Department of Applied Mathematics
Faculty of EEMCS

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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Two axiomatizations of the kernel of TU games: bilateral and converse reduced game properties

T.S.H. Driessen and C.-C.Hu[1]

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[1]Center for General Education, Southern Taiwan University of Technology, No. 1, Nantai St., Yung-Kang City, Tainan, Taiwan 710, ROC

Two axiomatizations of the kernel of TU games: bilateral and converse reduced game properties

Theo Driessen* Cheng-Cheng Hu†

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Abstract

We provide two axiomatic characterizations of the kernel of TU games by means of both bilateral consistency and converse consistency with respect to two types of 2-person reduced games. According to the first type, the worth of any single player in the 2-person reduced game is derived from the difference of player's positive (instead of maximum) surpluses. According to the second type, the worth of any single player in the 2-person reduced game either coincides with the 2-person max reduced game or refers to the constrained equal loss rule applied to an appropriate 2-person bankruptcy problem, the claims of which are given by the player's positive surpluses.

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1 Introduction

The kernel is one of the most important solution concepts of transferable utility(TU) games proposed by Davis and Maschler (1965). It has many interesting properties that reflect in various ways the structure of the game. To understand a solution concept of cooperative game theory, besides to observe the properties it preserves, one might focus on its definition and try to interprete it intuitively. Many attempts at providing an interpretation to the definition of the kernel

*Theo Driessen, Faculty Electrical Engineering, Mathematics and Computer Science, Department of Applied Mathematics, University of Twente, P.O. Box 217, NL–7500 AE Enschede, The Netherlands. E-mail: t.s.h.driessen@math.utwente.nl

†Cheng-Cheng Hu, Center for General Education, Southern Taiwan University of Technology, No. 1, Nantai St, Yung-Kang City, 71005, Tainan County, Taiwan. E-mail: cchu@mail.stut.edu.tw. This author is in charge of correspondence and gratefully acknowledges funding from the National Science Council (NSC) of Taiwan (project number 95-2911-I-218-001-2) and hospitality of the Department of Applied Mathematics of the University of Twente at Enschede, The Netherlands, during his research stay from January 18, till February 1, 2006, in the framework of bilateral scientific cooperation between NSC and NWO (The Netherlands Organization for Scientific Research).

seems to rely on "interpersonal comparison of utility" which is still an obscure notion for us.

The axiomatic approach is an acceptable way to justify a solution concept of TU games. Several famous solution concepts of TU games are characterized by convincing axioms, for instance, the core by Peleg (1986), the Shapley value by Shapley (1953), Hart and Mas-Colell (1989), and the prenucleolus by Sobolev (1975), etc. The prekernel is the auxiliary solution concept of the kernel and is characterized by Peleg (1986). Nevertheless, one can not use the same axioms characterizing the prekernel to justify the kernel.

In the paper, we construct two different bilateral reduced games. Two axiomatic characterizations are provided which are related to these two bilateral reduced games. These two collections of axioms are the same except the bilateral reduced games employed are different. In particular, The main axioms of these two characterizations are bilateral consistency and converse consistency. These two axioms play the same roles as max-consistency and converse max-consistency in Peleg's characterization of the prekernel.

Section 2 introduces definitions and conventions. In section 3, we construct the bilateral min reduced game and provide an axiomatic characterization of the kernel. In section 4, we use the constrained equal loss bankruptcy rule to create the bilateral CEL reduced game and provide another axiomatic characterization of the kernel.

2 Definitions and conventions

Let $\mathbb{N}$ be the set of potential *players*. A *coalition* is an nonempty finite subset of $\mathbb{N}$ and $\mathcal{N}$ denotes the class of all coalitions of $\mathbb{N}$.

A *transferable utility game* (*TU* game) with the player set $N \in \mathcal{N}$ is a characteristic function v that assigns to each $T \subseteq N$ a real number and $v(\emptyset)$ is assumed to be 0. The number $v(T)$ is called the *worth* of the coalition T. Let $\mathcal{G}$ represent the set of all TU games, that is, $\mathcal{G} = \cup_{N\in\mathcal{N}}\mathcal{G}^N$, where $\mathcal{G}^N$ denotes the class of TU games with the player set N. Let $|N|$ be the cardinality of N. Then $v \in \mathcal{G}^N$ is called a n-player TU game if $|N| = n$.

