Department of Applied Mathematics
Faculty of EEMCS

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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A characterization of extremal graphs without matching-cuts

P.S. Bonsma

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Paul Bonsma

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Abstract

A graph is called (matching-)immune if it has no edge cut that is also a matching. Farley and Proskurowski proved that for all immune graphs $G = (V, E)$, $|E| \geq \lceil 3(|V| - 1)/2 \rceil$, and constructed a large class of immune graphs attaining this lower bound for every value of $|V(G)|$, called ABC graphs. In this paper, we prove their conjecture that every immune graph that attains this lower bound is an ABC graph.

Keywords: matching-cut, matching immune, extremal graphs
AMS subject classifications: 05C35, 05C75

1 Introduction

Formally, a multi-graph (graph for short) is a triple $G = (V, E, \psi)$, where V and E are disjoint sets representing the vertices and edges, and ψ is an *incidence function* that associates with each edge of G an unordered pair of (not necessarily distinct) vertices. Since most of the graphs that are considered in this paper have few multi-edges and no loops, we will often omit the incidence funcion, and denote edges by their end vertices. $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges of a graph G. For two disjoint sets $S, T \subset V$, $[S, T]$ denotes the set of edges with exactly one end vertex in S and one end vertex in T. $M \subseteq E$ is called an *edge cut* for (or in) G if $M = [S, \overline{S}]$ for some $S \subset V$, $S \neq \emptyset$, $S \neq V$. Here $\overline{S}$ denotes $V \backslash S$. If a set $S \subseteq V$ exists such that $M = [S, \overline{S}]$ and $A \subseteq S$ and $B \subseteq \overline{S}$ ($u \in S$ and $v \in \overline{S}$)then the edge cut M is said to *separate* A from B (u from v). $M \subseteq E$ is a *matching* if no two edges in M share an end vertex. M is called a *matching-cut* if it is an edge cut and a matching. If a graph has no matching-cut, it is called *(matching-)immune*. Farley and Proskurowski [2] proved the following extremal result on immune graphs.

Theorem 1 (Farley and Proskurowski) *If $G = (V, E)$ is immune, then*

$$|E| \geq 3(|V| - 1)/2.$$

In addition, they constructed a large class of multi-graphs which we will call *ABC graphs*, that have the following properties:

- ABC graphs are immune.
- If $G = (V, E)$ is an ABC graph, then $|E| = \lceil 3(|V| - 1)/2 \rceil$.

For every integer $n \geq 1$, an ABC graph exists. For every integer $n \geq 1$, $n \neq 2$ a simple ABC graph exists. This shows that the lower bound from Theorem 1 is tight. It also inspires the following definition.

Definition 1 *An immune graph $G = (V, E)$ is called* extremal immune *if $|E| = \lceil 3(|V| - 1)/2 \rceil$.*

Instead of 'G is an ABC graph' we will often say 'G is ABC'. Farley and Proskurowski stated the following conjecture.

Conjecture 2 (Farley and Proskurowski) *Every extremal immune graph is ABC.*

We present a proof of Conjecture 2. The rest of the paper is organized as follows. We will start with some general definitions and results from [2] in Section 2. In Section 3 we give an overview of the proof. Definitions related to ABC graphs are stated in Section 4. Then in Section 5 the structure of ABC graphs is studied, and some properties are stated. In Section 6 a few types of matching-cuts that we will often use in the proof are introduced. In Section 7 the first part of the proof of the conjecture is given. In Section 8 and Section 9 two important cases of the proof are considered, and finally in Section 10 the proof of the conjecture is completed.

2 General definitions and preliminary observations

For any undefined terminology we refer to [1]. Throughout this section let $G = (V, E)$ denote a graph with vertex set V and edge set E. The *order* of G is $|V|$, and the *size* of G is $|E|$. $N_G(v)$ denotes the set of neighbors of a vertex v in G. If it is clear to which graph we refer, we simply write $N(v)$. The *degree* $d_G(v)$ of vertex v in G is $|N_G(v)|$, and similarly we write $d(v)$ if there is no cause for confusion. Let $S \subseteq V$. Then

$$G[S] = (S, \{uv \in E : u \in S \wedge v \in S\})$$

is the subgraph of G *induced by S*. If $M \subseteq E$, then

$$G[M] = (\{v \in V : \exists uv \in M\}, M)$$

is the subgraph of G *induced by M*. A subgraph H of G is called an *induced subgraph* of G if $H = G[S]$ for some $S \subseteq V$, and an *edge induced subgraph* of G if $H = G[M]$ for some $M \subseteq E$.

Let C_n denote the cycle on n vertices (*n-cycle*). So by C_2 we denote the loopless (multi-)graph with two vertices and two edges. P_n denotes the path on n vertices. K_n denotes the complete graph on n vertices. $K_3 = C_3$ is often called a *triangle*. $K_{m,n}$ denotes the complete bipartite graph with vertex sets of cardinality m and n.

If G is connected, a vertex $v \in V$ is a *cut vertex* if $G - v$ is not connected. A (connected) graph is *2-connected* if it has no cut vertices and is not equal to K_1 or K_2. A *block* of a graph is a maximal connected subgraph without cut vertices. So a block is 2-connected unless it is isomorphic to K_1 or K_2. Another way to characterize blocks is the following: u and v are part of the same block of G if and only if there is a cycle containing both u and v, or $uv \in E(G)$. The following property follows easily from this: if three different vertices u, v and w are part of the same block, there is a path (in this block) from u to w with v as an internal vertex. The next observation will be used repeatedly in this paper. In this paper observations denote statements that are well-known or easy to prove, and will be stated without proof.

Observation 3 *If $M \subseteq E(H)$ is an edge cut for a graph H, and H is a block of G, then M is an edge cut for G.*

Definition 2 *Let H be a subgraph of G, and $v \in V(H)$. If $d_H(v) = d_G(v)$ then v is called an* internal vertex *of H, otherwise v is called a* connection vertex *of H. H is called an* i-connection subgraph of G *if H has at most i connection vertices.*

This definition will be used often for induced triangles and 4-cycles. Note that if G is immune and an induced C_4 is a 2-connection subgraph of G with connection vertices u and v, then u and v cannot be neighbors. In this case the C_4 is called a 2-connection 4-cycle *between* u and v.

Observation 4 *If $G \neq K_1, K_2$ is connected but not 2-connected, then G has at least two 1-connection blocks.*

Using Observation 4, the following observation can be proved by induction.

Observation 5 *The order of a graph G is odd if and only if the number of even order blocks in G is even.*

2.1 Contraction and expansion operations

In order to properly define contractions and to show how new edges correspond to old edges, we denote graphs by the triple $G = (V, E, \psi)$ in this section, where ψ is an incidence function on the edges. The *contraction* of edge e with $\psi(e) = u_1u_2$ consists of the following steps: remove edge e, introduce a new vertex u, and in every pair of ψ replace u_1 and u_2 by u. Delete u_1 and u_2. So a contraction in a simple graph may result in parallel edges, and a contraction in a graph with parallel edges may result in loops. A contraction of a loop

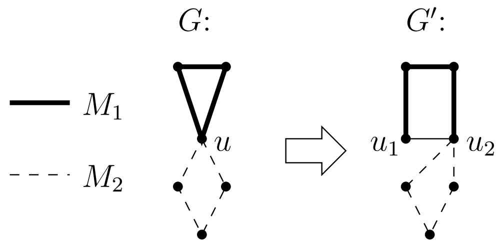

Figure 1: An example of splitting

corresponds to the deletion of this loop (and relabeling the vertex). An edge expansion can be seen as the reverse of a contraction. So the *edge expansion of* u *into* u_1u_2 consists of these steps: introducing vertices u_1 and u_2, introducing an edge e with $\psi(e) = u_1u_2$, replacing every occurrence of u in ψ by u_1 or u_2, and deleting u. Note that there is only one way to contract a particular edge of a graph (apart from the resulting vertex label), but in general there are many ways to expand a vertex into an edge. An edge expansion of u into u_1u_2 is called a *non-trivial edge expansion* if $d(u_1) \geq 2$ and $d(u_2) \geq 2$ in the resulting graph.

If a graph G' can be obtained from G by a series of edge contractions and edge deletions, then all edges of G' correspond to edges of G. We will often consider one particular edge set M and study its properties both in G' and in G. For instance, M can be a matching in G but not in G'.

Suppose G' can be obtained from G by a series of edge contractions and edge deletions. So G can be constructed from G' by a series of edge expansions and edge additions. In this case the following definition is useful:

Definition 3 *Suppose G can be obtained from G' be a series of edge expansions and edge additions. Then we say that an edge set $M \subseteq E(G')$ is* split *if $G'[M]$ and $G[M]$ are not isomorphic. In this case, the subgraph $G'[M]$ is also said to be split. Similarly, we say that two edges $e, f \in E(G')$ are split if $G'[\{e, f\}]$ and $G[\{e, f\}]$ are not isomorphic.*

In Figure 1, an example is shown: G' is obtained from G by an edge expansion of u into u_1u_2; $G[M_1]$ is split but $G[M_2]$ is not. In the remainder of the paper, we will use the conventional notations again, and omit the incidence function ψ from the definition of a graph. Edges will be denoted by their end vertices if there is no cause for confusion. It is important to keep in mind that if for instance G' is obtained from G by the contraction of u_1u_2 into u, edges $u_1v \in E(G)$ and $uv \in E(G')$ are considered to be the same edge.

For their proof of Theorem 1, Farley and Proskurowski [2] introduced four graph operations, named after the structure they reduce. The four operations are illustrated in Figure 2. Below are formal definitions, which show that all of these operations can be expressed by edge deletions and contractions.

C2 Let the vertices u and v induce a C_2 in the graph G. The C2 operation consists of deleting one of the edges of this C_2 and contracting the other.

