Fibonacci-like Differential Equations with a Polynomial Non-Homogeneous Part

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Abstract – We investigate non-homogeneous linear differential equations of the form $x''(t) + x'(t) - x(t) = p(t)$ where $p(t)$ is either a polynomial or a factorial polynomial in t. We express the solution of these differential equations in terms of the coefficients of $p(t)$, in the initial conditions, and in the solution of the corresponding homogeneous differential equation $y''(t) + y'(t) - y(t) = 0$ with $y(0) = y'(0) = 1$.

1. Introduction

In [1] and [2] we studied difference equations of the form

$$G_n = G_{n-1} + G_{n-2} + p(n) \tag{1}$$

where $G_0 = G_1 = 1$ and $p(n)$ is either a (ordinary or power) polynomial [1] or a factorial polynomial [2], i.e.,

$$p(n) = \sum_{j=0}^{k} \alpha_j n^j \quad \text{or} \quad p(n) = \sum_{j=0}^{k} \alpha_j n^{(j)} \tag{2}$$

respectively, where $n^{(j)} = n(n-1)(n-2)...(n-j+1)$ for $j \geq 1$ and $n^{(0)} = 1$. The main results established in [1] and [2] provide expressions for the solution of (1) in terms of the coefficients $\alpha_1, ..., \alpha_k$ and in the Fibonacci numbers F_n, i.e., in the solution of the homogeneous difference equation

$$F_n = F_{n-1} + F_{n-2} \tag{3}$$

where $F_0 = F_1 = 1$.

In this note we derive similar expressions for the family of differential equations corresponding to (1) and (2), viz. we consider differential equations of the form

$$x''(t) + x'(t) - x(t) = p(t) \tag{4}$$

where $x(0) = c$, $x'(0) = d$,

$$p(t) = \sum_{j=0}^{k} \alpha_j t^j \quad \text{or} \quad p(t) = \sum_{j=0}^{k} \alpha_j t^{(j)}$$

and we express the solution of (4) in terms of the coefficients $\alpha_1, ..., \alpha_k$ and in the solution of the homogeneous differential equation corresponding to (3), i.e., the solution of

$$y''(t) + y'(t) - y(t) = 0 \tag{5}$$

where $y(0) = y'(0) = 1$.

Essential in our approach is the following proposition in which $p(t)$ now need not be a (factorial) polynomial at all; it may be an arbitrary function which, however, gives rise to a particular solution $x_p(t)$ of (4).

Proposition 1.1. *Let $x_p(t)$ be a particular solution of (4) and let $y(t)$ be the solution of (5) where $y(0) = y'(0) = 1$. If $x(0) = c$ and $x'(0) = d$, then the solution of (4) can be expressed as*

$$x(t) = (d - x_p'(0))y(t) + (c - d - x_p(0) + x_p'(0))y'(t) + x_p(t). \tag{6}$$

Proof: Using standard methods (cf. e.g. [3]) it is straightforward to show that the solution $x_h(t)$ of the homogeneous equation corresponding to (4) equals

$$x_h(t) = C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t)$$

where $\lambda_1 = -\frac{1}{2}(1+\sqrt{5})$ and $\lambda_2 = -\frac{1}{2}(1-\sqrt{5})$.

Determining C_1 and C_2 from $x(0) = c$, $x'(0) = d$,

$$x(t) = x_h(t) + x_p(t) = C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) + x_p(t)$$

and

$$x'(t) = \lambda_1 C_1 \exp(\lambda_1 t) + \lambda_2 C_2 \exp(\lambda_2 t) + x_p'(t)$$

yields

$$C_1 = (\lambda_2 (c - x_p(0)) - (d - x_p'(0)))(\sqrt{5})^{-1},$$

$$C_2 = -(\lambda_1 (c - x_p(0)) - (d - x_p'(0)))(\sqrt{5})^{-1}.$$

Thus

$$x(t) = (\lambda_2 (c - x_p(0)) - (d - x_p'(0)))(\sqrt{5})^{-1} \exp(\lambda_1 t) + \\ - (\lambda_1 (c - x_p(0)) - (d - x_p'(0)))(\sqrt{5})^{-1} \exp(\lambda_2 t) + x_p(t). \qquad (7)$$