Let R be the set of real numbers and by R^N the set of all functions from N to R. We will think of members x of R^N as $|N|$-dimensional vectors whose coordinates are indexed by members of N; thus, when $i \in N$, we will write x_i for $x(i)$ and $x = (x_i)_{i\in N}$ is called a *payoff vector*.

Let N' be a subset of N. We write $x_{N'}$ for the restriction of x on $R^{N'}$, $x(N') = \sum_{i\in N'} x_i$, and $x(\emptyset)$ is defined to be 0. For convenience, we will express the notation $v(\{i,j,k\})$ to be $v(ijk)$ and write $N'\cup i$ instead of $N'\cup\{i\}$. Denote

$$b(N) = \{\{i,j\} : i,j \in N,\, i \neq j\}.$$

That is, $b(N)$ consists of all 2-player subsets of N.

Let $v \in \mathcal{G}^N$. We denote

$$X'(v) = \left\{x \in R^N : x(N) \leq v(N)\right\},$$

and

$$X(v) = \left\{x \in R^N : x(N) = v(N)\right\}.$$

Let $\mathcal{G}' \subseteq \mathcal{G}$. A solution σ on $\mathcal{G}'$ is a function which associates with each game $v \in \mathcal{G}'$ a subset $\sigma(v)$ in $X'(v)$.

σ is called *single-valued* $(SIVA)$ if $|\sigma(v)| = 1$ for every $v \in \mathcal{G}'$.

σ satisfies *Pareto optimality*(PO) if $\sigma(v) \subseteq X(v)$ for every $v \in \mathcal{G}'$.

σ satisfies *non-emptiness*(NE) if $\sigma(v) \neq \emptyset$ for every $v \in \mathcal{G}'$.

σ satisfies *individual rationality*(IR) if $x_i \geq v(i)$ for all $x \in \sigma(v)$ and for every $v \in \mathcal{G}'$.

We shall be primarily interested in the *kernel*. Let $v \in \mathcal{G}^N$. Given a payoff vector $x \in R^N$, x is called an *imputation* if

$$x \in X(v) \text{ and } x_i \geq v(i), \text{ for all } i \in N.$$

The set of all imputations is denoted to be $I(v)$. Let $\mathcal{G}_0$ be the collection of all TU games with nonempty imputation set.

Let $v \in \mathcal{G}^N$. Given a coalition $S \subseteq N$ and a payoff vector $x \in R^N$, the excess $e^v(S,x)$ with respect to v, S and x is defined to be

$$e^v(S,x) = v(S) - x(S),$$

For $i, j \in N$, $i \neq j$, we denote the set of coalitions containing i and not j by $\mathcal{F}_{ij}$. The *maximum surplus* of player i against player j at payoff vector x is given by

$$\mathcal{S}_{ij}(v,x) = \max_{S \in \mathcal{F}_{ij}} e^v(S,x).$$

An imputation x is said to be in the *kernel* of the game v, denoted to be $K(v)$, if for all $k, l \in N$ and $k \neq l$, either

$$\mathcal{S}_{kl}(v,x) = \mathcal{S}_{lk}(v,x), \tag{1}$$

or

$$\mathcal{S}_{kl}(v,x) > \mathcal{S}_{lk}(v,x) \text{ and } x_l = v(l).$$

It is known that the kernel satisfies non-emptiness on $\mathcal{G}_0$(see, e.g., Davis and Maschler (1965)).

As a new, but important tool we introduce the *positive surplus* of player i against player j at payoff vector x as follows:

$$\mathcal{S}^+_{ij}(v,x) = \mathcal{S}_{ij}(v,x) + x_i - v(i).$$

Note that

$$\mathcal{S}^+_{ij}(v,x) = \max_{Q \subseteq N \setminus \{i,j\}} [v(i \cup Q) - x(Q) - v(i)] \geq 0. \tag{2}$$

In words $\mathcal{S}^+_{ij}(v,x)$ is the largest net income that player i can achieve by cooperation with coalitions not containing player j, provided that players k different from i and j agree to receive their payoff x_k for their possible cooperative behavior.