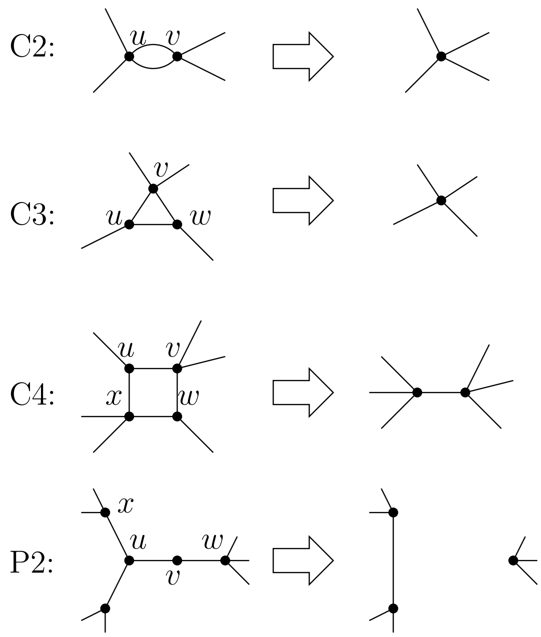

Figure 2: The four reduction operations

C3 Let the vertices u, v and w induce a C_3 in G. The C3 operation consists of deleting uv and contracting vw and wu.

C4 Let subgraph C of G be a 4-cycle with edge set $\{uv, vw, wx, ux\}$. The C4 operation consists of deleting uv and contracting ux and vw. Note that for one C_4 subgraph the C4 operation can have two different results.

Originally this operation was only defined for *induced* 4-cycles, but we remark that Lemma 6 below also holds when C is not induced. In this case, the resulting graph will have a multi-edge between the resulting two vertices.

P2 Let the vertices u and v be neighbors in G with $d(u) = 3$ and $d(v) = 2$. Let v have another neighbor $z \neq u$, and let u have another neighbor $x \neq v$. The P2 operation consists of deleting uv and contracting ux and vz.

It is clear that these operations consist of series of edge deletions and contractions, so Definition 3 of split subgraphs can be used for these operations. For instance we will say 'G' is obtained from G with a C4 operation such that the subgraph H of G' is split (when reconstructing G from G')'.

It can be checked that these four operations have the following properties.

Lemma 6 (Farley and Proskurowski) *Suppose G' can be obtained from G by a C2, C3, C4 or P2 operation. Then the following statements hold:*

- *If G is immune, then G' is immune.*
- *If $|E(G)| = \lceil 3(|V(G)| - 1)/2 \rceil$, then $|E(G')| \leq \lceil 3(|V(G')| - 1)/2 \rceil$. This inequality is always an equality for the C3, C4 and P2 operation.*

To prove Theorem 1, the following lemma was used.

Lemma 7 (Farley and Proskurowski) *If $G \neq K_1$ is an extremal immune graph, then one of the operations C2, C3, C4 or P2 can be applied to G.*

Our proof of Conjecture 2 is based on this lemma.

3 An overview of the proof

The proof of Conjecture 2 is by contradiction, so first we assume an extremal immune graph exists that is not an ABC graph. Then we consider a graph G with minimum number of vertices among all such graphs. This is called a *minimum counterexample*. We first consider the case that a minimum counterexample G contains a C_2, so a C2 operation can be applied, resulting in vertex v. After applying this C2 operation, another extremal immune graph G' is obtained (Lemma 6, Theorem 1). By our choice of G, G' must be an ABC graph. We consider a number of cases for G', for the choice of $v \in V(G')$ and for the possible graphs G that can correspond to this, and in every case we obtain one of the following contradictions: G has a matching-cut, G is also an ABC graph, or a smaller counterexample exists.

If a minimum counterexample G contains a triangle, we can apply operation C3 and find a contradiction in a similar way. If G contains a C_4, applying a C4 operation leads to a contradiction. Finally, we can show that if a P2 operation can be applied resulting in ABC graph G', then there always is a triangle or C_4 in G' that corresponds to a triangle or C_4 in G, so the previous cases can be applied. Since every operation leads to a contradiction, Lemma 7 shows that no counterexamples for the conjecture can exist.

Before we can state the proof, we need to study the structure of ABC graphs, which is done in the following three sections.

4 ABC Graphs: preliminaries

In [2] a set of graphs is defined which we will call ABC graphs. ABC graphs are named after the three graph operations that can be used to construct them, which are defined below. Whenever these operations are involved, graphs are assumed to be labeled. So if two graphs are said to be equal it means that the vertex labels are also equal, not just that the graphs are isomorphic. See Figure 3 for an example of these operations.

Definition 4 *An* A operation *on a vertex u introduces two new vertices v and w and the edges uv, uw and vw. $G' = A(G, u, v, w)$ is used to denote that G' is obtained by an A operation on a vertex u of G, introducing v and w.*

Definition 5 *A* B operation *on the edge uv introduces two new vertices w and x and the edges uw, vw, ux and vx, and removes the edge uv. $G' = B(G, uv, w, x)$ is used to denote that G' is obtained by a B operation on the edge uv of G, introducing w and x.*

Definition 6 *A* C operation *on the vertices u and v ($u = v$ is allowed) introduces a new vertex w and the edges uw and vw. $G' = C(G, u, v, w)$ is used to denote that G' is obtained by a C operation on the vertices u and v of G, introducing w.*

We say that a vertex v is *used in operation* x if x is an A or C operation on v, or a B operation on an edge incident with v. Note that the C operation is the only operation that can introduce parallel edges.

Definition 7 *An* AB graph *is a graph that can be obtained from K_1 by a sequence of A and B operations. An* ABC graph *is a graph that can be obtained from K_1 by a sequence of A and B operations and* at most one *C operation.*

If G is an AB(C) graph, a sequence of operations that constructs G is called a *decomposition* of G. Formally, a decomposition is a list of the form $G_0 = (\{u\}, \emptyset)$, $G_1 = A(G_0, u, v, w)$, $G_2 = B(G_1, vw, x, y)$ etc. (In this example, G_2 is isomorphic to $K_{2,3}$.). $G_0, \ldots, G_{i-1}$ are called *intermediate graphs* in this particular decomposition of G_i. In general, an ABC graph can have different decompositions, even decompositions where the intermediate graphs are not isomorphic. In the top part of Figure 3 an example of a decomposition of an ABC graph is shown.

Observation 8 *Suppose that in a decomposition of the ABC graph G, operation x and operation y are applied consecutively, x first. Now unless operation x introduces vertices that are used in operation y, the order in which x and y are applied can be reversed, giving another decomposition of G.*

The above observation will be used implicitly in a lot of proofs, just like the next observations.

Observation 9 *In a decomposition of the ABC graph G that does not start with vertex v, $d(v) = 2$ if and only if v is not used in any operation in the decomposition.*

For applying Observation 9 it is useful to note that for every vertex v in an ABC graph $G \neq K_1$, G has a decomposition that does not start with v.

Observation 10 *AB graphs are simple. In an ABC graph, only between one pair of vertices u and v parallel edges can exist, and only if the C operation is on u introducing v or on v introducing u, and no B operations are applied to these edges between u and v. There are at most two parallel edges between u and v.*

It can be checked that for ABC graphs the following two properties hold [2].

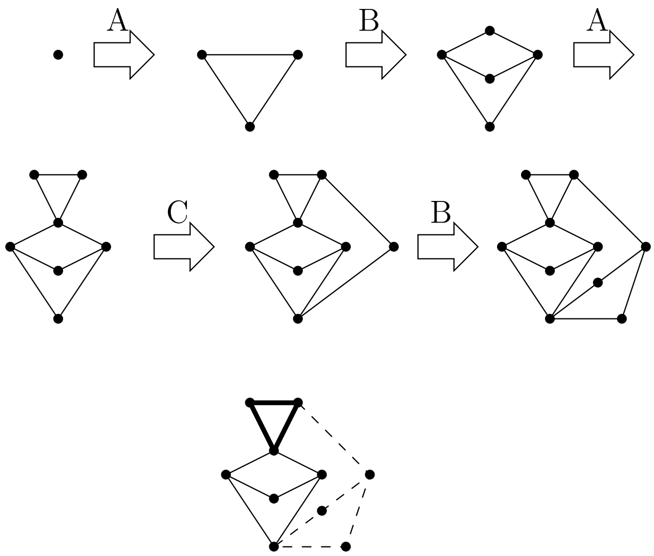

Figure 3: The construction of an ABC graph and corresponding edge partition

Theorem 11 (Farley and Proskurowski) *If G is an AB graph on n vertices with m edges, then n is odd, $m = 3(n-1)/2$ and G is immune.*

Theorem 12 (Farley and Proskurowski) *If G is an ABC graph on n vertices with m edges but G is not an AB graph, then n is even, $m = (3n-2)/2$ and G is immune.*

So ABC graphs are extremal immune graphs.

Observation 13 *Let $f : n \to \lceil 3(n-1)/2 \rceil$. $f(n+1) = f(n) + 1$ if n is even, and $f(n+1) = f(n) + 2$ if n is odd.*

4.1 Partitions of ABC graphs into H-components

In the next definition G represents a labeled graph (since it is obtained by B operations), but H is an unlabeled graph.

Definition 8 *A graph G that can be obtained from a graph H by assigning vertex labels and applying B operations is called an H-*component*.*

For an H-component G, the sequence of B operations that constructs G from a labeled graph isomorphic to H is called a *decomposition of G from H* (or *starting with H*). Note that G does not have to be an ABC graph, but it is an ABC graph if H is an ABC graph.

The following graphs will often be used for H in the contex of H-components:

K_2: a K_2-component is also called an *edge component*. If we consider a decomposition starting with a K_2 on vertices u and v, then this is called an edge component *between* u *and* v.

K_3: a K_3-component is also called a *triangle component*. If we consider a decomposition starting with a specific labeled copy of K_3, then the vertices of this K_3 are called the *triangle vertices* of this triangle component.

P_3: the end vertices of the P_3 are called the *end vertices* of the P_3-component.

C_2: both C_2 and P_3-components are associated with the C operation in a decomposition.

Next we prove the useful property that we can partition the edges of ABC graphs into edge induced subgraphs that are all H-components for $H = K_3$, C_2 or P_3.

Claim 14 *For any decomposition of an ABC graph G, we can partition the edges of G into sets $A_1, \dots, A_k$ and at most one set C such that for every i, $G[A_i]$ is a triangle component, and $G[C]$ is a C_2 or P_3-component.*

Proof: By induction on the number of operations. Observe that if G can be made from the ABC graph G' by an A or C operation, and the edge set M induces a graph H in G', then M induces the same graph H in G. The edges introduced by an A operation induce a K_3 (a triangle component), and the edges introduced by a C operation induce a P_3 or a C_2 (a P_3 or C_2-component). If G can be made from the ABC graph G' by a B operation, the statement is trivial. □

Triangle components that correspond to a decomposition of G in this way are called *A-components*, and the C_2 or P_3-component that corresponds to the C operation is called the *C-component*. See the bottom part of Figure 3 for an example of a partition of edges into two A-components and one C-component, corresponding to the given decomposition. For this graph it can be checked that any decomposition will give the same partition into A and C-components, but in general such a partition depends on the chosen decomposition.

