ON FUZZY CONCEPTS

Cornelis Hoede and Xin Wang*

University of Twente
Dept. of Applied Mathematics, Faculty of EEMCS
P.O. Box 217
7500 AE Enschede
The Netherlands

Abstract

In this paper we try to combine two approaches. One is the theory of knowledge graphs in which concepts are represented by graphs. The other is the axiomatic theory of fuzzy sets (AFS).

The discussion will focus on the idea of fuzzy concept. It will be argued that the fuzziness of a concept in natural language is mainly due to the difference in interpretation that people give to a certain word. As different interpretations lead to different knowledge graphs, the notion of fuzzy concept should be describable in terms of sets of graphs. This leads to a natural introduction of membership values for elements of graphs. Using these membership values we apply AFS theory as well as an alternative approach to calculate fuzzy decision trees, that can be used to determine the most relevant elements of a concept.

Key words: concept, fuzzy, graph, decision tree.

AMS classification: 03E72, 05C99

* On leave from Dalian Maritime University and Dalian University of Technology, Dalian, P.R. Chin

1. INTRODUCTION

Knowledge graphs have been studied at the University of Twente since 1982. The basic idea is to represent words, standing for concepts, by labeled vertices of a graph and connect these vertices by labeled links. The resulting graph can be seen as a proposition or as a concept itself.

For graph theory we refer to any of the many text books, e.g. that of Bondy and Murty[1], available on internet. For detailed discussion of the labels of the links we refer to Zhang [14], in which the use of knowledge graphs for natural language is discussed. In this paper we will only use unlabeled edges as links, as for our purpose the labels are irrelevant.

The axiomatic theory of fuzzy sets (AFS) has been developed at the Dalian Maritime University since 1995. We refer to the paper of Liu and Pedrycz [11] for an introduction to AFS theory. In that paper also fuzzy decision trees were developed. We will apply their theory to calculate some fuzzy decision trees in order to compare the outcomes with that of an other approach.

The concept of fuzziness was introduced by Zadeh [13] and has led to quite some discussion from the side of probability theory. The concept of fuzziness seems to be a fuzzy concept itself. We therefore should explain our own stand. We will do this in Section 2. In Section 3 we will discuss our definition of fuzzy concept and discuss the concept “democracy” as an example. We will also show how important elements can be found. In Section 4 we will discuss the same example in the context of AFS theory and show how fuzzy decision rules can be derived for determining important elements. In Section 5 an alternative approach is derived. In Section 6 we will discuss the concept “fuzzy” as illustration of the ideas developed in the earlier sections. The appendices contain the descriptions of the definitions of the two studied concepts.

2. ON FUZZINESS

We consider one of the standard examples of words often mentioned to illustrate the idea of fuzziness. It is the word “small”. We will consider the height of people and want to discuss “small”, next to “normal” and “tall”.

In our view there is no fuzziness involved in first instance. The word “normal” implies that norms are used. Consider a person P and ask him what “normal height” means. The answer of P may be that the height H is normal if 150 < H < = 180. If 0 < H <= 150 the considered height is small, and if 180 < H the considered height is tall where the numbers refer to centimeters. The concept “small” is now well defined and not “fuzzy” at all.

The situation changes if P_1, P_2, …., P_n are *n* persons, *jurors*, who give n pairs of values for determining the boundaries between “small”, “normal ”and “tall”. The two sets of boundary values, between “small” and “normal ”, respectively “normal” and “tall”, might be integers around 150 respectively 180 and can then be represented by histograms , as in Figure 1,

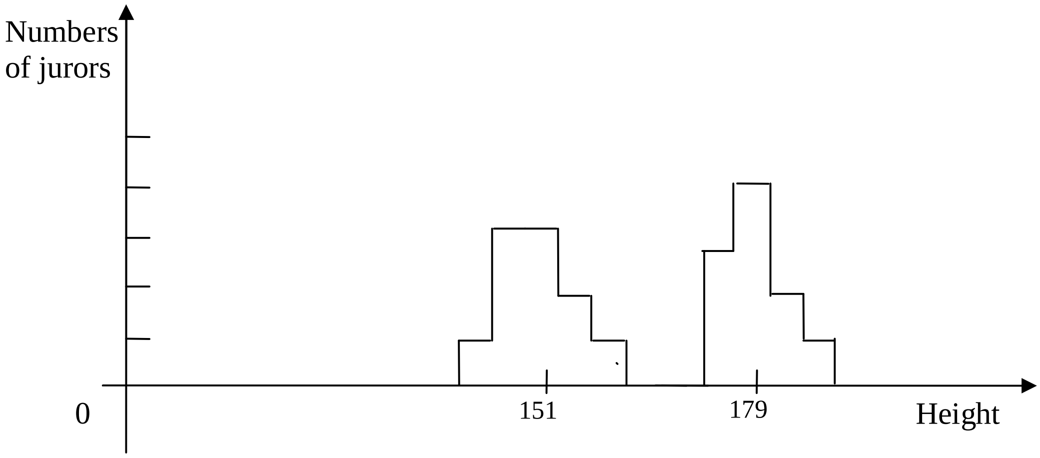


Figure 1
Distribution of boundaries

Whether a person, whose height is considered, is “small” now depends on the juror. A basketball coach might give 190 and 210 as boundary values! If the jurors agree on their average values, say 151 and 179, “small” is precisely defined again. If, however, the person is confronted with one of the jurors, the outcome gets a probabilistic character. In a certain number a of cases the person is considered to be small. But then the quotient *a*/*n* can be considered to express the probability that the person is considered small by a juror as well as to be the value of the membership function for the person with respect to the “set of small persons.”

However, the set of small persons has not been defined, so our discussion should lead to the conclusion that the adjective “small” will not always be used by jurors, who have to “adject” or “attribute” that word to a person. The concept “small” has a fuzzy character. Objections to the concept of “fuzzy set” are understandable.

For our further discussion it is important to note that people, our jurors in the example, give different interpretations to words. That is where the fuzziness comes in! Obtaining identical interpretations by discussions, as usual in science, in particular in natural science and mathematics, aims at removal of the fuzzy character. A definition that is agreed upon describes a concept that is not fuzzy at all, or is *crisp*.

But now we can ask ourselves whether this is always the case and find out that this is seldom the case. A striking example is “democracy”, for which an internet search for “definition of democracy” gives more than 10 million sites. In any research in which natural language plays an important role, often the ontology to use is strongly debated, by the very fact that most words allow different interpretations.

In the theory of knowledge graphs, a word has a corresponding word graph, depending on the interpreter, juror, who interprets the word. Differences in interpretation are similar to the differences in the boundary values of our example “small”. Consequently we have to consider **differences in word graphs** for discussing the fuzziness of the concept named by the word considered.

3. A KNOWLEDGE GRAPH ANALYSIS OF THE CONCEPT “DEMOCRACY”

In Appendix A we have listed 10 definitions of “democracy”, without reference to authors, as that is of no concern considering the many other definitions.

We can give extensive knowledge graphs for all 10 definitions, but that would not mark the point we want to make. Our procedure was simply this. First we determined concepts in the definitions, using our own background knowledge to identify, for example, “persons”, “people” and “citizens” and choosing “people” as word. Then we dropped all concepts occurring only once, which left 8 concept elements of “democracy”. For each of the 10 jurors, the elements occurring in their definition were represented by labeled vertices. The occurring vertices were then linked by an, unlabeled, edge whenever we considered them linked, on interpretation of the definition. This then led to 10 small graphs, we might call them *definition graphs*.

From the 10 definitions graphs a larger graph is easily constructed having 8 vertices and 14 edges. It is given in Figure 2.

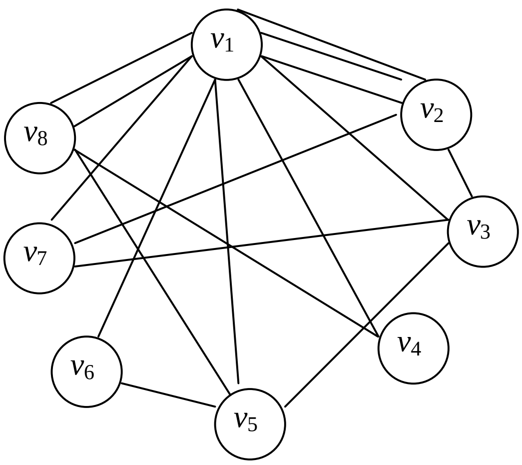


Figure 2
Combined definition graph for “democracy”

For the curious reader: v_1=people, v_2=decisions, v_3=institutions, v_4= voting, v_5=power, v_6= majority , v_7=negotiations , v_8= representation .