Let $v \in \mathcal{G}_0^N$, $N' = \{i,j\} \in b(N)$ and $x \in R^N$. Define *the bilateral min reduced game* $mr_{N'}^x(v)$ of v with respect to $N' = \{i,j\}$ and x as follows:

$$mr_{N'}^x(v)(N') = v(N) - x(N \backslash N'),$$

$$mr_{N'}^x(v)(i) = v(i) + \min\left\{v(N) - x(N\backslash N') - v(i) - v(j), \max\left[0, \mathcal{S}_{ij}^+(v,x) - \mathcal{S}_{ji}^+(v,x)\right]\right\},$$

and

$$mr_{N'}^x(v)(j) = v(j) + \min\left\{v(N) - x(N\backslash N') - v(i) - v(j), \max\left[0, \mathcal{S}_{ji}^+(v,x) - \mathcal{S}_{ij}^+(v,x)\right]\right\}.$$

A solution concept σ on $\mathcal{G}_0$ has *bilateral* $min - consistency$ if the following condition is satisfied: If $v \in \mathcal{G}_0^N$, $N' \in b(N)$ and $x \in \sigma(v)$, then $mr_{N'}^x(v) \in \mathcal{G}_0^{N'}$ and

$$x_{N'} \in \sigma\left(mr_{N'}^x(v)\right). \tag{3}$$

A solution concept σ on $\mathcal{G}_0$ has *converse* $min - consistency$ if the following condition is satisfied: If $v \in \mathcal{G}_0^N$, $x \in I(v)$ and for every $N' \in b(N)$ $mr_{N'}^x(v) \in \mathcal{G}_0^{N'}$ and $x_{N'} \in \sigma\left(mr_{N'}^x(v)\right)$, then $x \in \sigma(v)$.

Let τ be a solution defined on 2-player games, that is, $\cup_{|N|=2}\mathcal{G}^N$. τ is called *the standard solution* if for any $v \in \mathcal{G}^{\{i,j\}}$,

$$\tau_k(v) = v(k) + \frac{1}{2}(v(ij) - v(i) - v(j)),$$

where $k = i, j$.

A solution σ on $\mathcal{G}'$ satisfies $standardness(ST)$ if $\sigma = \tau$ on $\cup_{|N|=2}\mathcal{G}'^N$.

3 The bilateral min reduced game and the kernel

Let $v \in \mathcal{G}_0^N$, $x \in I(v)$ and $N' = \{i,j\} \in b(N)$. Considering the bilateral min reduced game $mr_{N'}^x(v)$, the worth $mr_{N'}^x(v)(k)$, $k \in N'$ could be interpreted under the following two situations.

Situation 1: Generally speaking players in $N\backslash N'$ are supposed to be paid according to the imputation x, so each player h in $N\backslash N'$ receives at least the amount x_h. Therefore, the bargaining range of i and j is at most

$$x_i + x_j - v(i) - v(j).$$

Situation 2: Suppose that there is a public project proposed by the government. For each $S \subseteq N$, $v(S)$ can be viewed as the profit the coalition S can earn by completing this public project. By (2), $\mathcal{S}_{ij}^+(v,x)$(or $\mathcal{S}_{ji}^+(v,x)$) is the largest net income that player i(player j) can gain by cooperating with coalitions not containing player j(player i). Without loss of generality, suppose that $\mathcal{S}_{ij}^+(v,x) \geq \mathcal{S}_{ji}^+(v,x)$. We assume that players i and j compete over the public project. Only one of them can get the public project. Player i could discount the payment received from the government by the amount $\mathcal{S}_{ji}^+(v,x)$. Since the largest net income of player i exceeds that of player j, player j has no ability to stop the government to give the public project to player i. Hence, players i, j will obtain the net income $\mathcal{S}^{+}_{ij}(v,x) - \mathcal{S}^{+}_{ji}(v,x)$, 0 respectively.

