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Memorandum No. 1798

Equivalence of consistency and bilateral consistency through converse consistency

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May, 2006

ISSN 0169-2690

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Equivalence of consistency and bilateral consistency through converse consistency

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April 27, 2006

Abstract

In the framework of (set-valued or single-valued) solutions for coalitional games with transferable utility, the three notions of consistency, bilateral consistency, and converse consistency are frequently used to provide axiomatic characterizations of a particular solution (like the core, prekernel, prenucleolus, Shapley value, and $EANSC$-value). Our main equivalence theorem claims that a solution satisfies consistency (with respect to an arbitrary reduced game) if and only if the solution satisfies both bilateral consistency and converse consistency (with respect to the same reduced game). The equivalence theorem presumes transitivity of the reduced game technique as well as difference independence on payoff vectors for two-person reduced games. Moulin's complement reduced game, Davis and Maschler's maximum reduced game and Yanovskaya and Driessen's linear reduced game versions are evaluated.

Keywords: coalitional TU-game, reduced game, (bilateral) consistency, converse consistency
2000 Mathematics Subject Classifications: 91A12

1 Introduction

Concerning the axiomatic approach to set-valued or single-valued solutions for coalitional games with transferable utility, we distinguish between two developments in accordance with a fixed or variable player set. For instance, the classical axiomatization of the Shapley value on the class of TU-games with a fixed player set involves the axioms of linearity, Pareto optimality, dummy player property, and substitution property. Opposite to this former approach, Sobolev's [10] axiomatization of the Shapley value on the class of all TU-games (with variable player sets) involves a so-called reduced game and its corresponding reduced game property as well. The latter property requires that, if a payoff vector belongs to the solution set of a TU-game, then its restriction to any subcoalition should belong to the solution set of the reduced game, of which the player set equals the relevant subcoalition and its game structure arises from an appropriate adaptation of both the initial game structure and the payoff vector. The weak reduced game property, or alternatively bilateral consistency property, refers to two-person reduced games (instead of reduced games with arbitrary player sets). In the same paper Peleg [6] introduced the converse reduced game property, or alternatively converse consistency property, which requires that, if any restriction of the payoff vector to any pair of players belongs to the solution set of the corresponding two-person reduced games, then the payoff vector itself belongs to the solution set of the initial TU-game.
In the framework of set-valued solutions, Peleg was successful in axiomatizing both the (nonempty)

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core and the prekernel. Peleg's axiomatization of the core involves, besides the three minor axioms non-emptiness, individual rationality, superadditivity, also the weak reduced game property (cf. Theorem 3.6.1, on page 52 in [7]). Peleg's axiomatization of the prekernel involves, besides the minor axiom of 1-standardness for two-person games, both the reduced game property and the converse reduced game property (cf. Theorem 5.4.2, on page 122 in [7]). Its uniqueness proof for the prekernel resembles very much the contents of Thomson's Elevator Lemma to be discussed in Section 4. Recall that Peleg's axiomatizations of the core and the prekernel refer to Davis-Maschler's reduced game, which was used also by Sobolev [11], [4] in his pioneer work to axiomatize the single-valued solution called pre-nucleolus (cf. Theorem 6.3.1 and its complicated proof, on pages 149–153 in [7]).

In the framework of single-valued solutions, the interrelationships between (bilateral) consistency and converse consistency (with respect to arbitrary reduced games) were studied in vain during the last two decades (1985–2005). Chang and Hu's [1] recent contribution states that bilateral consistency and converse consistency imply consistency itself (provided the underlying reduced game technique behaves transitive). The main goal of the present paper is to prove the converse of Chang and Hu's recent result. We claim that consistency implies, besides bilateral consistency, converse consistency as well, provided the reduced game fulfils a special requirement. To be exact, it is presumed that the difference of the worth of singletons in two-person reduced games does not depend on the underlying payoff vector.

The organization of the paper is as follows. Section 2 provides all mathematical definitions concerning reduced games, (bilateral) consistency and converse consistency properties for set-valued or single-valued solutions. Two types of reduced games (the so-called maximum and complement reduced game) are surveyed in order to illustrate the corresponding consistency and the converse consistency of both the core and the single-valued solution called "equal allocation of nonseparable contribution" value. Section 3 recalls Chang and Hu's recent result (cf. Theorem 3.1), next we add the converse statement as the main Theorem 3.4 of the present paper. Both theorems are combined as the Equivalence Theorem 3.5 about the equivalence between consistency and bilateral consistency through converse consistency, inclusive of the two requirements on the reduced game. Section 4 recalls the notion of a linear reduced game as an extension of the former complement reduced game. The main Theorem 3.4 and Thomson's Elevator Lemma are used to present three-fold consistency axiomatizations of "linear" values like the $EANSC$-value and the Shapley value.

2 Definitions and conventions

Since the present paper deals with a continuation of Chang and Hu's recent result about the converse consistency property for transferable utility games, we shall follow the mathematical notation of their paper [1] to a large extent.