Observation 15 *For any decomposition of an H-component G from a labeled copy of H, we can partition $E(G)$ into sets $\{E_{uv} : uv \in E(H)\}$ such that $G[E_{uv}]$ is an edge component between u and v. u and v are the only vertices of $G[E_{uv}]$ that can be connection vertices in G.*

The edge components $G[E_{uv}]$ from this observation will be denoted as $F(uv)$. For instance, for a triangle component with triangle vertices a, b and c (so a particular decomposition is chosen), we will often consider the subgraphs $F(ab)$, $F(ac)$ and $F(bc)$. $F(ab)$ is a 2-connection edge component with connection vertices a and b. Similarly, C_2 and P_3-components will be partitioned into two edge components.

Now that the main terminology and notations are defined, we will proceed by stating a large number of properties of ABC graphs, which we need in order to prove Conjecture 2 in Sections 7-10.

5 The structure of ABC graphs

5.1 Blocks in ABC graphs

The following lemma describes the block structure of ABC graphs.

Lemma 16 *An ABC graph G is connected and consists of odd order blocks and at most one even order block. For every decomposition of G, the odd order blocks are A-components, and the even order block B has the following structure: it contains the C-component C, and either $B = C$ and C is a C_2-component, or C is a P_3-component with end vertices x and y such that*

1. *There are $k \geq 1$ A-components $T_1, \ldots, T_k$ such that $E(B) = E(C) \cup E(T_1) \cup \ldots \cup E(T_k)$.*
2. *There are $k-1$ vertices $v_1, \ldots, v_{k-1}$ such that $V(T_i) \cap V(T_{i+1}) = \{v_i\}$, and no other A-components in B share vertices.*
3. *If $k = 1$ then $V(T_1) \cap V(C) = \{x, y\}$. If $k \geq 2$ then $V(T_1) \cap V(C) = \{x\}$ and $V(T_k) \cap V(C) = \{y\}$. No other A-components in B share vertices with C.*

Proof: Let G be an ABC graph. G is obviously connected. The proof is by induction on the number of operations in a decomposition of G. Consider a decomposition of G.

Suppose G is obtained by a B operation from an ABC graph G'. Observe that B operations do not change the block structure. This means that if $E' \subseteq E(G)$ induces a block and $e \in E'$, then a B operation on e will result in four edges that together with $E' - e$ form a block in the new graph, and all other blocks remain the same. Also all other properties in the lemma are maintained when B operations are applied: by definition an H-component is still an H-component after applying a B operation on one of its edges, all other H-components remain the same, the parity of the order of blocks does not change, and the vertices that two H-components have in common are unchanged by B operations.

Suppose G is obtained by an A operation from an ABC graph G'. Every A operation introduces a new block, consisting of the three new edges, and does not change the block structure of the rest of the graph. Again all other properties in the lemma are maintained by A operations. This already proves the statement for AB graphs.

Now we only have to prove the lemma for the case $G = C(G', x, y, z)$. For G' we know that the blocks correspond to A-components. Let $C = G[\{xz, yz\}]$. If $x = y$, then vertex x is a cut vertex in G, and the two new edges induce a new block in G. All other blocks remain the same. It is easy to check that the lemma holds.

If $x \neq y$, then consider a path P from x to y in G'. This path contains edges from A-components $T_1, \ldots, T_k$, numbered along the path P. Note that this path does not visit the same A-component twice, since in that case a cycle can be constructed in G' containing edges of multiple blocks (A-components) of G', which is a contradiction.

The edges of P together with xz and yz form a cycle in G, so $E(T_1), \ldots, E(T_k)$ and $E(C)$ are part of the same block B in G. Suppose there is a cycle K in G containing at least one edge from $E(T_1) \cup \ldots \cup E(T_k) \cup E(C)$ and at least one edge from another A-component of G'. Since K is not part of G', it must contain both xz and yz. Now replace xz and yz in K by the edges of P, and it can be checked that the resulting set of edges contains a cycle in G' through different A-components, a contradiction. We conclude that the block containing xz and yz is induced by $E(T_1) \cup \ldots \cup E(T_k) \cup E(C)$.

T_i and T_{i+1} clearly share a vertex ($1 \le i \le k-1$). Since they are blocks in G', they share at most one vertex. We call this vertex v_i. If T_i and T_j with $j > i+1$ share a vertex, then P can be used to construct a cycle through $T_i, T_{i+1}, \ldots, T_j$ in G', a contradiction.

If T_i with $1 < i < k$ contains x or y, then we can use P to construct a cycle through multiple A-components in G', a contradiction.

Finally, note that the number of vertices in the new block B is even, and that the C operation does not change parity of other blocks, or the fact that the other blocks are A-components. This concludes the proof. □

Note that there is an even order block in an ABC graph G if and only if G has even order (Observation 5).

We remark that a C_2-component that is not equal to a C_2 can be viewed as a combination of a triangle component and a P_3-component, so in this case the characterization of the even order block in Lemma 16 can be applied in two ways (see also Claim 24). But other than this case, the number of A-components is always the same in any decomposition of an ABC graph.

Loosely speaking, Lemma 16 says that blocks of an ABC graph are either A-components, or consist of a C-component connecting a path structure of A-components. We state some immediate corollaries of this lemma (without proofs). These will be used later, while referring to the above lemma:

- If an AB graph is 2-connected, it contains exactly one A-component.
- If the C operation is applied to two distinct vertices x and y of the same A-component, then the even order block has only one A-component.
- If the C operation is applied on x and y, then x is incident with only one A-component that is part of the even order block, and the same holds for y.
- If G is a 2-connected ABC graph such that the C operation is applied on x and y, then x and y are the only connection vertices of the C-component.

Corollary 17 *Let G be an even order ABC graph such that the C operation is applied on x and y. If M_1 is an edge cut for the C-component that separates x from y, and M_2 is an edge cut for an A-component T, then either M_2 or $M_2 \cup M_1$ is an edge cut for G.*

Proof: We use the notation from Lemma 16, but use the notation v_0 and v_k for x resp. y. If T is a block, the result follows directly from Observation 3. Otherwise, T is part of the even order block B, so $T = T_i$ for some i. If M_2 is not an edge cut for G, then $M_2 = [S_2, V(T_i)\backslash S_2]$ for some $S_2 \subset V(T_i)$, with $v_{i-1} \in S_2$, $v_i \notin S_2$. We know that $M_1 = [S_1, V(C)\backslash S_1]$ for some $S_1 \subset V(C)$, with $v_0 \in S_1$, $v_k \notin S_1$. Consider $S = S_1 \cup V(T_1) \cup \ldots \cup V(T_{i-1}) \cup S_2$. By Lemma 16, $[S, V(B)\backslash S] = M_1 \cup M_2$, and therefore $M_1 \cup M_2$ is an edge cut for G, since B is a block. □

The following corollary follows directly from Lemma 16 in combination with Observation 4.

Corollary 18 *Let $G \neq K_1$ be an ABC graph that is not 2-connected. Then G contains a 1-connection A-component. If G is an AB graph, then it contains at least two 1-connection A-components.*

We will also use the following observation about 2-connected ABC graphs.

Observation 19 *If G is a 2-connected ABC graph, then in any decomposition no A operation is applied after the C operation.*

5.2 Decompositions of triangle components

In this section we show that triangle components have many different decompositions. In particular, if edge uv is an edge in triangle component T, then T has a decomposition that starts with a triangle that contains edge uv. This is shown in Corollary 22. A similar statement appears in Claim 23.

Claim 20 *If an ABC graph G contains a 2-connection 4-cycle C, then a C4 operation on C yields another ABC graph G'. Moreover, if G is a triangle component, then G' is a triangle component.*

Proof: Consider a 2-connection 4-cycle C. As the main step in the proof, we first determine a decomposition of G such that all edges of C are introduced by the same B operation.

Consider a decomposition of G. If G is a triangle component, this decomposition starts with G_0 which is a triangle. Let G_i be the first intermediate graph that contains all edges of C (note that $i > 0$). Let operation a be the operation that is used to obtain G_i from G_{i-1}. So some of the edges of C are introduced by operation a.

Clearly, operation a is not an A operation.

Now suppose operation a is a B operation, say $G_i = B(G_{i-1}, uv, w, x)$. If $E(C) = \{uw, vw, ux, vx\}$, then the desired decomposition is found. Otherwise, C contains exactly two edges introduced by this operation, and w.l.o.g. $E(C) = \{uw, vw, uy, vy\}$ for some vertex y. Since $N_{G_i}(x) = N_{G_i}(y) = \{u, v\}$, we can switch the labels x and y throughout the decomposition of G_i, to get a decomposition of G_i where all edges of C are introduced by the B operation.

Then continue with the rest of the decomposition (without changing any labels). Apart from possibly the vertex labels, G_0 is still the same in the new decomposition.

If operation a is a C operation, then w.l.o.g. it is a C operation on u and v that introduces w, and $E(C) = \{uw, vw, ux, vx\}$. If x is introduced by a B operation that also introduces y, then this must be a B operation on uv since $d(x) = 2$ (Observation 9). Now instead let this B operation introduce w and x and immediately apply a C operation on u and v introducing y. Now proceed with the rest of the decomposition, which gives the desired decomposition. If x is introduced by an A operation, then w.l.o.g. this must be an A operation on u introducing x and v. Instead apply a C operation only on u introducing v, and immediately apply a B operation on one of the edges between u and v, introducing w and x. Observe that we always can find a decomposition where x is not the starting vertex, so this covers all cases for x.

