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A solution set for fine games

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Abstract

Bumb and Hoede have shown that a cooperative game can be split into two games, *the reward game* and *the fine game*, by considering the sign of quantities c_S^v in the c-diagram of the game. One can then define a solution x for the original game as $x = x_r - x_f$, where x_r is a solution for the reward game and x_f is a solution for the fine game. Due to the distinction of cooperation rewards and fines, for allocating the fines one may use another solution concept than for the rewards.

In this paper, fine vectors are introduced and a solution mapping for fine games is defined. We discuss the structure and properties of this mapping and show how the solution set is related to the Shapley value, the core and the Weber set. We also characterize the mapping as the unique mapping satisfying axioms of Efficiency, Pseudo-symmetry, Dummy Player Property and Additivity.

Key Words: c-diagram, fine games, fine vectors, solution mapping.
AMS classification: 91A44.

1 Introduction and fine games

A *cooperative game* with transferable utility (TU) is a pair $\langle N, v\rangle$, where N is a nonempty, finite set and $v : 2^N \to R$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. An element of N (notation: $i \in N$) and a nonempty subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) are called a *player* and *coalition* respectively, and the associated real number $v(S)$ is called the *worth* of coalition S. The size of coalition S is denoted by s. Particularly, n denotes the size of the player set N.

A *solution vector* of an n-person TU-game is an n-dimensional vector giving a payoff to any player $i \in N$. A *solution function* is a function x that assigns a solution vector $x(v) \in \mathbb{R}^n$ to any game $\langle N, v\rangle$. A solution function x is *efficient* if for any game the total payoff it assigns to the players is equal to the worth $v(N)$ of the grand coalition, *i.e.*, $\sum_{i\in N} x_i(v) = v(N)$ for any n-players game $v \in V$, where V denotes the set of all cooperative games. Most of the proposed solution concepts meet the *individual rationality* principle which requires $x_i(v) \geq v(i)$. An example of an efficient solution function is the Shapley value, and the value meets the individual rationality principle (see Shapley [9]), being the weighted average of so-called *marginal value vectors*.

A *solution mapping* is a mapping Ψ that assigns to every game $\langle N, v\rangle$ a set of solution vectors in $\mathbb{R}^n$. Well-known solution mappings are the core and the Weber set.

Given the player set N, with every subset $S \subseteq N, S \neq \emptyset$, there is associated its *unanimity game* $\langle N, u_S\rangle$ defined by

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T; \\ 0, & \text{otherwise.} \end{cases}$$

From the theory of cooperative games, one knows that any cooperative game $\langle N, v\rangle$ can be represented as a linear combination of the characteristic functions of the unanimity games. To be exact, it is well-known that

$$v(T) = \sum_{S \subseteq T, S \neq \emptyset} c_S^v \cdot u_S, \quad \text{where } c_S^v = \sum_{T \subseteq S} (-1)^{s-t} v(T) \text{ for all } S \subseteq N, S \neq \emptyset.$$

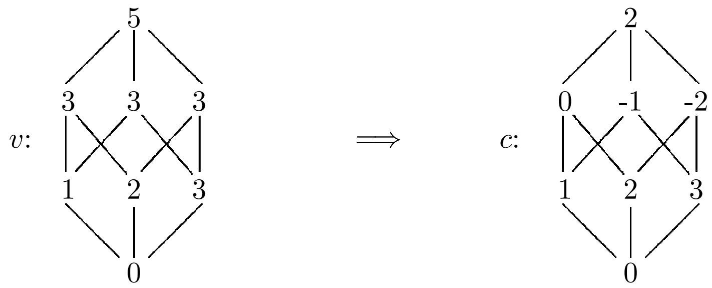
The quantities c_S^v are widely used in the theory of cooperative games. Recall that one of the classical proofs for the Shapley value satisfying four axioms, namely *efficiency, anonymity, dummy player and additivity* is done by using the above expression (see [9]). And, Harsanyi defined $\frac{c_S^v}{|S|}$ as *dividends* (see [6], [7]). Based on the definition, the solution set named Harsanyi set and related concepts have been introduced independently by Vasil'ev [11], [12], [13] and [14], and by Hammer, Peled and Sorensen [5]. Recently this set of solutions has been discussed by Derks, Haller and Peters [3] as the so-called selectope. The quantities c_S^v also proved to be essential in establishing the connection between set games and cooperative games (see [1]).