The graph in Figure 2 may be considered, by the 10 jurors, to be “the” definition of “democracy” and their agreement removes the fuzziness of the concept. However, we consider only v_1, v_2, v_3 and v_5 to be really basic aspects of “democracy”. Each of the 10 jurors considers a concept or a relationship basic or not. *It is in this respect that the concept is fuzzy.*

As determining the essential aspects of a concept is of extreme importance in many instances of use of words in natural language, we will now focus on the question how to determine important vertices and edges in the combined definition graph. We considered these to be the vertices and edges of the subgraph induced by the vertex set $\{v_1, v_2, v_3, v_5\}$. However, we would like to develop a more objective way of dealing, a *decision rule*.

We consider four parameters that capture some important aspects of the resulting combined definition graph.

1. The number of occurrences of the vertices as concepts in the definitions, e.g. v_1 “people” occurred in 8 definitions.
2. The number of occurrences of the links, e.g. the link $\{v_1, v_2\}$ occurred 3 times.
3. The number of links in which a vertex is involved taking into account the multiplicity of the links.
4. The number of other links in which the two vertices of a link are involved, again taking into account the multiplicity of links.

Multiplicity of links was indicated by multiple edges in Figure 2. The other links occurred only once. The numbers in 3 and 4 are called *vertex- degree* and *edge-degree*.

One objective way to determine important elements of the combined definition graph would be to consider the degrees. The important vertices are probably those that are somehow “central”. Centrality can be implemented in various ways, here we simply take the occurrence as even simpler measure for centrality. We can extend the set of vertices to be considered important by starting with vertices of highest degree and gradually adding vertices of lower degree in order of value of the degrees. Note that only those edges are to be added, whose both vertices belong to the constructed set.

It is clear from Figure 2 that v_1 and v_2 are the central elements of “democracy”, the “decisions” of “people”.

In the next section we will put this problem in the context of axiomatic fuzzy set theory.

4. APPLYING AXIOMATIC FUZZY SET THEORY

In this section, we will shortly describe the AFS theory. Then we will apply this theory to the example of “democracy”, and derive fuzzy decision rules.

In current fuzzy theories, the membership functions are often given by personal intuition and the logic operations are implemented by a kind of triangular norms, shortly norms, which are chosen beforehand and independent of the original data and facts. The large-scale intelligent systems in real-world applications are usually very large and complex, containing such a large number of concepts that it is impossible to define the membership functions by personal intuition and to choose a suitable norm from very many kinds of triangular norms to implement fuzzy logic systems.

In order to deal with the above discussed problems, AFS (Axiomatic Fuzzy Set) theory was firstly proposed in [3, 4] in 1995. In [5, 6], the mathematical properties of AFS algebras and AFS structures have been extensively investigated and discussed, and the fuzzy theory based on AFS algebras and AFS structures has been initially established. Then the topological structures on the AFS algebras and AFS structures were obtained in [7], and the preliminary combinatorial properties of AFS structures have been discussed in [8].

First we introduce notations and definitions, introduce several pertinent results concerning and discuss the basic ideas of AFS theory. In essence, the AFS framework supports studies on how to convert the information in training examples or databases to the membership functions and their fuzzy logic operations. AFS theory is made of AFS structures which are a special kind of combinatorial object [2] and AFS algebra which is a family of completely distributive lattices [11]. For detailed mathematical properties of AFS algebra, see [3,6,7,5,8].

In order to study the essential nature of fuzzy concepts and fuzzy logic, let X, M be two sets. In general M is the set of crisp or fuzzy concepts on X. For example, in “knowledge graph of democracy” data, $M=\{m_1, m_2, ..., m_8\}$, (where m_1=occurrence low, m_2=occurrence medium, m_3=occurrence high, m_4=degree low, m_5=degree medium, m_6=degree high, m_7=acceptance, m_8=non-acceptance} and X is the set of 14 or 8 training samples, given respectively by 14 edges and 8 vertices. Let

$$EM^* = \{\sum_{i\in I}A_i \mid A_i\subseteq M,\ i\in I,\ I \text{ is any non-empty indexing set}\}$$
$$EXM^* = \{\sum_{i\in I}a_iA_i \mid a_i\subseteq X,\ A_i\subseteq M,\ i\in I,\ I \text{ is any non-empty indexing set}\}.$$

In [5, 6], an equivalence relation R is defined on EM^* and EXM^* respectively, and we denote EM^*/R as EM and EXM^*/R as EXM. For two elements, the semantics of them (or the membership degrees they represent) are equivalent if they have relation R:
$(\sum_{i\in I}A_i, \sum_{j\in J}B_j)\in R \Leftrightarrow \forall i\in I, \exists k\in J$, such that $A_i\supseteq B_k$ and $\forall j\in J, \exists q\in I$ such that $B_j\supseteq A_q$.
$(\sum_{i\in I}a_iA_i, \sum_{j\in J}b_jB_j)\in R \Leftrightarrow \forall i\in I, \exists\ k\in J$ such that $a_i\subseteq b_k$, $A_i\supseteq B_k$, and $\forall j\in J, \exists q\in I$ such that $b_j\subseteq a_q$, $B_j\supseteq A_q$.

Definition 1: For any $\sum_{i\in I}A_i, \sum_{j\in J}B_j\in EM$,

$$\sum_{i\in I}A_i \vee \sum_{j\in J}B_j = \sum_{u\in U}C_u$$
$$\sum_{i\in I}A_i \wedge \sum_{j\in J}B_j = \sum_{i\in I, j\in J}A_i\cup B_j.$$

Here U is the union of I and J, for $u \in U$, $C_u=A_u$, if $u \in I$ and $C_u=B_u$, if $u \in J$. A more convenient way to write it is

$$\sum_{i\in I}A_i + \sum_{j\in J}B_j = \sum_{u\in U}C_u.$$

(EM, $\wedge$, $\vee$) is called the *EI algebra* over M .

Definition 2: If X and M are sets, 2^M is the power set of M, $\tau: X\times X \to 2^M$, then the triple (M, τ, X) is called an *AFS structure* if τ satisfies the following axioms:

A1: $\forall (x_1, x_2) \in X\times X$, $\tau(x_1, x_2) \subseteq \tau(x_1, x_1)$

A2: $\forall (x_1, x_2), (x_2, x_3) \in X\times X$, $\tau(x_1, x_2) \cap \tau(x_2, x_3) \subseteq \tau(x_1, x_3)$.

X is called the *universe of discourse*, M is called an *attribute set* and τ is called a *structure*.

We can verify that (M, τ, X) is an AFS structure if for each $m \in M$, τ is defined as $\tau(x_i, x_j) = \{m | m \in M, (x_i, x_j) \in R_m\}$, $x_i, x_j \in X$. That is, for any $m \in \tau(x, x)$, x belongs to attribute m to some degree and for any $m \in \tau(x, y)$, the degree of x belonging to m is larger than or equal to that of y. (M, τ, X) is the mathematical abstraction of the complicated relationships among objects in X under the attributes in M. This implies that the information contained in databases and human intuitions is transformed to (M, τ, X), from which we can obtain the fuzzy sets and fuzzy logic operations.

Theorem 1. [6] Let (M, τ, X) be an AFS structure. For $B\subseteq X$, $A\subseteq M$ we define the symbol:

$$\underline{A}(B)=\{y \mid y\in X, \tau(x, y)\supseteq A, \forall x\in B\}, \text{ and}$$

for any given $x\in X$, we define a mapping ϕ_x: $EM\to EXM$, $\forall \sum_{i\in I}A_i \in EM$,

$$\phi_x(\sum_{i\in I}A_i)=\sum_{i\in I}\underline{A_i}(\{x\})A_i \in EXM.$$

Then φ_x is a *homomorphism* from lattice (EM, $\wedge$, $\vee$) to lattice (EXM, $\wedge$, $\vee$).

By Theorem 1, we know that, for any given concept $\zeta = \sum_{i\in I}A_i \in EM$, we get a mapping $\zeta: X \to EXM$. In this way, for each $\zeta \in EM$, ζ is Lattice-fuzzy set on X and the membership degree of x ($x \in X$) belonging to fuzzy set ζ is $\sum_{i\in I}\underline{A_i}(\{x\})A_i \in EXM$. Further, in [8, 9], the logic operator $'$ (negation) is defined as: $\forall \eta=\sum_{i\in I}A_i\in EM$, $\eta' = \wedge_{i\in I}(\vee_{a\in A_i}\{a'\})$. ($EM$, $\wedge$, $\vee$, $'$) is called an AFS fuzzy logic system.

Now we introduce a special family of measures used to determine which algebra becomes norm lattice structure, such that we can convert represented membership degrees to [0,1] interval representations and to great extent preserve the information contained in the representing fuzzy sets.

Definition 3*:* (Continuous case) Let X be a set, $X\subseteq R^n$ and ρ: $X\to R^+=[0, \infty)$. ρ is integrable under Lebesgue measure and $0<\int_X \rho d\mu<\infty$. S ($S\subseteq 2^X$) is the set of all Borel sets. For all $A\in S$, we define a measure m over S:

$$m(A) = \frac{\int_A \rho d\mu}{\int_X \rho d\mu}.$$

(Discrete case) Let X be a set, S a σ-algebra on X and $\rho: X \to R^+ = [0, \infty)$, $0 < \sum_{x\in X}\rho(x) < \infty$. For any $A \in S$, m is a measure over the σ-algebra S, defined as follows:

$$m(A) = \frac{\sum_{x\in A} \rho(x)}{\sum_{x\in X} \rho(x)}.$$

In what follows, we define a measure m by which we can convert *EII* algebra membership degree to [0, 1], to preserve the information contained in the *EII* algebra to a significant extent.

