Robust Dynamic Cooperative Games

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Abstract

Classical cooperative game theory is no longer a suitable tool for those situations where the values of coalitions are not known with certainty. Recent works address situations where the values of coalitions are modelled by random variables. In this work we still consider the values of coalitions as uncertain, but model them as *unknown but bounded* disturbances. We do not focus on solving a specific game, but rather consider a family of games described by a polyhedron: each point in the polyhedron is a vector of coalitions' values and corresponds to a specific game. We consider a dynamic context where while we know with certainty the average value of each coalition on the long run, at each time such a value is unknown and fluctuates within the bounded polyhedron. Then, it makes sense to define "robust" allocation rules, i.e., allocation rules that bound, within a pre-defined threshold, a so-called complaint vector while guaranteeing a certain average (over time) allocation vector. We also present as motivating example a joint replenishment application.

Key words: cooperative games; dynamic games; joint replenishment.

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1 Introduction.

Classical cooperative game theory is no longer a suitable tool for those situations where the values of coalitions are not known with certainty. Recent works address situations where the values of coalitions are modelled by random variables (see, e.g., [8, 9, 11, 12]. In this work we also consider the values of coalitions as uncertain, but model them as *unknown but bounded* disturbances as in [1]. We do not focus on solving a specific game, but rather consider a family of games described by a polyhedron: each point in the polyhedron is a vector of coalitions' values and corresponds to a specific game. In doing this, we revisit the notion of the core with reference to a family of balanced games described in polyhedral form.

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The main contribution of this paper is the uncertain and dynamic framework within which we cast the cooperative game. Dynamic cooperative games dealt with in [2, 4] model the evolution of the coalitions' values given the current value and the allocated revenues. Differently, in this work the average value of each coalition over time, henceforth called *average coalitions' value*, is known with certainty but the instantaneous value is unknown and fluctuates within a bounded polyhedron. Actually, knowing the average coalitions' value is more realistic than knowing the fluctuations of the instantaneous values as the latter are much more subject to randomness. At each time, to each coalition a certain revenue is allocated which, in general, will not meet the actual instantaneous value of that coalition. A "complaint" vector stores the disagreement between the instantaneous value of each coalition and the sum of the allocated revenues to all its players. The complaint is then the state variable describing the past history of the dynamic system considered. Under the assumption that the only information available at each time is the complaint of the coalitions, it makes sense to define "robust" allocation rules, i.e., allocation rules that i) keep the complaint vector bounded within a pre-defined threshold ε at each time (we will refer to such rules as ε-stabilizing), while ii) guaranteeing a certain average allocation vector over time. Obviously, the average allocation vector must meet the average coalitions' values (we will also say that it must satisfy the *average constraints*). Summarizing, the problem of interest consists in finding ε-stabilizing allocation rules meeting the average constraints.

A further contribution of this work is a constructive method to solve the aforementioned problem. The dynamic allocation rule that we present allocates the revenues according to the values assumed by an opportunely designed augmented state variable. Such a state variable models the complaint level of each coalition combined with the deviation of the instantaneous allocation from the pre-defined average allocation of each coalition. With the given augmented state variable the problem reduces to simply finding an ε-stabilizing allocation rule for the augmented dynamic system. Actually, as it will be clearer later on, ε-stabilizing the augmented system implies both ε-stabilizing the complaint vector and meeting the average constraints.

This paper is organized as follows. In Section 2, we describe the joint replenishment application which motivates our research. Furthermore, we introduce the polyhedral description for a family of balanced games. In Section 3, we describe the system of interest and formulate the problem. In Section 4, we present the constructive method to find the dynamic allocation rule, solution of Problem 1. In Section 5, we present a numerical example. Finally, in Section 6 we deal with the Shapley value and the associated linear allocation rule.

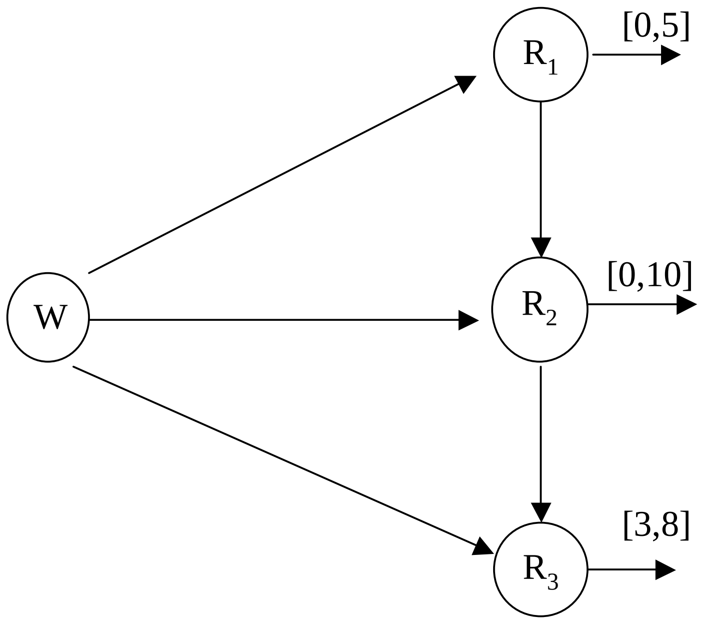

Figure 1: Example of a one warehouse W and three retailers R_1, R_2 and R_3. R_1 faces a demand in the interval $[0, 5]$, R_2 in the interval $[0, 10]$, and R_3 in the interval $[3, 8]$.