First, let us recall several results already stated in the papers of Bumb and Hoede (see [2]). They used the Hasse diagram to indicate the values of the coalitions in the cooperative game as well as the associated numbers c_S^v, see Figure 1 (Note that we have written c_S for c_S^v).

Figure 1: v-diagram and c-diagram of a 3-players cooperative game

Note that the sum of the numbers c_S equals v_{123}. Restriction of the c-diagram to subsets of a coalition S determines a sub-c-diagram with numbers that sum up to v_S, *i.e.* $v(S) = \sum_{T \subseteq S} c_S$. For example, $c_{13} + c_1 + c_3 = v_{13}$.

Example 1 *Consider the following 3-players game.*



∎

The c_S^v can be interpreted as a cooperation bonus or reward in case $c_S^v \geq 0$ or as a cooperation malus or fine in case the inequality in the other direction. Bumb and Hoede separated the c-diagram into two c-diagrams, one only having nonnegative numbers and another only having nonpositive numbers in the diagrams. As a c-diagram determines a v-diagram, *i.e.*, a game, it means that in a natural way one can split the game into two games; *the reward game* and *the fine game* now.

For Example 1, we can split the c-diagram into two c-diagrams c_r and c_f.

and the reward game and the fine game are

where the minus signs have been omitted.

So when we discuss the fine game and its solution concept, both $v_f(S)$ and $c_S^{v_f}$ should be considered to be nonnegative numbers.

A game $\langle N, v\rangle$ is *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$. By splitting a game v into a reward game v_r and a fine game v_f, both with c-diagrams containing non-negative numbers we have

Theorem 1.1 *The reward game v_r and the fine game v_f are convex games.*

Proof: We only prove this for the fine game, a similar proof can be given for the reward game. For all $S, T \subseteq N$, let $Q = S \cap T$.

$$\begin{aligned}
v_f(S) + v_f(T) &= \sum_{R \subseteq S} c_R^{v_f} + \sum_{R \subseteq T} c_R^{v_f} \\
&= \sum_{R \subseteq S \backslash Q} c_R^{v_f} + \sum_{R \subseteq Q} c_R^{v_f} + \sum_{Q \subset R \subset S} c_R^{v_f} + \sum_{S \backslash Q \subset R \subseteq S} c_R^{v_f} \\
&\quad + \sum_{R \subseteq T \backslash Q} c_R^{v_f} + \sum_{R \subseteq Q} c_R^{v_f} + \sum_{Q \subset R \subset T} c_R^{v_f} + \sum_{T \backslash Q \subset R \subseteq T} c_R^{v_f} \\
&\leq \sum_{R \subseteq S \backslash Q} c_R^{v_f} + \sum_{R \subseteq Q} c_R^{v_f} + \sum_{Q \subset R \subset S} c_R^{v_f} + \sum_{S \backslash Q \subset R \subseteq S} c_R^{v_f} \\
&\quad + \sum_{R \subseteq T \backslash Q} c_R^{v_f} + \sum_{R \subseteq Q} c_R^{v_f} + \sum_{Q \subset R \subset T} c_R^{v_f} + \sum_{T \backslash Q \subset R \subseteq T} c_R^{v_f} \\
&\quad + \sum_{S \subset R \subset S \cup T} c_R^{v_f} + \sum_{T \subset R \subseteq S \cup T} c_R^{v_f} \\
&= \sum_{R \subseteq (S \backslash Q) \cup (T \backslash Q) \cup Q} c_R^{v_f} + \sum_{R \subseteq Q} c_R^{v_f} \\
&= \sum_{R \subseteq S \cup T} c_R^{v_f} + \sum_{R \subseteq S \cap T} c_R^{v_f} \\
&= v_f(S \cup T) + v_f(S \cap T).
\end{aligned}$$