Definition 4: Let X and M be sets, (M, τ, X) be an AFS structure and S a σ-algebra over X. For each $a \in M$, there exists a function $\rho_a: X \to R^+ = [0, \infty)$, and let m_α be a measure defined by Definition 3 for each ρ_a. Then in a *semi-cognitive field* (M, τ, X, S), for a measurable fuzzy set $\sum_{i\in I}A_i \in EM$, we define its membership function as follows. For all x $\in$ X

$$\mu_{\sum_{i\in I} A_i}(x) = \mathrm{M}((\textstyle\sum_{i\in I} A_i)(x)) = \mathrm{M}(\textstyle\sum_{i\in I} \underline{A_i}(x)A_i) \in [0,1],$$

where M: $EXM \to [0, \infty)$, for each $\sum_{i\in I} a_i A_i \in EXM$, $a_i \in S$, and

$$\mathrm{M}(\textstyle\sum_{i\in I} a_i A_i) = \sup_{i\in I}(\prod_{\alpha\in A_i} m_\alpha(a_i)) \in [0,1], \ \forall\ i \in I.$$

Liu and Pedrycz [11] consider training examples in which for certain membership of "income" and "employment" a customer of a bank is given credit.

This decision on "credit" is strictly analogous to our decision to see a vertex or an edge as "acceptable" for a definition of a concept. Remember that we want to determine the important elements given a combined definition graph. Our personal decision for our "democracy" example was that $\{v_1, v_2, v_3, v_5\}$ was the set of acceptable vertices. The analogue of Table 1 in [11] now is

Table I(a): Data on vertices

	Occurrence	Degree	Acceptance
v_1	8	10	1
v_2	4	5	1
v_3	3	4	1
v_4	3	2	0
v_5	2	4	1
v_6	2	2	0
v_7	2	3	0
v_8	2	4	0

Table I(b): Data on edges

	Occurrence	Degree	Acceptance
$e_1 = v_1 - v_2$	3	9	1
$e_2 = v_1 - v_3$	1	12	1
$e_3 = v_1 - v_4$	1	10	0
$e_4 = v_1 - v_5$	1	12	1
$e_5 = v_1 - v_6$	1	10	0
$e_6 = v_1 - v_7$	1	11	0
$e_7 = v_1 - v_8$	2	10	0
$e_8 = v_2 - v_3$	1	7	1
$e_9 = v_2 - v_7$	1	6	0
$e_{10} = v_3 - v_5$	1	6	1
$e_{11} = v_3 - v_7$	1	5	0
$e_{12} = v_4 - v_8$	1	4	0
$e_{13} = v_5 - v_6$	1	4	0
$e_{14} = v_5 - v_8$	1	6	0

Some remarks are due here. First there is a semantic difference between credit and acceptance. If two persons have the same score on income and employment, their credit should be the same. Otherwise the data are inconsistent. If two vertices, or edges, have the same score on occurrence and degree, they need not both be acceptable or non-acceptable in a definition. Hence, the data cannot be called inconsistent then. Consequently a decision tree can not distinguish between them.

Secondly, we have essentially two tables, one for vertices and one for edges. The 12 attributes, mentioned before, should be split into two tables of 6 attributes. If a rule for the acceptance of vertices has been derived, that rule does not say anything about edges. However, if a rule for the acceptance of edges has been derived, that rule implies that certain vertices should be accepted, as an edge is just a pair of vertices. For non-acceptance of edges the situation is different, the vertices may well be acceptable, but there is no relationship, although in some definition that was mentioned.

Thirdly, an application of the theory to the data for vertices and to the data for edges yields two decision trees. As just mentioned, the results for the edges imply results for the vertices. Therefore the edges should be considered first and the results might be used to interpret the decision tree for the vertices.

We can now follow the AFS theory as described in [11], involving e.g. the choice of τ, and ρ_m almost literally. The data are first transformed through the function ρ_{m_i}.

ρ_{m_i} is obtained in the following way:

Occurrence low, ρ_{m_1}

$\rho_{m_1}(e_j) = h_1 - u_j^1$, $h_1 = \max\{u_1^1, u_2^1, \ldots, u_8^1\}$.

Occurrence medium, ρ_{m_2}

$\rho_{m_2}(e_j) = h_2 - |a - u_j^1|$, h_2=max $\{|a - u_j^1| \,|\, j=1, 2, \ldots, 8\}$, a=$\dfrac{u_1^1 + u_2^1 + \cdots + u_8^1}{8}$;

Occurrence high, ρ_{m_3}

$\rho_{m_3}(e_j) = u_j^1 - h_3$, h_3=min$\{u_1^1, u_2^1, \ldots, u_8^1\}$;

Acceptance $\rho_{m_7}(e_j) = 1$, when e_j is accepted; $\rho_{m_8}(e_j) = 0$, when e_j is not accepted.

Similarly, we can obtain ρ_{m_4} ρ_{m_5} ρ_{m_6}. In general, for each $m \in M$, ρ_m is a function derived from the original data or some training examples. $\rho_m(x) \geq \rho_m(y)$ $(x, y \in X)$ if and only if the degree of x belonging to attribute m is greater than or equal to that of y.

Here, some things deserve to be mentioned. The first one is that the *impurity measure* we used in this case is

$$I^N = -\sum_{v^c \in D_c} \frac{P_{v^c}^N}{p^N} \log_2 \frac{P_{v^c}^N}{p^N},$$

where $p^N = \sum_{v^c \in D_c} P_{v^c}^N$; $P_{v^c}^N = \sum_{j=1}^{|E|} \mu_{\beta^N \wedge v^c}(e_j)$;

D_c is the set of fuzzy terms for the decision variable, such as acceptance m_7 and non-acceptance m_8;

β^N is a fuzzy set in the *EI* algebra EM, and $\mu_{\beta^N}(e_j)$ is the membership degree of training sample e_j in node N of the decision tree, such as e.g. $\beta^N = \{m_2, m_8\}$;

E is the set of training examples, in this case consisting of 8 vertices and 14 edges.

The second thing is that the method we used to select attributes of child nodes of the decision tree is to maximize the information gain $G_{V_i}^N$, based on the impurity measure

$$G_{V_i}^N = I^N - I^{S_{V_i}^N}.$$

Here $S_{V_i}^N = \{(N|v_p^i) | v_p^i \in D_i^N\}$; $D_i^N = \{v_p^i \in D_i^N \,|\, \exists\, e_j \in E,\ \mu_{\beta^N \wedge v_p^i}(e_j) > \delta\}$;

$$I^{S_{V_i}^N} = \frac{1}{\sum_{v_p^i \in D_i^N} P^{N|v_p^i}} \sum_{v_p^i \in D_i^N} (P^{N|v_p^i} I^{N|v_p^i}),$$

$N|v_p^i$ denotes the particular child of node N created by the use of the fuzzy attribute V_i (V_i = Occurrence or Degree) to split N and following the edge $v_p^i \in D_i$;

D_i denotes the fuzzy term for the variable V_i, e.g. D_i ={Occurrence low}.

The third thing is the method of describing acceptance and non-acceptance with the nodes. We select the v^c with maximum value of $P^N_{v^c}$.

For both of the tables above, according to the value of G^{Occ} and G^{Deg}, first the parameter Degree was used to split, because G^{Deg} is larger than G^{Occ} for the parameter Occurrence.

By computation, we obtain the decision trees of the combined definition graph of Figure 4. Deg stands for “Degree” and Occ for “Occurrence”, L, M and H for “Low”, “Medium” and “High” respectively. χ_i denotes the membership degree of *i*.

