#\{S_n \Rightarrow_L^{\star} x \mid x \in L(G_n)\}$, where $\Rightarrow_L$ denotes the leftmost derivation relation. In particular this measure makes sense, when we generate a finite language by means of a λ-free grammar with bounded ambiguity.

Example 3.4. (1) For the grammars G_1 and G_2 of Section 2 we have $\nu(1) = \pi(1) = \delta(1) = 1$ and $\nu(2) = 3$, $\pi(2) = 4$ and $\delta(2) = 2$. Both G_1 and G_2 are unambiguous.

(2) Consider $G_3 = (V_3, \Sigma_3, P_3, S_3)$ with $S_3 = A_{123}$, $N_3 = \{A_{123}, A_{12}, A_{13}, A_{23}, A_1, A_2, A_3\}$ and $P_3 = \{A_{123} \to a_1A_{23} \mid a_2A_{13} \mid a_3A_{12},\ A_{12} \to a_1A_2 \mid a_2A_1,\ A_{13} \to a_1A_3 \mid a_3A_1,\ A_{23} \to a_2A_3 \mid a_3A_2,\ A_1 \to a_1,\ A_2 \to a_2,\ A_3 \to a_3\}$. Note that G_3 is regular, unambiguous and in Greibach 1-form.

Now $E_{3,3} = \{A_{123}\}$, $E_{3,2} = \{A_{12}, A_{13}, A_{23}\}$, $E_{3,1} = \{A_1, A_2, A_3\}$, $A_i \sqsubset A_{ij} \sqsubset S_3$ $(1 \leq i < j \leq 3)$, $D(3,3) = 1$, $D(3,2) = D(3,1) = 3$, $\nu(3) = 7$, $\pi(3) = 12$ and $\delta(3) = 6$. □

We conclude this section with a very simple family of grammars in GNF that generates $\{L_n\}_{n\geq 1}$. Starting point is the family of trivial grammars with a single nonterminal symbol S_n and the set of rules $\{S_n \to w \mid w \in L_n\}$. In order to obtain grammars in GNF we need a family of isomorphisms.

Let for each $n \geq 3$, $\varphi_n : \Sigma_n \to \{A_1, A_2, \ldots, A_n\}$ be the isomorphism defined by $\varphi_n(a_i) = A_i$ $(1 \leq i \leq n)$. As usual, φ_n is extended to words over Σ_n by

$$\varphi_n(\sigma_1\sigma_2\cdots\sigma_k) = \varphi_n(\sigma_1)\varphi_n(\sigma_2)\cdots\varphi_n(\sigma_k) \qquad (\sigma_i \in \Sigma_n,\ 1 \leq i \leq k)$$

and to languages L over Σ_n by

$$\varphi_n(L) = \{\varphi_n(w) \mid w \in L\}.$$

Definition 3.5. The family $\{G_n^T\}_{n\geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n\geq 1}$ with for $n \geq 3$,

- $N_n = V_n - \Sigma_n = \{S_n\} \cup \{A_i \mid 1 \leq i \leq n\}$,
- $P_n = \{S_n \to \sigma_1\varphi(\sigma_2\cdots\sigma_n) \mid \sigma_1\sigma_2\cdots\sigma_n \in L_n\} \cup \{A_i \to a_i \mid 1 \leq i \leq n\}$. □

We emphasize that the descriptional complexity measures ν, π and δ depend on n as well as on the family under consideration; so we use $\nu_\alpha(n)$, $\pi_\alpha(n)$ and $\delta_\alpha(n)$ in the context of a family $\{G_n^\alpha\}_{n\geq 1}$ of which the individual members are labeled by α.

Example 3.6. For $n = 3$, Definition 3.5 yields the grammars $G_3^T = (V_3, \Sigma_3, P_3, S_3)$ with

$N_3 = \{S_3, A_1, A_2, A_3\}$ and $P_3 = \{S_3 \to a_1A_2A_3 \mid a_1A_3A_2 \mid a_2A_1A_3 \mid a_2A_3A_1 \mid a_3A_1A_2 \mid a_3A_2A_1,\ A_1 \to a_1,\ A_2 \to a_2,\ A_3 \to a_3\}$. Clearly, G_3^T is an unambiguous grammar, it is in GNF and, as it happens, in Greibach 2-form (since in general G_n^T is in Greibach $(n-1)$-form).

Then $E_{3,3} = \{S_3\}$, $E_{3,2} = \varnothing$, $E_{3,1} = \{A_1, A_2, A_3\}$, $A_i \sqsubset S_3$ $(1 \leq i \leq 3)$, $D(3,3) = 1$, $D(3,2) = 0$, and $D(3,1) = 3$. Thus $\nu_T(3) = 4$, $\pi_T(3) = 9$ and $\delta_T(3) = 6$. □

The following result easily follows from Definition 3.5.

Proposition 3.7. *For the family $\{G_n^T\}_{n\geq 1}$ of Definition 3.5 we have for $n \geq 3$,*

(1) $D(n,n) = 1$, $D(n,k) = 0$ $(1 < k < n)$, *and* $D(n,1) = n$.

(2) $\nu_T(n) = n + 1$,

(3) $\pi_T(n) = n! + n$,

(4) $\delta_T(n) = n!$, *i.e., G_n^T is unambiguous.* □

4 A Lower Bound

From Definition 3.5 and Proposition 3.7 it is clear that the use of arbitrary GNF does not lead to very interesting results. Therefore we restrict ourselves in the remaining part of this paper to context-free grammars in Greibach m-form with $m = 1, 2$. Similar to [10] we establish for these grammars a lower bound on the number of nonterminal symbols. The proofs in this section are straightforward modifications of arguments from [10]; for completeness' sake they are included here as well.

Lemma 4.1. *Let $G = (V, \Sigma, P, S)$ be a context-free grammar in Greibach m-form $(m = 1, 2)$ and let $w \in L(G)$ with $|w| \geq 1$. Then for each derivation $S \Rightarrow^+ w$, there exists a nonterminal symbol A with*

(a) $S \Rightarrow^\star \alpha A\beta \Rightarrow^+ w$, *for some $\alpha, \beta \in V^\star$, and*

(b) *if u is the yield of A in this derivation of w, then $|w|/3 \leq |u| < 2|w|/3 + 1$.*

Proof. The case $|w| = 1$ is trivial: we take $A = S$ and, consequently, we have $u = w$ which satisfies (b).

So we may assume that $|w| > 1$. In the derivation tree of (a) according to G we follow a path from the root S down to a leaf, at each point choosing the nonterminal with the larger yield (whenever there is a choice). In the end we arrive at a nonterminal Z with a yield of length 1. As $|w| \geq 1$ we have for the yield u of this nonterminal Z that $|u| < 2|w|/3 + 1$.

Returning upwards in the direction of the root S we sooner or later meet a nonterminal A with yield u satisfying $|u| < 2|w|/3 + 1$, but for which its parent nonterminal B has yield z with $|z| \geq 2|w|/3 + 1$. At this point in the derivation tree a rule of the form (i) $B \to aAC$, (ii) $B \to aCA$ or (iii) $B \to aA$ (for some $a \in \Sigma$ and some $C \in V - \Sigma$) has been applied. In moving downwards along this path in the tree from S to Z we always chose the nonterminal with the larger yield. Therefore in the cases (i), (ii) and (iii) A is the desired nonterminal and for its yield u we have $|u| \geq |w|/3$. □

Notice that Lemma 4.1 holds for any context-free grammar in Greibach m-form ($m = 1, 2$), whereas the following result (Theorem 4.2) only holds for such context-free grammars that generate L_n; cf. Lemma 25 and Theorem 24 in [10], respectively.