Similarly, we have that

$$y(t) = (\lambda_2 - 1)(\sqrt{5})^{-1} \exp(\lambda_1 t) - (\lambda_1 - 1)(\sqrt{5})^{-1} \exp(\lambda_2 t)$$

and hence

$$y'(t) = \lambda_2 (\sqrt{5})^{-1} \exp(\lambda_1 t) - \lambda_1 (\sqrt{5})^{-1} \exp(\lambda_2 t), \qquad (8)$$

$$y''(t) = -(\sqrt{5})^{-1} \exp(\lambda_1 t) + (\sqrt{5})^{-1} \exp(\lambda_2 t) \qquad (9)$$

since $\lambda_1 \lambda_2 = -1$, $\lambda_1 + 1 = -\lambda_2$ and $\lambda_2 + 1 = -\lambda_1$.

From (7), (8) and (9) it follows that

$$x(t) = x_h(t) + x_p(t) = (c - x_p(0)) y'(t) + (d - x_p'(0)) y''(t) + x_p(t). \qquad (10)$$

Substituting $y(t) - y'(t)$ for $y''(t)$ in (10) now yields the desired equality. □

Expressions slightly different from (6) are easily obtained: we already saw (10) which also yields

$$x(t) = (c - x_p(0)) y(t) - (c - d - x_p(0) + x_p'(0)) y''(t) + x_p(t)$$

if we replace $y'(t)$ by $y(t) - y''(t)$.

Note that in all these expressions the coefficients of y, y' and y'' are linear combinations of $c - x_p(0)$ and $d - x_p'(0)$.

From Proposition 1.1 it is clear that we now need a particular solution of (4). As in [1] and [2] we distinguish two cases, viz. $p(t)$ is a polynomial (Section 2) and $p(t)$ is a factorial polynomial (Section 3).

2. Polynomials

Throughout this section we assume that $p(t)$ is an ordinary or power polynomial

$$p(t) = \sum_{j=0}^{k} \alpha_j t^j .$$

As a particular solution of (4) we try

$$x_p(t) = \sum_{i=0}^{k} A_i t^i$$

which yields

$$\sum_{i=0}^{k-2} A_{i+2}(i+1)(i+2)t^i + \sum_{i=0}^{k-1} A_{i+1}(i+1)t^i - \sum_{i=0}^{k} A_i t^i = \sum_{i=0}^{k} \alpha_i t^i .$$

From comparing the coefficients of t^i it follows that

$$\begin{aligned} A_k &= -\alpha_k \\ A_{k-1} &= -\alpha_{k-1} - k\,\alpha_k \\ A_i &= -\alpha_i + (i+1)(i+2)A_{i+2} + (i+1)A_{i+1} \qquad \text{for } 0 \le i \le k-2. \end{aligned}$$

Thus we can successively compute $A_k, A_{k-1}, \ldots, A_0$: A_i is a linear combination of $\alpha_i, \ldots, \alpha_k$. Therefore we write

$$A_i = -\sum_{j=i}^{k} a_{ij}\alpha_j$$

(cf. [1] and [2]) which gives

$$-\sum_{j=i}^{k} a_{ij}\alpha_j = -\alpha_i - (i+1)(i+2)\sum_{j=i+2}^{k} a_{i+2,j}\alpha_j - (i+1)\sum_{j=i+1}^{k} a_{i+1,j}\alpha_j .$$

Comparing the coefficients of α_j yields the following difference equation for each j ($1 \le j \le k$):

$$a_{ij} = (i+1)(i+2)a_{i+2,j} + (i+1)a_{i+1,j} \qquad \text{for } j-i \ge 2 \tag{11}$$

where $a_{jj} = 1$ and $a_{j-1,j} = j$. Substituting $b_{j-i} = j^{-(j-i)}a_{ij}$ – where $j^{-(n)}$ means $(j^{(n)})^{-1}$ – in (11) yields

$$\begin{aligned} j^{(j-i)}b_{j-i} &= (i+1)(i+2)j^{(j-i-2)}b_{j-i-2} + (i+1)j^{(j-i-1)}b_{j-i-1} = \\ &= j^{(j-i)}(b_{j-i-2} + b_{j-i-1}) \end{aligned}$$