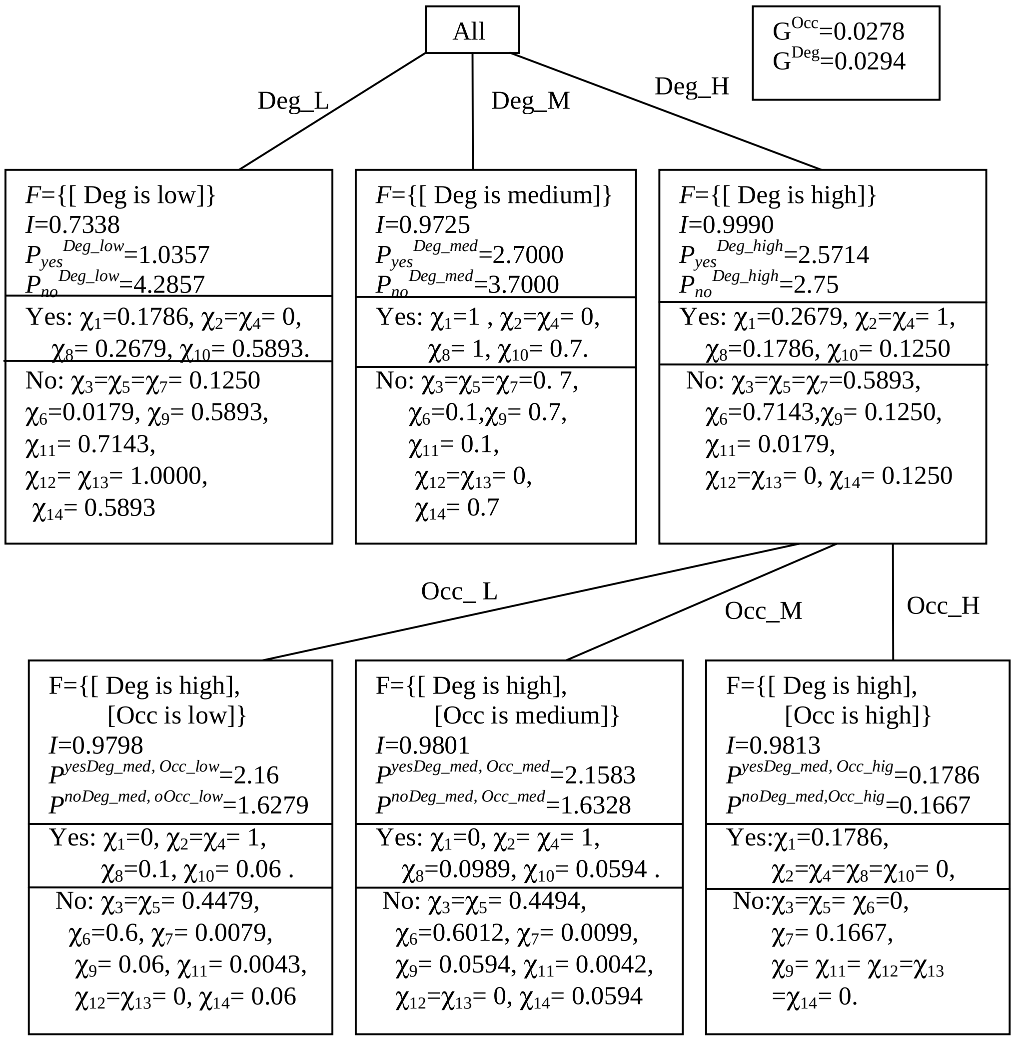


Figure 4(a): Decision tree for edges(δ=0)

The fuzzy rules derived from the decision tree for edges (δ=0) are:

Fuzzy rule 1: if e is in $\{m_4\}+\{m_5\}$, then e is No.(i.e. if an edge has "Degree is low" or "Degree is medium" then the decision on "acceptance" is No).

Fuzzy rule 2: if e is in $\{m_6, m_1\}+\{m_6, m_2\}+\{m_6, m_3\}$, then e is Yes. (i. e. if an edge has the properties "Degree is high and Occurrence is low" or "Degree is high and Occurrence is medium" or "Degree is high and Occurrence is high", then the decision on "acceptance" is Yes).

The fuzzy rules derived from the decision tree for vertices (δ=0) are:

Fuzzy rule 3: if v is in$\{m_5\} + \{m_6\}$, then v is Yes. (i. e. if a vertex has the properties "Degree is medium" or "Degree is high", then the decision on "acceptance" is Yes).

Fuzzy rule 4: if v is in$\{m_4, m_1\} + \{m_4, m_2\}+ \{m_4, m_3\}$then v is No. (i. e. if a vertex has "Degree is low and Occurrence is low" or "Degree is low and Occurrence is medium" or "Degree is low and Occurrence is high", then the decision on "acceptance" is No).

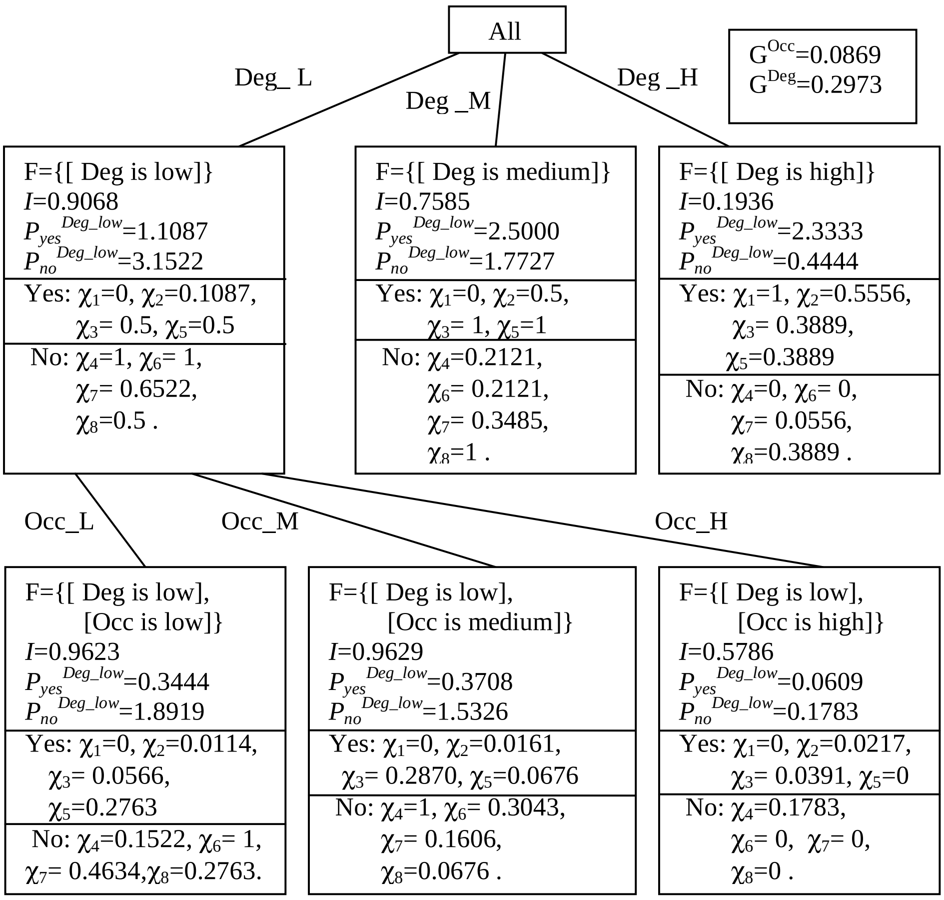


Figure 4(b): Decision tree for vertices (δ=0)

We should pay attention to a fact that we can use here: when attribute V_i is used to split a node N, some fuzzy terms $v^i \in D_i$, which satisfy $\mu_{\beta^N \wedge v_i}(e_j) \le \delta$ for any $e \in E$, are not used to create sub-trees. If we set δ=0.4, then we get the decision trees in Figure 5(a) and 5(b):

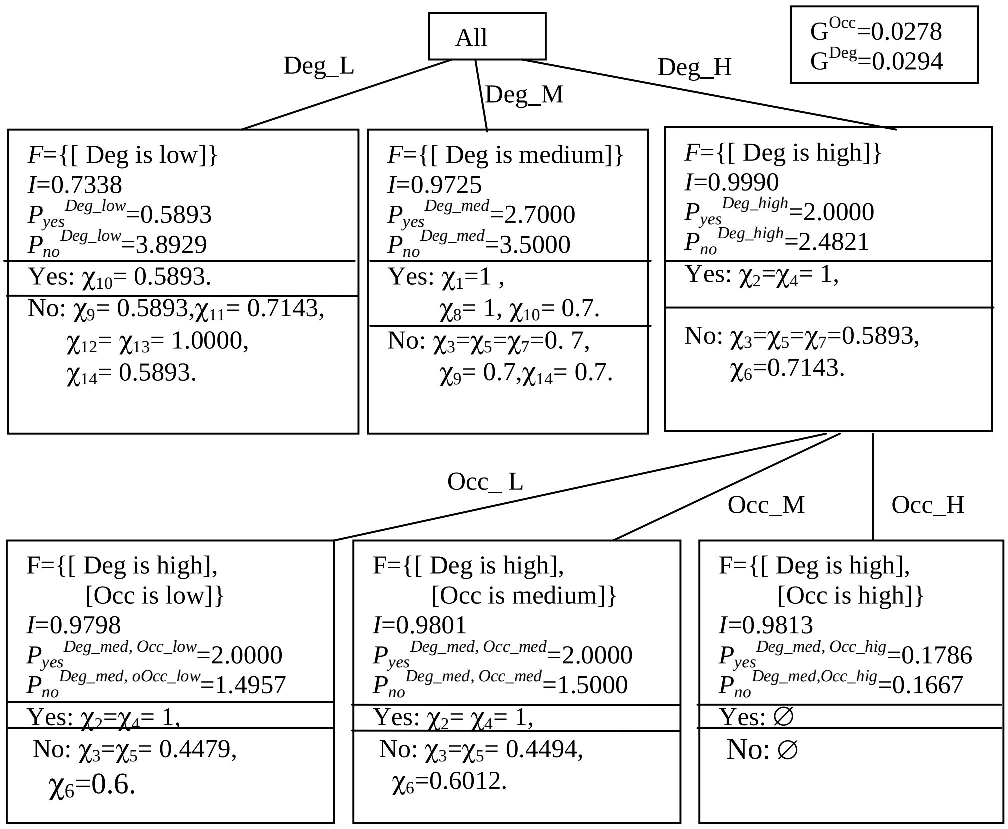


Figure 5(a): Decision tree for edges (δ=0.4)

We deleted the samples whose membership is less than δ . Also when we compute $P_{v^c}^N = \sum_{j=1}^{|E|} \mu_{\beta^N \wedge v^c}(e_j)$, the E changed into the set of samples in the node. We can see that some trivial detailed information occurring in Figure 4(a) is filtered out and we get a clearer tree. We can also let the value of δ rise in order to get a more satisfactory tree (this means that either it can have higher accuracy for the test examples or more meaningful rules).