Theorem 4.2. *Let $G_n = (V_n, \Sigma_n, P_n, S_n)$ be a context-free grammar in Greibach m-form ($m = 1, 2$) generating L_n. Then $\nu(n) \in \Omega(n^{-3/2} r^n)$ where $r = \frac{3}{2}\sqrt[3]{2} = 1.88988157\cdots$.*

Proof. With each word w in L_n we associate a pair (A, k) where A is a nonterminal symbol from $V_n - \Sigma_n$ and k is a natural number ($1 \leq k \leq n$) that represents a position in the string w. By Lemma 4.1 there exists such a nonterminal A that generates a subword u of w with $|w|/3 \leq |u| < 2|w|/3 + 1$. Since w is a permutation, this subword u occurs (or starts) at a uniquely determined position k in w; the resulting pair (A, k) will be associated with the word w.

Next we consider all such pairs (A, k) and determine the number of words that can be associated with a fixed pair (A, k). Following Proposition 3.1(1), A generates strings of a fixed length l, and by Lemma 4.1 we have $|w|/3 \leq l < 2|w|/3 + 1$. There are $l!$ different possibilities for the strings generated by A, and the $n - l$ remaining symbols (once the word generated by A is disregarded from w) give rise to at most $l!(n-l)!$ possible words to be associated with (A, k). Since there are $n!$ words in total, we have at least $n!/l!(n-l)! = C(n, l)$ distinct pairs (A, k). Because there are only n different positions in w (i.e., possible values for k), G_n must possess at least $n^{-1} \cdot C(n, l)$ different nonterminals.

In the interval $1 \leq l \leq \lfloor n/2 \rfloor$, $C(n, l)$ increases monotonically and under the restriction $\lceil n/3 \rceil \leq l < \lceil 2n/3 \rceil + 1$ it reaches its minimum value at $l = \lceil 2n/3 \rceil$. Therefore we have $\nu(n) \geq n^{-1} \cdot C(n, \lfloor n/3 \rfloor)$. Using Stirling's formula, we obtain for large values of n,

$$
\begin{aligned}
\nu(n) \;&\geq\; n^{-1} \cdot \binom{n}{\lfloor n/3 \rfloor} \;=\; \frac{n^{-1} n!}{\lfloor n/3 \rfloor ! \lceil 2n/3 \rceil !} \\
&\approx\; \frac{n^{-1}\sqrt{2\pi n}(n/e)^n (1 + c_1 n^{-1})}{\sqrt{2\pi \lfloor n/3 \rfloor}(\lfloor n/3 \rfloor / e)^{\lfloor n/3 \rfloor}(1 + c_2 n^{-1})\sqrt{2\pi \lceil 2n/3 \rceil}(\lceil 2n/3 \rceil / e)^{\lceil 2n/3 \rceil}(1 + c_3 n^{-1})} \\
&=\; \frac{3n^{-3/2}}{2\sqrt{\pi}} \cdot \frac{3^n}{2^{2n/3}} \cdot \frac{1 + c_1 n^{-1}}{(1 + c_2 n^{-1})(1 + c_3 n^{-1})}
\end{aligned}
$$

for some constants $c_1, c_2, c_3 > 0$; cf. Exercise 5.60 in [14]. Since this last factor tends to 1 as $n \to \infty$, we have asymptotically that $\nu(n) \in \Omega(n^{-3/2} r^n)$ with $r = \frac{3}{2}\sqrt[3]{2}$. □

It is likely that variations of Lemma 4.1 and Theorem 4.2 can be established for context-free grammars in Greibach m-form with $m > 2$, although the combinatorial arguments become more complicated. Certainly, they cannot be extended to context-free grammars in arbitrary GNF as the family of Definition 3.5 may serve as a counterexample to the conclusion of Theorem 4.2; cf. Proposition 3.7(2).

Of course, Theorem 4.2 does not indicate *how* to generate L_n by context-free grammars in Greibach m-form ($m = 1, 2$). The following sections are devoted to this problem.

5 Greibach m-form $(m = 1, 2)$ — Subsets

In this section we consider a few ways of generating $\{L_n\}_{n\geq 1}$ by a family of grammars in Greibach m-form ($m = 1, 2$). These grammars have the property that each nonterminal symbol corresponds to a nonempty subset of Σ_n in a unique fashion. First, we consider the case $m = 2$ (Definitions 5.1 and 5.4) and then we turn to a family with $m = 1$ (Definition 5.7).

Definition 5.1. The family $\{G^1_n\}_{n\geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n\geq 1}$ with for $n \geq 3$,

- $N_n = V_n - \Sigma_n = \{A_X \mid X \in \mathcal{P}_+(\Sigma_n)\}$,
- $P_n = \{A_{\{a\}\cup X\cup Y} \to aA_XA_Y \mid a \in \Sigma_n;\ X, Y \in \mathcal{P}(\Sigma_n),\ X \cap Y = \varnothing\}$, and
- $S_n = A_{\Sigma_n}$. □

We will identify $A_\varnothing$ with λ in this definition of P_n; in particular, this implies that $A_{\{a\}} \to a$ is in P_n for each a in Σ_n (viz. when $X = Y = \varnothing$). Note that $A_\varnothing \notin V_n$.

Clearly, $A_X \sqsubset A_Y$ [$A_X \sqsubseteq A_Y$, respectively] holds if and only if $X \subset Y$ [$X \subseteq Y$] for all X and Y in $\mathcal{P}_+(\Sigma_n)$.

In the sequel we use the notation $A \twoheadrightarrow aBC$ as an abbreviation for $A \to aBC \mid aCB$. The reader should always keep in mind that $A \twoheadrightarrow aBC$ counts for two productions.

Example 5.2. We consider the case $n = 3$ in detail; instead of subsets of Σ_3, we use subsets of $\{1, 2, 3\}$ as indices of nonterminals. Then we have $G^1_3 = (V_3, \Sigma_3, P_3, S_3)$ with $S_3 = A_{123}$, $N_3 = \{A_{123}, A_{12}, A_{13}, A_{23}, A_1, A_2, A_3\}$ and $P_3 = \{A_{123} \twoheadrightarrow a_1A_2A_3 \mid a_2A_1A_3 \mid a_3A_1A_2,\ A_{123} \to a_1A_{23} \mid a_2A_{13} \mid a_3A_{12}, A_{12} \to a_1A_2 \mid a_2A_1,\ A_{13} \to a_1A_3 \mid a_3A_1,\ A_{23} \to a_2A_3 \mid a_3A_2,\ A_1 \to a_1,\ A_2 \to a_2,\ A_3 \to a_3\}$.

Now $E_{3,3} = \{A_{123}\}$, $E_{3,2} = \{A_{12}, A_{13}, A_{23}\}$, $E_{3,1} = \{A_1, A_2, A_3\}$, $D(3, 3) = 1$, $D(3, 2) = D(3, 1) = 3$, $\nu_1(3) = 7$ and $\pi_1(3) = 18$. □

Proposition 5.3. *For the family $\{G^1_n\}_{n\geq 1}$ of Definition 5.1 we have for $n \geq 3$,*

(1) $D(n, k) = C(n, k)$ *with* $1 \leq k \leq n$,

(2) $\nu_1(n) = 2^n - 1$,

(3) $\pi_1(n) = n \cdot 3^{n-1} - n \cdot 2^{n-1} + n$.