∎

Theorem 1.1 shows that any game $\langle N, v \rangle$ can be split into two convex games, the reward game and the fine game. Being a convex game is important in game theory because of its good properties. Let x_r be a solution for the reward game and let x_f be a solution for the fine game. One can then allocate for the original game $x = x_r - x_f$. Designing a solution concept may be seen as deciding on how the cooperation rewards and cooperation fines should be allocated. The fair way to split a cooperation reward for some coalition, seems to split the reward into equal parts and allocate them to each of the members of the coalition. This would mean using the Shapley value for the reward game. Due to the distinction of cooperation rewards and fines, in fact for allocating the fines one may use another solution concept than for the rewards.

In this paper, we concentrate on solutions and their properties for fine games, so v always means a fine game and V_f denotes the set of all fine games if no confusion can occur. In Section 2 fine vectors are introduced and the solution set for fine games is defined. We discuss the structure and properties of this mapping and show how the solution set is related to the Shapley value, the core and the Weber set. In Section 3 we characterize the mapping as the unique mapping satisfying axioms of Efficiency, Pseudo-symmetry, Dummy Player Property and Additivity.

2 A solution set defined by fine vectors

There are many solution concepts proposed in the literature of cooperative games. All of them are given different ways for sharing the worth of all coalitions $v(T), T \subseteq N$. In terms of the c_S^v's a solution concept may be a rather complex expression. However, any efficient solution distributes $v(N)$ over the n players. As $v(N) = \sum_{S \subseteq N, S \neq \emptyset} c_S^v$ any such solution can be written as

$$x_i = \sum_{S \subseteq N} \lambda_{S,i} c_S^v. \tag{1}$$

This simply expresses that in efficient games every player i gets a certain share of each c_S^v.

The c_S^v can be interpreted as a cooperation bonus or reward in case $c_S^v \geq 0$ or as a cooperation malus or fine in case they are nonpositive. Then (1) can be seen as distributing cooperation rewards and fines.

In case $\lambda_{S,i} = \frac{1}{n}$, for each S and i, every player gets the same, namely $\frac{v(N)}{n}$. This allocation is called *the egalitarian value.* But if only for player 1 we have $\lambda_{S,1} = 1$ while $\lambda_{S,i} = 0$ for all $i, i \neq 1$, the allocation can be called the *unfair value.*

Solutions may therefore be studied or classified by considering the possibilities for $\lambda_{S,i}$. The well-known Shapley value is a solution concept where the allocation is according to $\lambda_{S,i} = \frac{1}{s}$, for all $i \in S$, and the allocation has an extremely simple expression in terms of the c_S^v.

The members of the coalitions with negative c_S^v should assume responsibility for the fines. For allocating the fines or for deciding on the $\lambda_{S,i}$, the *fine vectors* $\vec{f_N} = (f_1, f_2, \cdots, f_n)$ are introduced to describe the weight of responsibility for the fines.

Definition 1 *For a fine game, the vector* $\vec{f_N} = (f_1, f_2, \cdots, f_n)$ *has the following properties: (i)* $0 \leq f_i \leq 1$ *and (ii)* $\sum_{i \in N} f_i = 1$.

We will denote the set of fine vectors as

$$\mathbb{F}^n = \{\vec{f_N} | 0 \leq f_i \leq 1, \sum_{i \in N} f_i = 1\}.$$

Given the set $\mathbb{F}^n$ of fine vectors, a solution mapping set $\Phi : V_f \to \mathbb{R}^n$ for fine games is defined as follows:

Definition 2 *Let* $f(S) = \sum_{i \in S} f_i$. *The solution vector* $\phi^{\vec{f}} := \Phi(v)$ *of a fine game is*

$$\phi_i^{\vec{f}} = \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} c_S^v, \qquad \vec{f} \in \mathbb{F}^n,$$

where

$$c_S^v = \sum_{T \subseteq S, T \neq \emptyset} (-1)^{s-t} v(T), \qquad v(\emptyset) = 0.$$

In Definition 2, the quotient $\frac{f_i}{f(S)}$ may be undefined in case $f(S) = 0$, and therewith $f_i = 0$ for all $i \in S$. So we assume that $f_i = 0$ only if i does not belong to any S for which $c_S^v > 0$. The term $\frac{f_i}{f(S)} c_S^v$ is then still undefined, with also $c_S^v = 0$. We will define such a term to be zero.