Most nodes become empty now, the classification becomes clearer. So now we can rewrite Rule 2 and Rule 4 as:

Fuzzy rule 5: If e is in $\{m_6, m_1\} + \{m_6, m_2\}$, then e is Yes. (i. e. if an edge has the properties “Degree is high and Occurrence is low” or “Degree is high and Occurrence is medium”, then the decision on “acceptance” is Yes).

Fuzzy rule 6: If v is in $\{m_4, m_1\} + \{m_4, m_2\}$, then v is No. (i. e. if a vertex has the properties “Degree is low and Occurrence is low” or “Degree is low and Occurrence is medium”, then the decision on “acceptance” is No).

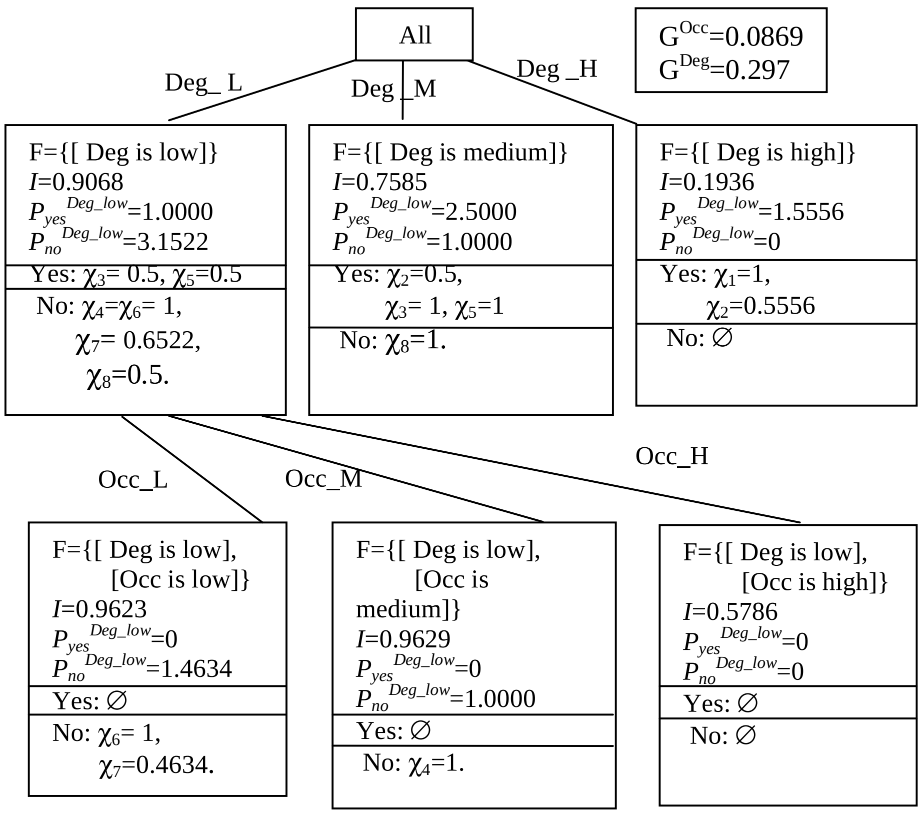


Figure 5(b): Decision tree for vertices (δ=0.4)

The decision tree for the edges of Figure 5(a), with δ=0.4, shows the following:

(1) Edge e_{10} has the same scores, 1 and 6, as edge e_9, that we gave acceptance 0, so e_{10} is taken as *noise*. In the case of credit we would speak of an inconsistency. So we can give the label of node {[Deg is low]} as that of class “NO”.

(2) The node{[Deg is high], [Occ is high]} has no examples, that are in agreement with the original data. Almost no training example has significant values for this fuzzy set.

(3) The fuzzy set of elements with medium edge-degree could not be separated, for e_1 and e_8 are in the class “Yes”. But, in fact, through observing the original data, we find a contradiction for e_1 and e_8, also they are a bit special, so we can also take them as noise.

But no matter whether they are noise or not, it did not effect our classification, because we take v^c with maximum value of $P_{v^c}^N$ as the label of the node. Similarly, node {[Deg is high], [Occ is low]} and {[Deg is high], [Occ is medium]} are also not separated, we can take e_3, e_5, and e_6 as noise. Here we also notice that the examples e_2 and e_4 are identical, and also e_3 and e_5. e_6 is quite similar to e_3 and e_5.

For Figure 5(b), the outcome is remarkable.

(1) The only noise is v_8 in the fuzzy set {[Deg is medium]}. Again there is the situation that v_5 has scores 2 and 4 and is acceptable, whereas v_8 has the same scores but was held not to be acceptable. So we can take the label of node {[Deg is medium]} as that of class "Yes".

(2) The node {[Deg is low], [Occ is high]} has no examples, in agreement with the original data. Almost no training example has significant values for this fuzzy set.

5. APPLYING KNOWLEDGE GRAPH THEORY

We recapitulate our ideas about fuzzy concepts by stating the following definition:

Definition 5: Given a set of jurors a concept is *fuzzy* if the word describing the concept is not interpreted by all jurors in the same way.

*Remark 1***:** As in knowledge graph theory a concept is a word graph, the interpretations of the word are graphs, these graphs may be combined into one definition graph. The elements of this definition graph are a set of vertices, the union of the sets of vertices of the graphs representing the interpretations of the jurors, and a set of links, likewise the union of link sets.

Remark 2: The combined definition graph may be generated by all jurors. Only in that case the concept is not fuzzy. However, suppose that all definitions of democracy would yield the induced subgraph of Figure 2 on the vertex set {v_1, v_2, v_3, v_5}. Then the concept would not be fuzzy according to our definition, but the elements occurring may be fuzzy. Jurors may, for example, agree on "people" and "decisions", but for "institutions" different interpretations might be given and "power" is a notoriously fuzzy concept. Also links may have different meaning for jurors.

As an introduction to the use of membership functions in our alternative approach let us consider the set of values {2, 3, 5, 8}. We want to consider a value to be "low", "medium" or "high". As we tried to make clear in Section 2 this depends on the jurors, however, it also depends on the scale. When we consider integers from -50 to +50 we are inclined to call all four "medium". If, however, the scale is from 0 to 100 all four seem "low". The presence of such a scale may not be given. In that case we only have the jurors. But these too may not be present.

We might act as jurors ourselves, e.g. say 2 and 3 are low values, 5 is a medium value and 8 is a high value. But we want a more objective procedure. One such procedure may be the following. We consider artificial jurors, that have to choose two boundaries. Let these boundaries be chosen between the values 2 up to 8, so from six possible values, say 2.5, 3.5, 4.5, 5.5, 6.5, 7.5. An artificial juror now chooses two different boundary values from these 6 values. This means that there can be $\binom{6}{2}$=15 different artificial jurors. Unlike in our discussion of “small”, “normal” and “tall” in Section 2, where we assumed human jurors, giving boundary values around 150 cm and 180 cm, we have no information on the artificial jurors, unless we give that ourselves by making assumptions.

The assumption we make here is that all artificial jurors are equally likely. The 15 possible combinations of two boundary values are given in Figure 3 by vertical lines. This procedure is similar to procedures in statistical mechanics where, due to incomplete information of the atoms, say of a gas, sums over all possible states are considered.

2 | 3| 5 8	2 3| |5 8	2 3 |5 | 8
2 | 3 |5 8	2 3| 5| 8	2 3 |5 |8
2 | 3 5| 8	2 3| 5 | 8	2 3 5| | 8
2 | 3 5 | 8	2 3| 5 |8	2 3 5| |8
2 | 3 5 |8	2 3 |5| 8	2 3 5 | |8

Figure 3: 15 artificial jurors

We can now count the numbers of times that a value is considered to be “low”, “medium” or “high”.