Proof. Definition 5.1 and $\nu_1(n) = \sum_{k=1}^n D(n, k) = \sum_{k=1}^n C(n, k) = 2^n - 1$ [14] imply immediately (1) and (2). For (3) we determine $\#P_n$: if the set $\{a\} \cup X \cup Y$ possesses k elements ($k \geq 3$), then the set $\{A_{\{a\}\cup X\cup Y} \to aA_XA_Y \mid X, Y \in \mathcal{P}(\Sigma_n),\ X \cap Y = \varnothing\}$ contains $k(2^{k-1} - 1)$ elements, because both cases $X = \varnothing$ and $Y = \varnothing$ result in the same production. For $k = 2$, we have k elements, which equals $k(2^{k-1} - 1)$ as well, but for $k = 1$ there is just one element. Then

$$\#P_n = \binom{n}{1} 1 + \sum_{k=2}^{n} \binom{n}{k} k(2^{k-1} - 1) \;=\; n + \sum_{k=1}^{n} \binom{n}{k} k(2^{k-1} - 1) \;=$$

$$= n + \sum_{k=1}^{n} \frac{n! \cdot k}{k!(n-k)!}(2^{k-1} - 1) \;=\; n + n \cdot \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}(2^{k-1} - 1) \;=$$

$$= n + n \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} (2^j - 1) \;=\; n + n \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} 2^j 1^{n-j-1} - n \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} \;=$$

$$= n(2+1)^{n-1} - n \cdot 2^{n-1} + n \;=\; n \cdot 3^{n-1} - n \cdot 2^{n-1} + n.$$

Consequently, we have $\pi_1(n) = \#P_n = n \cdot 3^{n-1} - n \cdot 2^{n-1} + n$. □

In order to reduce the number of productions, we will demand in the next family that in rules of the form $A \to aBC$ we have either $B = A_\varnothing = \lambda$ or $B = A_{\{b\}}$ for some $b \in \Sigma_n$.

Definition 5.4. The family $\{G_n^2\}_{n\geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n\geq 1}$ with for $n \geq 3$,

- $N_n = V_n - \Sigma_n = \{A_X \mid X \in \mathcal{P}_+(\Sigma_n)\}$,
- $P_n = \{A_{\{a\}\cup X\cup Y} \to aA_X A_Y \mid a \in \Sigma_n;\; X, Y \in \mathcal{P}(\Sigma_n),\; X \cap Y = \varnothing,\; \#X \leq 1\}$, and
- $S_n = A_{\Sigma_n}$. □

Example 5.5. As it happens, $G_3^2 = G_3^1$ holds; however, for $n \geq 4$, we have $G_n^2 \neq G_n^1$. E.g., $A_{1234} \to a_1 A_{34} A_2$ is a production of G_4^1, but not of G_4^2, while the corresponding rules $A_{1234} \to a_1 A_2 A_{34}$, $A_{1234} \to a_1 A_3 A_{24}$ and $A_{1234} \to a_1 A_4 A_{23}$ belong to both these grammars. In general we have for $n \geq 4$, $\pi_2(n) < \pi_1(n)$; cf. Propositions 5.3(3) and 5.6(3). □

Proposition 5.6. *For the family $\{G_n^2\}_{n\geq 1}$ of Definition 5.4 we have for $n \geq 3$,*

(1) *$D(n, k) = C(n, k)$ with $1 \leq k \leq n$,*

(2) *$\nu_2(n) = 2^n - 1$,*

(3) *$\pi_2(n) = n^2 \cdot 2^{n-2} + n \cdot 2^{n-2} - n^2 + n$.*

Proof. With respect to the previous proof, the only difference is (3): if the set $\{a\}\cup X\cup Y$ has k elements ($k \geq 3$), then now the set $\{A_{\{a\}\cup X\cup Y} \to aA_X A_Y \mid X, Y \in \mathcal{P}(\Sigma_n),\; X \cap Y = \varnothing,\; \#X \leq 1\}$ contains $k(k-1) + k$ elements: the first term corresponds to $\#X = 1$, the second one to $\#X = 0$. For $k = 2$ and $k = 1$, there are k elements and just a single element, respectively. Now we have

$$\#P_n = \binom{n}{1} + \sum_{k=2}^{n} \binom{n}{k} k + \sum_{k=3}^{n} \binom{n}{k} k(k-1) \;=\; n + \sum_{k=2}^{n} \frac{n! \cdot k}{k!(n-k)!} + \sum_{k=3}^{n} \frac{n! \cdot k(k-1)}{k!(n-k)!} \;=$$

$$= n + n \cdot \sum_{k=2}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} + n(n-1) \cdot \sum_{k=3}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} \;=$$

$$= n + n \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} - n \binom{n-1}{0} + n(n-1) \cdot \sum_{j=0}^{n-2} \binom{n-2}{j} - n(n-1) \binom{n-2}{0} \;=$$

$$= n + n \cdot 2^{n-1} - n + n(n-1) \cdot 2^{n-2} - n(n-1) \;=\; n^2 \cdot 2^{n-2} + n \cdot 2^{n-2} - n^2 + n,$$

i.e., $\pi_2(n) = \#P_n = n^2 \cdot 2^{n-2} + n \cdot 2^{n-2} - n^2 + n$. □

Finally, we replace the restriction "$\#X \leq 1$" in Definition 5.4 by "$\#X = 0$", i.e., we now consider grammars in Greibach 1-form or, equivalently, regular grammars for $\{L_n\}_{n\geq 1}$. From [2] we quote the following definition and results.

Definition 5.7. The family $\{G_n^3\}_{n\geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n\geq 1}$ with for $n \geq 3$,

- $N_n = V_n - \Sigma_n = \{A_X \mid X \in \mathcal{P}_+(\Sigma_n)\}$,
- $P_n = \{A_{\{a\}} \to a \mid a \in \Sigma_n\} \cup \{A_X \to aA_{X-\{a\}} \mid X \subseteq \Sigma_n,\ a \in X,\ \#X \geq 2\}$,
- $S_n = A_{\Sigma_n}$. □

For an example with $n = 3$ we refer to Example 3.4(2).

Proposition 5.8. [2] *For the family $\{G_n^3\}_{n\geq 1}$ of Definition 5.7 we have for $n \geq 3$,*

(1) $D(n,k) = C(n,k)$ *with* $1 \leq k \leq n$,

(2) $\nu_3(n) = 2^n - 1$,

(3) $\pi_3(n) = n \cdot 2^{n-1}$,

(4) $\delta_3(n) = n!$, *i.e., G_n^3 is unambiguous.* □

Although $\nu_1(n) = \nu_2(n) = \nu_3(n)$ for $n \geq 1$, we obtain $\pi_1(n) > \pi_2(n) > \pi_3(n)$ for $n \geq 4$. We can apply the idea of subsets of Σ_n to construct a grammar family with fewer nonterminals as well. It is rather straightforward to define a family with $D(n,1) = n$, and for $k \geq 2$, $D(n,k) =$ **if** $k \equiv n \pmod 2$ **then** $C(n,k)$ **else** 0. Then $\nu(n) = 2^{n-1}$ if n is odd, and $\nu(n) = 2^{n-1} + n - 1$ if n is even, but a closed form for $\pi(n)$ is less easy to derive.

6 Greibach 2-form — Grammatical Transformations

In this section we start with the grammars $G_1^4 = G_1$ and $G_2^4 = G_2$, defined in Section 2, together with an explicitly given grammar G_3^4, and then we proceed inductively to define $G_4^4, G_5^4, G_6^4, \cdots$ by means of a grammatical transformation T_1 that produces G_{n+1}^4 from G_n^4 $(n \geq 3)$. This transformation is based on the following observation: L_n with $L_n = L(G_n^4)$ is a language over Σ_n, whereas L_{n+1} is a language over Σ_{n+1}; so we may obtain the elements of L_{n+1} by inserting the new terminal symbol a_{n+1} at each available spot in the strings of L_n. In essence this is realized by our grammatical transformation T_1.

Definition 6.1. The family $\{G_n^4\}_{n\geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n\geq 1}$ with

(1) G_1^4 equals G_1 from Section 2.

(2) G_2^4 equals G_2 from Section 2.

(3) G_3^4 is defined by $G_3^4 = (V_3, \Sigma_3, P_3, S_3)$ with $N_3 = \{S_3, A_1, A_2, A_3\}$ and $P_3 = \{S_3 \to a_1A_2A_3 \mid a_2A_1A_3 \mid a_3A_1A_2,\ A_1 \to a_1\ A_2 \to a_2\ A_3 \to a_3\}$.