Theorem 2.1 *The solution vector $\phi^{\vec{f}}$ of a fine game can be expressed by marginal contributions as:*

$$\phi_i^{\vec{f}} = \sum_{T \subseteq N, T \ni i} f_i(T) m_i(T), \quad \textit{where} \;\; m_i(T) = v(T) - v(T \setminus i) \;\; \textit{and}$$

$$f_i(T) = \sum_{S | S \supseteq T} (-1)^{s-t} \frac{f_i}{f(S)}.$$

Proof:

$$\begin{aligned}
\phi_i^{\vec{f}} &= \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} c_S^v = \sum_{S \subseteq N, S \ni i} \Big(\frac{f_i}{f(S)} \sum_{T \subseteq S} (-1)^{s-t} v(T) \Big) \\
&= \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} \Big(\sum_{T \subseteq S, T \ni i} (-1)^{s-t} v(T) + \sum_{T \subset S, T \not\ni i} (-1)^{s-t} v(T) \Big) \\
&= \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} \Big[\sum_{T \subseteq S, T \ni i} \Big((-1)^{s-t} v(T) + (-1)^{s-t-1} v(T \setminus i) \Big) \Big] \\
&= \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} \Big[\sum_{T \subseteq S, T \ni i} (-1)^{s-t} \big(v(T) - v(T \setminus i) \big) \Big] \\
&= \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} \Big(\sum_{T \subseteq S, T \ni i} (-1)^{s-t} m_i(T) \Big) \\
&\overset{*}{=} \sum_{T \subseteq N, T \ni i} m_i(T) \Big(\sum_{S | S \supseteq T} (-1)^{s-t} \frac{f_i}{f(S)} \Big) \\
&= \sum_{T \subseteq N, T \ni i} f_i(T) m_i(T).
\end{aligned}$$

The equality $(*)$ holds because the marginal contribution $m_i(T)$ must be included by all coalitions $S (S \supseteq T)$ and appears as $(-1)^{s-t} \frac{f_i}{f(S)} \cdot m_i(T)$. ∎

Lemma 2.2 $\sum_{T \subseteq N, T \ni i} f_i(T) = 1.$

Proof:

$$\sum_{T \subseteq N, T \ni i} f_i(T) = \sum_{T \subseteq N, T \ni i} \sum_{S \supseteq T} (-1)^{s-t} \frac{f_i}{f(S)}.$$

Fix coalition S and choose $T \subseteq N$. The number of coalitions T for which $i \in T \subseteq S$, so $1 \leq t \leq s$, is $\binom{s-1}{t-1}$. Therefore

$$\begin{aligned}
\sum_{T \subseteq N, T \ni i} f_i(T) &= \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} \sum_{t=1}^{s} (-1)^{s-t} \binom{s-1}{t-1} \\
&\overset{k=t-1}{=\!=\!=} \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} \sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{s-1}{k} 1^k
\end{aligned}$$

$$= \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} \Big((-1) + 1 \Big)^{s-1}.$$

For $s \geq 2$ the contribution is 0. For $s = 1$, $f(S) = f_i$ and the contribution is 1, as follows directly from the first expression. ∎

A game is called *inessential* if $v(S) = \sum_{i \in S} v(i)$ for any $S \subseteq N, S \neq \emptyset$. For an inessential fine game $m_i(T) = v(T) - v(T \setminus i) = v(i)$. Then, by Lemma 2.2, $\phi_i^{\vec{f}} = m_i(T) = v(i)$.

For $\vec{f} = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ Definition 2 gives $\phi_i^{\vec{f}} = \sum_{S \subseteq N, S \ni i} \frac{1}{s} c_S^v$, which is the Shapley value. The more familiar expression for this value is confirmed by

Lemma 2.3 *If* $\vec{f} = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$, *then* $f_i(T) = \frac{(t-1)!(n-t)!}{n!}$.