In all cases we have determined a decomposition of G such that all edges of C are introduced by a B operation that introduces w and x. Since $d(w) = d(x) = 2$, by Observation 8 and Observation 9 we can assume that this is the last operation applied. So the graph G' obtained by a C4 operation on C is one of the intermediate graphs in this decomposition, and therefore is an ABC graph. In addition, if all operations in the original decomposition are B operations, then the initial graph G_0 is the same in our new decomposition, so if G is a triangle component, then G' is a triangle component. □

Claim 21 *If T is a triangle component with at least 5 vertices, then for every edge $e \in E(T)$, T contains a 2-connection 4-cycle that does not contain e.*

Proof: Observe that every triangle component on 5 vertices is a $K_{2,3}$, and for this graph the property holds. Now consider a triangle component T with at least 7 vertices, and a decomposition of T. For an edge e that is not one of the edges introduced by the last B operation, the statement is obvious. Otherwise, consider the triangle component T' from which T was constructed by a B operation on an edge e' (so T' does not contain e). By induction, T' contains a 2-connection 4-cycle C that does not contain e'. After the B operation on e', C still is a 2-connection 4-cycle in T, which proves the statement for T. Since triangle components have an odd number of vertices, this proves the claim by induction. □

By combining the previous two claims, we obtain a useful corollary.

Corollary 22 *If u is a vertex in a triangle component T, then a decomposition of T exists that starts on u. If uv is an edge in a triangle component T, then a decomposition of T exists where u and v are triangle vertices.*

Proof: We first prove the statement for the edge uv by induction. For $T = K_3$, the statement is clearly true. Otherwise, T contains a 2-connection 4-cycle K that does not contain edge uv (Claim 21). A C4 operation on K gives another triangle component T' (Claim 20), from which T can be constructed (assuming proper vertex labeling in T'). By induction, T' has a decomposition where u and v are triangle vertices.

The statement for the single vertex u follows immediately. □

Claim 23 *For a triangle component T and any two vertices $u, v \in V(T)$, a decomposition exists with triangle vertices u, a and b (and edge component $F(ab)$) such that $v \in V(F(ab))$ ($v = a$ or $v = b$ is possible).*

Proof: If $T = K_3$, then the statement is obvious. Now suppose $T \neq K_3$.

If a 2-connection 4-cycle K exists such that a C4 operation on K does not remove u or v, then apply this C4 operation. This gives another triangle component (Claim 20) with vertices u and v, and the statement follows by induction.

If no such 2-connection 4-cycle exists, then either u or v, say u, is a degree two vertex on a 2-connection 4-cycle. Choose an edge ua incident with u. By Corollary 22, a decomposition exists such that u, a and another vertex b are triangle vertices. Since $d(u) = 2$, both edge component $F(ua)$ and $F(ub)$ are single edges, so $v \in V(F(ab))$. □

5.3 Decompositions of ABC graphs

Claim 24 *If an ABC graph G is simple, then a decomposition of G exists such that every intermediate graph is simple.*

Proof: Consider a decomposition of a simple ABC graph G. The only way that parallel edges can be introduced is with a C operation on $x = y$, introducing z (Observation 10). Since G is simple, one of the edges between x and z must be used in a B operation, that introduces v and w. Now instead of the C operation, use an A operation on x that introduces v and z. Instead of the B operation, use a C operation on x and z that introduces w. This gives the desired decomposition of G. □

Claim 24 implies that for every ABC graph G there is a decomposition where the C-component is a P_3-component, unless the C-component consists of two parallel edges.

Claim 25 *If an ABC graph G contains a 1-connection A-component T with connection vertex u, then G has a decomposition in which $G[E(G)\backslash E(T)]$ is an intermediate graph.*

Proof: Consider a decomposition of G and a 1-connection A-component T with connection vertex u.

Suppose T is a triangle. Let $V(T) = \{u, v, w\}$ with $d(v) = d(w) = 2$. W.l.o.g. a decomposition exists that does not start with v or w, so v and w are not used in any operation in this decomposition (Observation 9). Therefore they are both introduced by an A operation on u, and we can assume that this operation is the last operation in the decomposition (Observation 8). So $G[E(G)\backslash E(T)]$ is an intermediate graph in this decomposition of G.

If T is not a triangle, choose a vertex v such that $uv \in E(T)$. By Claim 21, T contains a 2-connection 4-cycle C that does not contain uv. Therefore u is not one of the degree two vertices on C. Since T is a 1-connection A-component, C is also a 2-connection 4-cycle in G. By Claim 20, a C4 operation on C yields an ABC graph G'. G can be constructed from G' by a B operation. For G' the statement is true by induction on the size of T, and therefore the statement is true for G. □

Claim 26 *If T is an A-component of an AB graph G, then a decomposition of G exists in which T is an intermediate graph or $T = G$.*

Proof: We use induction on the number of A-components. If G has only one A-component, then the statement is trivial. Otherwise, G has a cut vertex (Lemma 16), and therefore at least two 1-connection A-components (Corollary 18). So there is a 1-connection A-component $T' = G[E']$ that is not equal to T. By Claim 25, G can be constructed from the AB graph $G' = G[E(G)\backslash E(T')]$. G' has fewer A-components than G (in any decomposition). By induction G' has a decomposition in which T is an intermediate graph, which proves the claim. □

Note that a similar statement is not true for ABC graphs: consider the ABC graph from Figure 3. Apply an A operation on the vertex introduced by the C operation. It can be checked that there is no decomposition of the resulting ABC graph in which the A-component introduced by the last A operation is an intermediate graph.

Claim 27 *If an ABC graph G contains a triangle T, then a decomposition exists in which T is an A-component.*

Proof: We consider a decomposition of G such that G is obtained from ABC graph G' with a single operation a, such that G' is simple if G is simple (Claim 24). Clearly, for every triangle in G that is also a triangle in G' the statement is true by induction. So if suffices to consider triangles in G that use edges introduced by operation a.

If operation a is an A operation, then for the new triangle the statement is true. (Note that no new edges combine with old edges to form a triangle.)

If $G = B(G', uv, w, x)$, and there is a triangle in G that contains edges from $C = G[u, v, w, x]$, then w.l.o.g. this triangle is equal to $G[u, v, w]$. This means that in G', two parallel edges exist between u and v, but G is simple (by Observation 10 there are at most two parallel edges between u and v and no other parallel edges exist in G'), a contradiction with our choice of the decomposition.

Finally consider the case that $G = C(G', x, y, z)$, and there is a triangle K in G that contains edge xz or yz. Then $K = G[x, y, z]$, and $xy \in E(G')$. xy is part of an A-component T of G' (since G' is an AB graph). By Claim 26, there is a decomposition of G' that starts with the construction of T. By Corollary 22, there is a decomposition of T that has x and y as triangle vertices. Let z' be the third triangle vertex in this decomposition. So a decomposition of G' exists that starts with a triangle with vertices x, y and z'. Now instead start with a triangle with vertices x, y and z, and apply a C operation on x and y introducing z'. Proceed with the rest of the decomposition of G'. This is a decomposition of G such that the triangle K is introduced by an A operation. □

6 Edge components and matching-cuts

In this section, we show various ways to find matching-cuts for edge components, and for graphs deduced from edge components, triangle components and P_3-components by an expansion operation.

Throughout this section, we use G_{uv} to denote an edge component with which we associate a decomposition from a copy of K_2 with vertex labels u and v. In the following proofs we use the fact that every edge component G_{uv} not equal to a single edge has at least one 2-connection 4-cycle C such that a C4 operation on C yields again an edge component between u and v. In a decomposition of G_{uv}, the edges introduced by the last B operation correspond to such a 2-connection 4-cycle. A C4 operation on this 2-connection 4-cycle allows us to use induction.

Claim 28 *For every edge component G_{uv} and every edge $e \in E(G_{uv})$, there is a matching-cut M that separates u from v with $e \in M$.*

Proof: If $E(G_{uv}) = \{uv\}$ then the statement is true. Otherwise, consider a 2-connection 4-cycle C between x and y. A C4 operation on C gives another edge component G'_{uv}. If $e \in E(G'_{uv})$, then consider a matching-cut M for G'_{uv} that contains e and separates u from v (induction). M can be turned into a matching-cut in G_{uv} with the desired properties, also if $xy \in M$. If $e \notin E(G'_{uv})$, then consider a matching-cut M for G'_{uv} that separates u from v with $xy \in M$ (induction). M can be turned into a matching-cut M' in G_{uv} that separates u from v with $e \in M'$. □

Claim 29 *For every edge component G_{uv} and every vertex $w \in V(G_{uv})\backslash\{v\}$, there is a matching-cut M that separates $\{u, w\}$ from $\{v\}$.*

Proof: If $E(G_{uv}) = \{uv\}$ then the statement is true. Otherwise, a 2-connection 4-cycle C between x and y exists such that a C4 operation on C yields another edge component G'_{uv}. If $w \in V(G'_{uv})$, then start with a matching-cut M for G'_{uv} that separates $\{v\}$ from $\{u, w\}$ (induction). M can be turned into a matching-cut for G_{uv} with the desired properties, also if $xy \in M$. If $w \notin V(G'_{uv})$, consider a matching-cut M for G'_{uv} that separates u from v with $xy \in M$ (Claim 28). This can be made into a matching-cut in G_{uv} with the desired properties. □

The proof of the following claim is illustrated in Figure 4.

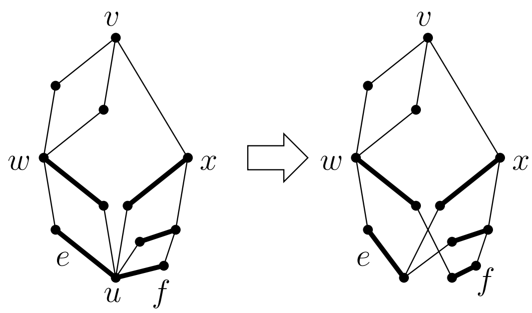

Figure 4: A matching-cut in a split edge component that is not incident with v

Claim 30 *If a graph G' can be made from an edge component G_{uv} by a non-trivial edge expansion of u and possibly deletion of the resulting edge, then there is an edge cut M in G_{uv} that separates u from v, is not incident with v, and is a matching-cut in G'.*

Proof: Since G' is obtained by a non-trivial edge expansion, in G_{uv} we have $d(u) \geq 2$. So in a decomposition of G_{uv} at least one B operation is applied. Let the first B operation introduce two vertices w and x. So the edges of G_{uv} can be partitioned into 2-connection edge components $F(uw)$, $F(ux)$, $F(vw)$ and $F(vx)$. Since the edge expansion is non-trivial, we can find $e \in E(F(uw))$ and $f \in E(F(ux))$ such that e and f are split by the edge expansion. Let $M_1 = [S_1, T_1]$ be a matching-cut for $F(uw)$ with $u \in S_1$, $w \in T_1$ and $e \in M_1$ (Claim 28). Let $M_2 = [S_2, T_2]$ be a matching-cut for $F(ux)$ with $u \in S_2$, $x \in T_2$ and $f \in M_2$. The only adjacent edges in $M_1 \cup M_2$ are e and f, so $M_1 \cup M_2$ becomes a matching in G'. Considering the vertices that the four edge components have in common, we see that $M_1 \cup M_2 = [S_1 \cup S_2, T_1 \cup T_2 \cup V(F(vw) \cup V(F(vx))]$ is an edge cut in G_{uv} and therefore also an edge cut in G'. □

Claim 31 *If the graph G' can be made from the edge component G_{uv} by a non-trivial edge expansion of u, then for any two vertices $w, x \in V(G_{uv})\backslash\{u, v\}$ an edge cut for G_{uv} exists that separates u from v, does not separate w from x and is a matching-cut in G'.*

Proof: Consider a decomposition of G_{uv}. If the first B operation in this decomposition introduces both w and x, then we can construct the same edge cut as in the proof of the previous claim (see Figure 4). This edge cut separates $\{u\}$ from $\{v, w, x\}$.