Let $\mathbb{N}$ be the set of potential *players*. A *coalition* is a nonempty finite subset of $\mathbb{N}$ and let $\mathcal{N} := \{N \mid N \subseteq \mathbb{N}\}$ denote the set of all coalitions of $\mathbb{N}$. A *coalitional game with transferable utility* (a so-called TU-game) is a pair $\langle N, v\rangle$, where the player set $N \in \mathcal{N}$ is a coalition and $v : 2^N \to \mathbb{R}$ is the so-called *characteristic function* that assigns to each subcoalition S of N a real number $v(S)$, that is $v(S) \in \mathbb{R}$ for all $S \in \mathcal{N}$ with $S \subseteq N$. The number $v(S)$ is called the *worth* of S in the game v, and it is always assumed $v(\emptyset) := 0$. Further, let $|N|$ denote the cardinality of the coalition $N \in \mathcal{N}$. This concludes the modelling part.

For the purpose of the solution part, recall that $\mathbb{R}^N$ denotes the set of all functions from the coalition $N \in \mathcal{N}$ to the set $\mathbb{R}$ of real numbers. In fact, we will consider functions $x \in \mathbb{R}^N$ as $|N|$-dimensional vectors whose coordinates are indexed by the members of the coalition N, that is we write x_i for $x(i)$ and $\vec{x} = (x_i)_{i\in N}$ is called a *payoff vector*. Given the payoff vector $\vec{x} = (x_k)_{k\in N}$ and subcoalition S of N, we write $\vec{x}(S) = \sum_{k\in S} x_k$, where $\vec{x}(\emptyset) := 0$.

With every TU-game $\langle N, v\rangle$, there is associated the set $X^*(N, v)$ of *feasible payoff vectors* and the set $X(N,v)$ of *Pareto optimal feasible payoff vectors* respectively by

$$X^*(N,v) = \{\vec{x} \in \mathbb{R}^N \mid \vec{x}(N) \leq v(N)\} \qquad \text{and} \qquad X(N,v) = \{\vec{x} \in \mathbb{R}^N \mid \vec{x}(N) = v(N)\}.$$

A *solution* on the class $\mathcal{G}$ of all TU-games is a function σ on $\mathcal{G}$ which associates with each TU-game $\langle N,v\rangle$ a subset $\sigma(N,v) \subseteq X^*(N,v)$ of feasible payoff vectors. We distinguish between set-valued solutions and single-valued solutions. Single-valued solutions are called *values* and their corresponding solution set $\sigma(N,v)$ consists of a singleton denoted by $(\sigma_i(N,v))_{i\in N} \in \mathbb{R}^N$.

A well-known set-valued solution is the *core*, of which the feasible payoff vectors are determined by the property that no coalition can improve them. Formally, the core $Core(N,v)$ of a TU-game $\langle N,v\rangle$ is given by

$$Core(N,v) := \{\vec{x} \in X^*(N,v) \mid \vec{x}(S) \geq v(S) \quad \text{for all } S \subseteq N,\ S \neq \emptyset\}.$$

TU-games with empty core do exist. On the class of TU-games with non-empty core, the core solution turns out to possess a so-called reduced game property in that the restriction of a core payoff vector of the initial game is once again a core payoff vector of any reduced game proposed by Davis and Maschler [2].
With any TU-game $\langle N,v\rangle$, any subcoalition $N' \subseteq N$, and any payoff vector $\vec{x} = (x_k)_{k\in N} \in \mathbb{R}^N$, there is associated the *maximum reduced game* $\langle N', mr^{\vec{x}}_{N'}(v)\rangle$ as follows: $(mr^{\vec{x}}_{N'}(v))(\emptyset) = 0$ and $(mr^{\vec{x}}_{N'}(v))(N') = v(N) - \sum_{k\in N\setminus N'} x_k$, whereas

$$(mr^{\vec{x}}_{N'}(v))(T) = \max_{Q\subseteq N\setminus N'} \left[v(T\cup Q) - \sum_{k\in Q} x_k \right] \quad \text{for all } \emptyset \neq T \subsetneq N'.$$

Davis and Maschler's reduced game is appropriately chosen to discover the so-called reduced game property of the core solution in that, if $\vec{x} \in Core(N,v)$ of the initial TU-game $\langle N,v\rangle$, then its restriction $(x_k)_{k\in N'}$ to any subcoalition $N' \subseteq N$ is also a core solution of the maximum reduced game, that is $\vec{x}_{N'} = (x_k)_{k\in N'} \in Core(N', mr^{\vec{x}}_{N'}(v))$. The validity of the so-called reduced game property of the (non-empty) core solution is simple to check.

A well-known single-valued solution is the *equal allocation of nonseparable contribution (EANSC) value*, which assigns the following Pareto optimal feasible payoff vector $(EANSC_i(N,v))_{i\in N} \in \mathbb{R}^N$ to a TU-game $\langle N,v\rangle$:

$$EANSC_i(N,v) = SC^v_i + \frac{1}{|N|} \cdot \left[v(N) - \sum_{k\in N} SC^v_k \right] \quad \text{for all } i \in N,$$

where the *separable contribution* SC^v_i of player $i \in N$ in the TU-game $\langle N,v\rangle$ is given by $SC^v_i = v(N) - v(N\backslash\{i\})$. On the class $\mathcal{G}$ of all TU-games, the EANSC-value turns out to possess a so-called reduced game property with respect to the reduced game proposed by Moulin [5].
With any TU-game $\langle N,v\rangle$, any subcoalition $N' \subseteq N$, and any payoff vector $\vec{x} = (x_k)_{k\in N} \in \mathbb{R}^N$, there is associated the *complement reduced game* $\langle N', cr^{\vec{x}}_{N'}(v)\rangle$ as follows: $(cr^{\vec{x}}_{N'}(v))(\emptyset) = 0$, whereas

$$(cr^{\vec{x}}_{N'}(v))(T) = v(T\cup(N\backslash N')) - \sum_{k\in N\setminus N'} x_k \quad \text{for all } \emptyset \neq T \subseteq N'.$$