Proof:

$$\begin{aligned}
f_i(T) &= \sum_{S | S \supseteq T} (-1)^{s-t} \frac{f_i}{f(S)} = \sum_{S | S \supseteq T} (-1)^{s-t} \frac{1}{s} \\
&= \sum_{s=t}^{n} (-1)^{s-t} \frac{1}{s} \binom{n-t}{s-t} \\
&\overset{k=s-t}{=\!=} \sum_{k=0}^{n-t} (-1)^k \frac{1}{k+t} \binom{n-t}{k} \\
&= \sum_{k=0}^{n-t} (-1)^k \binom{n-t}{k} \int_0^1 x^{k+t-1} dx \\
&= \int_0^1 \sum_{k=0}^{n-t} (-1)^k x^{k+t-1} \binom{n-t}{k} dx \\
&= \int_0^1 x^{t-1} \sum_{k=0}^{n-t} (-x)^k \binom{n-t}{k} 1^{n-t-k} dx \\
&= \int_0^1 x^{t-1} (1-x)^{n-t} dx.
\end{aligned}$$

We now let $f' = (1-x)^{n-t}$, $g = x^{t-1}$, and use $\int f'g = fg - \int fg'$, repeatedly. This leads to

$$\begin{aligned}
f_i(T) &= -\frac{1}{n-t+1} (1-x)^{n-t+1} x^{t-1} \Big|_0^1 - \int_0^1 \frac{-1}{n-t+1} (1-x)^{n-t+1} dx^{t-1} \\
&= \frac{t-1}{n-t+1} \int_0^1 (1-x)^{n-t+1} (x)^{t-2} dx \\
&= \cdots \\
&= \frac{(t-1)(t-2)\cdots 1}{(n-t+1)(n-t+2)\cdots(n-1)} \cdot \int_0^1 (1-x)^{n-1} (x)^0 dx \\
&= \frac{(t-1)!}{(n-t+1)(n-t+2)\cdots(n-1)} \cdot \int_0^1 -(1-x)^{n-1} d(1-x) \\
&= \frac{(t-1)!}{(n-t+1)(n-t+2)\cdots(n-1)n} \cdot -(1-x)^n \Big|_0^1 \\
&= \frac{(t-1)!(n-t)!}{n!}
\end{aligned}$$

∎

For a game $\langle N, v\rangle$, the set $I(v)$ of *imputations* is the set of all individually rational payoff vectors that efficiently distribute the payoff $v(N)$ of the grand coalition amongst its members, *i.e.*,

$$I(v) = \{x \in \mathbb{R}^n \big| x(N) = v(N), x_i \geq v(i),\ i \in N\}.$$

and the *core*, introduced in game theory by Gillies [4], is the solution mapping $C : V \to \mathbb{R}^n$ defined by

$$C(v) = \{x \in I(v) \big| x(S) \geq v(S),\ \text{for all } S \subseteq N\}.$$

It is well-known that the Shapley value is the barycenter of the core when v is convex, see Shapley [10] and Ichiishi [8].

The *Weber mapping*, see Weber [15], is the solution defined as the convex hull of the marginal value vectors, *i.e.*

$$W(v) = Conv\{m^\pi(v) \big| \pi \in \Pi\}.$$

where Π is the set of all permutations on N. Furthermore, it is known that $C(v) = W(v)$ when v is convex.

Lemma 2.4 *Let $\phi^{\vec{f}}$ be a solution of a fine game defined by fine vector $\vec{f}$, then*

$$\phi^{\vec{f}} \in C(v) = W(v).$$

Proof: By Theorem 1.1, $C(v) = W(v)$ is obvious. For $0 \leq f_i \leq 1$ and c_S^v nonnegative we have

$$\phi_i^{\vec{f}} = \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} c_S^v \geq \frac{f_i}{f_i} c_i^v = v(i). \qquad (a)$$