2 Joint replenishment: motivation and polyhedral description.

Consider a single-period one-warehouse multi-retailer inventory system (see, e.g., [3, 5, 6]. Fig. 1 displays a warehouse W serving three retailers R_1, R_2 and R_3. Each retailer faces a demand bounded by a minimum and a maximum value. For instance R_1 faces a demand d_1 in the interval $[d_1^-, d_1^+] = [0, 5]$, R_2 faces a demand d_2 in the interval $[d_2^-, d_2^+] = [0, 10]$, and R_3 faces a demand d_3 in the interval $[d_3^-, d_3^+] = [3, 8]$.

The retailers do not hold any private inventory, therefore to fulfill the demand, they must reorder from the central warehouse at a fixed transportation cost. Let this cost be $K = 7$ in the current example. The value of d_i is known by retailer R_i who selects his best decision regarding whether to reorder or not. In case of joint replenishment retailers are served by a single truck and share the transportation cost as established by the warehouse holder. The warehouse holder must in turn choose how to allocate costs among retailers based on the only knowledge of their demand intervals $[d_i^-, d_i^+]$. In other words, from the warehouse holder standpoint, demand d_i is unknown but bounded. Also, the decision on cost allocation takes place before demand is realized. Because, after demand is realized, the retailers will place their orders or not and from this the warehouse may deduce information about the demand d_i. We wish to find allocation rules that let all retailers benefit from joint replenishment at least on the long run.

We model the problem as a cooperative inventory game in coalitional form. In the example we have a set of three players $N = \{1, 2, 3\}$, namely the three retailers. If player i plays alone, the cost of reordering coincides with the full transportation cost (a single truck serves him alone) whereas the cost of not reordering is the cost of unfulfilled demand, that is, lost demand. Assume the latter cost is one unit per unit of unfulfilled demand.

To be more specific, for retailer R_1 in the example, the cost of reordering is $K = 7$ whereas the cost of not reordering varies in the bounded range $[0, 5]$. For R_1 the best decision is "no reordering" independently of the realization of d_1 and the associated cost $c(\{1\}) \in \left[\min\{d_1^-, K\}, \min\{d_1^+, K\}\right] = [0, 5]$.

For R_2 the best decision is "reordering" if d_2 is between 7 and 10 and "no reordering" if d_2 is between 0 and 7. In the first case he reorders and pays K, whereas in the second case he does not reorder and pays d_2. Then $c(\{2\}) \in \left[\min\{d_2^-, K\}, \min\{d_2^+, K\}\right] = [0, 7]$. Note that for $d_2 = 7$ the incurred cost is 7 independently of the decision.

If two players form a coalition they are forced to select a joint decision ("both reorder" or "both do not reorder"). The cost of reordering for the coalition is still the total transportation cost which, this time, must be shared between the two players. The cost of not reordering is the sum of the unfulfilled demands of both players. For instance, the cost of coalition $S = \{1, 2\}$ is $c(\{1, 2\}) \in \left[\min\{(d_1^- + d_2^-), K\}, \min\{(d_1^+ + d_2^+), K\}\right] = [0, 7]$.

In general, denote by $\mathbb{R}^+$ the set of nonnegative reals and let $d_i \in \mathbb{R}^+$ be the unknown but bounded demand faced by retailer i and varying between d_i^- and d_i^+, that is $d_i \in [d_i^-, d_i^+]$. Let $N = \{1, \ldots, n\}$ be the set of players and apply the same reasoning as above to compute the cost of all subcoalitions of N except for the empty set $\emptyset$. Henceforth, for the sake of notation, the inclusion $S \subseteq N$ means "all subcoalitions of N except the empty set $\emptyset$". For any given coalition $S \subseteq N$ it holds

$$\min(K, \sum_{i \in S} d_i^-) \leq c(S) \leq \min(K, \sum_{i \in S} d_i^+). \tag{1}$$

Hence, the joint replenishment model results in a family of cost-games $< N, \mathcal{C} >$ where $\mathcal{C}$ is a polyhedron defined by

$$\mathcal{C} = \left\{ c \in \mathbb{R}^{2^n - 1} : \min(K, \sum_{i \in S} d_i^-) \leq c(S) \leq \min(K, \sum_{i \in S} d_i^+), \quad \text{for all } S \subseteq N \right\}.$$

Observe that each point $c \in \mathcal{C}$ is a vector of coalitions' costs and corresponds to a specific cost-game.

Given the cost of a coalition S as in (1), we can compute the cost savings $v(S)$ of this coalition.

$$\begin{aligned} v(S) &= \sum_{i\in S} c(\{i\}) - c(S) & (2)\\ &\in \sum_{i\in S}\left[\min(K,d_i^-),\min(K,d_i^+)\right] - \left[\min(K,\sum_{i\in S} d_i^-),\min(K,\sum_{i\in S} d_i^+)\right] & (3)\\ &= \left[\sum_{i\in S}\min(K,d_i^-) - \min(K,\sum_{i\in S} d_i^-), \sum_{i\in S}\min(K,d_i^+) - \min(K,\sum_{i\in S} d_i^+)\right]. & (4)\end{aligned}$$

For example, the cost savings of coalition $S = \{1,2\}$ in the example are $v(\{1,2\}) = c(\{1\}) + c(\{2\}) - c(\{1,2\}) \in [0,5]$.

It turns out that the cost savings, or value, of each coalition is bounded by a minimum and a maximum value, i.e., $v_{min}(S) \leq v(S) \leq v_{max}(S)$ with fixed bounds $v_{min}(S)$ and $v_{max}(S)$. Thus, the family of cost games implies a family of cost-saving games $< N, \mathcal{V} >$ with polyhedron

$$\mathcal{V} = \left\{v \in \mathbb{R}^{2^n-1} : v_{min}(S) \leq v(S) \leq v_{max}(S), \text{for all } c \in \mathcal{C} \text{ and } S \subseteq N\right\} \quad (5)$$

and each value $v(S)$ is computed according to (2).