Obviously, the separable contributions are invariant under the complement reduced game, that is $SC^{cr^{\vec{x}}_{N'}(v)}_k = SC^v_k$ for all $k \in N'$. As a direct consequence, the EANSC-value of the complement reduced game is determined as follows:

$$EANSC_k(N', cr^{\vec{x}}_{N'}(v)) = SC^v_k + \frac{1}{|N'|} \cdot \sum_{\ell\in N'} (x_\ell - SC^v_\ell) \quad \text{for all } k \in N', \text{ all } \vec{x} \in X(N,v). \text{ So,}$$

$$EANSC_k(N', cr^{\vec{y}}_{N'}(v)) = EANSC_k(N, v) \quad \text{for all } k \in N', \text{ where } \vec{y} := EANSC(N, v).$$

In view of the latter equality, it is said the EANSC-value possesses the reduced game property with respect to the complement reduced game. Following this introductory part about types of reduced games and the reduced game property for set- or single-valued solutions (like the core and the EANSC-value), we present the formal model of these fundamental notions.

With any TU-game $\langle N, v\rangle$, any subcoalition $N' \subseteq N$, and any payoff vector $\vec{x} = (x_k)_{k\in N} \in \mathbb{R}^N$, there is associated a *reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ of any type, provided the worth for the empty set and its grand coalition are determined by $(r^{\vec{x}}_{N'}(v))(\emptyset) = 0$ and $(r^{\vec{x}}_{N'}(v))(N') = v(N) - \sum_{k\in N\setminus N'} x_k$. In this framework, non-members of the subcoalition N' are supposed to leave the initial game and it is assumed that all members of the initial player set N agree that the non-members of N' are paid according to the payoff vector $\vec{x}$. The members of N' form the player set of the reduced game and in order to describe the worth of a non-trivial coalition $T \subseteq N'$ in the reduced game, the members of T may consider various options to cooperate with possible subsets consisting of non-members of N' (subject to the foregoing agreement about payments).

Definition 2.1. A set-valued solution σ on the class $\mathcal{G}$ of all TU-games is said to possess the *reduced game property with respect to a specific type of reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ if for all TU-games $\langle N, v\rangle$ with $|N| \geq 3$, any subcoalition $N' \subseteq N$, the following holds:

$$\vec{x} = (x_k)_{k\in N} \in \sigma(N, v) \quad \text{implies} \quad \vec{x}_{N'} = (x_k)_{k\in N'} \in \sigma(N', r^{\vec{x}}_{N'}(v)).$$

As a special case, a single-valued solution σ on the class $\mathcal{G}$ of all TU-games possesses the *reduced game property with respect to a specific type of reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ if for all TU-games $\langle N, v\rangle$ with $|N| \geq 3$, any subcoalition $N' \subseteq N$, the following holds:

$$\sigma_k(N', r^{\vec{y}}_{N'}(v)) = \sigma_k(N, v) \quad \text{for all } k \in N', \text{ where } \vec{y} := \sigma(N, v).$$

Sobolev [11] introduced the maximum reduced game property for single-valued solutions in order to axiomatize the prenucleolus (on the set of all TU-games) by means of single-valuedness, covariance, anonymity, and the maximum reduced game property. Sobolev's complicated proof can be found in Peleg and Sudhölter's introductory book [7] about cooperative game theory (cf. Theorem 6.3.1 and its proof on pages 149–153). The set-valued extension of the maximum reduced game property is due to Peleg [6] who axiomatized the set-valued solutions of the core and prekernel. In fact, the reduced game property is a condition of *consistency* in that the solution of the initial TU-game is reconfirmed as the solution of the reduced game (subject to the foregoing agreement that payments to non-members of the reduced game are settled according to the solution of the initial game). Throughout the remainder of this paper, we prefer the term "consistency property" to "reduced game property" (as done in Chang and Hu's paper [1] with reference to maximum consistency and complement consistency).

Remark 2.2. For future purposes, recall the common knowledge that, for single-valued solutions, consistency implies Pareto optimality. That is, if the value σ satisfies consistency with respect to an arbitrary reduced game $\langle N', r^{\vec{x}}_{N'}(v)\rangle$, then it holds $\sigma(N, v) \in X(N, v)$ for all TU-games $\langle N, v\rangle$ with $|N| \geq 3$ (presuming that Pareto optimality of σ is already valid for two-person TU-games). The inductive proof of Pareto optimality proceeds by induction on the player size $|N|$, $|N| \geq 3$. Fix $i \in N$. The consistency of σ yields that $\sigma_k(N\setminus\{i\}, r^{\vec{y}}_{N\setminus\{i\}}(v)) = \sigma_k(N, v)$ for all $k \in N\setminus\{i\}$, where $\vec{y} := \sigma(N, v)$. From this and the induction hypothesis applied to the reduced game $\langle N\setminus\{i\}, r^{\vec{y}}_{N\setminus\{i\}}(v)\rangle$, it follows

$$\begin{aligned}\sum_{k\in N} \sigma_k(N, v) &= \sigma_i(N, v) + \sum_{k\in N\setminus\{i\}} \sigma_k(N, v) = \sigma_i(N, v) + \sum_{k\in N\setminus\{i\}} \sigma_k(N\setminus\{i\}, r^{\vec{y}}_{N\setminus\{i\}}(v)) \\ &= \sigma_i(N, v) + (r^{\vec{y}}_{N\setminus\{i\}}(v))(N\setminus\{i\}) = \sigma_i(N, v) + v(N) - y_i = v(N).\end{aligned}$$

□

A solution is said to possess the *bilateral consistency property* if the above consistency property applies in the framework of *all pairs of players*, i.e., the above consistency property holds for all two-person subcoalitions $N' \subseteq N$. Clearly, consistency implies bilateral consistency, but the converse is not true in general. It remains to add one more fundamental property due to Peleg [6].