2: 15 times “low”,
3: 10 times “low”, 5 times “medium”,
5: 3 times “low”, 9 times “medium”, 3 times “high”,
8: 15 times “low”.

It is now easy to define a membership function for the four values with respect to the three “fuzzy sets” (we will use this terminology too, having given our interpretation earlier). We simply consider the quotients of the counted numbers and the number 15 of artificial jurors. For the fuzzy set of “low” values, the membership function μ_{low} is:

$\mu_{low}(2)=15/15=1$, $\mu_{low}(3)=10/15=2/3$, $\mu_{low}(5)=3/15=1/5$, $\mu_{low}(8)=0/15=0$.

Analogously we find:

$\mu_{med}(2)=0$, $\mu_{med}(3)=5/15=1/3$, $\mu_{med}(5)=9/15=3/5$, $\mu_{med}(8)=0/15=0$.

$\mu_{hig}(2)=0$, $\mu_{hig}(3)=0$, $\mu_{hig}(5)=3/15=1/5$, $\mu_{hig}(8)=15/15=1$.

The most interesting value is 5 as this value has membership values that are not zero for all three fuzzy sets:

$\mu_{\text{low}}(5)=3/15=1/5$, $\mu_{\text{med}}(5)=9/15=3/5$, $\mu_{\text{hig}}(5)=3/15=1/5$.

Note that here the sum of membership degrees is always 1, due to our choice of producing them. This need not always be the case.

Let us now consider the alternative procedure to the one described in Section 4. The method of using artificial jurors immediately yields membership values for the three attributes of one parameter for each element, these values sum up to 1 and allow a probabilistic interpretation. This is not the case in the first procedure. For two parameters we then have, for our example, the problem of determining membership values for all fuzzy sets. If $\mu_{\text{low}}(\rho_1, e)$, $\mu_{\text{med}}(\rho_1, e)$, $\mu_{\text{hig}}(\rho_1, e)$ are these values, for element e and parameter ρ_1, we have:

$\mu_{\text{low}}(\rho_1, e)+ \mu_{\text{med}}(\rho_1, e)+ \mu_{\text{hig}}(\rho_1, e)=1.$

Likewise we have with respect to the second parameter ρ_2

$\mu_{\text{low}}(\rho_2, e)+ \mu_{\text{med}}(\rho_2, e)+ \mu_{\text{hig}}(\rho_2, e)=1.$

The rather natural procedure to determine the membership values for the fuzzy sets is to multiply the two left hand sides and consider the 9 products of the μ's. So, e.g., $\mu_{\text{low}}(\rho_1, e)\cdot \mu_{\text{hig}}(\rho_2, e)$ gives the membership value for the element with respect to the fuzzy set of low occurrence and high degree. These 9 values sum up to 1 and allow a probabilistic interpretation.

From here to the decision tree we now proceed as follows. Each element *E* will have a product that is highest (in the unlikely case that there are two or more equal values we can just choose one or carry on for each choice). This "defuzzifies" the 9 fuzzy sets as now each element belongs to precisely one set. We only face the problem of choosing the order in which $\rho_1$and ρ_2 are used for constructing the decision tree.

As each element is in precisely one of the nine sets, one might think that the order in which the parameters are used to obtain the full decision tree is irrelevant. However, the goal is to find rules to separate the elements with acceptance 1 from elements with acceptance 0. Suppose one parameter separates into three sets and within each set only elements with the same acceptance occur. Then this parameter alone would be sufficient for our purpose. The other parameter might determine three sets in which acceptance and non-acceptance elements are mixed. We then prefer to use the first parameter first in the construction.

In order to grasp this distinction mathematically we introduce the notion of "noise". Let elements be partitioned into c sets and let them have label 1or 0: let $n_i(1)$ and $n_i(0)$ denote the number of elements in set i with label 1 and label 0, respectively.

Definition 6: The *noise in set i* is

$$n_i(1) * n_i(0).$$

Definition 7: The *noise of a partition* is

$$\sum_{i=1}^{c} n_i(1) * n_i(0).$$

If one of $n_i(1)$ and $n_i(0)$ is zero then the noise is zero and there is perfect separation in set *i*. If the noise of a partition is zero the separation of elements with label 1 or label 0 is perfect.

Note that we consider "noise" to be a number here. However, the word can also be used to distinguish between elements in a class. Let {1, 1, 0, 0, 0} be a class with two elements with label 1 and three with label 0. Then the noise in this class is 2∗3=6. But one may also say that the elements with label 1 form the noise in a class with primarily elements with label 0.

We consider noises for crisp sets. However the notion of noise is easily extended to fuzzy sets. If in our small example, the membership values of the two elements with label 1 are $\mu(e_1)$ and $\mu(e_2)$ and of the three elements with label 0 are $\mu(e_3)$, $\mu(e_4)$ and $\mu(e_5)$ then we could define the noise as

$$[\mu(e_1) + \mu(e_2)] *[\mu(e_3) + \mu(e_4) + \mu(e_5)].$$

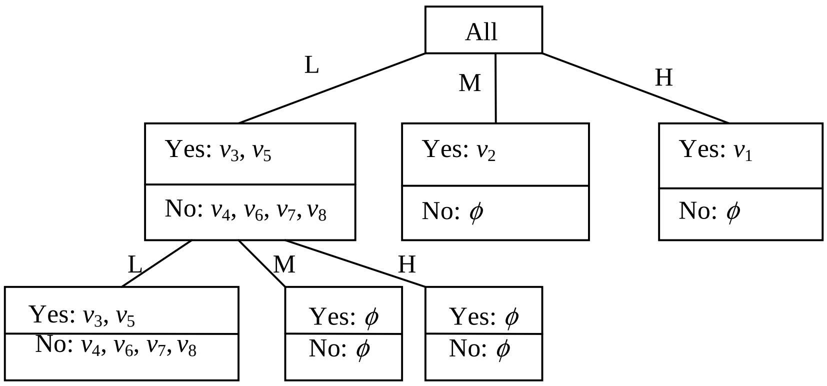
For crisp sets all μ are 1 and we regain the definition given.

So, we partition the elements in two ways. Once we determine the partition according to the attributes of ρ_1 and a second time according to the attributes of ρ_2. Then we first partition according to the parameter that gives lowest noise. In the exceptional case that both give the same noise we choose arbitrarily.

We shortly give the intermediate results of this simple procedure for our democracy example. For the vertices we find:

$v_1 \in$ (H, H), $v_2 \in$ (M, M), v_3, v_4, v_5, v_6, v_7, $v_8 \in$ (L, L)

where, e. g., (M, M) denotes the set with medium occurrence and medium degree. Both parameters give the same noise. So the choice between ρ_1 and ρ_2 is irrelevant. We find

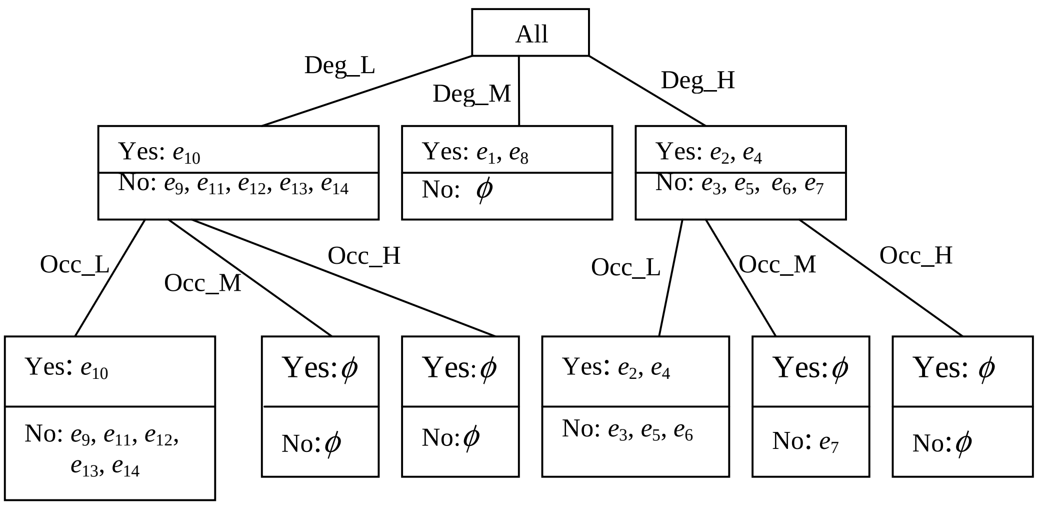


for the decision tree,{v_1, v_2} as accepted set of vertices and {v_3, v_5} as noise. Note that the splitting on the second parameter does not give any further information.

For the edges we find $e_1 \in$ (H, M), e_2 ,e_3, e_4, e_5, $e_6 \in$ (L, H), $e_7 \in$ (M, H), $e_8 \in$ (L, M), e_9 ,e_{10}, e_{11}, e_{12}, e_{13}, $e_{14} \in$ (L, L).

Now the choice between ρ_1 and ρ_2 does give a difference. First the parameter edge-degree is used because it gives the lowest noise.