We are now in a position to show that the family of games (5) satisfies an interesting property. Namely, each vector $v \in \mathcal{V}$ corresponds to a balanced game, or in other words, (5) is a family of balanced games. For this, denote by 2^N the family of subsets of N. We recall the definition of a balanced map and a balanced game (see, e.g., [10, Def. 11.5]).

Definition 1 *(Balanced map) A map* $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}^+$ *is called a* balanced map *if*

$$\sum_{S\subseteq N} \lambda(S) e^S = e^N.$$

Here, $e^S \in \mathbb{R}^n$ is the *characteristic vector* for coalition S with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \in N \setminus S$.

Definition 2 *(Balanced game) An n-person game* $< N, v >$ *game is called a* balanced game *if for each balanced map* $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}^+$,

$$\sum_{S\subseteq N} \lambda(S) v(S) \leq v(N). \quad (6)$$

With the above definitions in mind, we can now prove that each point of the polyhedron (5) is corresponds to a balanced game. To do this, let $S_a^+ = \{i \in S : d_i^+ < K\}$ and $S_b^+ = \{i \in S : d_i^+ \geq K\}$ for all subsets S of N (similarly replace d_i^+ with d_i^- to define S_a^- and S_b^-). Now equation (4) can be rewritten as

$$v(S) \in [0,0] \quad \text{if } \textstyle\sum_{i\in S} d_i^+ < K \tag{7}$$

$$v(S) \in \left[0, \sum_{i\in S_a^+} d_i^+ + \left(|S_b^+|-1\right)K\right] \text{if} \sum_{i\in S} d_i^+ \geq K \text{ and } \sum_{i\in S} d_i^- < K \tag{8}$$

$$v(S) \in \left[\sum_{i\in S_a^-} d_i^- + \left(|S_b^-|-1\right)K, \sum_{i\in S_a^+} d_i^+ + \left(|S_b^+|-1\right)K\right] \text{if} \sum_{i\in S} d_i^- \geq K. \tag{9}$$

These expressions are used to prove the following Lemma.

Lemma 1 *The polyhedron (5) describes a family of balanced games.*

Proof We prove that condition (6) holds for $v(S)$ as in (7)-(9). Observe that the latter equations include the generic term

$$\sum_{i\in S_a} d_i + (|S_b|-1)\,K.$$

Also note that condition (6) holds true if the above generic term satisfies

$$\sum_{i\in N_a} d_i + (|N_b|-1)K \geq \sum_{S\subseteq N} \lambda(S)\left[\sum_{i\in S_a} d_i + (|S_b|-1)K\right].$$

After manipulating the right-hand side, the above condition can be rewritten as

$$\sum_{i\in N_a} d_i + (|N_b|-1)K \geq \sum_{i\in N_a} d_i \underbrace{\sum_{S\subseteq N} \lambda(S)e_i^S}_{=e_i^N} + \sum_{S\subseteq N} \lambda(S)(|S_b|-1)K.$$

Now, observe that the first term in the left-hand side is equal to the first term in the right-hand. Then it suffices to prove that

$$(|N_b|-1)K \geq \sum_{S\subseteq N} \lambda(S)(|S_b|-1)K.$$

But this is easy to see, as the right-hand side can be rewritten as

$$\begin{aligned}
\sum_{S\subseteq N} \lambda(S)(|S_b|-1)K &= \sum_{S\subseteq N} \lambda(S)\left[e^{N_b^T} e^S - 1\right]K \\
&= e^{N_b^T} \sum_{S\subseteq N} \lambda(S)e^S K - \sum_{S\subseteq N} \lambda(S)K \\
&= e^{N_b^T} e^N K - \sum_{S\subseteq N} \lambda(S)K \\
&= \left(|N_b| - \sum_{S\subseteq N} \lambda(S)\right)K \\
&\leq (|N_b|-1)K
\end{aligned}$$

where e^{N_b} is the characteristic vector of N_b and $e^{N_b^T}$ is its transpose. $\square$

Let us generalize what we have done for the joint replenishment example and introduce a polyhedral description for addressing families of balanced games. The underlying idea is to use a polyhedron to describe the infinite set of admissible coalitions' values. The last part of this section addresses the notions of the core and an allocation rule. In doing so, we point out that any allocation rule in the core can be obtained by solving a set of linear equalitions.

Define a *family of games* $< N, \mathcal{V} >$ as the set of games $< N, v >$ obtained when v varies within a polyhedron

$$\mathcal{V} = \left\{ v \in \mathbb{R}^{2^n - 1} : v_{min}(S) \leq v(S) \leq v_{max}(S), \quad \text{for all } S \subseteq N \right\},$$

where the bounds $v_{min}(S)$ and $v_{max}(S)$ are given. For sake of simplicity, throughout this paper we always assume $v \geq 0$. Also, let $n = |N|$, where $|X|$ is the cardinality of the set X, and let $m = 2^n - 1$.