Definition 2.3. A set-valued solution σ on the class $\mathcal{G}$ of all TU-games is said to possess the *converse reduced game property with respect to a specific type of reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ if for all TU-games $\langle N, v\rangle$ with $|N| \geq 3$, all Pareto optimal feasible payoff vectors $\vec{x} = (x_k)_{k\in N} \in X(N, v)$, the following holds:

$$\text{If } \vec{x}_{N'} = (x_k)_{k\in N'} \in \sigma(N', r^{\vec{x}}_{N'}(v)) \text{ for all two-person subcoalitions } N' \subseteq N, \text{ then } \vec{x} \in \sigma(N, v).$$

As a special case, a single-valued solution σ on the class $\mathcal{G}$ of all TU-games possesses the *converse reduced game property with respect to a specific type of reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ if for all TU-games $\langle N, v\rangle$ with $|N| \geq 3$, all Pareto optimal feasible payoff vectors $\vec{x} = (x_k)_{k\in N} \in X(N, v)$, the following holds:

$$\text{If } \vec{x}_{N'} = \sigma(N', r^{\vec{x}}_{N'}(v)) \text{ for all two-person subcoalitions } N' \subseteq N, \text{ then } \vec{x} = \sigma(N, v).$$

Once again, we prefer the term "converse consistency property" to "converse reduced game property". As an example, we illustrate the converse consistency property of the EANSC-value with respect to the complement reduced game. Since the EANSC-value of a complement reduced game has been determined before, the assumption $\vec{x}_{N'} = EANSC(N', cr^{\vec{x}}_{N'}(v))$ for any two-person subcoalition $N' = \{i, j\}$ simplifies to $x_i = SC^v_i + \frac{1}{2}\cdot\left[x_i - SC^v_i + x_j - SC^v_j\right]$. The equivalent formula $x_i - SC^v_i = x_j - SC^v_j$ for any pair $i, j \in N$, $i \neq j$, together with $\vec{x}(N) = v(N)$, yields that $\vec{x} = EANSC(N, v)$. Peleg [6] showed that the core possesses the converse consistency property with respect to the maximum reduced game.

3 Equivalence Theorem

This section is devoted to interrelationships between the consistency, bilateral consistency, and converse consistency properties for a single-valued solution. Obviously, consistency implies bilateral consistency. Roughly speaking, we claim that consistency and bilateral consistency are equivalent through converse consistency. In other words, we claim that, under certain circumstances, consistency holds if and only if both bilateral consistency and converse consistency hold. The "if" statement is due to a recent result by Chang and Hu [1]. The validity of their conclusion is strongly based on a natural assumption, called *transitivity*, of the underlying reduced game technique. The reduced game $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ is said to behave *transitive* if the repeated use of the reduced game does not depend on the consecutive order of removal of players, i.e., $r^{\vec{x}_{N'}}_{N''}(r^{\vec{x}}_{N'}(v)) = r^{\vec{x}}_{N''}(v)$ for all TU-games $\langle N, v\rangle$, all subcoalitions $\emptyset \neq N'' \subseteq N' \subseteq N$ and all payoff vectors $\vec{x} \in \mathbb{R}^N$. The transitivity of Moulin's complement reduced game is simple to check, whereas the transitivity of Davis and Maschler's maximum reduced game requires some straightforward calculations (cf. Lemma 7 in [1]). For the sake of completeness, we recall Chang and Hu's main result and repeat its elegant short proof.

Theorem 3.1. (cf. "Equivalence" Lemma 18 in [1])
Let σ be a set-valued solution on the class $\mathcal{G}$ of all TU-games satisfying both the bilateral consistency and the converse consistency with respect to a transitive reduced game technique. Then σ satisfies consistency (with respect to the same reduced game) as well.

Proof. Let $\langle N, v\rangle$ be a TU-game with $|N| \geq 3$, and a subcoalition $N' \subseteq N$. In order to establish the consistency of σ, suppose $\vec{x} \in \sigma(N, v)$. We aim to prove the "inclusion" $\vec{x}_{N'} \in \sigma(N', r^{\vec{x}}_{N'}(v))$.