(4) G_{n+1}^4 is obtained from G_n^4 $(n \geq 3)$ by the grammatical transformation T_1 described in steps (a), (b), (c), (d) and (e); T_1 properly extends P_n to P_{n+1} by adding new productions.

(a) If $A \to aBC$ is in P_n, then $A \to aBC$ and $A' \to aB'C \mid aBC'$ are in P_{n+1}.

(b) If $A \to aB$ is in P_n, then $A \to aB$ and $A' \to aB'$ are in P_{n+1}.

(c) If $A \to a$ is in P_n, then $A \to a$ and $A' \to aA_{n+1}$ are in P_{n+1}.

(d) We add $\nu_4(n) + 1$ new productions $A' \to a_{n+1}A$ $(A \in N_n)$ and $A_{n+1} \to a_{n+1}$ to P_{n+1}.

(e) Finally, each occurrence of S_n' in G_{n+1}^4 will be replaced by S_{n+1}, i.e., by the initial nonterminal symbol of G_{n+1}^4. □

In step (c) there is no need to add productions of the form $A' \to a_{n+1}A$, as they will be introduced in step (d).

A primed symbol in a derivation according to G_n^4 indicates that in the subtree rooted by that symbol an occurrence of the terminal symbol a_{n+1} should be inserted. A similar remark applies to the initial symbol S_{n+1}; cf. step (e) in Definition 6.1(3).

Example 6.2. (1) Note that $\nu_4(3) = 4 < \nu_i(3)$ and $\pi_4(3) = 9 < \pi_i(3)$ for $i = 1, 2, 3$.

(2) We will construct G_4^4 from G_3^4 by means of T_1 as defined in Definition 6.1: $G_4^4 = (V_4, \Sigma_4, P_4, S_4)$ with $N_4 = \{S_4, S_3, A_4\} \cup \{A_i, A_i' \mid 1 \le i \le 3\}$ and P_4 consists of the rules

$$
\begin{array}{ll}
S_3 \twoheadrightarrow a_1A_2A_3 \mid a_2A_1A_3 \mid a_3A_1A_2,\ A_1 \to a_1\ A_2 \to a_2\ A_3 \to a_3, & P_3 \\
S_4 \twoheadrightarrow a_1A_2'A_3 \mid a_1A_2A_3' \mid a_2A_1'A_3 \mid a_2A_1A_3' \mid a_3A_1'A_2 \mid a_3A_1A_2', & \text{(a)} \\
\text{——} & \text{(b)} \\
A_1' \to a_1A_4,\ A_2' \to a_2A_4,\ A_3' \to a_3A_4, & \text{(c)} \\
S_4 \to a_4S_3,\ A_1' \to a_4A_1,\ A_2' \to a_4A_2,\ A_3' \to a_4A_3,\ A_4 \to a_4. & \text{(d)}
\end{array}
$$

Then we have $E_{4,4} = \{S_4\}$, $E_{4,3} = \{S_3\}$, $E_{4,2} = \{A_1', A_2', A_3'\}$, $E_{4,1} = \{A_1, A_2, A_3, A_4\}$, $A_i \sqsubset S_3 \sqsubset S_4$, $A_i \sqsubset A_i' \sqsubset S_4$, $A_4 \sqsubset A_i'$ $(1 \le i \le 3)$, $\nu_4(4) = 9$ and $\pi_4(4) = 29$.

(3) It is an illustrative exercise to construct G_5^4 from G_4^4 in a similar way. However, before starting to do so the reader should rename some nonterminals —for instance A_i' by B_i— in order to avoid confusion caused by double primes. □

Proposition 6.3. *For the family* $\{G_n^4\}_{n\ge1}$ *of Definition 6.1 we have*

$$
\begin{array}{lll}
(1) & D(n,n) = 1, \quad D(n,1) = n & (n \ge 1), \\
 & D(3,2) = 0, & \\
 & D(n,k) = D(n-1,k) + D(n-1,k-1) & (n \ge 4; \quad 2 \le k \le n-1), \\
(2) & \nu_4(n) = 5 \cdot 2^{n-3} - 1 & (n \ge 3), \\
(3) & \pi_4(n) = 2 \cdot 3^{n-2} + 5n \cdot 2^{n-4} - 9 \cdot 2^{n-4} & (n \ge 3).
\end{array}
$$

Proof. (1) Obviously, $D(n,n) = 1$ and $D(n,1) = n$ since $E_{n,n} = \{S_n\}$ and $E_{n,1} = \{A_1, \ldots, A_n\}$ because $A_i \to a_i$ are the only rules in P_n with terminal right-hand sides. The fact that $D(3,2) = 0$ and the recurrence relation easily follow from Definition 6.1(3) and the grammatical transformation T_1, respectively.

(2) From Definition 6.1(4) it follows that for the new set of nonterminal symbols N_{n+1} of G_{n+1}^4 we have

$$N_{n+1} = N_n \cup \{A' \mid A \in N_n\} \cup \{A_{n+1}\}$$

with $S_{n+1} = S_n'$. Then we have $\nu_4(n+1) = 2\cdot\nu_4(n)+1$ for $n \ge 3$. Solving the corresponding homogeneous difference equation yields $\nu_{4,H}(n) = c\cdot 2^n$, whereas $\nu_{4,P}(n) = -1$ is a particular solution. Now $\nu_4(n) = \nu_{4,H}(n) + \nu_{4,P}(n) = c \cdot 2^n - 1$ which with initial condition $\nu_4(3) = 4$ results in $c = 5/8$ and $\nu_4(n) = 5 \cdot 2^{n-3} - 1$.

(3) Let $p_i(n)$ $(i = 1, 2, 3)$ be the number of productions in P_n of the form $A \to a$, $A \to aB$ and $A \to aBC$, respectively. Then we have by the definition of T_1:

	$D(n,k)$									
n	$k=1$	2	3	4	5	6	7	8	9	10
1	1									
2	2	1								
3	3	0	1							
4	4	3	1	1						
5	5	7	4	2	1					
6	6	12	11	6	3	1				
7	7	18	23	17	9	4	1			
8	8	25	41	40	26	13	5	1		
9	9	33	66	81	66	39	18	6	1	
10	10	42	99	147	147	105	57	24	6	1

Table 1: $D(n,k)$ for G_n^4 $(1 \leq n \leq 10)$.

(3.1) $p_1(n) = n$, since $E_{n,1} = \{A_1, \ldots, A_n\}$,

(3.2) $p_2(n+1) = 2 \cdot p_2(n) + \nu_4(n) + n = 2 \cdot p_2(n) + 5 \cdot 2^{n-3} + n - 1, \quad p_2(3) = 0$,

(3.3) $p_3(n+1) = 3 \cdot p_3(n), \quad p_3(3) = 6$.

From (3.3) we obtain $p_3(n) = 2 \cdot 3^{n-2}$ for $n \geq 3$. The solution of the homogeneous version of (3.2) is $p_{2,H}(n) = c \cdot 2^n$. A candidate particular solution $p_{2,P}(n)$ of the form $p_{2,P}(n) = An \cdot 2^n + Bn + C$ —cf. §4.5 in [20] for the details of this approach— results in $A = 5/16$, $B = -1$ and $C = 0$; consequently, $p_{2,P}(n) = 5 \cdot 2^{n-4} - n$ and $p_2(n) = p_{2,H}(n) + p_{2,P}(n) = c \cdot 2^n + 5 \cdot 2^{n-4} - n$. From $p_2(3) = 0$, we infer that $c = -9/16$, and hence $p_2(n) = 5n \cdot 2^{n-4} - 9 \cdot 2^{n-4} - n$.