Define a *family of balanced games* $< N, \mathcal{V}_b >$ as the set of games $< N, v >$ obtained when v varies within a polyhedron

$$\mathcal{V}_b = \{ v \in \mathcal{V} : \text{condition (6) holds} \}.$$

Now, let us revisit the notions of the core and an allocation rule for the above family of balanced games in polyhedral form. Indicate with Δ^n the simplex in $\mathbb{R}^n$ and remind that a game is balanced if and only if the core is nonempty (see, e.g., [10, Theorem 11.7]). By definition each game $< N, v >$ with $v \in \mathcal{V}_b$ is balanced, and so the core $C(v)$,

$$C(v) = \{ a \in \mathbb{R}^n : \frac{a}{v(N)} \in \Delta^n, \sum_{i \in S} a_i \geq v(S) \, \text{for all } S \subseteq N \},$$

is nonempty. This means that there exists an allocation $a \in C(v)$ such that no coalition has an incentive to split off from the coalition N. Now, the idea is to find an allocation rule $a(v)$ such that for all games $v \in \mathcal{V}_b$ it holds that $a(v) \in C(v)$. To do this, first observe that the core is a convex set described by linear equations and inequalities. For our purpose it is useful to change all inequalities into equations. To do this, we first introduce a vector of nonnegative surplus variables $s = (s_1, \ldots, s_{m-1})^T$. Each surplus variable corresponds to a coalition of players and describes the difference between the allocated value and the coalitional value, $\sum_{i \in S} a_i - v(S)$. Notice that we only need $m-1$ surplus variables because $\sum_{i \in N} a_i = v(N)$ due to the efficiency condition of the core. Further, we introduce an incidence matrix $B \in \mathbb{R}^{m \times n}$ with the characteristic vectors e^S as rows, and an augmented matrix $A \in \mathbb{R}^{m \times n + (m-1)}$ defined by

$$A = \left[B \left| \begin{array}{c} -I \\ ----- \\ 0 \ldots 0 \end{array} \right. \right], \tag{10}$$

where I is the $(m-1)$-dimensional identity matrix. Finally, define a vector $u \in \mathbb{R}^{n+(m-1)}$, henceforth called *allocation vector*, by

$$u = \begin{bmatrix} a \\ s \end{bmatrix}.$$

Now, for all $v \in \mathcal{V}_b$ and for any vector $a \in C(v)$, there is an allocation vector $u \in \mathbb{R}^{n+m-1}$ that satisfies the following set of linear equations

$$Au = v, \tag{11}$$

$$u \geq 0. \tag{12}$$

For instance, if $n = 3$ condition (11) becomes

$$\underbrace{\left[\begin{array}{ccc|cccccc} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{A} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ s_1 \\ s_2 \\ s_3 \\ s_{12} \\ s_{13} \\ s_{23} \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} v(\{1\}) \\ v(\{2\}) \\ v(\{3\}) \\ v(\{1,2\}) \\ v(\{1,3\}) \\ v(\{2,3\}) \\ v(N) \end{bmatrix}}_{v}.$$

Let $\mathcal{U}$, be the set of solutions satisfying (11)-(12) and observe that, in general, $\mathcal{U}$ is a polyhedron of dimension $n-1$. This means that finding an allocation vector in the core reduces to solving the set of equalities (11)-(12) for u. Notice that two different solutions of (11) describe different allocations of $v(N)$ among the players, which in turn means a higher satisfaction for those coalitions with larger surplus variables.

3 Dynamic system.

In this Section we consider a dynamic context where at each time the vector of coalitions' values fluctuates within a bounded polyhedron. While such fluctuations are unknown, it is realistic to assume that we know with certainty the average value of each coalition on the long run. The problem of interest, formulated at the end of this section, consists in finding "robust" allocation rules that bound a so-called complaint vector and guarantee a certain average allocation vector.

Consider, the dynamic system

$$x(k+1) = x(k) + Au(k) - v(k), \quad v(k) \in \mathcal{V}_b, \quad k = 1, 2, \ldots \tag{13}$$

in which the state variable $x(k)$ describes the complaint levels of all subcoalitions at time k, i.e., the disagreement between the sum of the allocated revenues to the players of the coalitions, expressed by the product $Au(k)$, and the vector of coalitions' values $v(k)$. The condition $u(k) \geq 0$ is omitted for sake of notation. More generally, system (13) also holds if the coalitions' values change slowly in comparison to the rate at which revenues are allocated. In that case, it suffices to redefine the vector of coalitions' values by

$$v(k) = v(t)\Theta,\ t = \lfloor k\Theta \rfloor,\ v(t) \in \mathcal{V}_b,\ k = 1, 2, \ldots, \tag{14}$$

where $\lfloor y \rfloor$ is the greatest integer less than or equal to y and the sample interval Θ with $0 \leq \Theta \leq 1$ describes the length of the time interval between two successive allocations. Hence, if $v(t)$ is the vector of coalitions' values in the interval $(t-1, t]$, then $v(k) = v(t)\Theta$ is the portion of value in the time interval $[k\Theta, (k+1)\Theta]$ for $k = \frac{t-1}{\Theta}, \ldots, \frac{t}{\Theta} - 1$. A dynamic system of type (13) suggests the following interpretation (see, e.g., multi-inventory systems in [1]): $x(k)$ is a vector whose components are the buffer levels (each buffer level describes the level of complaint of a single coalition), $u(k)$ is the controlled flow vector, A is the controlled process matrix, and $v(k)$ is an (uncontrolled) exogenous input, typically modelling demand.

Now, given a vector function $y : \mathbb{R}^+ \longrightarrow \mathbb{R}^m$, we define the *average* of y by

$$\bar{y} = \lim_{T \longrightarrow \infty} \frac{1}{T} \sum_{t=0}^{T} y(t).$$

If we assume that one knows the average value of each coalition, the following lemma recalls a result obtained in [1].