In case $|N'| = 2$, the inclusion holds by bilateral consistency of σ applied to $\vec{x} \in \sigma(N,v)$. Let $|N'| \geq 3$. We claim the relevant inclusion to be supported by the converse consistency of σ applied to the reduced game $\langle N', r^{\vec{x}}_{N'}(v)\rangle$. For that purpose it suffices to check the following:

$$\vec{x}_{N''} \in \sigma(N'', r^{\vec{x}_{N'}}_{N''}(r^{\vec{x}}_{N'}(v)) \quad \text{for all two-person subcoalitions } N'' \subseteq N'.$$

By the transitivity of the reduced game technique, it holds $r^{\vec{x}_{N'}}_{N''}(r^{\vec{x}}_{N'}(v)) = r^{\vec{x}}_{N''}(v)$. Hence, the latter condition is equivalent to the following:

$$\vec{x}_{N''} \in \sigma(N'', r^{\vec{x}}_{N''}(v)) \quad \text{for all two-person subcoalitions } N'' \subseteq N',$$

which is guaranteed to hold (for all two-person subcoalitions $N'' \subseteq N$) because of the bilateral consistency of σ applied to $\vec{x} \in \sigma(N,v)$. Thus, σ satisfies consistency. □

The main purpose of the paper is to establish the converse of Chang and Hu's recent result. We claim that consistency implies, besides bilateral consistency, converse consistency as well. As a counterpart of Chang and Hu's requirement for reduced games to behave transitive, we introduce a new requirement for two-person reduced games as follows.

Definition 3.2. The *two-person reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ is said to fulfil the *difference independence on payoff vectors* if the difference of the worth of the singletons in the two-person reduced game does not depend on the payoff vector $\vec{x} \in \mathbb{R}^N$, i.e., for all $i \in N$, $j \in N$, $i \neq j$, the difference $(r^{\vec{x}}_{ij}(v))(\{i\}) - (r^{\vec{x}}_{ij}(v))(\{j\})$ is the same for all payoff vectors $\vec{x} \in \mathbb{R}^N$.

Moulin's complement two-person reduced game $\langle N', cr^{\vec{x}}_{N'}(v)\rangle$ satisfies difference independence on payoff vectors since the difference $(cr^{\vec{x}}_{ij}(v))(\{i\}) - (cr^{\vec{x}}_{ij}(v))(\{j\}) = v(N\backslash\{j\}) - v(N\backslash\{i\})$ is the same for all payoff vectors $\vec{x} \in \mathbb{R}^N$. Generally speaking, Davis and Maschler's maximum two-person reduced game $\langle N', mr^{\vec{x}}_{N'}(v)\rangle$ does not satisfy difference independence on payoff vectors due to its structure $(mr^{\vec{x}}_{ij}(v))(\{i\}) = \max_{Q\subseteq N\backslash\{i,j\}}\left[v(Q\cup\{i\}) - \sum_{k\in Q} x_k\right]$. As a minor requirement, the single-valued solution is supposed to behave standard for two-person games. The common notion of 1-standardness for two-person games is extended to the general notion of λ-standardness for two-person games. In words, given that each of the two players receives the same fraction (denoted by $\lambda \in \mathbb{R}$) of the individual worth, the surplus is divided equally among both of them.

Definition 3.3. Let $\lambda \in \mathbb{R}$. A single-valued solution σ on the class $\mathcal{G}$ of all TU-games is said to satisfy *λ-standardness for two-person games* if the following holds for any two-person game $\langle\{i,j\}, v\rangle$ $(i \neq j)$:

$$\sigma_k(\{i,j\}, v) = \lambda \cdot v(\{k\}) + \tfrac{1}{2} \cdot \Big[v(\{i,j\}) - \lambda \cdot v(\{i\}) - \lambda \cdot v(\{j\})\Big] \quad \text{for } k \in \{i,j\}.$$

Theorem 3.4. Let $\lambda \in \mathbb{R}$. Let σ be a single-valued solution on the class $\mathcal{G}$ of all TU-games satisfying both λ-standardness for two-person games and consistency with respect to a specific type of reduced game $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ such that the two-person reduced game fulfils the difference independence on payoff vectors. Then σ satisfies converse consistency (with respect to the same reduced game) as well.

Proof. Let $\langle N, v\rangle$ be a TU-game with $|N| \geq 3$. In order to establish the converse consistency of σ, suppose $\vec{x} \in X(N,v)$ satisfies $\vec{x}_{N'} = \sigma(N', r^{\vec{x}}_{N'}(v))$ for all two-person subcoalitions $N' \subseteq N$. We aim to prove $\vec{x} = \sigma(N,v)$. Fix the two-person subcoalition N' by writing $N' = \{i,j\}$.
Part one. Firstly, we exploit the basic assumption on $\vec{x}$ jointly with the λ-standardness of σ applied to the two-person (reduced) game $\langle\{i,j\}, r^{\vec{x}}_{ij}(v)\rangle$. From this we deduce the following:

$$x_i = \sigma_i(\{i,j\}, r^{\vec{x}}_{ij}(v)) = \lambda \cdot \alpha^v_i(\vec{x}) + \tfrac{1}{2} \cdot \Big[(r^{\vec{x}}_{ij}(v))(\{i,j\}) - \lambda \cdot \alpha^v_i(\vec{x}) - \lambda \cdot \alpha^v_j(\vec{x})\Big],$$

where $\alpha_k^v(\vec{x}) := (r_{ij}^{\vec{x}}(v))(\{k\})$ for $k \in \{i,j\}$. Recall that the worth of the grand coalition of a reduced game is determined by $(r_{ij}^{\vec{x}}(v))(\{i,j\}) = v(N) - \sum_{k \in N \setminus \{i,j\}} x_k = x_i + x_j$ where the last equality holds because of the assumption $\vec{x} \in X(N,v)$. Hence, the former equality reduces to

$$x_i = \lambda \cdot \alpha_i^v(\vec{x}) + \tfrac{1}{2} \cdot \Big[x_i + x_j - \lambda \cdot \alpha_i^v(\vec{x}) - \lambda \cdot \alpha_j^v(\vec{x})\Big] \quad \text{or equivalently,}$$

$$x_i - \lambda \cdot \alpha_i^v(\vec{x}) = x_j - \lambda \cdot \alpha_j^v(\vec{x}), \qquad \text{that is,} \qquad x_i - x_j = \lambda \cdot \Big[\alpha_i^v(\vec{x}) - \alpha_j^v(\vec{x})\Big].$$