$$\begin{aligned}\phi^{\vec{f}}(N) &= \sum_{i \in N} \phi_i^{\vec{f}} = \sum_{i \in N} \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} c_S^v \\ &= \sum_{S \in N} \sum_{i \in S} \frac{f_i}{f(S)} c_S^v = \sum_{S \in N} c_S^v = v(N). \qquad (b)\end{aligned}$$

$$\begin{aligned}\phi^{\vec{f}}(T) &= \sum_{i \in T} \phi_i^{\vec{f}} = \sum_{i \in T} \sum_{S \subseteq N, S \ni i} \frac{f_i}{f(S)} c_S^v \\ &\geq \sum_{i \in T} \sum_{S \subseteq T, S \ni i} \frac{f_i}{f(S)} c_S^v = \sum_{S \subseteq T} \sum_{i \in S} \frac{f_i}{f(S)} c_S^v \\ &= \sum_{S \subseteq T} c_S^v = v(T). \qquad (c)\end{aligned}$$

From (a) and (b), the solution $\phi^{\vec{f}}$ is individually rational and efficient, so $\phi^{\vec{f}} \in I(v)$, and then $\phi^{\vec{f}} \in C(v)$ follows for (c). ∎

3 Axiomatization of the solution set

In this section we provide an axiomatization of the solution mapping set $\Phi : V_f \to \mathbb{R}^n$ defined by fine vectors, *i.e.*, the value mapping assigning the solution set $\Phi(v)$ to any fine game $\langle N, v\rangle$. To do so, first we state four reasonable axioms to be satisfied by a value mapping $\Psi : V_f \to \mathbb{R}^n$.

Axiom System A

1. A value mapping Ψ has the *Efficiency* property when for any fine game $v \in V_f$ holds $\sum\limits_{i\in N} \psi_i(v) = v(N)$.

2. A value mapping Ψ has the *Pseudo-symmetry* property, if for any fine game $v \in V_f$ and any permutation $\pi : N \to N$ with $\pi v \in V_f$: $\psi_{\pi(i)}^{\pi(\vec{f})}(\pi v) = \psi_i^{\vec{f}}(v)$. This means that any permutation of N should keep the same proportion of values in the solution vector.

3. A value mapping Ψ has the *Dummy Player* property, if for any fine game $v \in V_f$ and any dummy player $i \in N$, $\psi_i(v) = v(i)$. Here player i is called a *dummy* in the game $\langle N, v\rangle$ if $v(S \cup \{i\}) - v(S) = v(i)$, for all $S \subseteq N \setminus \{i\}$.

4. A value mapping Ψ has the *Additivity* property, if for any fine game v and u, $\Psi(v + u) = \Psi(v) + \Psi(u)$.

Axiom 2 was introduced because we wanted to give axioms in terms of the coalition worths $v(S)$. In terms of the c_S^v, another property can be demanded to fix the solution concept, once Additivity, Dummy Player property and Efficiency property have reduced the possibilities to those of dividing c_S^v over the players belonging to coalition S. For the Shapley value the Equal Treatment property can be considered, stating that $\phi_i(v) = \phi_j(v)$ when players i and j determine the same marginal values for all coalitions S to which they are added. On the level of elementary games this implies that each player $i, i \in S$, is allocated $\frac{c_S^v}{s}$. The allocation according to a fine vector might be called the Proportional Treatment property. In terms of coalition worths $v(S)$ this property is not easily formulated. Hence our choice of the Pseudo-symmetry property.

Theorem 3.1 *The solution mapping $\Phi : V_f \to \mathbb{R}^n$ for fine games defined by fine vectors is the unique mapping which satisfies* **Axiom System A**.

Proof: We first prove that the solution mapping satisfies all axioms in the Axiom System A. The axiom of Efficiency has been proved in Lemma 2.4. Since

$$\phi_{\pi(i)}^{\pi(\vec{f})}(\pi v) = \sum_{\pi(S)\subseteq N, \pi(S)\ni\pi(i)} \frac{f_{\pi(i)}}{f(\pi(S))} c_{\pi(S)}^{\pi v} = \sum_{S\subseteq N, S\ni i} \frac{f_i}{f(S)} c_S^v = \phi_i^{\vec{f}}(v),$$

we have the Pseudo-symmetry property.