Otherwise, we can actually construct a matching-cut in G_{uv} (instead of in G') that separates u and v but does not separate w and x. The proof is again by induction. Let C be the 2-connection 4-cycle in G_{uv} that corresponds to the last B operation in the decomposition. A C4 operation on C yields an edge e in an edge component G'_{uv}. Since the first B operation does not introduce both w

and x, we know that G_{uv} is not equal to a 4-cycle and therefore G'_{uv} does not consist of a single edge. W.l.o.g. we can consider three cases:

1. The C4 operation removes w, but not x. Consider a matching-cut $M = [S, \overline{S}]$ for G'_{uv} that separates u from v and includes e (Claim 28). If $x \in S$ ($x \in \overline{S}$) then M is easily turned into a matching-cut $[S', \overline{S}']$ for G_{uv} with $\{w, x\} \subseteq S'$ ($\{w, x\} \subseteq \overline{S}'$).

2. The C4 operation removes both w and x. In G'_{uv}, let f be an edge that is adjacent to the edge e. Consider a matching-cut M for G'_{uv} such that $f \in M$ (Claim 28). Now $e \notin M$, so M can be made into a matching-cut for G_{uv} with the desired properties.

3. The C4 operation neither removes w nor x. Recall that in the decomposition of G'_{uv} we consider, w and x are not introduced by the first B operation. So by induction, G'_{uv} has a matching-cut M that separates u from v but does not separate w from x. M is easily turned into a matching-cut in G_{uv} with the same properties.

Note that if G'_{uv} is a C_4, then case 1 or 2 applies since we assume that w and x are not introduced by the first B operation. This proves the induction base.

All matching-cuts constructed above are also matching-cuts in G'. □

The following two lemmas are useful for determining matching-cuts in graphs made from ABC graphs by an expansion operation.

Lemma 32 *Let T' be a graph that can be made from a triangle component T with triangle vertices u, v and w by a non-trivial edge expansion of v.There is an edge cut M in T that is not incident with any vertex in $V(F(uw))\backslash\{u, w\}$ and that is a matching-cut in T'.*

Proof: The edges of T can be partitioned into 2-connection edge components $F(uv)$, $F(uw)$ and $F(vw)$ (Observation 15). Since the edge expansion is non-trivial, we can find $e \in E(F(uv))$ and $f \in E(F(vw))$ such that e and f are split by the expansion. Let $M_1 = [S_1, T_1]$ be a matching-cut for $F(uv)$ with $v \in S_1$, $u \in T_1$ and $e \in M_1$ (Claim 28). Similarly, let $M_2 = [S_2, T_2]$ be a matching-cut for $F(vw)$ with $v \in S_2$, $w \in T_2$ and $f \in M_2$. The only adjacent edges in $M_1 \cup M_2$ are e and f, so the edges of $M_1 \cup M_2$ form a matching in T'. It can be checked that $M_1 \cup M_2 = [S_1 \cup S_2, T_1 \cup T_2 \cup V(F(uw))]$ is an edge cut in T and therefore also an edge cut in T'. Since M_1 and M_2 contain only edges from $F(uv)$ and $F(vw)$, for every $a \in V(F(uw))\backslash\{u, w\}$, a is not incident with edges from $M_1 \cup M_2$. □

Lemma 33 *Let P' be a graph that can be made from a P_3-component P with end vertices x and y by a non-trivial edge expansion of x, and possibly an edge expansion of y. In P an edge cut M exists that does not separate x and y and that is a matching-cut in P'.*

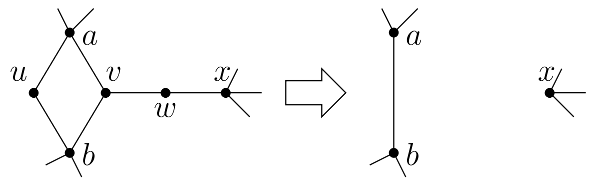

Figure 5: The operation from the proof of Claim 35

Proof: The edges of P can be partitioned into 1-connection edge components $F(xz)$ and $F(yz)$ that only have z in common. Since x is only incident with edges from $F(xz)$, the edge expansion of P corresponds to a non-trivial edge expansion of $F(xz)$ into F'. Now let M_1 be an edge cut for $F(xz)$ that separates x from z, contains no edges incident with z, and that is a matching in F' (Claim 30). Let M_2 be any matching-cut for $F(yz)$ that separates y from z. $M_1 \cup M_2$ forms the desired edge cut. Since M_1 is a matching in F' and contains no edges incident with z, this is a matching-cut in P'. □

7 A proof by contradiction: properties of minimum counterexamples

We want to prove that every extremal immune graph is an ABC graph. Our proof is by contradiction, so first we assume an extremal immune graph exists that is not an ABC graph. Then we consider a graph with minimum size among all such graphs, and derive a contradiction by exploring the properties of this possible counterexample. This explains the following definition.

Definition 9 *A graph G is a* minimum counterexample *if it is extremal immune, it is not an ABC graph, and has minimum size among all such graphs.*

Claim 34 *A minimum counterexample G contains no 2-connection 4-cycle.*

Proof: If there is a 2-connection 4-cycle C, apply a C4 operation on C. This yields an extremal immune graph G' (Lemma 6). If G' is ABC, then G is ABC (use a B operation). Otherwise, G is not a minimum counterexample. □

In Figure 5 the following claim and proof are illustrated.

Claim 35 *A minimum counterexample G contains no vertices u, v and w with $N(u) = \{a, b\}$, $N(v) = \{a, b, w\}$ and $d(w) = 2$.*

Proof: Consider $G' = G - u - v - w + ab$. If $ab \in E(G)$ then G' will have two parallel edges between a and b. $|V(G')| = |V(G)| - 3$ and $|E(G')| \leq |E(G)| - 5$. Therefore $|E(G')| \leq 3(|V(G')| - 1)/2$ and this inequality is strict if $|V(G')|$ is even (Observation 13). It can be checked that G' is immune again, so $|E(G')| \geq 3(|V(G')| - 1)/2$ (Theorem 1). We conclude that $|V(G')|$ is odd and G' is an extremal immune graph. Since G is a minimum counterexample, G' is an ABC graph, and since $|V(G')|$ is odd it is an AB graph. In G, let $N(w) = \{v, x\}$ ($x = a$ or $x = b$ is possible). In G', apply a B operation on the new edge between a and b, introducing u and v. Now a C operation can be applied on v and x introducing w. This way G is obtained, so G is ABC, a contradiction. □

Claim 36 *A minimum counterexample G contains no i-connection subgraph H that is an AB graph with $|V(H)| > 3$ and $i \leq 3$.*

Proof: Suppose G contains such a subgraph H.

Choose three vertices u, v and w in $V(H)$ such that $y \in V(H)\backslash\{u, v, w\}$ implies that y is only incident with edges in H (y is not a connection vertex of H).

Replace H with a triangle T with vertices u, v and w, such that none of the edges outside of H are destroyed. Call the new graph G'. (Formally $G' = ((V(G)\backslash V(H)) \cup \{u, v, w\}, (E(G)\backslash E(H)) \cup \{uv, vw, uw\})$.) It can be checked that G' is again extremal immune ($|V(G)| - |V(G')| = 2k$ for some k, and $|E(G)| - |E(G')| = 3k$). Since G is a minimum counterexample, G' is an ABC graph. By Claim 27, a decomposition of G' exists such that all edges of T are introduced by the same A operation. W.l.o.g., this is an A operation on u introducing v and w. By Corollary 22 and Claim 26, H has a decomposition that starts with u (without C operations). In the decomposition of G', use this decomposition of H instead of the A operation introducing T. This is a decomposition of G, so G is an ABC graph, a contradiction. □

Claim 37 *A minimum counterexample G is simple.*

Proof: If G has vertices u and v with at least three parallel edges between them, then one of these edges can be deleted and the resulting graph is still immune, contradicting Theorem 1.

Now suppose there are two parallel edges between u and v. Suppose that a C2 operation on u and v gives an ABC graph G' with vertex u resulting from the contraction. Since G' is immune (Lemma 6), G' has odd order (Observation 13, Theorem 1) and thus is an AB graph.

First assume that two edges e and f that are part of the same A-component T are split in the construction of G from G'. Then a matching-cut for G can be obtained: consider an edge set $M \subset E(T)$ that is an edge cut in T and a matching in G (Lemma 32). Since T is a block in G' (Lemma 16), M is also an edge cut in G' (Observation 3), and therefore an edge cut in G.

So now we may assume that no A-component is split. We complete the proof by showing how an ABC decomposition of G can be obtained: by Claim 26 and Corollary 22, G' has a decomposition that starts with u. Now first apply a C operation on u introducing v, and proceed with the rest of the decomposition. If an A-component T is introduced by an A operation on u, then $G[E(T)]$ is a triangle component that is incident with only one of u or v in G, since T is not split. Apply an A operation on either u or v instead in this new decomposition.