or

$$b_{j-i} = b_{j-i-2} + b_{j-i-1}$$

where $b_0 = b_1 = 1$. But this means that $b_{j-i} = F_{j-i}$ and hence

$$a_{ij} = j^{(j-i)}F_{j-i} \qquad \text{for } 0 \le i \le j \le k.$$

Consequently,

$$x_p(t) = -\sum_{i=0}^{k}\sum_{j=i}^{k} \alpha_j j^{(j-i)}F_{j-i}t^i = -\sum_{j=0}^{k} \alpha_j \left(\sum_{i=0}^{j} j^{(j-i)}F_{j-i}t^i \right)$$

which implies

$$x_p(0) = -\sum_{j=0}^{k} j!\,F_j\alpha_j ,$$

$$x_p'(t) = -\sum_{j=1}^{k} \alpha_j \left(\sum_{i=1}^{j} j^{(j-i)}F_{j-i}\,i\,t^{i-1} \right)$$

and

$$x_p'(0) = -\sum_{j=1}^{k} j\,!F_{j-1}\alpha_j$$

(Recall that $j^{(j)} = j\,!$).

These equalities together with Proposition 1.1 yield the following proposition.

Proposition 2.1. *The solution of (4) with* $x(0)=c,\ x'(0)=d$ *and*

$$p(t) = \sum_{j=0}^{k} \alpha_j t^j$$

can be expressed as

$$x(t) = (d + L_k)y(t) + (c - d + l_k)y'(t) - \sum_{j=0}^{k} \alpha_j p_j(t)$$

where L_k *and* l_k *are linear combinations of* $\alpha_0, \ldots, \alpha_k$, $y(t)$ *is the solution of (5) with* $y(0)=y'(0)=1$, *and for each* j $(0 \le j \le k)$, $p_j(t)$ *is a polynomial of degree* j:

$$L_k = \sum_{j=1}^{k} j\,!F_{j-1}\alpha_j, \quad l_k = \alpha_0 + \sum_{j=2}^{k} j\,!F_{j-2}\alpha_j, \quad p_j(t) = \sum_{i=0}^{j} j^{(j-i)}F_{j-i}t^i. \qquad \square$$

The polynomials $p_j(t)$ are given in Table 1 for $j = 0, 1, 2, ..., 9$.

j	$p_j(t)$
0	1
1	$t+1$
2	t^2+2t+4
3	$t^3+3t^2+12t+18$
4	$t^4+4t^3+24t^2+72t+120$
5	$t^5+5t^4+40t^3+180t^2+600t+960$
6	$t^6+6t^5+60t^4+360t^3+1800t^2+5760t+9360$
7	$t^7+7t^6+84t^5+630t^4+4200t^3+20160t^2+65520t+105840$
8	$t^8+8t^7+112t^6+1008t^5+8400t^4+53760t^3+262080t^2+846720t+$ $+1370880$
9	$t^9+9t^8+144t^7+1512t^6+15120t^5+120960t^4+786240t^3+3810240t^2+$ $+12337920t+19958400$

Table 1.

The coefficients of α_j in L_k and l_k are independent of k; cf. [1] and [2]. They give rise to two infinite sequences L and l of natural numbers (not mentioned in [4]) as k tends to infinity. The first few elements of these new sequences are

L: 0, 1, 2, 12, 72, 600, 5760, 65520, 846720, 12337920, ...
l: 1, 0, 2, 6, 48, 360, 3600, 40320, 524160, 7620480, ...

3. Factorial Polynomials

In order to try

$$x_p(t) = \sum_{i=0}^{k} A_i t^{(i)} \tag{12}$$

as a particular solution of (4) we first ought to determine the derivative of $t^{(n)}$.