The worth of 1-person coalition of the bilateral min reduced game is derived under these two situations. In the following, we will use the bilateral min reduced game to characterize the kernel. First, we introduce that the kernel satisfies standardness. It is easy to derive the result, we omit the proof.

Lemma 1 *$K(v) = \tau(v)$ for all $v \in \cup_{|N|=2}\mathcal{G}^{N}_{0}$.*

Lemma 2 *If $v \in \mathcal{G}^{N}_{0}$ and $x \in I(v)$, then $mr^{x}_{\{i,j\}}(v) \in \mathcal{G}^{\{i,j\}}_{0}$ for every $i,j \in N$, $i \neq j$.*

Proof. Let $N' = \{i,j\}$. We suppose without loss of generality, due to the interchangeable roles of player i and player j, that $\mathcal{S}^{+}_{ij}(v,x) \geq \mathcal{S}^{+}_{ji}(v,x)$. Then it follows by construction of the reduced game that it holds that $mr^{x}_{N'}(v)(j) = v(j)$, whereas

$$\begin{aligned} mr^{x}_{N'}(v)(i) &\leq v(i) + x_i + x_j - v(i) - v(j) \\ &= x_i + x_j - v(j). \end{aligned}$$

Consequently, it holds in general

$$mr^{x}_{N'}(v)(i) + mr^{x}_{N'}(v)(j) \leq x_i + x_j - v(j) + v(j) = mr^{x}_{N'}(v)(N'),$$

so the imputation set of the bilateral reduced game $mr^{x}_{N'}(v)$ is always non-empty. The proof is complete. ■

We will show that the kernel satisfies bilateral min-consistency and converse min-consistency. These two axioms play the main roles to characterize the kernel.

Lemma 3 *The kernel satisfies bilateral min-consistency.*

Proof. Let $v \in \mathcal{G}^{N}_{0}$, $N' = \{i,j\} \in b(N)$ and $x \in K(v)$. By Lemma 2, $mr^{x}_{N'}(v) \in \mathcal{G}^{N'}_{0}$. We shall show that $x_{N'} \in K(mr^{x}_{N'}(v))$. There are two possibilities to be considered:

Case 1 $\mathcal{S}_{ij}(v,x) > \mathcal{S}_{ji}(v,x)$ and $x_j = v(j)$. It holds that

$$\mathcal{S}^{+}_{ij}(v,x) - \mathcal{S}^{+}_{ji}(v,x) > x_i - v(i) \geq 0,$$

$mr^{x}_{N'}(v)(i) = x_i$ and $mr^{x}_{N'}(v)(j) = v(j) = x_j$. Hence, $I(mr^{x}_{N'}(v))$ is the singleton $\{(x_i,x_j)\}$ and $x_{N'} \in K(mr^{x}_{N'}(v))$.

Case 2 $\mathcal{S}_{ij}(v,x) = \mathcal{S}_{ji}(v,x)$. We suppose without loss of generality, due to the interchangeable roles of player i and player j, that $\mathcal{S}^{+}_{ij}(v,x) \geq \mathcal{S}^{+}_{ji}(v,x)$. Then $mr^{x}_{N'}(v)(j) = v(j)$ and

$$\begin{aligned} \mathcal{S}^{+}_{ij}(v,x) - \mathcal{S}^{+}_{ji}(v,x) &= x_i - v(i) - [x_j - v(j)] \\ &\leq x_i + x_j - v(i) - v(j). \end{aligned}$$

We derive that $mr^x_{N'}(v)(i) = x_i - x_j + v(j)$. By standardness of the kernel,

$$K(mr^x_{N'}(v)) = \tau(mr^x_{N'}(v)) = \{x_{N'}\}. \tag{4}$$

We obtain the desired result.

From Cases 1 and 2, the proof is complete. ■

Lemma 4 *The kernel satisfies converse min-consistency.*

Proof. Let $v \in \mathcal{G}_0^N$, $x \in I(v)$, for every $N' \in b(N)$ $mr^x_{N'}(v) \in \mathcal{G}_0^{N'}$ and $x_{N'} \in K(mr^x_{N'}(v))$. We will show that $x \in K(v)$. If not, there exists $N' = \{i, j\} \in b(N)$ such that $\mathcal{S}_{ij}(v, x) > \mathcal{S}_{ji}(v, x)$ and $x_j > v(j)$. There are two cases to be discussed.