Finally, we obtain $\pi_4(n) = p_1(n) + p_2(n) + p_3(n) = 2 \cdot 3^{n-2} + 5n \cdot 2^{n-4} - 9 \cdot 2^{n-4}$. □

The recurrence relation in Proposition 6.3(1) is identical to the one for the binomial coefficients $C(n,k)$, although the fact that $D(3,2) = 0$ results in a different Pascal-like triangle; cf. Table 1.

Although the family $\{G_n^4\}_{n\geq 1}$ is rather efficient with respect to the number of nonterminals as compared to the families $\{G_n^1\}_{n\geq 1}$, $\{G_n^2\}_{n\geq 1}$ and $\{G_n^3\}_{n\geq 1}$ —the number of productions is a different story; cf. §8— its degree of ambiguity is rather high. To illustrate this point consider a subderivation according to G_n^4 of the form $A \Rightarrow aBC \Rightarrow^\star aw_Bw_C$ with $B \Rightarrow^\star w_B$ and $C \Rightarrow^\star w_C$. Applying T_1 to G_n^4 yields a grammar G_{n+1}^4 according to which the substring $aw_Ba_{n+1}w_C$ can be obtained by $A' \Rightarrow aB'C \Rightarrow^\star aw_Ba_{n+1}w_C$ or by $A' \Rightarrow aBC' \Rightarrow^\star aw_Ba_{n+1}w_C$.

Next we will modify T_1 of Definition 6.1 into a grammatical transformation T_2 in such a way that the first subderivation is not possible, because the occurrence of a_{n+1} will always be introduced to the left of the terminal symbols $a_1, a_2, \cdots, a_n$.

Definition 6.4. The family $\{G_n^5\}_{n\geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n\geq 1}$ with

(1) G_1^5 equals G_1 from Section 2.

(2) G_2^5 equals G_2 from Section 2.

(3) G_3^5 equals G_3^4 from Definition 6.1.

(4) G_{n+1}^5 is obtained from G_n^5 $(n \geq 3)$ by the grammatical transformation T_2 described in steps (a), (b), (c), (d) and (e); T_2 properly extends P_n to P_{n+1} by adding new productions.

(a) If $A \to aBC$ is in P_n, then $A \to aBC$, $A' \to aB'C \mid aBC'$ and $A^\circ \to aBC^\circ$ are in P_{n+1}.

(b) If $A \to aB$ is in P_n, then $A \to aB$, $A' \to aB'$ and $A^\circ \to aB^\circ$ are in P_{n+1}.

(c) If $A \to a$ is in P_n, then $A \to a$ and $A^\circ \to aA_{n+1}$ are in P_{n+1}.

(d) We add $\nu_5(n) + 1$ new productions $A' \to a_{n+1}A$ $(A \in N_n)$ and $A_{n+1} \to a_{n+1}$ to P_{n+1}.

(e) Finally, each occurrence of S_n' and of S° in G_{n+1}^5 will be replaced by S_{n+1}, i.e., by the initial nonterminal symbol of G_{n+1}^5. □

Example 6.5. We apply T_2 to G_3^5 in order to obtain $G_4^5 = (V_4, \Sigma_4, P_4, S_4)$ with $N_4 = \{S_4, S_3, A_4\} \cup \{A_i, A_i', A_i^\circ \mid 1 \leq i \leq 3\}$ and P_4 consists of the rules

$$
\begin{array}{ll}
S_3 \twoheadrightarrow a_1A_2A_3 \mid a_2A_1A_3 \mid a_3A_1A_2,\ A_1 \to a_1\ A_2 \to a_2\ A_3 \to a_3, & P_3 \\
S_4 \twoheadrightarrow a_1A_2'A_3 \mid a_1A_2A_3' \mid a_2A_1'A_3 \mid a_2A_1A_3' \mid a_3A_1'A_2 \mid a_3A_1A_2', & \text{(a)} \\
S_4 \to a_1A_2A_3^\circ \mid a_1A_3A_2^\circ \mid a_2A_1A_3^\circ \mid a_2A_1A_3^\circ \mid a_3A_1A_2^\circ \mid a_3A_2A_1^\circ & \text{(a)} \\
\text{——} & \text{(b)} \\
A_1^\circ \to a_1A_4,\ A_2^\circ \to a_2A_4,\ A_3^\circ \to a_3A_4, & \text{(c)} \\
S_4 \to a_4S_3,\ A_1' \to a_4A_1,\ A_2' \to a_4A_2,\ A_3' \to a_4A_3,\ A_4 \to a_4. & \text{(d)}
\end{array}
$$

For G_4^5 we obtain $E_{4,4} = \{S_4\}$, $E_{4,3} = \{S_3\}$, $E_{4,2} = \{A_1', A_2', A_3', A_1^\circ, A_2^\circ, A_3^\circ\}$, $E_{4,1} = \{A_1, A_2, A_3, A_4\}$, $A_i \sqsubset S_3 \sqsubset S_4$, $A_i \sqsubset A_i' \sqsubset S_4$, $A_i \sqsubset A_i^\circ \sqsubset S_4$, $A_4 \sqsubset A_i'$, $A_4 \sqsubset A_i^\circ$ $(1 \leq i \leq 3)$, $\nu_4(4) = 12$ and $\pi_4(4) = 35$. □

Proposition 6.6. *For the family* $\{G_n^5\}_{n\geq 1}$ *of Definition 6.4 we have*

(1) $D(n,n) = 1, \quad D(n,1) = n \qquad (n \geq 1),$

$D(3,2) = 0,$

$D(n,k) = D(n-1,k) + 2 \cdot D(n-1,k-1) \qquad (n \geq 4; \quad 2 \leq k \leq n-1),$

(2) $\nu_5(n) = 4 \cdot 3^{n-3} \qquad (n \geq 3),$

(3) $\pi_5(n) = 6 \cdot 4^{n-3} + 4n \cdot 3^{n-4} - \frac{1}{4} \cdot 3^{n-1} + \frac{1}{2}n - \frac{1}{4} \qquad (n \geq 3),$

(4) $\delta_5(n) = n!$, *i.e.,* G_n^5 *is unambiguous.*

Proof. The proof is similar to the one of Proposition 6.3; so (1) follows from the definitions of G_3^5 and T_2; see also Table 2.

(2) Definition 6.4(4) implies that the new set of nonterminals N_{n+1} of G_{n+1}^5 satisfies

$$N_{n+1} = N_n \cup \{A', A^\circ \mid A \in N_n\} \cup \{A_{n+1}\}$$

with $S_{n+1} = S_n' = S_n^\circ$. Then $\nu_5(n+1) = 3 \cdot \nu_5(n) - 1 + 1 = 3 \cdot \nu_5(n)$ for $n \geq 3$ with $\nu_5(3) = 4$. Solving this homogeneous difference equation yields $\nu_5(n) = 4 \cdot 3^{n-3}$.

	D(n, k)									
n	$k = 1$	2	3	4	5	6	7	8	9	10
1	1									
2	2	1								
3	3	0	1							
4	4	6	1	1						
5	5	14	13	3	1					
6	6	24	41	29	7	1				
7	7	36	89	111	65	15	1			
8	8	50	161	289	287	145	31	1		
9	9	66	261	611	865	719	321	63	1	
10	10	84	399	1133	2087	2449	1759	705	127	1

Table 2: $D(n, k)$ for G_n^5 $(1 \le n \le 10)$.

(3) From the definition of T_2 we obtain for $p_i(n)$ $(i = 1, 2, 3)$, i.e., the number of productions in P_n of the form $A \to a$, $A \to aB$ and $A \to aBC$, respectively:

(3.1) $p_1(n) = n$, since $E_{n,1} = \{A_1, \ldots, A_n\}$,

(3.2) $p_2(n + 1) = 3 \cdot p_2(n) + n + \nu_5(n) = 3 \cdot p_2(n) + 4 \cdot 3^{n-3} + n, \quad p_2(3) = 0$,

(3.3) $p_3(n + 1) = 4 \cdot p_3(n), \quad p_3(3) = 6$.