Part two. Secondly, we exploit the bilateral consistency of σ jointly with the λ-standardness of σ applied to the two-person (reduced) game $\langle \{i,j\}, r_{ij}^{\vec{y}}(v)\rangle$, where $\vec{y} := \sigma(N,v)$. From this we deduce the following:

$$\sigma_i(N,v) = \sigma_i(\{i,j\}, r_{ij}^{\vec{y}}(v)) = \lambda \cdot \alpha_i^v(\vec{y}) + \tfrac{1}{2} \cdot \Big[(r_{ij}^{\vec{y}}(v))(\{i,j\}) - \lambda \cdot \alpha_i^v(\vec{y}) - \lambda \cdot \alpha_j^v(\vec{y})\Big].$$

Notice that $(r_{ij}^{\vec{y}}(v))(\{i,j\}) = v(N) - \sum_{k \in N \setminus \{i,j\}} y_k = y_i + y_j$ where the last equality holds because of the Pareto optimality of $\vec{y} = \sigma(N,v)$ (as a result of the consistency of σ, see Remark 2.2). Hence, the former equality reduces to

$$y_i = \lambda \cdot \alpha_i^v(\vec{y}) + \tfrac{1}{2} \cdot \Big[y_i + y_j - \lambda \cdot \alpha_i^v(\vec{y}) - \lambda \cdot \alpha_j^v(\vec{y})\Big] \quad \text{or equivalently,}$$

$$y_i - \lambda \cdot \alpha_i^v(\vec{y}) = y_j - \lambda \cdot \alpha_j^v(\vec{y}), \qquad \text{that is,} \qquad y_i - y_j = \lambda \cdot \Big[\alpha_i^v(\vec{y}) - \alpha_j^v(\vec{y})\Big].$$

Part three. From parts one and two, together with the difference independence of the two-person reduced game with respect to payoff vectors, we deduce the following:

$$x_i - x_j = \lambda \cdot \Big[(r_{ij}^{\vec{x}}(v))(\{i\}) - (r_{ij}^{\vec{x}}(v))(\{j\})\Big] = \lambda \cdot \Big[(r_{ij}^{\vec{y}}(v))(\{i\}) - (r_{ij}^{\vec{y}}(v))(\{j\})\Big] = y_i - y_j.$$

That is, $x_i - x_j = y_i - y_j$, or equivalently, $x_i - y_i = x_j - y_j$ for all $i \in N$, $j \in N$, $i \neq j$. From this, together with $\sum_{k \in N} x_k = v(N) = \sum_{k \in N} y_k$, it follows immediately that $\vec{x} = \vec{y}$. Thus, $\vec{x} = \sigma(N,v)$ as was to be shown. □

Theorem 3.5. Equivalence Theorem.
Let σ be a single-valued solution on the class $\mathcal{G}$ of all TU-games that is λ-standard for two-person games (for some $\lambda \in \mathbb{R}$). With respect to any transitive reduced game technique that is difference independent on payoff vectors for two-person reduced games, the solution σ satisfies consistency if and only if σ satisfies both bilateral consistency and converse consistency. In other words, under these two special circumstances, consistency and bilateral consistency are equivalent through converse consistency.

Notice that the equivalence theorem is applicable to the EANSC-value with respect to Moulin's complement reduced game. As noted before, the complement reduced game behaves transitive and is difference independent on payoff vectors for two-person reduced games, whereas the EANSC-value itself is 1-standard for two-person games. The next section is devoted to another type of reduced games fulfilling both requirements.

4 The linear reduced game

Definition 4.1. With any TU-game $\langle N, v\rangle$, any subcoalition $N' \subseteq N$, and any payoff vector $\vec{x} = (x_k)_{k\in N} \in \mathbb{R}^N$, Yanovskaya and Driessen [14] proposed the *linear reduced game* $\langle N', lr^{\vec{x}}_{N'}(v)\rangle$ as follows: $(lr^{\vec{x}}_{N'}(v))(\emptyset) = 0$ and $(lr^{\vec{x}}_{N'}(v))(N') = v(N) - \sum_{k\in N\setminus N'} x_k$, whereas

$$(lr^{\vec{x}}_{N'}(v))(T) = \sum_{Q\subseteq N\setminus N'} \alpha_{|N|,|N'|,|T|,|Q|} \cdot \left[v(T\cup Q) - \sum_{k\in Q} x_k\right] \quad \text{for all } \emptyset \neq T \subsetneq N',$$

where the non-negative real numbers $\alpha_{n,n',t,q}$ are arbitrary, except for $\sum_{q=0}^{n-n'} \binom{n-n'}{q} \cdot \alpha_{n,n',t,q} = 1$ for all $n' \in \{1, 2, \ldots, n-1\}$ and all $t \in \{1, 2, \ldots, n'-1\}$. Here the sizes of sets N, N', T, Q are denoted by n, n', t, q, respectively. For the probabilistic interpretation of the worth $(lr^{\vec{x}}_{N'}(v))(T)$ of subcoalition T in the reduced game, see [14], page 603.