Case 1 $\mathcal{S}^+_{ij}(v, x) \geq \mathcal{S}^+_{ji}(v, x)$. Then, $mr^x_{N'}(v)(j) = v(j)$. There are two subcases to be discussed.

Subcase 1 $\mathcal{S}^+_{ij}(v, x) - \mathcal{S}^+_{ji}(v, x) \geq x_i + x_j - v(i) - v(j)$. Then $mr^x_{N'}(v)(i) = x_i + x_j - v(j)$. We derive that $I(mr^x_{N'}(v)) = \{(x_i + x_j - v(j), v(j))\}$ and

$$K(mr^x_{N'}(v)) = \{(x_i + x_j - v(j), v(j))\} \neq \{x_{N'}\},$$

by the assumption that $x_j > v(j)$. We obtain the desired contradiction.

Subcase 2 $\mathcal{S}^+_{ij}(v, x) - \mathcal{S}^+_{ji}(v, x) < x_i + x_j - v(i) - v(j)$. Then,

$$\begin{aligned} mr^x_{N'}(v)(i) &= v(i) + \mathcal{S}^+_{ij}(v, x) - \mathcal{S}^+_{ji}(v, x) \\ &= \mathcal{S}_{ij}(v, x) - \mathcal{S}_{ji}(v, x) + x_i - x_j + v(j). \end{aligned}$$

It holds that,

$$\tau_i(mr^x_{N'}(v)) = x_i + \frac{1}{2}(\mathcal{S}_{ij}(v, x) - \mathcal{S}_{ji}(v, x)) > x_i \tag{5}$$

and

$$\tau_j(mr^x_{N'}(v)) = x_j - \frac{1}{2}(\mathcal{S}_{ij}(v, x) - \mathcal{S}_{ji}(v, x)) < x_j. \tag{6}$$

It violates the assumption that $x_{N'} \in K(mr^x_{N'}(v)) = \tau(mr^x_{N'}(v))$.

Case 2 $\mathcal{S}^+_{ij}(v, x) < \mathcal{S}^+_{ji}(v, x)$. Then, $mr^x_{N'}(v)(i) = v(i)$, and

$$\begin{aligned} \mathcal{S}^+_{ji}(v, x) - \mathcal{S}^+_{ij}(v, x) &= \mathcal{S}_{ji}(v, x) - \mathcal{S}_{ij}(v, x) + x_j - v(j) - (x_i - v(i)) \\ &< x_i + x_j - v(i) - v(j). \end{aligned}$$

Hence, $mr^x_{N'}(v)(j) = \mathcal{S}_{ji}(v, x) - \mathcal{S}_{ij}(v, x) + x_j - x_i + v(i)$. We derive that (5) and (6) hold. We obtain the desired contradiction. By Cases 1 and 2, the proof is complete. ■

Theorem 5 *The kernel is the unique solution on $\mathcal{G}_0$ satisfying Pareto optimality, non-emptiness, individual rationality, standardness, bilateral min-consistency and converse min-consistency.*

Proof. It is known that the kernel satisfies PO, NE, IR. By Lemmas 1, 3 and 4, the kernel satisfies standardness, bilateral min-consistency and converse min-consistency. We will show the uniqueness part.

Let σ be a solution on $\mathcal{G}_0$ satisfying PO, NE, IR, ST, bilateral min-consistency and converse min-consistency. Let $v \in \mathcal{G}_0$. There are three cases to be considered:

Case 1 v is a 1-player game. By PO and NE, it holds that $K(v) = \sigma(v)$.

Case 2 v is a 2-player game. By ST, $K(v) = \tau(v) = \sigma(v)$.