Lemma 2 *(Average constraint) Consider an average vector of coalitions' values $\bar{v}$ and a pre-defined average allocation vector $\bar{u}$ such that*

$$A\bar{u} = \bar{v}.$$

There exists an allocation rule $f : \mathbb{R}^m \longrightarrow \mathbb{R}^{n+(m-1)}$ such that for $u(k) = f(v(k))$, with $v(k)$ as in (14),

$$Au(k) = v(k)$$

and whenever the average coalitions' value tends to $\bar{v}$, then the average allocation vector tends to $\bar{u}$, if and only if there exists a matrix $D \in \mathbb{R}^{n+(m-1)\times m}$ that satisfies

$$AD = I \in \mathbb{R}^{m\times m} \tag{15}$$

$$Dv + \bar{u} \geq 0 \quad \forall v \in \mathcal{V}_b. \tag{16}$$

The allocation rule is linear on v, that is

$$u(k) = \bar{u} + Dv(k). \tag{17}$$

Observe that the linear allocation rule (17) requires perfect knowledge of the coalition values at each sample time. Differently, consider the case where revenues at time k must be allocated without a-priori knowledge of the coalitions' values $v(k)$. We are interested in finding dynamic allocation rules that keep the complaint vector bounded within a pre-specified threshold while satisfying the condition that if the average coalitions's value is $\bar{v}$ then the average allocation is $\bar{u}$. For this we need the following definition, see [1]. For $\xi \in \mathbb{R}^m$, let ξ_i denote the ith component of ξ, and define

$$|\xi| = \max_i |\xi_i|.$$

Let $\mathbb{Z}$ denote the set of integers, and $\mathbb{Z}^+$ the set of nonnegative integers. Let $f = \{f(0), f(1), f(2), \ldots\}$ be any bounded one-sided sequence in $\mathbb{R}^m$, and define

$$\|f(k)\| = \sup_{k \in \mathbb{Z}^+} |f(k)|.$$

Our dynamic allocation rule is defined as follows.

Definition 3 *Given $\varepsilon > 0$ and a reference value $\bar{x}$ for system (13), an* ε-stabilizing allocation rule *is a feedback rule for which there exists a continuous positive function $\phi(k)$, monotonically decreasing and converging to 0 as $k \longrightarrow \infty$ such that for all $x(0)$, the following condition holds true*

$$\|x(k) - \bar{x}\| \leq \max\{\|x(0)\|\phi(k), \varepsilon\}.$$

The problem of interest can be stated as follows.

Problem 1 *Given an average vector of coalitions' values $\bar{v}$ and a pre-defined average allocation vector $\bar{u}$ such that $A\bar{u} = \bar{v}$, find an ε-stabilizing allocation rule such that whenever the average coalitions' value tends to $\bar{v}$ then the average allocation vector tends to $\bar{u}$.*

In the next section we present a dynamic allocation rule that solves Problem 1.

4 Dynamic Allocation Rule.

The dynamic allocation rule that we propose as a solution to Problem 1 allocates the revenues according to the values assumed by an opportunely designed augmented state variable. Such a state variable models the complaint level of each coalition combined with the deviation of the instantaneous allocation from the pre-defined average allocation of each coalition. With the given augmented state variable Problem 1 reduces to simply finding an ε-stabilizing allocation rule for the augmented dynamic system. Actually, as it will be clearer later on, ε-stabilizing the augmented system implies both ε-stabilizing the complaint vector and meeting the average constraints.

From a standard property of linear algebra, see also the appendix, we can find two matrices C and F which "square" A and D and satisfy

$$\begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} D & F \end{bmatrix} = I. \tag{18}$$

Consider the following augmented system

$$\begin{aligned} x(k+1) &= x(k) + Au(k) - v(k), \\ y(k+1) &= y(k) + Cu(k), \end{aligned} \tag{19}$$

where $v(k)$ is as in (14). The additional dynamic variable $y(k)$ keeps track of the deviation between the instantaneous and the average allocation of each player. Define the augmented state variable $z \in \mathbb{R}^{n+m-1}$ as

$$z(k) = \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}, \quad \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} z(k).$$

This variable satisfies the equation

$$\begin{aligned} z(k+1) &= \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k+1) \\ y(k+1) \end{bmatrix} \\ &= \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} u(k) - \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} v(k) \\ 0 \end{bmatrix} \\ &= z(k) + u(k) - Dv(k). \end{aligned} \tag{20}$$

This indicates that the allocation rule $u(k) = -z(k)$, which is linear in z, solves the problem.

Theorem 1 *Consider system (20) with $v(k)$ as in (14). The allocation rule in feedback form*

$$u(k) = -z(k) \tag{21}$$

is such that the following condition on z holds true

$$\|z(k)\| \leq \|Dv(k)\| \tag{22}$$

and if the average coalitions' value is $\bar{v}$ then the average allocation vector is $\bar{u}$.

Proof To prove (22) let us substitute the allocation rule (21) in the dynamics of (20). Then, we obtain $z((k+1)) = Dv(k)$ for all k, which implies (22). For the rest of the proof, by summing (20) for different $k = 1, 2, \ldots$, we have that

$$\frac{1}{T}\sum_{k=0}^{T-1} u(k) - \frac{1}{T}\sum_{k=0}^{T-1} Dv(k) = \frac{z(T) - z(0)}{T} \to 0$$

as $T \to \infty$ (actually the numerator is a finite quantity whereas the denominator tends to infinity). Therefore $\bar{u} = D\bar{v}$, which concludes the proof. $\square$

For fixed ε we wish to find the maximum time interval Θ^* such that $\|Dv(k)\| \leq \varepsilon$. Trivially, such a value is $\Theta^* = \frac{\varepsilon}{\delta}$ where $\delta = \max_{v \in \mathcal{V}_b} |Dv|$. Then we have the following corollary.

Corollary 1 *Consider system (20) with $v(k)$ as in (14). For any ε and corresponding Θ^*, if one chooses $\Theta \leq \min\{\Theta^*, 1\}$, then the allocation rule in feedback form*

$$u(k) = -z(k), \tag{23}$$

is ε-stabilizing.