From (3.3) we infer that $p_3(n) = 6 \cdot 4^{n-3}$ for $n \ge 3$. The solution of the homogeneous equation corresponding to (3.2) is $p_{2,H}(n) = c \cdot 3^n$. A particular solution of the form $p_{2,P}(n) = An \cdot 3^n + Bn + C$ yields $A = 4/81$, $B = -1/2$ and $C = -1/4$, i.e., $p_{2,P}(n) = 4n \cdot 3^{n-4} - \frac{1}{2}n - \frac{1}{4}$. So $p_2(n) = p_{2,H}(n) + p_{2,P}(n) = c \cdot 3^n + 4n \cdot 3^{n-4} - \frac{1}{2}n - \frac{1}{4}$ and $p_2(3) = 0$ results in $c = -1/12$, i.e., $p_2(n) = 4n \cdot 3^{n-4} - \frac{1}{4} \cdot 3^{n-1} - \frac{1}{2}n - \frac{1}{4}$. Consequently, we have $\pi_5(n) = p_1(n) + p_2(n) + p_3(n) = 6 \cdot 4^{n-3} + 4n \cdot 3^{n-4} - \frac{1}{4} \cdot 3^{n-1} + \frac{1}{2}n - \frac{1}{4}$ for $n \ge 3$.

(4) The argument is by induction on n and analogous to the proof of Proposition 7.3 in [2]; viz. we distinguish two cases: (i) the string to be derived ends in a_{n+1} (and each nonterminal sentential form in that derivation contains a single "circled nonterminal symbol" and no "primed nonterminal symbol"), and (ii) the string to be derived does not end in a_{n+1} (and each nonterminal sentential form possesses a single "primed nonterminal symbol" and no "circled nonterminal symbol"). The detailed proof is left as an exercise to the interested reader. □

The price we have to pay for unambiguous grammars in Greibach 2-form is rather high. Comparing Propositions 6.3 and 6.6 yields: $\nu_5(n) > \nu_4(n)$ and $\pi_5(n) > \pi_4(n)$ for $n \ge 4$; cf. also Tables 1 and 2.

Notice that the grammatical transformations T_i $(i = 1, 2)$ of Definitions 6.1 and 6.4 are of general interest in the following way: given *any* context-free grammar G_n in Greibach

2-form that generates L_n, then T_i yields a context-free grammar G_{n+1} in Greibach 2-form for L_{n+1}. We will apply this observation in Section 8.

7 Greibach 2-form — Divide and Conquer

In the previous sections we studied families of grammars with the property that $E_{n,k} \neq \varnothing$ for all k ($1 \leq k \leq n$) with an exception of $E_{3,2} = \varnothing$. The family $\{G_n^6\}_{n\geq 1}$ to be introduced in this section is a divide-and-conquer variant of the family $\{G_n^1\}_{n\geq 1}$ of Section 5: rather than dividing the set $X \cup Y$ in all possible disjoint nonempty subsets X and Y, we only split $X \cup Y$ into almost equally sized X and Y; cf. Definitions 5.1 and 7.1. This results in grammars G_n^6 with $E_{n,k} = \varnothing$ for some values of k, provided we have $n \geq 4$. Among others these values of k always include the ones that satisfy $\lceil (n+1)/2 \rceil \leq k < n$.

$E_{n,1} := \{A_{\{a\}} \mid a \in \Sigma_n\}$;
$N_n := E_{n,1}$;
$M := \{A_{\Sigma_n}\}$;
$P_n := \{A_{\{a\}} \to a \mid a \in \Sigma_n\}$;
while $M - E_{n,1} \neq \varnothing$ [i.e., $\exists A_X \in M :\ X \subseteq \Sigma_n$ and $\#X \geq 2$] **do**
  begin
  if $\#X \geq 3$ **then**
    begin
    $S(X) := \{(a, Y, Z) \mid a \in X,\ Y \subset X - \{a\},\ \#Y = \lceil \frac{1}{2}\#(X - \{a\}) \rceil,$
      $Z = X - \{a\} - Y\}$;
    $P_n := P_n \cup \{A_X \to aA_YA_Z \mid (a, Y, Z) \in S(X)\}$;
    $M := (M - \{A_X\}) \cup \{A_Y, A_Z \mid (a, Y, Z) \in S(X)\}$
    end
  else [i.e., $\#X = 2$]
    begin
    $S(X) := \{(a, Y) \mid a \in X,\ Y = X - \{a\}\}$;
    $P_n := P_n \cup \{A_X \to aA_Y \mid (a, Y) \in S(X)\}$;
    $M := (M - \{A_X\}) \cup \{A_Y \mid (a, Y) \in S(X)\}$
    end;
  $N_n := N_n \cup \{A_X\}$
  end

Figure 1: Algorithm to determine N_n and P_n of G_n^6.

	$D(n,k)$									
n	$k=1$	2	3	4	5	6	7	8	9	10
1	1									
2	2	1								
3	3	0	1							
4	4	6	0	1						
5	5	10	0	0	1					
6	6	15	20	0	0	1				
7	7	0	35	0	0	0	1			
8	8	28	56	70	0	0	0	1		
9	9	36	0	126	0	0	0	0	1	
10	10	45	0	210	252	0	0	0	0	1

Table 3: $D(n,k)$ for G_n^6 $(1 \leq n \leq 10)$.

Definition 7.1. The family $\{G_n^6\}_{n\geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n\geq 1}$ with

- $S_n = A_{\Sigma_n}$, and
- the sets $N_n = V_n - \Sigma_n$ and P_n are determined by the algorithm in Figure 1. □

Example 7.2. (1) For $n = 4$, Definition 7.1 yields the grammar G_4^6 with $S_4 = A_{1234}$, $N_4 = E_{4,1} \cup E_{4,2} \cup E_{4,3} \cup E_{4,4}$, $E_{4,1} = \{A_1, A_2, A_3, A_4\}$, $E_{4,2} = \{A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}\}$, $E_{4,3} = \varnothing$, $E_{4,4} = \{A_{1234}\}$, $P_4 = \{A_{1234} \to a_1A_{23}A_4 \mid a_1A_{24}A_3 \mid a_1A_{34}A_2 \mid a_2A_{13}A_4 \mid a_2A_{14}A_3 \mid a_2A_{34}A_1 \mid a_3A_{12}A_4 \mid a_3A_{14}A_2 \mid a_3A_{24}A_1 \mid a_4A_{12}A_3 \mid a_4A_{13}A_2 \mid a_4A_{23}A_1\} \cup \{A_{ij} \to a_iA_j,\ A_{ij} \to a_jA_i \mid 1 \leq i < j \leq 4\} \cup \{A_i \to a_i \mid 1 \leq i \leq 4\}$, $\nu_6(4) = 11$ and $\pi_6(4) = 28$.

(2) Similarly, for $n = 7$ we obtain G_7^6 with $S_7 = A_{1234567}$, $E_{7,6} = E_{7,5} = E_{7,4} = E_{7,2} = \varnothing$, $N_7 = E_{7,7} \cup E_{7,3} \cup E_{7,1}$, $E_{7,7} = \{A_{1234567}\}$, $E_{7,3} = \{A_{ijk} \mid 1 \leq i < j < k \leq 7\}$ and $E_{7,1} = \{A_i \mid 1 \leq i \leq 7\}$. We leave it to reader to write down all elements of P_7 and to verify that $\nu_6(7) = 43$ and $\pi_6(7) = 357$.