We obtain



for the decision tree, {e_1, e_8} as accepted set of edges and {e_2, e_4}as noise, not separable from the other elements by splitting according to the attributes of ρ_1, occurrence.

We find as rule for acceptance of edges rule A, and as rule for acceptance of vertices rule B.

Rule A: If “edge-degree of *e* is medium”, then edge *e* is accepted.

Rule B: If “vertex-degree of *v* is medium or high”, then vertex *v* is accepted.

Due to the occurrence of noise only node (M, H) contains, only one, non-accepted element.

We have discussed just one concept: “democracy”, and found a rule to determine the important elements of that concept. The usefulness of these rules will have to be tested on other examples.

6. APPLICATION TO THE CONCEPT “FUZZINESS”

As a summary of this paper we shortly repeat the followed alternative procedure for the concept “fuzziness”. That concept has led to much discussion and therefore is a fuzzy concept. The definitions used are given in Appendix B, together with their definition graphs, data survey and our, subjective, acceptance. Also the procedure of growing graphs according to decreasing degrees is illustrated.

The example of fuzziness can be considered in three ways.

First we can simply apply the rules A and B found from the “democracy” example, to the combined definition graph, for which the crisp sets containing the elements are

$v_1 \in$ (H, H), v_2, $v_6 \in$ (M, H), v_3, v_4, v_5, $v_7 \in$ (L, L)
e_1, $e_2 \in$ (H, M), $e_3 \in$ (M, H), e_4, $e_5 \in$ (M, L), , $e_6 \in$ (L, M), e_7, e_8, e_9, $e_{10} \in$ (L, L).

Rules A gives acceptance for $\{e_1, e_2, e_6\}$ and hence for $\{v_1, v_2, v_4, v_6\}$.

Rules B gives acceptance for $\{v_1, v_2, v_6\}$.

Secondly, we can use the example itself for deriving a decision rule, first determining our subjective acceptance:
$\{v_1, v_2, v_3, v_4, v_6\}$ and $\{e_1, e_2, e_3, e_5, e_6, e_8, e_{10}\}$ were considered essential elements of the definition.

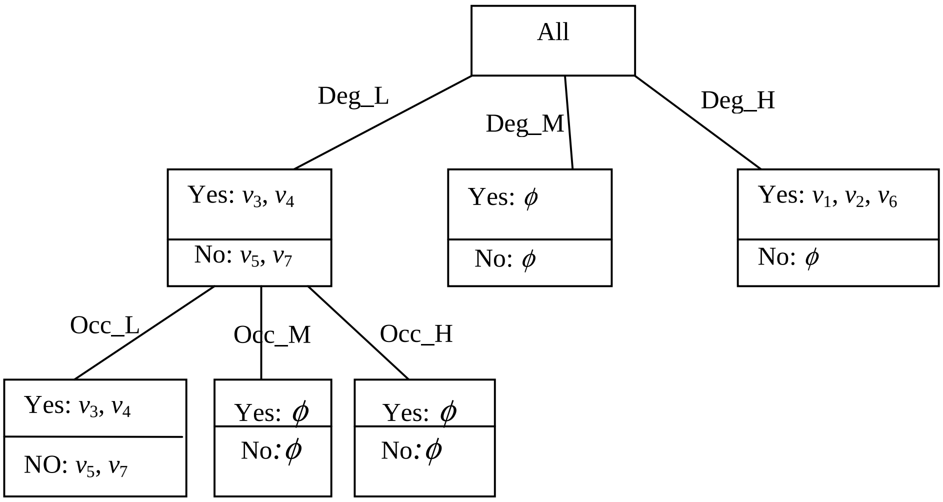
After calculation of the membership values we find for occurrence:

$\{v_3, v_4, v_5, v_7\} \subset$ L and $\{e_6, e_7, e_8, e_9, e_{10}\} \subset$ L
$\{v_2, v_6\} \subset$ M $\{e_3, e_4, e_5\} \subset$ M
$\{v_1\} \subset$ H, $\{e_1, e_2\} \subset$ H

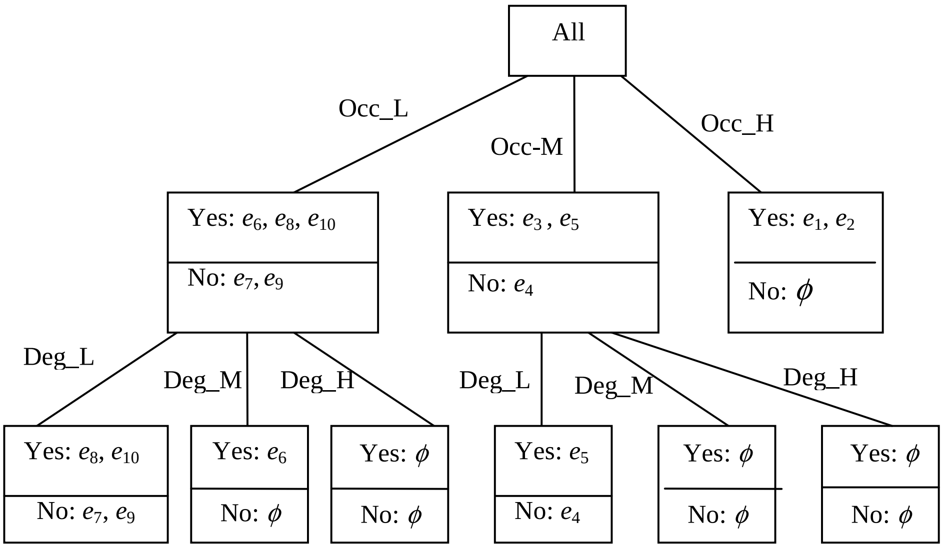
For degree we find:

$\{v_3, v_4, v_5, v_7\} \subset$ L and $\{e_4, e_5, e_7, e_8, e_9, e_{10}\} \subset$ L,
$\phi \subset$ M $\{e_1, e_2, e_6\} \subset$ M
$\{v_1, v_2, v_6\} \subset$ H $\{e_3\} \subset$ H

Calculation of the noise makes us decide for occurrence as first parameter for the construction of the decision tree for edges and the choice between both paramters is irrelevant for vertices. Pruning the tree if no further separation is possible, this gives:



for the vertices and,



for the edges.

The decision rules, read off from these trees, would be:

Rule A*: if "edge –degree of *e* is high or medium'' accept the edge *e*.

Rule B*: if "degree of *v* is high", accept the vertex *v*.

As third consideration we remark that the concepts involved in the graph are in excellent agreement with the view on fuzziness developed in this paper.

Two more important results are the following. First, the use of artificial jurors to obtain membership values and, second, the alternative method to get at a decision tree.

There are some problems we did not go into. Transforming a definition into a graph is called *structural parsing*, see Zhang[14] and Zhang and Hoede[15], where a much more precise treatment of the types of edges or arcs is discussed. After the accepted elements have been determined the graph with these elements has to be "brought under words", which is called *uttering*, see Zhang[14].

Finally, we would like to have a measure of fuzziness of a concept. Consider the quotient of occurrence of E and number of definitions. If all definitions contain E, then this quotient is 1. We then say that E is *crisp* with respect to the concept.

Definition 6: The *fuzziness* of an element E with respect to a concept is

$$F(E) = 1 - (\text{occurrence}(E) / \text{number of definitions})$$

If there are n definitions and E does not occur $F(E) = 1$. For occurring elements the highest fuzziness is $1 - 1/n$.

Definition 7: The *fuzziness* of a concept C is

$$F(C) = \frac{1}{|C|}\sum_{E \in C} F(E)$$, where C is the set of elements, occurring in the concept.

If all definitions are the same each element has fuzziness zero and C has fuzziness zero. If all elements occurring occur only once, each element has fuzziness $1\text{-}1/n$ and so has C, the highest value possible, approaching 1 with increasing number of definitions. The measure given is appropriately normed:

$$0 \leq F(C) < 1 .$$

7. DISCUSSION

Summarizing the decision rules obtained we have for acceptance of edges and vertices, with respect to the parameter combinations (Occ, Deg):

Fuzzy rule 5: $e \in (L, H)$ or $e \in (M, H)$

Rules A: $e \in (L, M)$ or $e \in (M, M)$ or $e \in (H, M)$

Rule A*: $e \in$ (L, M) or $e \in$ (M, M) or $e \in$ (H, M) or $e \in$ (L, H) or $e \in$ (M, H) or $e \in$ (H, H),

respectively:

Fuzzy rule3: $v \in$ (L, M) or $v \in$ (M, M) or $v \in$ (H, M) or $v \in$ (L, H) or $v \in$ (M, H) or $v \in$ (H, H)

Rule B: $v \in$ (L, M) or $v \in$ (M, M) or $v \in$ (H, M) or $v \in$ (L, H) or $v \in$ (M, H) or $v \in$ (H, H)

Rule B*: $v \in$ (L, H) or $v \in$ (M, H) or $v \in$ (H, H).