Proof It is easy to show that

$$\|z(k)\| \leq \|Dv(t)\Theta\| \leq \|Dv(t)\Theta^*\| \leq \max_{v \in \mathcal{V}_b} |Dv\Theta^*| \leq \varepsilon.$$

$\square$

Remark 1 *A side effect of $\|z\| \leq \varepsilon$ is that also $\|u\| \leq \varepsilon$ as $u = -z$. This means that the smaller ε the smaller the maximum allocation (in magnitude).*

5 Numerical example.

We return to the example in Section 2. In short, there is a set of retailers $N = \{1, 2, 3\}$, a transportation cost $K = 7$, minimum and maximum demands for each retailer $d_1 \in [0, 5]$, $d_2 \in [0, 10]$ and $d_3 \in [3, 8]$. The family of cost games is

$$c(\{1\}) \in [0,5] \quad c(\{2\}) \in [0,7] \quad c(\{3\}) \in [3,7]$$
$$c(\{1,2\}) \in [0,7] \quad c(\{1,3\}) \in [3,7] \quad c(\{2,3\}) \in [3,7] \quad c(N) \in [3,7],$$

and the associated family of cost-saving games is defined by

$$v(\{1\}) = 0 \quad v(\{2\}) = 0 \quad v(\{3\}) = 0$$
$$v(\{1,2\}) \in [0,5] \quad v(\{1,3\}) \in [0,5] \quad v(\{2,3\}) \in [0,7] \quad v(N) \in [0,12].$$

Let the average vector of coalitions' values be

$$\bar{v} = [0, 0, 0, 2, 3, 4, 10]^T$$

and the pre-defined average allocation vector

$$\bar{u} = [3, 5, 2, 3, 5, 2, 6, 2, 3]^T.$$

Note that $A\bar{u} = \bar{v}$. As for the threshold for the complaint vector we choose $\varepsilon = 0.5$

First we calculate D (details on how to formulate a linear programming problem to find D are in [1]) and obtain

$$D = \begin{bmatrix} 0.22222 & -0.11111 & -0.11111 & 0.11111 & 0.11111 & -0.22222 & 0.33333 \\ -0.11111 & 0.22222 & -0.11111 & 0.13695 & -0.18346 & 0.16279 & 0.46253 \\ -0.11111 & -0.11111 & 0.22222 & -0.24806 & 0.072351 & 0.059432 & 0.20413 \\ -0.77778 & -0.11111 & -0.11111 & 0.11111 & 0.11111 & -0.22222 & 0.33333 \\ -0.11111 & -0.77778 & -0.11111 & 0.13695 & -0.18346 & 0.16279 & 0.46253 \\ -0.11111 & -0.11111 & -0.77778 & -0.24806 & 0.072351 & 0.059432 & 0.20413 \\ 0.11111 & 0.11111 & -0.22222 & -0.75194 & -0.072351 & -0.059432 & 0.79587 \\ 0.11111 & -0.22222 & 0.11111 & -0.13695 & -0.81654 & -0.16279 & 0.53747 \\ -0.22222 & 0.11111 & 0.11111 & -0.11111 & -0.11111 & -0.77778 & 0.66667 \end{bmatrix}.$$

Then we compute the matrices C and F that square B and D using the method explained in detail in appendix A. For the maximum sample time we get $\Theta^* > 0.1$ and choose $\Theta = 0.1$. It is left to implement the dynamic allocation rule (23) in feedback form to simulate the evolution of the augmented state variable $z(.)$. Fig. 2 displays all nine components of vector $z(.)$ (remind z and u have the same dimension). From the plot we can see that each component $z_i(.)$ always lies in between -0.5 and 0.5.

6 The Shapley value as linear allocation rule.

In this Section we study the Shapley value as a special linear allocation rule of the form (17). In particular, we show that there is a matrix Φ that satisfies (15).

The *Shapley value* ϕ, which is defined in [7], equals $\phi = \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m^\sigma$ where $\Pi(N)$ is the set of all permutations of N and m^σ is the marginal vector corresponding to the permutation σ.

Theorem 2 *The* Shapley value ϕ *is linear on* v*, i.e.,*

$$\phi = Lv, \tag{24}$$

where the matrix $L \in \mathbb{R}^{n\times m}$ *is defined by*

$$L_{ij} = \frac{1}{n!}\cdot\begin{cases} -s!(n-(s+1))! & \text{if } i \notin S \\ (s-1)!(n-s)! & \text{if } i \in S. \end{cases} \tag{25}$$

if column j *corresponds to coalition* S *with* $s = |S|$.

Proof The proof follows immediately from the definition of the Shapley value in [7]. □

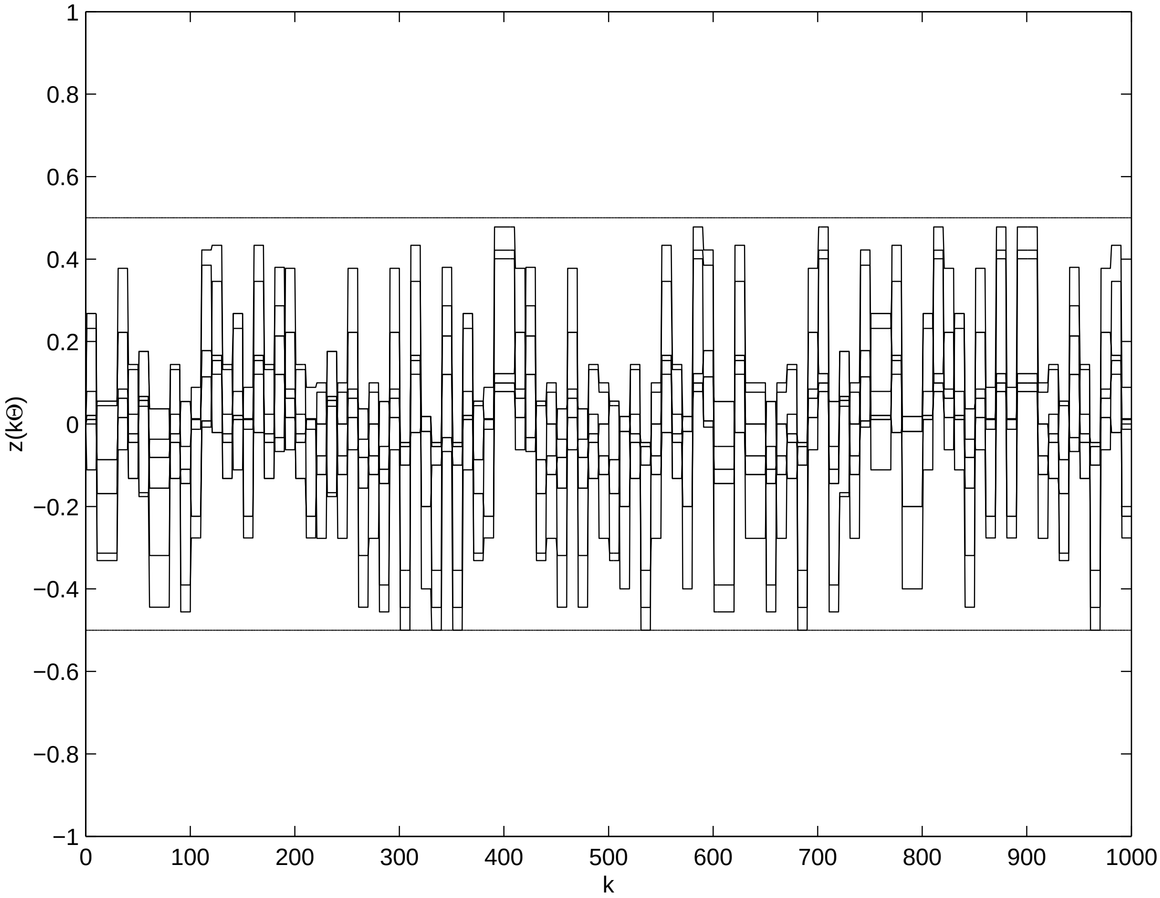

Figure 2: Time plot of the augmented state variable $z(.)$. Variable $z(.)$ is always comprised between -0.5 and 0.5.

For instance, if $n = 3$ we have

$$L = \frac{1}{6} \begin{bmatrix} 2 & -1 & -1 & 1 & 1 & -2 & 2 \\ -1 & 2 & -1 & 1 & -2 & 1 & 2 \\ -1 & -1 & 2 & -2 & 1 & 1 & 2 \end{bmatrix}. \tag{26}$$

Let $s(\phi)$ be the vector of surplus variables when revenues are allocated according to the Shapley value ϕ. The idea is now to express $s(\phi)$ linearly in v.

Theorem 3 *The vector of surplus variables is linear on v, i.e.,*

$$s(\phi) = Qv, \tag{27}$$

where $Q \in \mathbb{R}^{m-1 \times m}$ has row i associated to a surplus variable (a subcoalition $S \subset N$), column j associated to a subcoalition $M \subseteq N$, and generic ijth element

$$Q_{ij} = \begin{cases} \sum_{p \in S} L_{pj} & \text{if } i \neq j \\ \sum_{p \in S} L_{pj} - 1 & \text{if } i = j. \end{cases} \tag{28}$$

Proof First, consider the coalition of only player 1 and let $L_{i\bullet}$ be the generic ith row of L. The associated surplus variable is

$$s_1(\phi) = \phi_1 - v(\{1\}) = L_{1\bullet}v - v(\{1\}) = (L_{11} - 1)v(\{1\}) + L_{12}v(\{2\}) + \ldots + L_{1m}v(N).$$

The latter equation yields $Q_{1\bullet} = [(L_{11} - 1)\, L_{12} \ldots L_{1m}]$, which is in according with (28).

If we repeat the same reasoning for a generic subcoalition $M \subset N$, the surplus variable is

$$s_M(\phi) = \sum_{i \in M} \phi_i - v(M) = \sum_{i \in M} L_{i\bullet}v - v(M).$$

Remind j is the column associated to coalition M. Then, the latter equation yields $Q_{jk} = \sum_{i \in M} L_{ik}$ if $k \neq j$ and $Q_{jj} = \sum_{i \in M} L_{ij} - 1$ which is in accordance with (28). □

An example of matrix Q for the case $n = 3$, is the following one (see, e.g., matrix L in (26) for the case $n = 3$)

$$Q = \frac{1}{6} \begin{bmatrix} -4 & -1 & -1 & 1 & 1 & -2 & 2 \\ -1 & -4 & -1 & 1 & -2 & 1 & 2 \\ -1 & -1 & -4 & -2 & 1 & 1 & 2 \\ -1 & -1 & -2 & -4 & -1 & -1 & 4 \\ -1 & -2 & -1 & -1 & -4 & -1 & 4 \\ -2 & -1 & -1 & -1 & -1 & -4 & 4 \end{bmatrix}. \tag{29}$$

Using the fact that ϕ and $s(\phi)$ are linear functions of v, we define the allocation vector associated to the Shapley value by

$$u(\phi) = \begin{bmatrix} \phi \\ s(\phi) \end{bmatrix}.$$

Corollary 2 *There exists a matrix* $\Phi \in \mathbb{R}^{n+m-1\times m}$*, defined by* $\Phi = \begin{bmatrix} L \\ Q \end{bmatrix}$ *such that*

$$u(\phi) = \Phi v.$$

Furthermore Φ is a pseudo inverse of A, i.e.,

$$A\Phi = I. \tag{30}$$

Proof For the first part, from Theorem 2 and 3 we have

$$\begin{bmatrix} \phi \\ s(\phi) \end{bmatrix} = \begin{bmatrix} L \\ Q \end{bmatrix} v$$

which finishes this part.