(3) For $n = 15$ the algorithm of Definition 7.1 produces a grammar G_{15}^6 with $N_{15} = E_{15,15} \cup E_{15,7} \cup E_{15,3} \cup E_{15,1}$ whereas the other $E_{15,k}$'s are empty; see Example 7.3 below. Now we have $\nu_6(15) = 6906$ and $\pi_6(15) = 955125$. □

In order to formulate the next result concisely (cf. Proposition 7.4) we need an indicator function $I : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ defined recursively by

- $I(1) = \{1\}$,
- $I(2) = \{1, 2\}$,
- $I(2n+1) = \{2n+1\} \cup I(n)$, and
- $I(2n+2) = \{2n+2\} \cup I(n+1) \cup I(n)$.

Example 7.3 $I(3) = \{1, 3\}$, $I(4) = \{1, 2, 4\}$, $I(5) = \{1, 2, 5\}$, $I(6) = \{1, 2, 3, 6\}$, $I(7) =$

$\{1,3,7\}$, $I(8) = \{1,2,3,4,8\}$, $I(14) = \{1,2,3,6,7,14\}$, $I(15) = \{1,3,7,15\}$, $I(16) = \{1,2,3,4,7,8,16\}$ and for $j \geq 1$, we have $I(2^j - 1) = \{2^i - 1 \mid 1 \leq i \leq j\}$. □

The next equalities easily follow from the structure of the algorithm in Definition 7.1; cf. Figure 1.

Proposition 7.4. *For the family $\{G_n^6\}_{n\geq 1}$ of Definition 7.1 we have*

(1) $D(n,k)$ = **if** $k \in I(n)$ **then** $C(n,k)$ **else** 0,

(2) $\nu_6(n) = \sum_{k=1}^{n} D(n,k)$,

(3) $\pi_6(n) = \sum_{k=1}^{n} D(n,k) \cdot k \cdot C(k-1, \lceil (k-1)/2 \rceil)$. □

The values of $D(n,k)$ for $1 \leq n \leq 10$ are in Table 3. As usual a closed form for $D(n,k)$, $\nu_6(n)$ and $\pi_6(n)$ is very hard or even impossible to obtain; a situation met frequently in analyzing such divide-and-conquer approaches; cf. e.g. pp. 62–78 in [22], [23] or [2]. For a numerical evaluation of the complexity measures $\nu_6(n)$ and $\pi_6(n)$ together with a comparison to earlier measures we refer to Section 8.

8 Concluding Remarks

In this paper we investigated some ways to generate the set of all permutations of an alphabet of n symbols by context-free grammars in Greibach normal form. Since the arbitrary Greibach normal form does not yield very interesting results (cf. Proposition 3.7), we mainly restricted our attention to the Greibach m-form with $m = 1, 2$. This resulted in grammar families $\{G_n^i\}_{n\geq 1}$ $(1 \leq i \leq 6)$ of which we studied the descriptional complexity measures $\nu_i(n)$ (i.e., the number of nonterminal symbols) and $\pi_i(n)$ (i.e., the number of productions). An overview of the actual values for $1 \leq n \leq 16$ of these complexity measures is shown in Tables 4 and 5. Of course, these numerical values confirm that all functions ν_i and π_i show the exponential growth that has been predicted by Theorem 4.2.

With respect to the measures ν we observe that for $n \geq 9$, $\nu_6(n) < \nu_i(n)$ with $1 \leq i \leq 5$. As far as the measure π is concerned, we ignore the family $\{G_n^3\}_{n\geq 1}$ whose members are in Greibach 1-form. So restricting our attention to the Greibach 2-form we have that for $n \geq 4$, $\pi_6(n) < \pi_i(n)$ with $1 \leq i \leq 5$ and $i \neq 3$. But this does not mean that $\{G_n^6\}_{n\geq 1}$ is minimal with respect to both these measures since the following tiny local improvement to that family is possible.

Looking more closely to Tables 4 and 5 we see that in case $n = 2^k - 1$ for some $k \geq 2$, both $\nu_6(n)$ and $\pi_6(n)$ are rather small compared to the values of ν_6 and π_6 respectively, for the next two arguments 2^k and $2^k + 1$. This allows us to define a slightly improved family $\{G_n^7\}_{n\geq 1}$ as follows:

- $G_n^7 = G_n^6$ for all $n \geq 3$ with $n \neq 2^k$ for some $k \geq 2$,
- $G_n^7 = T_1(G_{n-1}^6)$, if $n = 2^k$ for some $k \geq 2$,

where T_1 is the grammatical transformation introduced in Definition 6.1. Remember that T_1 is applicable to any grammar G_n in Greibach 2-form that generates L_n, and that the resulting grammar $T_1(G_n)$ —which generates L_{n+1}— is in Greibach 2-form as well; a similar

n	$\nu_1(n) = \nu_2(n) = \nu_3(n)$	$\nu_4(n)$	$\nu_5(n)$	$\nu_6(n)$
1	1	1	1	1
2	3	3	3	3
3	7	4	4	4
4	15	9	12	11
5	31	19	36	16
6	63	39	108	42
7	127	79	324	43
8	255	159	972	163
9	511	319	2916	172
10	1023	639	8748	518
11	2047	1279	26244	529
12	4095	2559	78732	2015
13	8191	5119	236196	2094
14	16383	10239	708588	6905
15	32767	20479	2125764	6906
16	65535	40959	6377292	26827

Table 4: $\nu_i(n) \quad (1 \leq i \leq 6; \quad 1 \leq n \leq 16)$.

remark applies to the transformation T_2 of Definition 6.4. Then for $n = 2^k$ with $k \geq 2$, we obtain

$$\nu_7(n) = 2 \cdot \nu_6(n-1) + 1,$$
$$\pi_7(n) = \pi_6(n-1) + 4 \cdot 3^{n-3} + (5n-4) \cdot 2^{n-5};$$

cf. Tables 6 and 7, where the $r(X_6, n)$ with $X_6 = \nu_6, \pi_6$ are the ratios defined by $r(X_6, n) = X_6(n)/X_6(2^k - 1)$ and k is determined by $2^k - 1 \leq k < 2^{k+1} - 1$. We observe that $\nu_7(2^k) < \nu_6(2^k)$ for $k \geq 2$, but the price we have to pay for this improvement is an increase in the number of productions: $\pi_7(2^k) > \pi_6(2^k)$; cf. Tables 6 and 7.

One is tempted to apply T_1 twice, i.e., defining a family $\{G_n^7\}_{n\geq 1}$ by

- $G_n^8 = G_n^7$ for all $n \geq 3$ with $n \neq 2^k + 1$ for some $k \geq 2$,
- $G_n^8 = T_1(G_{n-1}^7)$, if $n = 2^k + 1$ for some $k \geq 2$,

but this turns out not to be an improvement upon $\{G_n^7\}_{n\geq 1}$: $\nu_8(2^k + 1) > \nu_7(2^k + 1)$ and $\pi_8(2^k + 1) > \pi_7(2^k + 1)$; cf. Tables 6 and 7.

Applying the transformation T_2 from Definition 6.4 instead of T_1 in the very similar way —resulting into two other families of grammars $\{G_n^9\}_{n\geq 1}$ and $\{G_n^{10}\}_{n\geq 1}$, respectively— is not of much use either: we lose rather than gain some descriptional efficiency. The recurrence relations corresponding to T_2 are

n	$\pi_1(n)$	$\pi_2(n)$	$\pi_3(n)$	$\pi_4(n)$	$\pi_5(n)$	$\pi_6(n)$
1	1	1	1	1	1	1
2	4	4	4	4	4	4
3	18	18	12	9	9	9
4	80	68	32	29	35	28
5	330	220	80	86	138	55
6	1272	642	192	246	542	216
7	4662	1750	448	694	2113	357
8	16480	4552	1024	1954	8193	1520
9	56754	11448	2304	5526	31688	2223
10	191720	28080	5120	15746	122548	11440
11	638286	67474	11264	45254	474687	16753
12	2101200	159612	24576	131154	1843511	86208
13	6855498	372580	53248	382966	7182118	116857
14	22205848	859978	114688	1125346	28073994	687064
15	71498790	1965870	245760	3323814	110096381	955125
16	229058240	4456208	524288	9856754	433078189	5333616

Table 5: $\pi_i(n) \quad (1 \leq i \leq 6; \quad 1 \leq n \leq 16)$.