For the dummy player, we have $m_i(T) = v(i)$. To show the Dummy Player property, let us recall the expression of the solution in Theorem 2.1

and the result of Lemma 2.2. We have

$$\phi_i^{\vec{f}} = \sum_{T\subseteq N, T\ni i} f_i(T)m_i(T) = m_i(T) \sum_{T\subseteq N, T\ni i} f_i(T) = m_i(T) = v(i).$$

Considering the linear combination representation of the unanimity games for any game $\langle N, v\rangle$, we know $c_S^v = \sum_{T\subseteq S}(-1)^{s-t}v(T)$. So for fine games v and u,

$$\begin{aligned} c_S^{v+u} &= \sum_{T\subseteq S}(-1)^{s-t}(v+u)(T) = \sum_{T\subseteq S}(-1)^{s-t}\Big(v(T)+u(T)\Big) \\ &= \sum_{T\subseteq S}(-1)^{s-t}v(T) + \sum_{T\subseteq S}(-1)^{s-t}u(T) \\ &= c_S^v + c_S^u. \end{aligned}$$

therefore

$$\begin{aligned} \phi_i^{\vec{f}}(v+u) &= \sum_{S\subseteq N, S\ni i} \frac{f_i}{f(S)} c_S^{v+u} = \sum_{S\subseteq N, S\ni i} \frac{f_i}{f(S)} \Big(c_S^v + c_S^u\Big) \\ &= \sum_{S\subseteq N, S\ni i} \frac{f_i}{f(S)} c_S^v + \sum_{S\subseteq N, S\ni i} \frac{f_i}{f(S)} c_S^u \\ &= \phi_i^{\vec{f}}(v) + \phi_i^{\vec{f}}(u). \end{aligned}$$

The Additivity property follows.

Let us turn to the unicity proof. We consider the solution of unanimity games $\{u_S \big| S \subseteq N, S \neq \emptyset\}$. If player i is not in S, then i is a dummy player of the unanimity game u_S, and $u_S(i) = 0$. From Axiom 3, we know $\phi_i(u_S) = 0$. Now, if π is any permutation of N which carries S to itself, it is clear that $\pi u_S = u_S$. Hence, by Axiom 2, $\phi_i(u_S) : \phi_j(u_S) = f_i : f_j$, where $f_i : f_j$ is the proportion. By Axiom 1 and the Dummy Player property, it follows that $\sum_{i\in N} \phi_i(u_S) = \sum_{i\in S} \phi_i(u_S) = 1$. So, we have

$$\phi_i(u_S) = \frac{f_i}{\sum\limits_{j\in S} f_j} = \frac{f_i}{f(S)}.$$

By the linear combination representation of the unanimity games for any fine game

$$v = \sum_{S\subseteq N, S\neq\emptyset} c_S^v \cdot u_S,$$

and, by Axiom 4, we have

$$\phi_i(v) = \sum_{S\subseteq N, S\ni i} c_S^v \cdot \phi_i(u_S) = \sum_{S\subseteq N, S\ni i} c_S^v \cdot \frac{f_i}{f(S)}.$$

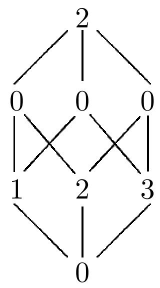
∎

Remark 1 *Theorem 3.1 means that the solution concept satisfies three of the four axioms for the Shapley value, yet of course is not identical with it. For obtaining the Shapley value we can add the Symmetry property. For characterizing the Fine value we have introduced the Pseudo-symmetry property.*

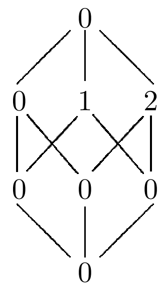
Choosing a reasonable fine vector means a good solution for a fine game. How to choose the fine vector is very important for the solution mapping Φ. For example, we can choose the number of times player i belongs to an S with negative c^v_S. This determines a vector $\vec{a} = (a_1, a_2, \cdots, a_n)$ and if $a = \sum_{i=1}^n a_i$, f_i could be taken to be $\frac{a_i}{a}$.