Case 3 $v \in \cup_{|N|>2}\mathcal{G}_0^N$. Let $x \in K(v)$. By bilateral min-consistency, for each $N' = \{i,j\} \in b(N)$,

$$x_{N'} \in K\left(mr_{N'}^{x}(v)\right) = \tau\left(mr_{N'}^{x}(v)\right) = \sigma\left(mr_{N'}^{x}(v)\right) \tag{7}$$

Since σ satisfies converse min-consistency, $x \in \sigma(v)$. We obtain that $K(v) \subseteq \sigma(v)$.

Let $x \in \sigma(v)$. By IR and PO, $x \in I(v)$. By bilateral min-consistency, for each $N' = \{i,j\} \in b(N)$, (7) holds. Since the kernel satisfies converse min-consistency, $x \in K(v)$. We obtain that $\sigma(v) \subseteq K(v)$, and $\sigma(v) = K(v)$.

From Cases 1,2 and 3, we complete the proof. ■

4 The constrained equal loss rule and the kernel

In this section, we will construct a bilateral reduced game by means of the *constrained equal loss*(CEL) rule. It is called the *bilateral CEL reduced game*. We will show that the kernel can be characterized by the same axioms in Theorem 5 except that the bilateral min reduced game is replaced by the bilateral CEL reduced game. First, let us recall the constrained equal loss rule for bankruptcy problems.

An n-person *bankruptcy problem* is a pair (E,d) with a finite set of claimants $N = \{1,2,\cdots,n\}$, where the estate $E \geq 0$ and the nonnegative claim-vector $d = (d_1, d_2, \cdots, d_n)$ satisfy $d(N) > E$. The constrained equal loss rule is defined by:

$$CEL_k(E,d) = \max[0, d_k - \lambda],$$

for all $k \in N$, where λ solves the equation $\sum_{k\in N} \max[0, d_k - \lambda] = E$. This rule is very ancient. For reference, please see Thomson(2003).

In the sequel we make use of the following two properties of the CEL rule for 2-person bankruptcy problems $(E,(d_i,d_j))$ where $d_i \geq d_j$:

$(P1)$ CEL$(E,(d_i,d_j)) = (E,0)$ whenever $d_i - d_j \geq E$.

$(P2)$ CEL$(E,(d_i,d_j)) = (d_i - d_j + \alpha, \alpha)$ whenever $d_i - d_j \leq E$, where $\alpha = (E - (d_i - d_j))/2$.

Let $x \in R^N$ be a payoff vector and $v \in \mathcal{G}^N$. *The max reduced game* $r_{N'}^{x}(v)$ of v with respect to $N' \subset N$ and x proposed by Davis and Maschler(1965) is defined in the following:

$$r^x_{N'}(v)(T) = \begin{cases} v(N) - x(N \backslash N'), & T = N', \\ \max\limits_{Q \subseteq N'^C} [v(T \cup Q) - x(Q)], & \emptyset \neq T \subsetneq N', \\ 0, & T = \emptyset. \end{cases} \tag{8}$$

Note that

$$\mathcal{S}_{ij}(v, x) = r^x_{\{i,j\}}(v)(i) - x_i$$

and

$$\mathcal{S}_{ji}(v, x) = r^x_{\{i,j\}}(v)(j) - x_j.$$

Let $N' = \{i, j\} \in b(N)$. By the definition of the bilateral max reduced game, it holds that $r^x_{N'}(v)(k) - v(k) \geq 0$ for $k = i, j$.

Notice that the imputation set of the bilateral max reduced game $r^x_{N'}(v)$ is non-empty if and only if $v(N) - x(N \backslash N') \geq r^x_{N'}(v)(i) + r^x_{N'}(v)(j)$. If $I(r^x_{N'}(v)) = \emptyset$, it holds that

$$\begin{aligned} v(N) - x(N \backslash N') - v(i) - v(j) &< r^x_{N'}(v)(i) - v(i) + r^x_{N'}(v)(j) - v(j) \\ &= \mathcal{S}^+_{ij}(v, x) + \mathcal{S}^+_{ji}(v, x). \end{aligned} \tag{9}$$

In case $I(r^x_{N'}(v)) = \emptyset$, we adapt the individual worths in the bilateral CEL reduced game by applying the CEL rule to the following 2-person bankruptcy problem:

$$B_{ij}(v, x) := \left(v(N) - x(N \backslash N') - v(i) - v(j), (\mathcal{S}^+_{ij}(v, x), \mathcal{S}^+_{ji}(v, x))\right).$$

The 2-person bankruptcy problem $B_{ij}(v, x)$ is well-defined due to (9) and $\mathcal{S}^+_{ij}(v, x), \mathcal{S}^+_{ji}(v, x)$ are non-negative.