Let us now prove that $A\Phi = I$. It suffices to show that $A_{i\bullet}\Phi_{\bullet j} = 1$ if $i = j$ and zero otherwise. By observing that the generic row i of A, denoted by $A_{i\bullet} \in \mathbb{R}^{1\times n+(m-1)}$, is associated to a generic subcoalition $M \subseteq N$, whereas the generic column j of Φ, denoted $\Phi_{\bullet j} \in \mathbb{R}^{n+(m-1)\times 1}$, is associated to a generic subcoalition $S \subseteq N$, the condition $i = j$ corresponds to $M = S$.

Returning to $A_{i\bullet}$, it has the first n elements associated to players $p = 1 \dots n$, and the last $m-1$ elements associated to all subcoalitions $R \subset N$ (recall the structure of A described in (10)). Using index p to scan all players and R to scan all subcoalitions, we can sketch its structure as follows

$$A_{i\bullet} = [\dots \underbrace{1}_{\forall p \in M} \dots\dots \underbrace{0}_{\forall p \notin M} \dots\dots \underbrace{-1}_{R=M} \dots\dots \underbrace{0}_{\forall R \neq M} \dots]. \tag{31}$$

Analogously, $\Phi_{\bullet j}$ has the first n elements associated to players $p = 1 \dots n$, and the last $m-1$ elements associated to all subcoalitions $R \subset N$ (remind the structure of Q and L from (25) and (28)).

Now, keeping in mind the structures of $A_{i\bullet}$ in (31), if $i = j$, namely, $M = S$ we have

$$A_{i\bullet}\Phi_{\bullet j} = \sum_{p\in S} L_{pj} - (\sum_{p\in S} L_{pj} - 1) = 1.$$

On the contrary if $i \neq j$, namely, $M \neq S$, we have

$$A_{i\bullet}\Phi_{\bullet j} = \frac{1}{n!}[\sum_{p\in M} L_{pj} - \sum_{p\in M} L_{pj}] = 0.$$

□

If $n = 3$, putting together (26) and (29), we have

$$\Phi = \frac{1}{6} \begin{bmatrix} 2 & -1 & -1 & 1 & 1 & -2 & 2 \\ -1 & 2 & -1 & 1 & -2 & 1 & 2 \\ -1 & -1 & 2 & -2 & 1 & 1 & 2 \\ -- & -- & -- & -- & -- & -- & -- \\ -4 & -1 & -1 & 1 & 1 & -2 & 2 \\ -1 & -4 & -1 & 1 & -2 & 1 & 2 \\ -1 & -1 & -4 & -2 & 1 & 1 & 2 \\ -1 & -1 & -2 & -4 & -1 & -1 & 4 \\ -1 & -2 & -1 & -1 & -4 & -1 & 4 \\ -2 & -1 & -1 & -1 & -1 & -4 & 4 \end{bmatrix}.$$

7 Conclusions.

Inspired by a joint replenishment application, we have considered a dynamic cooperative game where while we know with certainty the average value of each coalition on the long run, at each time such a value is unknown and fluctuates within a bounded polyhedron. In this context, we have presented a constructive method to find "robust" allocation rules, i.e., allocation rules that bound, within a pre-defined threshold, a so-called complaint vector while guaranteeing a certain average allocation vector.

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A Computation of C and F.

We briefly show how to compute C and F given A and D. To simplify notation let $n+m-1=r$. With this aim, note that the following conditions must hold:

$$AD = I \tag{32}$$

$$AF = 0 \tag{33}$$

$$CD = 0 \tag{34}$$

$$CF = I. \tag{35}$$

Let us now rewrite the matrices A, C, D and F as follows.

- $A = [\ A_0\ A_1\ ]$ where A_0 is a $m \times r-m$ matrix and A_1 is an $m \times m$ non singular matrix.
- $C = [\ C_0\ C_1\ ]$ where C_0 is a $r-m \times r-m$ matrix and C_1 is an $r-m \times m$ matrix.
- $D = \begin{bmatrix} D_0 \\ D_1 \end{bmatrix}$ where D_0 is an $r-m \times m$ matrix and D_1 is an $m \times m$ non singular matrix.
- $F = \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}$ where F_0 is a $r-m \times r-m$ matrix and F_1 is an $m \times r-m$ matrix.

Now, we derive the following relationships among the different components of the above matrices:

- from (33), we obtain $A_1F_1 = -A_0F_0$. Whence $F_1 = -A_1^{-1}A_0F_0$;
- from (34), we obtain $D_1C_1 = -C_0D_0$. Whence $C_1 = -C_0D_0D_1^{-1}$;
- from (35), we obtain $C_0F_0 = I - C_1F_1 = I + C_0D_0D_1^{-1}F_1 = I - C_0D_0D_1^{-1}A_1^{-1}A_0F_0$.

Imposing, e.g., $C_0 = I$ we have $F_0 = (I + D_0D_1^{-1}A_1^{-1}A_0)^{-1}$. Consequently,

$$C = [\ I \mid -D_0D_1^{-1}\ ]$$

$$F = \begin{bmatrix} (I + D_0D_1^{-1}A_1^{-1}A_0)^{-1} \\ -A_1^{-1}A_0(I + D_0D_1^{-1}A_1^{-1}A_0)^{-1} \end{bmatrix}.$$