Following Example 1, the c-diagrams of its reward game and fine game are

c_r:



$-c_f$:



The Shapley value of the reward game is $x_r = (\frac{5}{3}, \frac{8}{3}, \frac{11}{3})$. We choose the frequency based vector $\vec{a}$ as fine vector , i.e., $\vec{f} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, so the solution for the fine game is $x_f = (\frac{1}{3}, \frac{2}{3}, 2)$. The solution of the original game is then $x = x_r - x_f = (\frac{4}{3}, 2, \frac{5}{3})$

The Shapley value for the c_f-diagram is $(\frac{1}{2}, 1, \frac{3}{2})$ and the Shapley value for the original game is therewith $(\frac{7}{6}, \frac{5}{3}, \frac{13}{6})$. We see that the fining procedure gives a punishment to player 3, whose allocation goes from $\frac{13}{6}$ to $\frac{5}{3}$. His loss $\frac{1}{2}$ goes to player 1 : $\frac{1}{6}$ and player 2 : $\frac{1}{3}$.

For a game in which a cost is to be shared, the c-diagram can be split into two diagrams again, but the interpretation is then changing. The positive c_S's are now costs, resulting from joint activities, whereas the negative c_S's can be seen as savings on the costs, due to cooperation. The names *cost game* and *saving game* are proposed. A Cost and Saving method exchanges the ways of dealing with the two games, now the Shapley value seems fair for the saving game, whereas the method used for the fine game might be chosen for the cost game. After all, rewards and savings are typically cooperation bonuses, whereas fines and costs are cooperation maluses.

References

[1] A. Bumb and C. Hoede, *The standard set game of a cooperative game.* University of Twente, Memorandum 1672(2003).

[2] A. Bumb and C. Hoede, *C-diagrams, shifts and solidarity values.* University of Twente, Memorandum 1692(2003).

[3] J. Derks, H. Haller and H. Peters, The selectope for cooperative games, *International Journal of Game Theory* 29 (2000) 23-28.

[4] D.B. Gillies, *Some theorems on n-person games,* Ph-D Thesis. Princeton: Princeton University Press, NJ, 1953.

[5] P.L. Hammer, U.N. Peled and S. Sorensen, Pseudo-boolean functions and game theory. I. Core elements and Shapley value. *Cahiers du CERO*, 19 (1977) 159-176.

[6] J.C. Harsanyi, A bargaining model for the cooperative n-person game. *Annals of Mathematics Study*, 40 (1959) 325-355.

[7] J.C. Harsanyi, A simplified bargaining model for the n-person game. *International Economic Review*, 4 (1963) 194-220.

[8] T. Ichiishi, Super-modularity: applications to convex games and the greedy algorithm for LP. *Journal of Economic Theory*, 25 (1981) 283-286.

[9] L.S. Shapley, A value for N-person games, in: *Contributions to the theory of games II.* (H.W. Kuhn and A.W. Tucker, eds.) Princeton: Princeton University Press, (1953) 307-317.

[10] L.S. Shapley, Cores of convex games. *International Journal of Game Theory*, 1 (1971) 11-26.

[11] V.A. Vasil'ev, The Shapley value for cooperative games of bounded polynomial variation, *Optimizacija Vyp*, 17 (1975) 5-27 (in Russian).

[12] V.A. Vasil'ev, Support function of the core of a convex game, *Optimizacija Vyp*, 21 (1978) 30-35 (in Russian).

[13] V.A. Vasil'ev, On a class of imputations in cooperative games, *Soviet Math. Dokl.*, 23 (1981) 53-57.

[14] V.A. Vasil'ev and G. Laan, *The Harsanyi set for cooperative TU-games*, Tinbergen Institute Discussion Paper, TI 2001-004/1.

[15] R.J. Weber, *Probabilistic values for games*, in: The Shapley value, Essays in honor of L.S. Shapley, (A.E. Roth, ed.) pp. 101-119. Cambridge: Cambridge University Press, (1988).