The bilateral CEL reduced game $cr^x_{N'}$ with respect to v, x and $N' = \{i, j\} \in b(N)$ is defined as follows: $cr^x_{N'}(v)(N') = v(N) - x(N \backslash N')$ and for $k \in N'$,

$$cr^x_{N'}(v)(k) = \begin{cases} CEL_k(B_{ij}(v, x)) + v(k), & \text{if } I(r^x_{N'}(v)) = \emptyset, \\ r^x_{N'}(v)(k) & \text{o.w.} \end{cases}$$

Remark 6 *Let $x \in I(v)$. If $I(r^x_{N'}(v)) \neq \emptyset$, then $cr^x_{N'}(v) = r^x_{N'}(v) \in \mathcal{G}^{N'}_0$. If $I(r^x_{N'}(v)) = \emptyset$, then $cr^x_{N'}(v)(i) + cr^x_{N'}(v)(j) = x_i + x_j$. Hence the imputation set $I(cr^x_{N'}(v))$ shrinks to a singleton $\left\{(cr^x_{N'}(v)(k))_{k \in \{i,j\}}\right\}$, and $cr^x_{N'}(v) \in \mathcal{G}^{N'}_0$.*

Bilateral $CEL-consistency$ and *converse $CEL-consistency$* are defined as the same as bilateral min-consistency and converse min-consistency except that the bilateral CEL reduced game is employed.

The following Lemma is easy to derive, we omit the proof.

Lemma 7 *Let $v, v' \in \mathcal{G}^{\{i,j\}}$. If $v(\{i, j\}) = v'(\{i, j\})$ and $v(i) - v'(i) = v(j) - v'(j)$ then $\tau(v) = \tau(v')$.*

The following Lemma states a strong relation between the bilateral min reduced game and the bilateral CEL reduced game.

Lemma 8 *Suppose that* $v \in \mathcal{G}_0^N$, $N' = \{i,j\} \in b(N)$ *and* $x \in I(v)$. *Then* $cr^x_{N'}(v)(N') = mr^x_{N'}(v)(N')$ *and*

$$cr^x_{N'}(v)(i) - mr^x_{N'}(v)(i) = cr^x_{N'}(v)(j) - mr^x_{N'}(v)(j). \tag{10}$$

Proof. It is easy to see that $cr^x_{N'}(v)(N') = mr^x_{N'}(v)(N') = x_i + x_j$. We will show that (10) holds. There are two cases to be discussed.

Case 1 $I(r^x_{N'}(v)) \neq \emptyset$. Then $r^x_{N'}(v)(i) + r^x_{N'}(v)(j) \leq x_i + x_j$, and

$$\begin{aligned} \mathcal{S}^+_{ij}(v,x) + \mathcal{S}^+_{ji}(v,x) &= r^x_{N'}(v)(i) - v(i) + r^x_{N'}(v)(j) - v(j) \\ &\leq x_i - v(i) + x_j - v(j). \end{aligned}$$

We suppose without loss of generality, due to the interchangeable roles of player i and player j, that $\mathcal{S}^+_{ij}(v,x) \geq \mathcal{S}^+_{ji}(v,x)$. Then $mr^x_{N'}(v)(j) = v(j)$ and

$$\begin{aligned} mr^x_{N'}(v)(i) &= v(i) + \mathcal{S}^+_{ij}(v,x) - \mathcal{S}^+_{ji}(v,x) \\ &= \mathcal{S}_{ij}(v,x) + x_i - \mathcal{S}_{ji}(v,x) - x_j + v(j) \\ &= r^x_{N'}(v)(i) - r^x_{N'}(v)(j) + mr^x_{N'}(v)(j) \\ &= cr^x_{N'}(v)(i) - cr^x_{N'}(v)(j) + mr^x_{N'}(v)(j). \end{aligned}$$

Hence, (10) holds.