Note that every A-component T that is incident with u in G' is introduced by an A operation on u in the chosen decomposition of G', so by changing the operations this way, we obtain an ABC decomposition of G. □

Claim 38 *A minimum counterexample G is 2-connected.*

Proof: Suppose G has a cut vertex v. Let $Q_1, \ldots, Q_k$ be the components of $G - v$. Define $G_1 = G[V(Q_1) \cup \{v\}]$ and $G_2 = G[V(Q_2) \cup \ldots \cup V(Q_k) \cup \{v\}]$. Let $n = |V(G)|$, $n_1 = |V(G_1)|$, $n_2 = |V(G_2)|$, $m = |E(G)|$, $m_1 = |E(G_1)|$ and $m_2 = |E(G_2)|$. Observe that $n = n_1 + n_2 - 1$, $m = m_1 + m_2$. Let $f : n \to \lceil 3(n-1)/2 \rceil$. Then $m = f(n)$, and clearly $m_1 \geq f(n_1)$ and $m_2 \geq f(n_2)$ (Theorem 1).

$$m = m_1 + m_2 \geq \lceil 3(n_1 - 1)/2 \rceil + \lceil 3(n_2 - 1)/2 \rceil \geq$$

$$\lceil 3(n_1 - 1)/2 + 3(n_2 - 1)/2 \rceil = \lceil 3(n - 1)/2 \rceil = m.$$

We conclude that both inequalities are equalities, and therefore $m_1 = f(n_1)$ and $m_2 = f(n_2)$, so both G_1 and G_2 are smaller extremal immune graphs. Hence both are ABC. Since the second inequality above is also an equality, we know that at least one of n_1 and n_2 is odd. W.l.o.g. n_1 is odd, so G_1 is an AB graph. By Corollary 22 and Claim 26, a decomposition of G_1 can start with any vertex. Now consider a decomposition of G_2, and add a decomposition of G_1 that starts with v. Together this is a decomposition of G, a contradiction. □

8 A minimum counterexample contains no C_4

In this section, we prove the following lemma:

Lemma 39 *A minimum counterexample G contains no C_4.*

The proof is by contradiction. Suppose subgraph C of G is a 4-cycle with edge set $\{u_1u_2, u_2v_1, v_2v_2, v_2u_1\}$. W.l.o.g., assume that $d(u_1) \geq 3$ and $d(v_1) \geq 3$, for if two neighbors on C have degree 2 then a matching-cut is immediate. Applying operation C4 on C gives a new graph G'. The resulting edge in G' will be called uv (vertex u results from the contraction of u_1u_2, and v results from the contraction of v_1v_2). G' is extremal immune (Lemma 6), and therefore G' is an ABC graph.

Consider a decomposition of G' such that edge sets $E_1, \ldots, E_k$ induce the A-components $T_1, \ldots, T_k$, and if the order of G' is even, let the edge set F induce the C-component P (Claim 14). These edge sets correspond to edge sets $E'_1, \ldots, E'_k$ and F' of G such that if $uv \in E_i$ (or F), then $E(C) \subseteq E'_i$ (resp. F'). Observe that $\{E'_1, \ldots, E'_k, F'\}$ partitions $E(G)$. These edge sets induce subgraphs $T'_1, \ldots, T'_k$ and P' of G.

At first it may be confusing that parts of G' are denoted without primes and parts of G are denoted with primes, but note that G' is deduced from G,

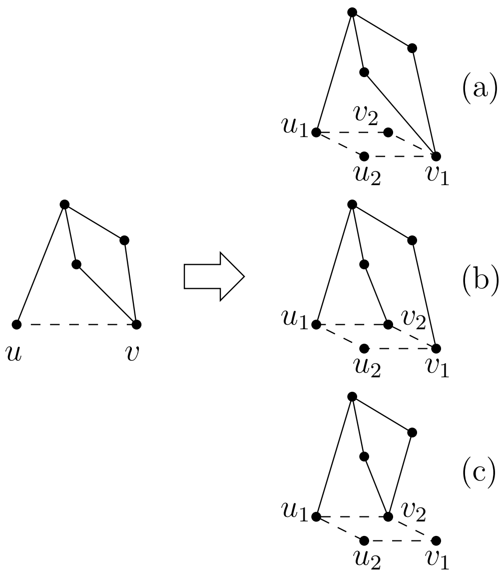

Figure 6: Three different ways to reverse a C4 operation

$E_1, \ldots, E_k$ and F are a natural partition of the edges of ABC graph G', and $E'_1, \ldots, E'_k$ and F' are deduced from these sets.

Since the C4 operation consists of two edge contractions and one edge deletion, we can use the notion of splitting from Section 2.1 for edge induced subgraphs of G'. However, note that the edge in G that corresponds to uv in G' can be u_1v_2 or u_2v_1, and on this arbitrary choice it can depend whether a subgraph $G'[M]$ with $uv \in M$ is split. Therefore we will slightly abuse the definition in this section, and say that $G'[M]$ *is split if* $G'[M - uv]$ *is not isomorphic to* $G[M - uv]$.

In Figure 6, some examples are shown. Only in case (b) the triangle component is split. Observe that it is possible that a triangle component $G'[M]$ is not split, while $G[M]$ is not isomorphic to a triangle component (Figure 6(c)). Note that in this case we may assume w.l.o.g. that the case in Figure 6(c) holds: if a triangle component T_i with triangle vertices u, v and w is not split, then in G no edges of edge component $F(uw)$ are incident with u_1 or none are incident with u_2, and a similar statement is true for $F(vw)$. Suppose no edges of $F(uw)$ are incident with u_2 (u_1) in G, and no edges of $F(vw)$ are incident with v_2 (v_1) in G (Figure 6(a)). Then T'_i can be obtained from T_i by a B operation on uv, and renaming the resulting vertices. Thus if T_i is not split and T'_i is not a triangle component, then edges of $F(uw)$ are not incident with u_2 (u_1) in G, and edges of $F(vw)$ are not incident with v_1 (v_2) in G, so w.l.o.g. the case in Figure 6(c)) holds.

If the order of G' is even, one C operation is used in every decomposition of G'. In the decomposition we consider, x and y will denote the vertices in G'

on which the C operation is applied, and z will denote the vertex introduced by the C operation. The notation $F(xz)$ and $F(yz)$ will be used to denote the edge components between x and z resp. between y and z of which P consists.

Note that vertices in G' directly correspond to vertices in G, except for u and v. So if for instance $u = x$, then there is not necessarily a unique vertex in G that corresponds to x. However, if $uv \notin F$ and no edges of $F(xz)$ are incident with u_2 (u_1) in G (so $F(xz)$ is not split), then u_1 (u_2) is called vertex x in G. For the cases $x = v$, $y = u$ and $y = v$ the notation is similar.

With G, G', u, v etc. defined as above, we first state a number of claims before Lemma 39 can be proved.

Claim 40 *G' has a decomposition with at least one A-component.*

Proof: We consider a decomposition of G' such that every intermediate graph is simple if G' is simple (Claim 24). Suppose no A operations are applied in the decomposition. So a decomposition of G' starts with a C operation on $x = y$. If at least one B operation is applied, then G' is simple (Observation 10), a contradiction with our choice of the decomposition. We conclude that $G' = C_2$. It is easy to check that G either is ABC or has a matching-cut, a contradiction. □

Claim 41 *If an A-component T_i of G' is split then the order of G' is even, and T_i contains both x and y and $x \neq y$. In addition, if G' is 2-connected then T_i is the only A-component of G'.*

Proof: Let T_i be an A-component that is split. W.l.o.g. $u \in V(T_i)$. First we construct a matching-cut M in T_i'.

If $v \notin V(T_i)$, then T_i' can be obtained from T_i by a non-trivial edge expansion of u into u_1u_2, and deleting u_1u_2. By Lemma 32, an edge cut M can be constructed in T_i that is a matching-cut in T_i'.

If $v \in V(T_i)$ then $uv \in E(T_i)$. T_i has a decomposition such that u, v and another vertex w are triangle vertices (Corollary 22). So $T_i - uv$ can be seen as a P_3-component. This P_3-component is split, so an edge cut M can be constructed in T_i that is a matching-cut in T_i' (Lemma 33).

In both cases an edge cut M in T_i is constructed that is a matching-cut in T_i', and that does not contain uv. If T_i is a block of G', then M is an edge cut in G' (Observation 3). Note that an edge cut in G' that does not contain uv is an edge cut in G, since the C4 operation can be reversed by adding a parallel edge to uv and expanding u and v, and edge expansions do not destroy edge cuts. So of T_i is a block of G', M is a matching-cut in G. Thus T_i is part of an even order block, and therefore the order of G' is even and $x \neq y$ (Lemma 16).

If $x \notin V(T_i)$, then consider any matching-cut M' for $F(xz)$ that separates x from z. Because of the block structure of G', either M or $M \cup M'$ is an edge cut in G' (Corollary 17). Because $uv \notin M$ and $uv \notin M'$, this edge cut in G' is also an edge cut in G. Since $x \notin V(T_i)$, we know that $V(T_i) \cap V(F(xz)) = \emptyset$ (Lemma 16), so $M \cup M'$ is also a matching in G, and we have found a matching-cut in G.

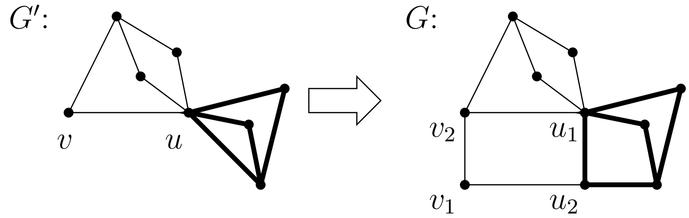

Figure 7: An example where G' has a cut vertex

If $x \in V(T_i)$ but $y \notin V(T_i)$, the matching-cut construction is similar. So since G is immune, the order of G' is even, $x \in V(T_i)$ and $y \in V(T_i)$. If G' is 2-connected, then T_i is the only A-component in G' (Lemma 16). □

Claim 42 *G' is 2-connected, has even order and $x \neq y$.*

Proof: Suppose G' has a cut vertex. Since G cannot have a cut vertex (Claim 38), this cut vertex must be u or v. W.l.o.g., u is a cut vertex in G'. G' contains a 1-connection A-component (Corollary 18). If G' contains a 1-connection A-component that does not contain uv, let T_i be this A-component, otherwise let T_i be the only 1-connection A-component.

If T_i is split, then it contains x and y, and $x \neq y$ (Claim 41). In this case x and y are two different connection vertices of T_i, a contradiction. So T_i is not split. Then if $uv \notin E(T_i)$ then the connection vertex of T_i in G' also corresponds to a cut vertex in G, a contradiction (Claim 38).