$\nu_9(n) = \nu_6(n-1) + 8 \cdot 3^{n-4}$,

$\pi_9(n) = \pi_6(n-1) + 18 \cdot 4^{n-4} + (8n - \frac{19}{2}) \cdot 3^{n-5} + \frac{1}{2}$;

which should enable the interested reader to construct the analogues of Tables 6 and 7 for the families $\{G_n^9\}_{n\geq 1}$ and $\{G_n^{10}\}_{n\geq 1}$.

In describing the complexity of a pushdown automaton (or PDA) frequently used measures are the number σ of states and the number γ of stack symbols [12, 13]. Applying the standard construction for transforming a context-free grammar G into an equivalent PDA $A(G)$ —e.g., Theorem 5.4.1 and its proof in [16]— results in a single-state PDA: $\sigma_{A(G)} = 1$. Therefore we will use the number τ of possible transitions of $A(G)$ rather than σ.

When we apply that standard construction to our grammars in CNF [2] or in GNF (Sections 5–8) for $\{L_n\}_{n\geq 1}$ we end up with families of single-state PDA's of which the transition relation δ is defined by

(a) $\delta(q, \lambda, A) = \{(q, \alpha^R) \mid A \to \alpha \in P_n\}$ for each $A \in N_n$, and

(b) $\delta(q, a, a) = \{(q, \lambda)\}$ for each $a \in \Sigma_n$,

where R is the reversal or mirror operation on strings; cf. Theorem 5.4.1 in [16]. This implies immediately that $\gamma(n) = \nu(n) + n$ and $\tau(n) = \pi(n) + n$. However, in case of Greibach normal form we may replace (a) and (b) by

		$\nu_i(n)$		
n	$r(\nu_i, n)$	$i = 6$	$i = 7$	$i = 8$
3	1.000	4	4	4
4	2.750	11	9	9
5	4.000	16	16	19
7	1.000	43	43	43
8	3.791	163	87	87
9	4.000	172	172	175
15	1.000	6906	6906	6906
16	3.885	26827	13813	13813
17	3.986	27524	27524	27627
31	1.000	303174297	303174297	303174297
32	3.895	1180728715	606348595	606348595
33	3.909	1185006252	1185006252	1212697191

Table 6: $\nu_i(n)$ $\quad (6 \le i \le 8; \quad 2^k - 1 \le n \le 2^k + 1, \quad 2 \le k \le 5)$.

$\delta(q, a, A) = \{(q, \alpha^R) \mid A \to a\alpha \in P_n\}$ for each $A \in N_n$ and each $a \in \Sigma_n$,

and then we obtain $\gamma(n) = \nu(n)$ and $\tau(n) = \pi(n)$. Consequently, the quest of a family of minimal single-state PDA's for $\{L_n\}_{n\ge1}$ is as tightly connected as possible to the search of a family of minimal context-free grammars in GNF generating $\{L_n\}_{n\ge1}$, provided we use γ and τ as descriptional complexity measures for PDA's. This latter condition sounds reasonable in the context of single-state PDA's.

References

1. B. Alspach, P. Eades & G. Rose, A lower-bound for the number of productions for a certain class of languages, *Discrete Appl. Math.* **6** (1983) 109–115.

2. P.R.J. Asveld, Generating all permutations by context-free grammars in Chomsky normal form, *Theoret. Comput. Sci.* **354** (2006) 118–130.

3. P.R.J. Asveld, Generating all circular shifts by context-free grammars in Chomsky normal form, CTIT Technical Report No. 05-23 (2005), ISSN 1381-3625, University of Twente, Enschede, the Netherlands. To appear in *Journal of Automata, Languages and Combinatorics* **11** (2006) 147–159.

n	$r(\pi_i, n)$	$\pi_i(n)$		
		$i = 6$	$i = 7$	$i = 8$
3	1.000	9	9	9
4	3.111	28	29	29
5	6.111	55	55	86
7	1.000	357	357	357
8	4.258	1520	1617	1617
9	6.227	2223	2223	5189
15	1.000	955125	955125	955125
16	5.584	5333616	7488065	7488065
17	7.390	7058519	7058519	26951717
31	1.000	15476986049221	15476986049221	15476986049221
32	5.881	91023676672384	290019433474321	290019433474321
33	7.764	120158370033735	120158370033735	1113627179961333

Table 7: $\pi_i(n) \quad (6 \leq i \leq 8; \quad 2^k - 1 \leq n \leq 2^k + 1, \quad 2 \leq k \leq 5)$.

4. P.R.J. Asveld, Generating all circular shifts by context-free grammars in Greibach normal form, *Internat. J. Found. of Comput. Sci.* **18** (2007) 1139–1149.

5. P.R.J. Asveld & A. Nijholt, The inclusion problem for some subclasses of context-free languages, *Theoret. Comput. Sci.* **230** (2000) 247–256.

6. W. Bucher, A note on a problem in the theory of grammatical complexity, *Theoret. Comput. Sci.* **14** (1981) 337–344.

7. W. Bucher, H.A. Maurer & K. Culik II, Context-free complexity of finite languages, *Theoret. Comput. Sci.* **28** (1984) 277–285.

8. W. Bucher, H.A. Maurer, K. Culik II & D. Wotschke, Concise description of finite languages, *Theoret. Comput. Sci.* **14** (1981) 227–246.

9. J. Dassow, On the circular closure of languages, *EIK* **15** (1979) 87–94.

10. K. Ellul, B. Krawetz, J. Shallit & M.-w. Wang, Regular expressions: new results and open problems, *J. Autom. Lang. Comb.* **9** (2004) 233–256.

11. S. Ginsburg & H.G. Rice, Two families of languages related to ALGOL, *J. Assoc. Comp. Mach.* **9** (1962) 350–371.

12. J. Goldstine, J.K. Price & D. Wotschke, A pushdown automaton or a context-free grammar — which is more economical? *Theoret. Comput. Sci.* **18** (1982) 33–40.

13. J. Goldstine, J.K. Price & D. Wotschke, On reducing the number of states in a PDA, *Math. Systems Theory* **15** (1982) 315–321.

14. R.L. Graham, D.E. Knuth & O. Patashnik, *Concrete Mathematics* (1989), Addison-Wesley, Reading, MA.

15. J. Gruska, Some classifications of context-free languages, *Inform. Contr.* **14** (1969) 152–179.

16. M.A. Harrison, *Introduction to Formal Language Theory* (1978), Addison-Wesley, Reading, MA.

17. V.A. Iljuškin, The complexity of the grammatical description of context-free languages, *Dokl. Akad. Nauk SSSR* **203** (1972) 1244–1245 / *Soviet Math. Dokl.* **13** (1972) 533–535.

18. A. Kelemenová, Complexity of normal form grammars, *Theoret. Comput. Sci.* **28** (1984) 299–314.

19. C.L. Liu, *Introduction to Combinatorial Mathematics* (1968), McGraw-Hill, New York, etc.

20. R.E. Mickens, *Difference Equations — Theory and Applications* (1987), Second Edition (1990), Chapman & Hall, New York, London.

21. G. Satta, personal communication (2002).

22. R. Sedgewick & Ph. Flajolet, *An Introduction to the Analysis of Algorithms* (1996), Addison-Wesley, Reading, Ma.

23. X. Wang & Q. Fu, A frame for general divide-and-conquer recurrences, *Inform. Process. Lett.* **59** (1996) 45–51.