Case 2 $I(r^x_{N'}(v)) = \emptyset$. We suppose without loss of generality, due to the interchangeable roles of player i and player j, that $\mathcal{S}^+_{ij}(v,x) \geq \mathcal{S}^+_{ji}(v,x)$. Then $mr^x_{N'}(v)(j) = v(j)$. There are two subcases to be considered.

Subcase 1 If $\mathcal{S}^+_{ij}(v,x) - \mathcal{S}^+_{ji}(v,x) \geq x_i + x_j - v(i) - v(j)$, then by property $(P1)$ of the CEL rule it follows that

$$(CEL_k(B_{ij}(v,x)))_{k \in N'} = (x_i + x_j - v(i) - v(j), 0).$$

We derive that

$$\begin{aligned} mr^x_{N'}(v)(i) &= x_i + x_j - v(j) \\ &= CEL_i(B_{ij}(v,x)) + v(i) \\ &= cr^x_{N'}(v)(i) \end{aligned}$$

and $mr^x_{N'}(v)(j) = v(j) = cr^x_{N'}(v)(j)$. We obtain that (10) holds.

Subcase 2 If $\mathcal{S}^+_{ij}(v,x) - \mathcal{S}^+_{ji}(v,x) < x_i + x_j - v(i) - v(j)$, then by property $(P2)$ of the CEL rule it follows that

$$(CEL_k(B_{ij}(v,x)))_{k \in N'} = \left(\mathcal{S}^+_{ij}(v,x) - \mathcal{S}^+_{ji}(v,x) + \alpha, \alpha\right),$$

where $\alpha = \frac{1}{2}\left(x_i + x_j - v(i) - v(j) - \left(\mathcal{S}^+_{ij}(v,x) - \mathcal{S}^+_{ji}(v,x)\right)\right)$. Then

$$cr^x_{N'}(v)(i) = v(i) + \mathcal{S}^+_{ij}(v,x) - \mathcal{S}^+_{ji}(v,x) + \alpha = mr^x_{N'}(v)(i) + \alpha$$

and

$$cr^x_{N'}(v)(j) = v(j) + d = mr^x_{N'}(v)(j) + \alpha.$$

Hence, (10) holds.

From Cases 1 and 2, we complete the proof. ■

Next, we shall show that the kernel satisfies bilateral CEL-consistency and converse CEL-consistency.

Lemma 9 *The kernel satisfies bilateral CEL-consistency.*

Proof. Let $v \in \mathcal{G}_0^N$, $N' = \{i,j\} \in b(N)$ and $x \in K(v)$. By Remark 6, $cr^x_{N'}(v) \in \mathcal{G}_0^{N'}$. We shall show that $x_{N'} \in K(cr^x_{N'}(v))$. Since the kernel satisfies bilateral min-consistency, it holds that

$$x_{N'} \in K(mr^x_{N'}(v)) = \tau(mr^x_{N'}(v)) = \tau(cr^x_{N'}(v)) = K(cr^x_{N'}(v))$$

by Lemmas 7 and 8. ■

Lemma 10 *The kernel satisfies converse CEL-consistency.*

Proof. Let $v \in \mathcal{G}_0^N$, $x \in I(v)$, for every $N' \in b(N)$ $cr^x_{N'}(v) \in \mathcal{G}_0^{N'}$ and $x_{N'} \in K(cr^x_{N'}(v))$. We will show that $x \in K(v)$. It holds that

$$x_{N'} \in K(cr^x_{N'}(v)) = \tau(cr^x_{N'}(v)) = \tau(mr^x_{N'}(v)) = K(mr^x_{N'}(v))$$

for every $N' \in b(N)$. Since the kernel satisfies converse min-consistency, we derive that $x \in K(v)$. ■

Using the same arguments of Theorem 5, we have the following result.

Theorem 11 *The kernel is the unique solution on $\mathcal{G}_0$ satisfying Pareto optimality, non-emptiness, individual rationality, standardness, bilateral CEL-consistency and converse CEL-consistency.*

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