Notice that Moulin's complement reduced game arises from the linear reduced game by choosing $\alpha_{n,n',t,n-n'} = 1$ and $\alpha_{n,n',t,q} = 0$ for all $q \in \{0, 1, 2, \ldots, n-n'-1\}$. Furthermore, by the particular choice $\alpha_{n,n-1,t,1} = \frac{t}{n-1}$ and $\alpha_{n,n-1,t,0} = \frac{n-1-t}{n-1}$ for all $t \in \{1, 2, \ldots, n-2\}$, it turns out that the linear reduced game agrees with Sobolev's [10] pioneer proposal to axiomatize the Shapley value [9].
Linear two-person reduced games $\langle N', lr^{\vec{x}}_{N'}(v)\rangle$ satisfy difference independence on payoff vectors since the difference $(lr^{\vec{x}}_{ij}(v))(\{i\}) - (lr^{\vec{x}}_{ij}(v))(\{j\}) = \sum_{Q\subseteq N\setminus\{i,j\}} \alpha_{|N|,2,1,|Q|} \cdot \left[v(Q\cup\{i\}) - v(Q\cup\{j\})\right]$ is the same for all payoff vectors $\vec{x} \in \mathbb{R}^N$. Moreover, by straightforward calculations, it turns out that the linear reduced game behaves transitive if and only if the following path-independence condition holds (write $w_{n,t} := \alpha_{n,n-1,t,0}$):

$$w_{n-1,t} \cdot (1 - w_{n,t}) = (1 - w_{n-1,t}) \cdot w_{n,t+1} \quad \text{for all } n \geq 3 \text{ and all } t \in \{1, 2, \ldots, n-2\}.$$

Theorem 4.2. (cf. Theorem 1.2 in [14], page 604).
Let $\lambda \in \mathbb{R}$. Let σ be a single-valued solution on the class $\mathcal{G}$ of all TU-games satisfying both λ-standardness for two-person games and consistency with respect to the linear reduced game. Then σ satisfies efficiency (Pareto optimality), anonymity, as well as linearity (that is, $\sigma(N, av + bw) = a\sigma(N, v) + b\sigma(N, w)$ for all $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}$ and all $a, b \in \mathbb{R}$).
Moreover, σ is a unique value satisfying λ-standardness for two-person games and "linear" consistency. In fact, for any TU-game $\langle N, v\rangle$, its value $(\sigma_i(N, v))_{i\in N}$ is given by

$$\sigma_i(N, v) = \frac{v(N)}{n} + \frac{\lambda}{n} \cdot \left[\sum_{\substack{S\subsetneq N,\\ i\in S}} (n-s) \cdot \alpha_{n,2,1,s-1} \cdot v(S) - \sum_{\substack{\emptyset\neq S\subsetneq N,\\ i\notin S}} s \cdot \alpha_{n,2,1,s-1} \cdot v(S)\right]. \tag{4.1}$$

In order to add two equivalent axiomatizations of the (linear) value σ of Theorem 4.2, we first recall a powerful tool called the "Elevator Lemma" proposed by Thomson in the context of a general treatment about consistency. Because of its importance we include Thomson's short proof.

Lemma 4.3. Elevator Lemma ([12], [13], cf. Lemma 16 in [1]).
Let σ and ψ be two single-valued Pareto optimal solutions on the class $\mathcal{G}$ of all TU-games such that σ satisfies bilateral consistency on $\mathcal{G}$ (with respect to an arbitrary reduced game), ψ satisfies converse consistency on $\mathcal{G}$ (with respect to the same reduced game), and $\sigma = \psi$ for two-person games. Then $\sigma(N, v) = \psi(N, v)$ for all TU-games $\langle N, v\rangle \in \mathcal{G}$.

Proof. Let $\langle N, v\rangle$ be a TU-game with $|N| \geq 3$. Put $\vec{x} := \sigma(N, v)$. Since σ satisfies bilateral consistency on $\mathcal{G}$, it holds $\vec{x}_{N'} = \sigma(N', r^{\vec{x}}_{N'}(v))$ for all two-person subcoalitions $N' \subseteq N$. Since $\sigma = \psi$ for any two-person game, it follows that $\vec{x}_{N'} = \psi(N', r^{\vec{x}}_{N'}(v))$ for all two-person subcoalitions $N' \subseteq N$. Because ψ satisfies converse consistency on $\mathcal{G}$, it holds $\vec{x} = \psi(N, v)$. Thus, $\sigma(N, v) = \psi(N, v)$ for all TU-games $\langle N, v\rangle$. □

Theorem 4.4. (extension of Theorem 4.2).
Let $\lambda \in \mathbb{R}$. Let σ be the unique value on the class $\mathcal{G}$ of all TU-games satisfying both λ-standardness for two-person games and "linear" consistency, that is, consistency with respect to a specific type of linear reduced game $\langle N', lr^{\vec{x}}_{N'}(v)\rangle$. Then the next two axiomatizations of σ hold true:
(1) σ is the unique value on $\mathcal{G}$ satisfying λ-standardness for two-person games and bilateral linear consistency, that is bilateral consistency with respect to the same linear reduced game;
(2) σ is the unique value on $\mathcal{G}$ satisfying λ-standardness for two-person games and converse linear consistency, that is converse consistency with respect to the same linear reduced game.