The rules found for determining whether an element *E*, an edge *e* or a vertex *v*, should be accepted in the definition of a concept, are not very spectacular. Accept *E* when it has medium or high scores on occurrence and degree, which is something rather obvious. However, it is somewhat surprising that in particular the medium edge degrees determine edges, that are incident with vertices that one would like to accept.

If we choose a restrictive rule for acceptance, say edge-degree should be medium and vertex-degree should be high, we only find $\{e_1, e_8\}$ and $\{v_1\}$ for the “democracy” example, so the graph induced by $\{v_1, v_2, v_3\}$ in Figure 2. We then do not find v_5= power, that we, subjectively, would accept. For the “fuzziness” example, we find $\{e_1, e_2, e_6\}$ and $\{v_1, v_2, v_6\}$, so the graph induced by the vertex set $\{v_1, v_2, v_4, v_6\}$, missing out v_3= variation, that we, subjectively, would accept as important aspect of fuzziness.

An important question is how these rules should be used. A bank considering a credit for a single customer, should have a way to determine the attributes of the parameters “income” and “employment”. In case a group of customers is considered, the data of the members of this group could be used to determine the attributes, for example by artificial jurors.

In the case of the analysis of some concept, the vertices and edges of the combined definition graph form the elements on which such an analysis can be carried out.

In a future investigation we want to make an extensive comparison and generalization of the two ways to derive decision rules as they were presented here. Also the way of applying the rules to new records should be discussed in detail.

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APPENDIX A

Definitions of "democracy"

1. set of methods to coordinate the decisions of persons and institutions in hierarchically equal positions, not influenced by the market.

2. production of all that is wanted: personal rights, human welfare, collective preference.

3. protection of minority against majority and of majority against minority, by horizontal spreading of political power over more institutions.

4. the people being a majority and law being the decision of the people.

5. institutional arrangement to obtain political decision, in which individuals get the power to decide by means of a competition for the votes of citizens.

6. negotiations between institutions and persons, to obtain a decision acceptable to as many as possible.

7. results of negotiations being justified by voting.

8. government in which the supreme power is vested in the people and exercised by them directly or indirectly through representation.

9. voting process guaranteeing to all citizens an a priori equal representation.

10. political system in which the people, not monarchs or aristocracies, rule.

The words occurring these in 10 definitions with frequency greater than 1:

v_1: people: 8
v_2: decision: 4
v_3: institutions: 3
v_4: voting: 3
v_5: power: 2
v_6: majority: 2
v_7: negotiations: 2
v_8: representation: 2

Links occurring between these vertices in these 10 definitions:

e_1:$v_1 - v_2$: 3	e_2:$v_2 - v_3$: 1
e_3:$v_1 - v_3$: 1	e_4:$v_2 - v_7$: 1
e_5:$v_1 - v_4$: 1	e_6:$v_3 - v_5$: 1
e_7:$v_1 - v_5$: 1	e_8:$v_3 - v_7$: 1
e_9:$v_1 - v_6$: 1	e_{10}:$v_4 - v_8$: 1
e_{11}:$v_1 - v_7$: 1	e_{12}:$v_5 - v_6$: 1
e_{13}:$v_1 - v_8$: 2	e_{14}:$v_5 - v_8$: 1.

APPENDIX B

Definitions of "fuzziness" or "fuzzy"

1. not clear: indistinctness, the quality of being indistinct and without sharp outlines.

2. ill-definedness, ill definition.

3. fuzziness is about a model of human estimation of real objects.

4. fuzziness describes the situation where the reference of an expression is not unambiguously determined, even when the complete context is given.

5. complete fuzziness merely signifies that any interpretation is as likely as any other one.

6. fuzziness results from lack of information about the thing being described.

7. a fuzzy concept is a concept if which the content or boundaries of application vary according to content or conditions. It does have a meaning, or multiple meanings, which however can become clearer only through further elaboration and specification.

8. fuzziness, though it applies primarily to what is cognitive, is a conception applicable to every kind of representation. A representation is vague when the reaction of the representing system to the represented system is not one-one, but one-many.

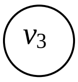
9. All definitions have a degree of fuzziness that requires intelligent application: what does "planet" really mean?

10. a proposition is fuzzy when there are possible states of things concerning which is uncertain whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition.

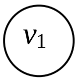
We distinguish, after identification,
v_1 = interpretation, v_2 = concept, v_3 = variation,v_4 = juror, v_5 = meaning, v_6 = content, v_7 = context.

The 10 definition graphs then are:

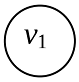
G_1 :



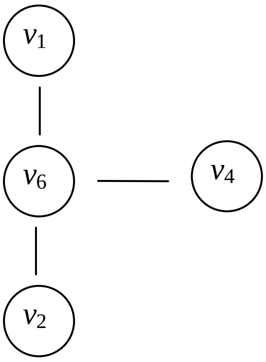
G_2 :



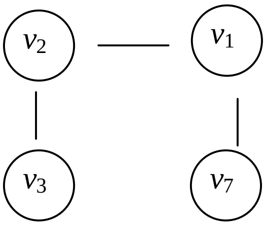
G_5 :



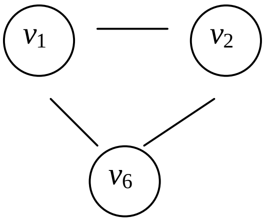
G_3 :



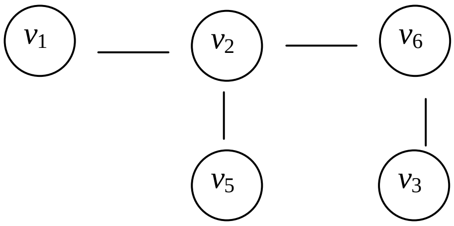
G_4 :



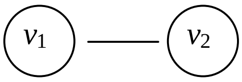
G_6 :



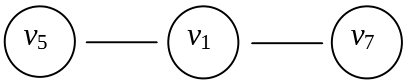
G_7 :



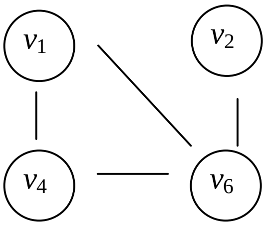
G_8 :



G_9 :



G_{10} :



The combined definition graph is

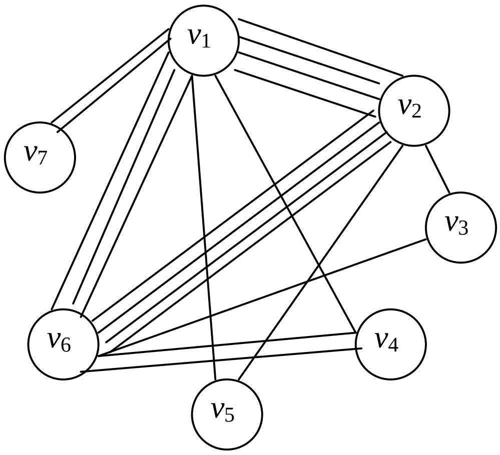


Table A

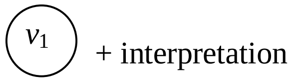
	Occurrence	Degree	Acceptance
v_1	9	11	1
v_2	6	10	1
v_3	2	2	1
v_4	2	3	1
v_5	2	2	0
v_6	4	10	1
v_7	2	2	0

Table B

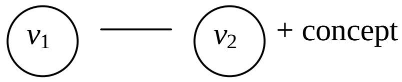
	Occurrence	Degree	Acceptance
$e_1 = v_1 - v_2$	4	13	1
$e_2 = v_2 - v_6$	4	11	1
$e_3 = v_1 - v_6$	3	15	1
$e_4 = v_1 - v_7$	2	9	0
$e_5 = v_4 - v_6$	2	9	1
$e_6 = v_1 - v_4$	1	12	1
$e_7 = v_1 - v_5$	1	10	0
$e_8 = v_2 - v_3$	1	10	1
$e_9 = v_2 - v_5$	1	10	0
$e_{10} = v_3 - v_6$	1	10	1

Using degrees to determine basic, important, central elements, we consecutively find the graphs

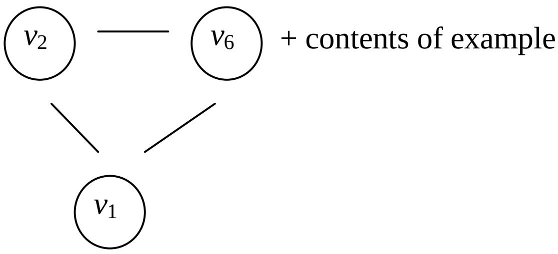
G_1:



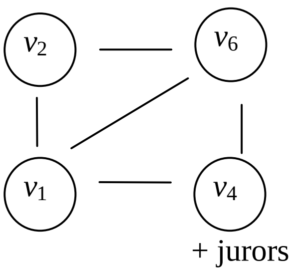
G_2:



G_3:



G_4:



G_5: