Fibonacci-like Differential Equations with a Polynomial Non-Homogeneous Part*

Peter R.J. Asveld

Department of Computer Science, Twente University of Technology
P.O. Box 217, 7500 AE Enschede, The Netherlands

Abstract — We investigate non-homogeneous linear differential equations of the form $x''(t) + x'(t) - x(t) = p(t)$ where $p(t)$ is either a polynomial or a factorial polynomial in t. We express the solution of these differential equations in terms of the coefficients of $p(t)$, in the initial conditions, and in the solution of the corresponding homogeneous differential equation $y''(t) + y'(t) - y(t) = 0$ with $y(0) = y'(0) = 1$.

1. Introduction

In [1] and [2] we studied difference equations of the form

$$G_n = G_{n-1} + G_{n-2} + p(n) \tag{1}$$

where $G_0 = G_1 = 1$ and $p(n)$ is either a (ordinary or power) polynomial [1] or a factorial polynomial [2], i.e.,

$$p(n) = \sum_{i=0}^{k} \alpha_i n^i \quad \text{or} \quad p(n) = \sum_{i=0}^{k} \alpha_i n^{(i)} \tag{2}$$

respectively, where $n^{(i)} = n(n-1)(n-2)...(n-i+1)$ for $i \geq 1$ and $n^{(0)} = 1$. The main results established in [1] and [2] provide expressions for the solution of (1) in terms of the coefficients $\alpha_1,...,\alpha_k$ of (2) and in the Fibonacci numbers F_n, i.e., in the solution of the homogeneous difference equation

$$F_n = F_{n-1} + F_{n-2} \tag{3}$$

where $F_0 = F_1 = 1$; cf. also [5].

In this note we derive similar expressions for the family of differential equations corresponding to (1) and (2), viz. we consider differential equations of the form

$$x''(t) + x'(t) - x(t) = p(t) \tag{4}$$

where $x(0) = c,\ x'(0) = d$,

$$p(t) = \sum_{i=0}^{k} \alpha_i t^i \quad \text{or} \quad p(t) = \sum_{i=0}^{k} \alpha_i t^{(i)}$$

and we express the solution of (4) in terms of the coefficients $\alpha_1,...,\alpha_k$ and in the solution of the homogeneous differential equation corresponding to (3), i.e., the solution of

$$y''(t) + y'(t) - y(t) = 0 \tag{5}$$

where $y(0) = y'(0) = 1$.

Essential in our approach is the following proposition in which $p(t)$ now need not be a (factorial) polynomial at all; it may be an arbitrary function which, however, gives rise to a particular solution $x_p(t)$ of (4).

* This paper appeared in *The Fibonacci Quarterly* **27** (1989) 303-309.

Let $F_{-1} = 0$, and $F_{-n} = (-1)^n F_{n-2}$ for each $n \geq 2$.

Proposition 1.1. *Let $x_p(t)$ be a particular solution of (4). If $x(0) = c$ and $x'(0) = d$, then the solution of (4) can be expressed as*

$$x(t) = (c - x_p(0))\left(\sum_{n=0}^{\infty} F_{-n} \frac{t^n}{n!}\right) + (d - x_p'(0))\left(\sum_{n=0}^{\infty} F_{-n-1} \frac{t^n}{n!}\right) + x_p(t). \tag{6}$$

Proof: Using standard methods (cf. e.g. [3]) we first determine the solution $x_h(t)$ of the homogeneous equation corresponding to (4). To this end we solve (5) with $y(0) = y'(0) = 1$:

$$y(t) = -(1+\phi_2)(\sqrt{5})^{-1}\exp(-\phi_1 t) + (1+\phi_1)(\sqrt{5})^{-1}\exp(-\phi_2 t),$$

where $\phi_1 = ½(1+\sqrt{5})$ and $\phi_2 = ½(1-\sqrt{5})$. Then we obtain

$$y(t) = -(1+\phi_2)(\sqrt{5})^{-1}\left(\sum_{n=0}^{\infty} \frac{(-\phi_1 t)^n}{n!}\right) + (1+\phi_1)(\sqrt{5})^{-1}\left(\sum_{n=0}^{\infty} \frac{(-\phi_2 t)^n}{n!}\right) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{\sqrt{5}}(\phi_1^{n-2} - \phi_2^{n-2})\frac{t^n}{n!} = \sum_{n=0}^{\infty} F_{-n+1}\frac{t^n}{n!},$$

since $(1+\phi_2)\phi_1^2 = 1$ and $(1+\phi_1)\phi_2^2 = 1$. Notice that

$$y'(t) = \sum_{n=0}^{\infty} F_{-n}\frac{t^n}{n!} \qquad \text{and} \qquad y''(t) = \sum_{n=0}^{\infty} F_{-n-1}\frac{t^n}{n!}. \tag{7}$$

Now it is straightforward to show that for the solution $x(t)$ of (4) we have

$$x(t) = x_h(t) + x_p(t) = (c - x_p(0))y'(t) + (d - x_p'(0))y''(t) + x_p(t),$$

which yields together with (7) the desired equality (6). □

From Proposition 1.1 it is clear that we now need a particular solution of (4). As in [1] and [2] we distinguish two cases, viz. $p(t)$ is a polynomial (Section 2) and $p(t)$ is a factorial polynomial (Section 3).

2. Polynomials

Throughout this section we assume that $p(t)$ is an ordinary or power polynomial

$$p(t) = \sum_{i=0}^{k} \alpha_i t^i.$$

As a particular solution of (4) we try

$$x_p(t) = \sum_{i=0}^{k} A_i t^i.$$

For $p(t)$ and $x_p(t)$ we write

$$p(t) = \sum_{i=0}^{k} \beta_i \frac{t^i}{i!} \qquad \text{and} \qquad x_p(t) = \sum_{i=0}^{k} B_i \frac{t^i}{i!},$$

respectively, where $\beta_i = i!\,\alpha_i$ and $B_i = i!\,A_i$ for each i $(0 \leq i \leq k)$. Then (4) yields

$$\sum_{i=0}^{k-2} B_{i+2}\frac{t^i}{i!} + \sum_{i=0}^{k-1} B_{i+1}\frac{t^i}{i!} - \sum_{i=0}^{k} B_i \frac{t^i}{i!} = \sum_{i=0}^{k} \beta_i \frac{t^i}{i!}.$$

From comparing the coefficients of $t^i/i!$ it follows that

$$\begin{aligned} B_k &= -\beta_k \\ B_{k-1} &= -\beta_{k-1} - \beta_k \\ B_i &= -\beta_i + B_{i+2} + B_{i+1} \qquad \text{for } 0 \le i \le k-2. \end{aligned}$$

Thus we can successively compute $B_k, B_{k-1}, \ldots, B_0$: B_i is a linear combination of $\beta_i, \ldots, \beta_k$. Therefore we write

$$B_i = -\sum_{j=i}^{k} a_{ij}\beta_j$$

(cf. [1] and [2]) which gives

$$-\sum_{j=i}^{k} a_{ij}\beta_j = -\beta_i - \sum_{j=i+2}^{k} a_{i+2,j}\beta_j - \sum_{j=i+1}^{k} a_{i+1,j}\beta_j.$$

Comparing the coefficients of β_j yields the following difference equation for each j ($1 \le j \le k$):

$$a_{ij} = a_{i+2,j} + a_{i+1,j} \qquad \text{for } j-i \ge 2$$

where $a_{jj} = a_{j-1,j} = 1$. But this means that $a_{ij} = F_{j-i}$ for $0 \le i \le j$, and hence

$$x_p(t) = \sum_{i=0}^{k} B_i \frac{t^i}{i!} = -\sum_{i=0}^{k}\sum_{j=i}^{k} F_{j-i}\, j!\,\alpha_j \frac{t^i}{i!} = -\sum_{j=0}^{k}\alpha_j \left(\sum_{i=0}^{j} j^{(j-i)} F_{j-i} t^i\right)$$

which implies

$$x_p(0) = B_0 = -\sum_{j=0}^{k} j!\, F_j \alpha_j \qquad \text{and} \qquad x_p'(0) = B_1 = -\sum_{j=1}^{k} j!\, F_{j-1}\alpha_j .$$

These equalities together with Proposition 1.1 yield the following proposition.

Proposition 2.1. *The solution of (4) with* $x(0)=c$, $x'(0)=d$ *and*

$$p(t) = \sum_{i=0}^{k} \alpha_i t^i$$

can be expressed as

$$x(t) = (c+L_k)\left(\sum_{n=0}^{\infty} F_{-n}\frac{t^n}{n!}\right) + (d+l_k)\left(\sum_{n=0}^{\infty} F_{-n-1}\frac{t^n}{n!}\right) - \sum_{j=0}^{k}\alpha_j p_j(t)$$

where L_k *and* l_k *are linear combinations of* $\alpha_0, \ldots, \alpha_k$, *and for each* j ($0 \le j \le k$), $p_j(t)$ *is a polynomial of degree j:*

$$L_k = \sum_{j=0}^{k} j!\, F_j \alpha_j, \quad l_k = \sum_{j=1}^{k} j!\, F_{j-1}\alpha_j, \quad p_j(t) = \sum_{i=0}^{j} j^{(j-i)} F_{j-i} t^i. \qquad \square$$

The polynomials $p_j(t)$ are given in Table 1 for $j = 0, 1, 2, \ldots, 9$.

The coefficients of α_j in L_k and l_k are independent of k; cf. [1] and [2]. They give rise to two infinite sequences L and l of natural numbers (not mentioned in [4]) as k tends to infinity. The first few elements of these new sequences are

L : 1, 1, 4, 18, 120, 960, 9360, 105840, 1370880, 19958400, $\cdots$,
l : 0, 1, 2, 12, 72, 600, 5760, 65520, 846720, 12337920, $\cdots$.

j	$p_j(t)$
0	1
1	$t+1$
2	t^2+2t+4
3	$t^3+3t^2+12t+18$
4	$t^4+4t^3+24t^2+72t+120$
5	$t^5+5t^4+40t^3+180t^2+600t+960$
6	$t^6+6t^5+60t^4+360t^3+1800t^2+5760t+9360$
7	$t^7+7t^6+84t^5+630t^4+4200t^3+20160t^2+65520t+105840$
8	$t^8+8t^7+112t^6+1008t^5+8400t^4+53760t^3+262080t^2+846720t+$ $+1370880$
9	$t^9+9t^8+144t^7+1512t^6+15120t^5+120960t^4+786240t^3+3810240t^2+$ $+12337920t+19958400$

Table 1.

3. Factorial Polynomials

This section is devoted to the case in which $p(t)$ is a factorial polynomial

$$p(t) = \sum_{i=0}^{k} \alpha_i t^{(i)}.$$

In order to try

$$x_p(t) = \sum_{i=0}^{k} A_i t^{(i)} \tag{8}$$

as a particular solution of (4) we first ought to determine the derivative of $t^{(n)}$.

Lemma 3.1.

$$\frac{dt^{(n)}}{dt} = \sum_{k=0}^{n-1} \binom{n}{k} t^{(k)} (-1)^{(n-k-1)}$$

Proof: The argument is by induction on n, the basis of which $(n=1)$ is trivial. Suppose the equality holds for $n-1$:

$$\frac{dt^{(n-1)}}{dt} = \sum_{k=0}^{n-2} \binom{n-1}{k} t^{(k)} (-1)^{(n-k-2)}. \tag{9}$$

To perform the induction step, consider

$$dt^{(n)}/dt = d(t(t-1)^{(n-1)})/dt = (t-1)^{(n-1)} + t.d((t-1)^{(n-1)})/dt.$$

Now by the Chain Rule we have $d((t-1)^{(n-1)})/dt = d((t-1)^{(n-1)})/d(t-1)$. Applying the Binomial Theorem from [2] to $(t-1)^{(n-1)}$ and the induction hypothesis (9) yields

$$\frac{dt^{(n)}}{dt} = \sum_{k=0}^{n-1} \binom{n-1}{k} t^{(k)} (-1)^{(n-k-1)} + t.\sum_{k=0}^{n-2} \binom{n-1}{k} (t-1)^{(k)} (-1)^{(n-k-2)} =$$

$$= (-1)^{(n-1)} + \sum_{k=1}^{n-1} \binom{n-1}{k} t^{(k)} (-1)^{(n-k-1)} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} t^{(k)} (-1)^{(n-k-1)} =$$