So $uv \in E(T_i)$, and because of our choice of T_i, we conclude that T_i is the only 1-connection A-component. Since u is the only cut vertex in G' in this case, G' consist of two blocks: T_i and an even order block B (Lemma 16). Note that B is an ABC graph (Claim 25), and therefore extremal immune.

If T_i' is again a triangle component (see Figure 6(a)), then G contains a triangle component subgraph on at least five vertices with at most two connection vertices, a contradiction with Claim 36. So since T_i is not split, w.l.o.g. u_2 and v_1 have degree two in T_i' (see Figure 6(c)). B is split, otherwise u_1 or u_2 is a cut vertex in G. Consider $B' = G[E(B) \cup \{u_1u_2\}]$. This case is illustrated in Figure 7, where B and B' are shown by the bold edges. If B' has a matching-cut M, then M or $M \cup \{v_1v_2\}$ is a matching-cut in G, so B' is immune. B' has one more vertex and one more edge than B, and the order of B is even, so B' is also extremal immune. Since B' has odd order and is smaller than G, B' is an AB graph. If $|V(B')| > 3$, then Claim 36 leads to a contradiction. So B' is a triangle on vertices u_1, u_2 and another vertex w.

Now we can show that G is ABC. Start with a decomposition of T_i. Rename u as u_1, and v as v_2. Apply an A operation on u_1 introducing w and u_2. Apply a C operation on u_2 and v_2 introducing v_1. Now graph G is obtained.

In every case a contradiction is obtained, so we conclude that G' is 2-connected. Suppose that the order of G' is odd. Then G' consists of only one triangle component T_1 (Lemma 16). If T_1' is again a triangle component, then G is an ABC graph, a contradiction. But T_1 is not split (Claim 41), so w.l.o.g. $d(u_1) = d(v_2) = 2$, and G has a matching-cut (see Figure 6(c)). We conclude that the order of G' is even, and since there is at least one A-component (Claim 40), it follows that $x \neq y$ (Lemma 16). □

Note that since $x \neq y$, G' must be simple (Observation 10), so we can conclude that C is an *induced* 4-cycle in G.

For the following claim recall that P is the C-component of G'.

Claim 43 *If $uv \notin E(P)$ then P consists only of the edges xz and yz.*

Proof: Assume the C-component P is split. We know that x and y are the only connection vertices of P (Claim 42, Lemma 16). Since we assume that uv is not part of P, w.l.o.g. $u = x$, and the C-component is split at x. So P' can be obtained from a P_3-component with end vertices x and y by a non-trivial edge expansion of x (and deletion of the resulting edge), and possibly also an edge expansion of y if $y = v$. By Lemma 33, an edge cut M for P exists that does not separate x and y and that is a matching-cut in P'. Since x and y are the only connection vertices of P, M is also an edge cut in G' and a matching-cut in G.

So the C-component is not split. Now if at least one B operation is applied to P, then G' contains a 2-connection 4-cycle which is also a 2-connection 4-cycle in G, a contradiction with Claim 34. □

So if $uv \notin E(P)$, the C-component is not split, and therefore we can unambiguously identify two vertices x and y in G corresponding to x and y in G'.

Claim 44 *If an A-component T_i of G' is not split, then T_i' in G is also a triangle component.*

Proof: Suppose this is not true. Note that since at least one A-component is not split, no A-component is split (Claim 41, Claim 42). If $uv \notin E(T_i)$ and T_i is not split then T_i' is isomorphic to T_i by definition, and therefore a triangle component. So $uv \in E(T_i)$. Consider a decomposition of T_i with triangle vertices u, v and w (Corollary 22). Suppose T_i is not split and T_i' is not a triangle component. Then w.l.o.g. in G no edges of edge component $F(uw)$ are incident with u_2 in G, and no edges of $F(vw)$ are incident with v_1 in G (see Figure 6(c)).

Consider $G - u_1u_2 - v_1v_2$. Observe that in T_i', $\{u_1u_2, v_1v_2\}$ is an edge cut. Now suppose there exists a path R in G from u_1 or v_2 to u_2 or v_1 that only uses edges of A-components in G' (R contains no edges of the C-component of G' and contains no edges of the 4-cycle $G[u_1, u_2, v_1, v_2]$). Since $\{u_1u_2, v_1v_2\}$ is an edge cut in T_i' separating u_1 and v_2 from u_2 and v_1, R contains edges of other A-components. Since no A-components are split, for every $j \neq i$, T_i' and T_j' share at most one vertex. Therefore this path contains edges of at least two A-components. In G', $E(R)$ or $E(R) \cup \{uv\}$ induces a cycle. This is a cycle through different A-components that does not use edges of the C-component,

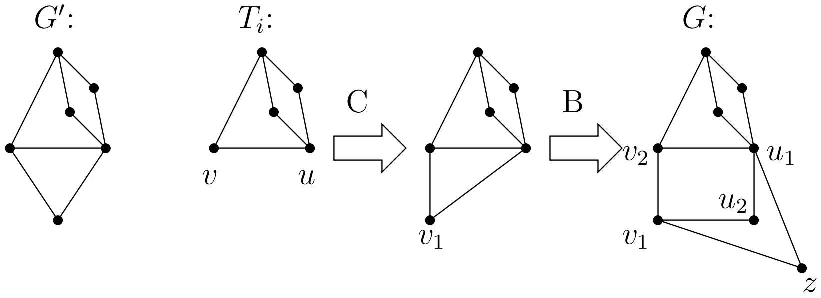

Figure 8: A decomposition of G from T_i

a contradiction with Lemma 16. We conclude that such a path does not exist, and therefore $\{u_1u_2, v_1v_2\}$ is an edge cut in $G[E'_1, \ldots, E'_k]$.

Let $\{S, T\}$ be the partition of the vertices of $G[E'_1, \ldots, E'_k]$ such that $[S, T] = \{u_1u_2, v_1v_2\}$. Since G is immune, these edges are not part of a matching-cut in G. Because G' is 2-connected, no A operations are applied after the C operation (Observation 19). So there is a decomposition of G' that ends with a C operation on vertices x and y in the AB graph $G'[E_1, \ldots, E_k]$ (Observation 8, Claim 43). We conclude that w.l.o.g. $x \in S$ and $y \in T$ and x and y are incident with $[S, T]$. Since $x \neq y$ in G', w.l.o.g. $x = u_1$ and $y = v_1$ in G. This implies that $x = u$ and $y = v$ in G', and therefore T_i is the only A-component of G' (Lemma 16).

Consider the following decomposition (see Figure 8). Start with a decomposition of triangle component T_i, and rename u as u_1 and v as v_2. Apply a C operation on u_1 and v_2 introducing v_1. Apply a B operation on u_1v_1, introducing z and u_2. This is an ABC decomposition of G. This final contradiction proves the claim. □

Claim 45 *There is a decomposition of G' such that uv is part of an A-component.*

Proof: Suppose G' only has decompositions for which uv is part of the C-component.

Consider the case where there is an A-component T_1 that has a decomposition such that x and y are triangle vertices. Because G' is 2-connected (Claim 42), T_1 is the only A-component (Lemma 16). Let z' be the third triangle vertex in this decomposition of T_1. Now there is a decomposition of G' that starts with a triangle on x, y and z', then applies a C operation on x and y introducing z, and ends with a number of B operations. Now if instead we start with a triangle on x, y and z and apply a C operation on x and y introducing z', a decomposition of G' is obtained such that uv is part of an A-component, a contradiction. So there is no A-component that has a decomposition where x and y are triangle vertices.

Now suppose that an A-component T_i is split. In this case, w.l.o.g. $u = x$ and $u \in V(T_i)$ since x and y are the only connection vertices of P. By Claim 41, $\{x, y\} \in V(T_i)$. Now T'_i can be obtained with a non-trivial edge expansion of u

and deleting the resulting edge. Consider a decomposition of T_i with u, a and b as triangle vertices such that $y \in V(F(ab))$ (Claim 23). We have shown that $y \neq a, b$. By Claim 32, an edge cut M_1 exists in T_i that is a matching-cut in T_i' such that y is not incident with edges from M_1. If y is separated from u by M_1 (in fact this is always true), M_1 can be made into an edge cut for G' and G by adding the edges of a matching-cut M_2 for $F(yz)$ that separates y from z. $M_1 \cup M_2$ is a matching in G since none of the vertices in $F(yz)$ are incident with edges of M_1.

This contradiction shows that we may assume that no A-component is split. At least one A-component exists (Claim 40). If G' has a single A-component and this is a triangle, then x and y are triangle vertices of the same A-component, a contradiction. Therefore, $G'' = G'[E_1 \cup \ldots \cup E_k]$ has at least five vertices, and is an AB graph (we may assume that the chosen decomposition ends with a C operation and a number of B operations on the C-component, so G'' is an intermediate graph). In G', the subgraph G'' has two connection vertices (x and y). No A-component is split, and x and y are both incident with exactly one A-component (Lemma 16). It follows that G'' is also a 2-connection AB subgraph of G, a contradiction (Claim 36). □

Claim 46 *At least one A-component of G' is split.*

Proof: Assume no A-components are split. Then if T_i is the A-component containing uv (such an A-component exists by Claim 45), T_i' can be obtained from T_i with a B operation (Claim 44). Since G' is 2-connected, a decomposition of G' exists such that no A operation is applied after the C operation. So using Claim 43 and Claim 45, $G' - z$ is an AB graph. uv is part of A-component T_i in $G' - z$. By Claim 26, there is a decomposition of $G' - z$ that starts with the construction of T_i. There is a decomposition of T_i that starts with a triangle on u, v and another vertex w (Corollary 22). Instead of starting only with this triangle, start with this triangle and apply a B operation on edge uv, introducing u_2 and v_2, and rename u and v as u_1 resp. v_1. Continue with the rest of the decomposition. If an A-component T_j is introduced by an A operation on u, then T_j' is only incident with u_1 or u_2 in G, since T_j is not split. So apply an A operation on u_1 resp. u_2 instead. This is similar for v and for the C-component. This gives an ABC decomposition of G. □

We summarize the above results in the following Corollary.