Proof. Obviously, σ satisfies bilateral linear consistency, and moreover, by Theorem 3.4, σ satisfies converse linear consistency (since the difference independence on payoff vectors applies in the framework of two-person linear reduced games). To prove the uniqueness part, let ψ be a value on $\mathcal{G}$ satisfying λ-standardness for two-person games as well as bilateral (respectively converse) linear consistency. Thus, $\psi = \sigma$ for two-person games by λ-standardness. In addition, by the Elevator Lemma 4.3, $\psi(N,v) = \sigma(N,v)$ for any TU-game $\langle N,v\rangle$, either because of the bilateral linear consistency of ψ and the converse linear consistency of σ, or the bilateral linear consistency of σ and the converse linear consistency of ψ. □

Recall from the introductory Section 2 that the EANSC-value on $\mathcal{G}$ satifies both the (bilateral) consistency and the converse consistency with respect to the complement reduced game (which behaves transitive and is difference independent on payoff vectors). To conclude with, we summarize three axiomatizations of the EANSC-value in terms of one or another consistency notion.

Corollary 4.5. (1) The *EANSC*-value is the unique value on $\mathcal{G}$ satisfying 1-standardness for two-person games and consistency with respect to the complement reduced game (see [5]);
(2) The *EANSC*-value is the unique value on $\mathcal{G}$ satisfying 1-standardness for two-person games and bilateral consistency with respect to the complement reduced game (cf. Theorem 21 in [1]);
(3) The *EANSC*-value is the unique value on $\mathcal{G}$ satisfying 1-standardness for two-person games and converse consistency with respect to the complement reduced game.

A similar three-fold consistency axiomatization of the well-known Shapley value [9] is valid with respect to the "linear" reduced game proposed by Sobolev [10]. Recall that the Shapley value is 1-standard for two-person games.
We conclude the paper with some remarks about single-valued solutions satisfying efficiency (Pareto optimality), symmetry (anonymity), and linearity. Ruiz et al. ([8], Lemma 9, page 117) proved that a value σ on the class $\mathcal{G}$ of all TU-games verifies efficiency, symmetry, and linearity if and only if there exists a (unique) collection of constants $\{\rho^n_s \mid n \in \{2,3,\ldots\},\ s \in \{1,2,\ldots,n-1\}\}$ such that, for every n-person game $\langle N,v\rangle$ with at least two players, its value $(\sigma_i(N,v))_{i\in N} \in \mathbb{R}^N$ is of the following form:

$$\sigma_i(N,v) = \frac{v(N)}{n} + \sum_{\substack{S\subsetneq N,\\ S\ni i}} \frac{\rho^n_s}{s}\cdot v(S) - \sum_{\substack{S\subseteq N,\\ S\not\ni i}} \frac{\rho^n_s}{n-s}\cdot v(S) \qquad \text{for all } i \in N, \tag{4.2}$$

or equivalently,

$$\sigma_i(N,v) = \sum_{S\subseteq N\setminus\{i\}} \frac{1}{n\cdot\binom{n-1}{s}}\cdot\Big[b^n_{s+1}\cdot v(S\cup\{i\}) - b^n_s\cdot v(S)\Big] \qquad \text{for all } i \in N, \tag{4.3}$$

by choosing the corresponding collection of constants $\mathcal{B} = \{b^n_s \mid n \in \{2,3,\ldots\},\ s \in \{1,2,\ldots,n\}\}$ as $b^n_s = \binom{n}{s}\cdot\rho^n_s$ for all $s \in \{1,2,\ldots,n-1\}$ and $b^n_n = 1$. Whenever we deal with the unit collection, that is $b^n_s = 1$ for all $s \in \{1,2,\ldots,n\}$, the expression at the right hand of (4.3) reduces to the so-called *Shapley value* $Sh_i(N,v)$ of player i in the n-person game $\langle N,v\rangle$ ([9]). Generally speaking, the right hand of (4.3) equals the Shapley value payoff $Sh_i(N,\mathcal{B}v)$ of player i in the $\mathcal{B}$-scaled game $\langle N,\mathcal{B}v\rangle$, defined by $(\mathcal{B}v)(S) := b^n_s\cdot v(S)$ for all $S\subseteq N$, $S\neq\emptyset$. That is, the (non-negative) constant b^n_s acts as a scaling factor. In summary, a value σ is efficient, symmetric, and linear if and only if

$\sigma(N,v) = Sh(N,\mathcal{B}v)$, that is the σ-value of a game coincides with the Shapley value of the $\mathcal{B}$-scaled game. This equivalence theorem turns out to be very useful throughout the development of new contributions about values, but it is outside the scope of this paper. For instance, by Theorem 4.2, if a value σ on the class $\mathcal{G}$ of all TU-games satisfies both 1-standardness for two-person games and consistency with respect to the linear reduced game, then the equivalence $\sigma(N,v) = Sh(N,\mathcal{B}v)$ holds by choosing $b_s^n = \frac{s(n-s)}{n} \cdot \binom{n}{s} \cdot \alpha_{n,2,1,s-1}$ for all $s \in \{1,2,\ldots,n-1\}$ (cf. (4.1) and (4.2)).

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