$$= (-1)^{(n-1)} + \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} t^{(k)} (-1)^{(n-k-1)} = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} t^{(k)} (-1)^{(n-k-1)}$$

which completes the induction. □

From Lemma 3.1, (4) and (8) we obtain

$$\sum_{i=2}^{k} A_i \left(\sum_{m=1}^{i-1} \begin{bmatrix} i \\ m \end{bmatrix} \left(\sum_{l=0}^{m-1} \begin{bmatrix} m \\ l \end{bmatrix} t^{(l)} (-1)^{(m-l-1)} \right) (-1)^{(i-m-1)} \right) +$$

$$+ \sum_{i=1}^{k} A_i \left(\sum_{m=0}^{i-1} \begin{bmatrix} i \\ m \end{bmatrix} t^{(m)} (-1)^{(i-m-1)} \right) - \sum_{i=0}^{k} A_i t^{(i)} = \sum_{i=0}^{k} \alpha_i t^{(i)}.$$

Comparing the coefficients of $t^{(i)}$ yields

$$A_k = -\alpha_k$$
$$A_{k-1} = -\alpha_{k-1} + k\alpha_k$$
$$A_i = -\alpha_i + \sum_{n=i+1}^{k} A_n \begin{bmatrix} n \\ i \end{bmatrix} (-1)^{(n-i-1)} + \sum_{n=i+2}^{k} A_n \left(\sum_{m=i+1}^{n-1} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ i \end{bmatrix} (-1)^{(m-i-1)} (-1)^{(n-m-1)} \right)$$

for each i $(0 \le i \le k-2)$. As $(-x)^{(n)} = (-1)^n (x+n-1)^{(n)}$ and $n^{(n)} = n!$, this latter recurrence can be rewritten to

$$A_i = -\alpha_i + \sum_{n=i+1}^{k} A_n (-1)^{n-i-1} \frac{n^{(n-i)}}{n-i} + \sum_{n=i+2}^{k} A_n \left(\sum_{m=i+1}^{n-1} (-1)^{n-i-2} \frac{n^{(n-i)}}{(n-m)(m-i)} \right)$$

or

$$A_i = -\alpha_i + (i+1)A_{i+1} + \sum_{n=i+2}^{k} \zeta_{in} A_n \tag{10}$$

where

$$\zeta_{in} = (-1)^{n-i-1} n^{(n-i)} \left((n-i)^{-1} - \sum_{m=i+1}^{n-1} (n-m)^{-1} (m-i)^{-1} \right).$$

Now (10) enables us to compute $A_k, \ldots, A_0$: A_i is a linear combination of $\alpha_i, \ldots, \alpha_k$. Thus

$$A_i = -\sum_{j=i}^{k} b_{ij} \alpha_j$$

and (10) becomes

$$\sum_{j=i}^{k} b_{ij} \alpha_j = \alpha_i + (i+1) \cdot \sum_{j=i+1}^{k} b_{i+1,j} \alpha_j + \sum_{n=i+2}^{k} \zeta_{in} \sum_{j=n}^{k} b_{nj} \alpha_j.$$

From the coefficients of α_j it follows that

$$b_{ii} = 1$$
$$b_{i,i+1} = i+1$$
$$b_{ij} = (i+1) b_{i+1,j} + \sum_{n=i+2}^{j} \zeta_{in} b_{nj} \qquad \text{for } j \ge i+2.$$

Hence

$$x_p(t) = -\sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} \alpha_j t^{(i)} = -\sum_{j=0}^{k} \alpha_j \left(\sum_{i=0}^{j} b_{ij} t^{(i)} \right)$$

and

$$x_p'(t) = -\sum_{j=1}^{k} \alpha_j \left[\sum_{i=1}^{j} b_{ij} \left[\sum_{l=0}^{i-1} \binom{i}{l} t^{(l)} (-1)^{(i-l-1)} \right] \right].$$

Since

$$x_p(0) = -\sum_{j=0}^{k} b_{0j}\alpha_j \qquad \text{and} \qquad x_p'(0) = -\sum_{j=1}^{k} \left[\sum_{i=1}^{j} (-1)^{(i-1)} b_{ij} \right] \alpha_j$$

we have the following result.

Proposition 3.2. *The solution of (4) with* $x(0)=c$, $x'(0)=d$ *and*

$$p(t) = \sum_{i=0}^{k} \alpha_i t^{(i)}$$

can be expressed as

$$x(t) = (c+M_k)\left[\sum_{n=0}^{\infty} F_{-n} \frac{t^n}{n!} \right] + (d+m_k)\left[\sum_{n=0}^{\infty} F_{-n-1} \frac{t^n}{n!} \right] - \sum_{j=0}^{k} \alpha_j \pi_j(t)$$

where M_k *and* m_k *are linear combinations of* $\alpha_0,\ldots,\alpha_k$, *and for each* j $(0 \le j \le k)$, $\pi_j(t)$ *is a factorial polynomial of degree* j:

$$M_k = \sum_{j=0}^{k} b_{0j}\alpha_j, \qquad m_k = \sum_{j=1}^{k} \left[\sum_{i=1}^{j} (-1)^{(i-1)} b_{ij} \right] \alpha_j, \qquad \pi_j(t) = \sum_{i=0}^{j} b_{ij} t^{(i)}. \qquad \square$$

j	$\pi_j(t)$
0	1
1	$t^{(1)}+1$
2	$t^{(2)}+2t^{(1)}+3$
3	$t^{(3)}+3t^{(2)}+9t^{(1)}+8$
4	$t^{(4)}+4t^{(3)}+18t^{(2)}+32t^{(1)}+50$
5	$t^{(5)}+5t^{(4)}+30t^{(3)}+80t^{(2)}+250t^{(1)}+214$
6	$t^{(6)}+6t^{(5)}+45t^{(4)}+160t^{(3)}+750t^{(2)}+1284t^{(1)}+2086$
7	$t^{(7)}+7t^{(6)}+63t^{(5)}+280t^{(4)}+1750t^{(3)}+4494t^{(2)}+14602t^{(1)}+11976$
8	$t^{(8)}+8t^{(7)}+84t^{(6)}+448t^{(5)}+3500t^{(4)}+11984t^{(3)}+58408t^{(2)}+ +95808t^{(1)}+162816$
9	$t^{(9)}+9t^{(8)}+108t^{(7)}+672t^{(6)}+6300t^{(5)}+26964t^{(4)}+175224t^{(3)}+ +431136t^{(2)}+1465344t^{(1)}+1143576$

Table 2.

For $j = 0, 1, ..., 9$, the factorial polynomials $\pi_j(t)$ are displayed in Table 2.

As in the previous section and [1,2] the coefficients of α_j in M_k and m_k are independent of k. The first few elements of the limit sequences (not mentioned in [4]) M and m (obtained from M_k and m_k for $k \to \infty$) are

M : 1, 1, 3, 8, 50, 214, 2086, 11976, 162816, 1143576, $\cdots$,
m : 0, 1, 1, 8, 16, 224, 608, 13320, 41760, 1366152, $\cdots$.

Finally, we remark that the coefficients b_{ij} (and hence the elements of the sequences M and m) can also be computed from a_{ij} by means of

$$b_{ij} = \sum_{m=i}^{j} S(i,m) \left(\sum_{l=m}^{j} a_{ml} s(l,j) \right) \qquad (i \leq j)$$

where $s(l,j)$ and $S(i,m)$ are the Stirling numbers of the first and of the second kind, respectively.

Acknowledgment. I am indebted to Bert Jagers for some useful discussions and in particular to the referee for suggesting an alternative approach which resulted in a more concise and less complicated version of this paper.

References

1. P.R.J. Asveld: A family of Fibonacci-like sequences, *The Fibonacci Quarterly* **25** no. 1 (1987): 81-83.
2. P.R.J. Asveld: Another family of Fibonacci-like sequences, *The Fibonacci Quarterly* **25** no. 4 (1987): 361-364.
3. M. Braun: *Differential Equations and Their Applications.* (3rd Edition), Applied Mathematical Sciences **15**. New York - Heidelberg - Berlin: Springer-Verlag, 1983.
4. N.J.A. Sloane: *A Handbook of Integer Sequences.* New York: Academic Press, 1973.
5. J.C. Turner: Note on a Family of Fibonacci-like Sequences, *The Fibonacci Quarterly* (to appear).