Lemma 3.1.

$$\frac{dt^{(n)}}{dt} = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} t^{(k)} (-1)^{(n-k-1)}$$

Proof: The argument is by induction on n, the basis of which ($n=1$) is trivial. Suppose the equality holds for $n-1$:

$$\frac{dt^{(n-1)}}{dt} = \sum_{k=0}^{n-2} \begin{bmatrix} n-1 \\ k \end{bmatrix} t^{(k)} (-1)^{(n-k-2)}. \tag{13}$$

To perform the induction step, consider

$$dt^{(n)}/dt = d\,(t\,(t-1)^{(n-1)})/dt = (t-1)^{(n-1)} + t.d\,((t-1)^{(n-1)})/dt.$$

Now by the Chain Rule we have $d\,((t-1)^{(n-1)})/dt = d\,((t-1)^{(n-1)})/d\,(t-1)$. Applying the Binomial Theorem from [2] to $(t-1)^{(n-1)}$ and the induction hypothesis (13) yields

$$\frac{dt^{(n)}}{dt} = \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} t^{(k)} (-1)^{(n-k-1)} + t.\sum_{k=0}^{n-2} \begin{bmatrix} n-1 \\ k \end{bmatrix} (t-1)^{(k)} (-1)^{(n-k-2)} =$$

$$= (-1)^{(n-1)} + \sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} t^{(k)} (-1)^{(n-k-1)} + \sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} t^{(k)} (-1)^{(n-k-1)} =$$

$$= (-1)^{(n-1)} + \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} t^{(k)} (-1)^{(n-k-1)} = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} t^{(k)} (-1)^{(n-k-1)}$$

which completes the induction. □

From Lemma 3.1, (4) and (12) we obtain

$$\sum_{i=2}^{k} A_i \left(\sum_{m=1}^{i-1} \begin{bmatrix} i \\ m \end{bmatrix} \left(\sum_{l=0}^{m-1} \begin{bmatrix} m \\ l \end{bmatrix} t^{(l)} (-1)^{(m-l-1)} \right) (-1)^{(i-m-1)} \right) +$$

$$+ \sum_{i=1}^{k} A_i \left(\sum_{m=0}^{i-1} \begin{bmatrix} i \\ m \end{bmatrix} t^{(m)} (-1)^{(i-m-1)} \right) - \sum_{i=0}^{k} A_i t^{(i)} = \sum_{i=0}^{k} \alpha_i t^{(i)}.$$

Comparing the coefficients of $t^{(i)}$ yields

$$A_k = -\alpha_k$$

$$A_{k-1} = -\alpha_{k-1} + k\,\alpha_k$$

$$A_i = -\alpha_i + \sum_{n=i+1}^{k} A_n \begin{bmatrix} n \\ i \end{bmatrix} (-1)^{(n-i-1)} + \sum_{n=i+2}^{k} A_n \left(\sum_{m=i+1}^{n-1} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ i \end{bmatrix} (-1)^{(m-i-1)} (-1)^{(n-m-1)} \right)$$

for each i ($0 \le i \le k-2$).

As $(-x)^{(n)} = (-1)^n (x+n-1)^{(n)}$ and $n^{(n)} = n\,!$, this latter recurrence can be rewritten to

$$A_i = -\alpha_i + \sum_{n=i+1}^{k} A_n (-1)^{n-i-1} \frac{n^{(n-i)}}{n-i} + \sum_{n=i+2}^{k} A_n \left(\sum_{m=i+1}^{n-1} (-1)^{n-i-2} \frac{n^{(n-i)}}{(n-m)(m-i)} \right)$$

or

$$A_i = -\alpha_i + (i+1) A_{i+1} + \sum_{n=i+2}^{k} \zeta_{in} A_n \tag{14}$$

with

$$\zeta_{in} = (-1)^{n-i-1} n^{(n-i)} \left((n-i)^{-1} - \sum_{m=i+1}^{n-1} (n-m)^{-1} (m-i)^{-1} \right).$$

Now (14) enables us to compute $A_k,...,A_0$: A_i is a linear combination of $\alpha_i,...,\alpha_k$. Thus

$$A_i = -\sum_{j=i}^{k} b_{ij}\alpha_j$$

and (14) becomes

$$\sum_{j=i}^{k} b_{ij}\alpha_j = \alpha_i + (i+1).\sum_{j=i+1}^{k} b_{i+1,j}\alpha_j + \sum_{n=i+2}^{k} \zeta_{in} \sum_{j=n}^{k} b_{nj}\alpha_j.$$

From the coefficients of α_j it follows that

$$b_{ii} = 1$$
$$b_{i,i+1} = i+1$$
$$b_{ij} = (i+1) b_{i+1,j} + \sum_{n=i+2}^{j} \zeta_{in} b_{nj} \qquad \text{for } j \geq i+2.$$

Hence

$$x_p(t) = -\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}\alpha_j t^{(i)} = -\sum_{j=0}^{k} \alpha_j \left(\sum_{i=0}^{j} b_{ij} t^{(i)} \right)$$

and

$$x_p'(t) = -\sum_{j=1}^{k} \alpha_j \left(\sum_{i=1}^{j} b_{ij} \left(\sum_{l=0}^{i-1} \binom{i}{l} t^{(l)} (-1)^{(i-l-1)} \right) \right).$$

Since

$$x_p(0) = -\sum_{j=0}^{k} b_{0j}\alpha_j \qquad \text{and} \qquad x_p'(0) = -\sum_{j=1}^{k} \left(\sum_{i=1}^{j} (-1)^{(i-1)} b_{ij} \right) \alpha_j$$

we have the following result.

Proposition 3.2. *The solution of (4) with* $x(0)=c,\ x'(0)=d$ *and*

$$p(t) = \sum_{j=0}^{k} \alpha_j t^{(j)}$$

can be expressed as

$$x(t) = (d+M_k) y(t) + (c-d+m_k) y'(t) - \sum_{j=0}^{k} \alpha_j \pi_j(t)$$

where M_k *and* m_k *are linear combinations of* $\alpha_0,...,\alpha_k$, $y(t)$ *is the solution of (5) with* $y(0)=y'(0)=1$, *and for each* j $(0 \leq j \leq k)$, $\pi_j(t)$ *is a factorial polynomial of degree* j*:*

$$M_k = -\sum_{j=1}^{k} \left(\sum_{i=1}^{j} (-1)^{(i-1)} b_{ij} \right) \alpha_j, \qquad m_k = \sum_{j=0}^{k} b_{0j}\alpha_j, \qquad \pi_j(t) = \sum_{i=0}^{j} b_{ij} t^{(i)}. \qquad \square$$

For $j = 0, 1, ..., 9$, the factorial polynomials $\pi_j(t)$ are displayed in Table 2.

As in the previous section and [1,2] the coefficients of α_j in M_k and m_k are independent of k. The first few elements of the limit sequences (not mentioned in [4]) M and m (obtained from M_k and m_k for $k \to \infty$) are

j	$\pi_j(t)$
0	1
1	$t^{(1)}+1$
2	$t^{(2)}+2t^{(1)}+3$
3	$t^{(3)}+3t^{(2)}+9t^{(1)}+8$
4	$t^{(4)}+4t^{(3)}+18t^{(2)}+32t^{(1)}+50$
5	$t^{(5)}+5t^{(4)}+30t^{(3)}+80t^{(2)}+250t^{(1)}+214$
6	$t^{(6)}+6t^{(5)}+45t^{(4)}+160t^{(3)}+750t^{(2)}+1284t^{(1)}+2086$
7	$t^{(7)}+7t^{(6)}+63t^{(5)}+280t^{(4)}+1750t^{(3)}+4494t^{(2)}+14602t^{(1)}+11976$
8	$t^{(8)}+8t^{(7)}+84t^{(6)}+448t^{(5)}+3500t^{(4)}+11984t^{(3)}+58408t^{(2)}+$ $+95808t^{(1)}+162816$
9	$t^{(9)}+9t^{(8)}+108t^{(7)}+672t^{(6)}+6300t^{(5)}+26964t^{(4)}+175224t^{(3)}+$ $+431136t^{(2)}+1465344t^{(1)}+1143576$

Table 2.

M : 0, 1, 1, 8, 16, 224, 608, 13320, 41760, 1366152, ...
m : 1, 0, 2, 0, 34, –10, 1478, –1344, 121056, –222576, ...

Finally, we remark that the coefficients b_{ij} (and hence the elements of the sequences M and m) can also be computed from a_{ij} by means of

$$b_{ij} = \sum_{m=i}^{j} S(i,m) \left(\sum_{l=m}^{j} a_{ml} s(l,j) \right) \qquad (i \le j)$$

where $s(l,j)$ and $S(i,m)$ are the Stirling numbers of the first and of the second kind, respectively.

Acknowledgment. I am indebted to Bert Jagers for some useful discussions.

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