Corollary 47 *G' consists of a single split A-component containing uv and a C-component consisting only of the edges xz and yz.*

Proof: Consider a decomposition such that uv is part of an A-component (Claim 45). At least one A-component T_i must be split (Claim 46). A split A-component T_i contains both x and y and $x \neq y$ (Claim 41). So because of the block structure and the 2-connectedness of G' (Lemma 16, Claim 42), T_i is the only A-component, so uv is part of T_i. By Claim 43, the C-component consists only of edge xz and yz. □

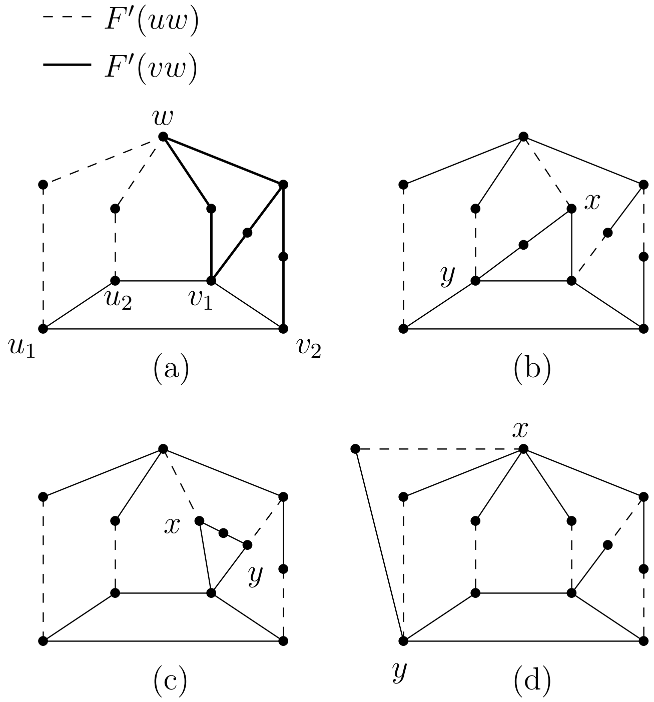

Figure 9: An example of $G - z$ and matching-cuts for three different choices of x and y

Claim 48 *In G, either $d(v_2) = 2$ or $d(u_2) = 2$.*

Proof: Let T be the only A-component of the decomposition of G' we consider, and let $uv \in E(T)$ (Corollary 47). Consider a decomposition of T with u and v as triangle vertices (Corollary 22), and let the third triangle vertex be w. Let $F'(uw)$ and $F'(vw)$ denote the subgraphs of G induced by the edge sets of $F(uw)$ resp. $F(vw)$. Let $T' = G - z$. See Figure 9(a) for an example of T'.

The outline of the proof is as follows: we will first assume that both $F(uw)$ and $F(vw)$ are split. Then, for every choice of x and y we will point out a matching-cut M in T' such that either M does not separate x from y, or x and y are not both incident with M. In the first case M is also a matching-cut in G, and in the second case $M \cup \{xz\}$ or $M \cup \{yz\}$ is a matching-cut in G. Hence we may conclude that $F(uw)$ or $F(vw)$ is not split and the statement will follow.

Assume that both $F(uw)$ and $F(vw)$ are split. Now we construct matching-cuts for $F'(uw)$ and $F'(vw)$ similar to the previous matching-cuts for a P_3-component (see Lemma 33), but this time taking the position of x and y into consideration.

First, suppose that $x \in V(F(vw)) \backslash \{v, w\}$, $y \notin V(F(vw)) \backslash \{v, w\}$. See Figure 9(b) for an example. Since $F(uw)$ is split at u, we can find an edge cut $[S_1, S_2]$ for $F(uw)$ with $u \in S_1$ and $w \in S_2$, that is not incident with

w and is a matching-cut in $F'(uw)$ (Claim 30). Also add v to S_1. Since $y \notin V(F(vw))\backslash\{v,w\}$, either $y \in S_1$ or $y \in S_2$. Suppose $y \in S_1$. Now for $F(vw)$ a matching-cut $[S_1', S_2']$ exists with $\{x, v\} \subseteq S_1'$ and $w \in S_2'$ (Claim 29). Since none of the edges in $[S_1, S_2]$ are incident with vertices in $F(vw)$, together these edge sets form a matching in G. These edge sets form an edge cut in T, since $[S_1', S_2'] \cup [S_1, S_2] = [S_1' \cup S_1, S_2' \cup S_2]$. Since $y \in S_1$ and $x \in S_1'$ this is also an edge cut in G' that does not contain uv, and therefore an edge cut in G. If $y \in S_2$, the matching-cut construction is similar.

This construction can easily be generalized to prove that if one of x and y is an internal vertex of one of the edge components and the other vertex is not an internal vertex of the same edge component, then a matching-cut in G can be found.

Now consider the case that both x and y are internal vertices of the same edge component, w.l.o.g. $\{x, y\} \subseteq V(F(vw))\backslash\{v, w\}$. See Figure 9(c). Since $F(vw)$ is split at v, for $F'(vw)$ a matching-cut $[S_1, S_2]$ exists with $\{v_1, v_2\} \subseteq S_1$ and $w \in S_2$, and either $\{x, y\} \subseteq S_1$ or $\{x, y\} \subseteq S_2$ (Claim 31). Similar to the previous case, we can combine this with a matching-cut $[S_1', S_2']$ for $F'(uw)$ such that w is not incident with edges of $[S_1', S_2']$ (Claim 30), and a matching-cut in G is found.

Finally, if x and y are both not internal vertices of any of the two edge components, then in G, $\{x, y\} \subset \{u_1, u_2, v_1, v_2, w\}$. Considering the previously constructed matching-cuts, it is clear that w.l.o.g. $x = w$. See Figure 9(d). Now let $M_1 = [S_1, S_2]$ be a matching-cut for $F'(uw)$ such that $\{u_1, u_2\} \in S_1$, $w \in S_2$ and w is not incident with edges in M_1 (Claim 30). Let $M_2 = [S_1', S_2']$ be a matching-cut for $F'(vw)$ such that $\{v_1, v_2\} \in S_1'$, $w \in S_2'$ and w is not incident with edges in M_2. Let $S = S_1 \cup S_1' \cup \{z\}$. Now $[S, \overline{S}] = M_1 \cup M_2 \cup \{xz\}$ is a matching-cut in G.

We have found matching-cuts for G in all cases, so this shows that not both $F(uw)$ and $F(vw)$ can be split. W.l.o.g. $F(uw)$ is split but $F(vw)$ is not, and edges of $F'(vw)$ are not incident with v_2, only with v_1. In this case we prove $v_2 \neq x, y$. Consider a matching-cut $[S_1, S_2]$ for the split P_3-component $G[E(F'(uw)) \cup E(F'(vw))]$ with $\{u_1, u_2, v_1\} \subseteq S_1$ (Lemma 33). v_2 is not part of $F'(uw)$ or $F'(vw)$, and therefore not incident with M. If $y = v_2$ ($x = v_2$) then either M or $M \cup \{yz\}$ ($M \cup \{xz\}$) is a matching-cut in G. Therefore $d_G(v_2) = 2$. □

Now we are ready to finish the proof of Lemma 39.

Proof of Lemma 39: Corollary 47 shows that in the decomposition of G' we consider, there is only one A-component T, $uv \in E(T)$, and T is split. We also know that $G' - z = T$. By Claim 48, w.l.o.g. $d(v_2) = 2$ in G. T' is the subgraph of G that corresponds to T ($T' = G - z$). Throughout this proof we use the fact that if $[S, \overline{S}]$ is a matching-cut in T', it is easily extended to a matching-cut in G unless w.l.o.g. $x \in S$, $y \in \overline{S}$ and x and y are both incident with edges from $[S, \overline{S}]$.

Consider a decomposition of T such that u, v and another vertex w are triangle vertices (Corollary 22). We make the following observations:

1. Every 2-connection 4-cycle C in T that does not contain uv is split. If not, then in G there is also a 2-connection 4-cycle (a contradiction with Claim 34), unless w.l.o.g. x is equal to one of the degree 2 vertices of C. If this is the case, and y is not equal to the other degree 2 vertex of C, then the forbidden structure from Claim 35 is present, a contradiction. If y is the other degree 2 vertex, then in G a 2-connection $K_{2,3}$ is present, which is a 2-connection AB graph on 5 vertices, again a contradiction (Claim 36).

2. Since every 2-connection 4-cycle in T that does not contain uv is split, every 2-connection 4-cycle is incident with u. This is because $d_G(v_2) = 2$ (Claim 48). Therefore edge component $F(vw)$ consists of a single edge.

3. In a decomposition of edge component $F(uw)$ (this decomposition follows from the decomposition of T), every B operation is applied on an edge incident with u, otherwise in $F(uw)$ 2-connection 4-cycles will exist that are not incident with u (consider the last such B operation).

4. Therefore, if in a decomposition of T a B operation is applied to ua, then $d_T(a) = 3$.

5. In an edge component, no triangles or multi-edges are present. Therefore, if C is a 2-connection 4-cycle in $F(uw)$, then in any decomposition of $F(uw)$, all edges of C are introduced by the same B operation, so 2-connection 4-cycles correspond to B operations. (Observe that this is not necessarily true for 2-connection 4-cycles in T.) So a C4 operation on any 2-connection 4-cycle in $F(uw)$ yields again an edge component.

6. In $F(uw)$, no two edge disjoint 2-connection 4-cycles exist. Proof: Suppose in $F(uw)$ two edge disjoint 2-connection 4-cycles C_1 and C_2 exist. In $F(uw)$, if we apply C4 operations on C_1 and C_2, the resulting graph has no multi-edges since it is again an edge component. Therefore C_1 and C_2 have at most one vertex in common, and $V(C_1) \cap V(C_2) = \{u\}$. Let $V(C_1) = \{u, a_1, a_2, a_3\}$ and $V(C_2) = \{u, b_1, b_2, b_3\}$, such that $ua_1 \notin E(T)$ and $ub_1 \notin E(T)$. When we consider again the edge component resulting from the C4 operations on these two cycles, and the fact that this edge component does not have triangles, we can also deduce that a_1 and b_1 are not neighbors. Since $d_T(a_1) = d_T(b_1) = 3$ and both 2-connection 4-cycles are split, vertex sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ both yield a matching-cut in T'. Since $[A, \overline{A}]$ does not extend to a matching-cut in G, w.l.o.g. $x \in A$. In this case, x is not incident with edges from $M = [B, \overline{B}]$ (since a_1 and b_1 are not adjacent), and therefore either M or $M \cup \{xz\}$ is a matching-cut in G, a contradiction.

7. Consider a decomposition of $F(uw)$. If in any intermediate graph two edge disjoint 2-connection 4-cycles exist, then in $F(uw)$ at least two edge dis-

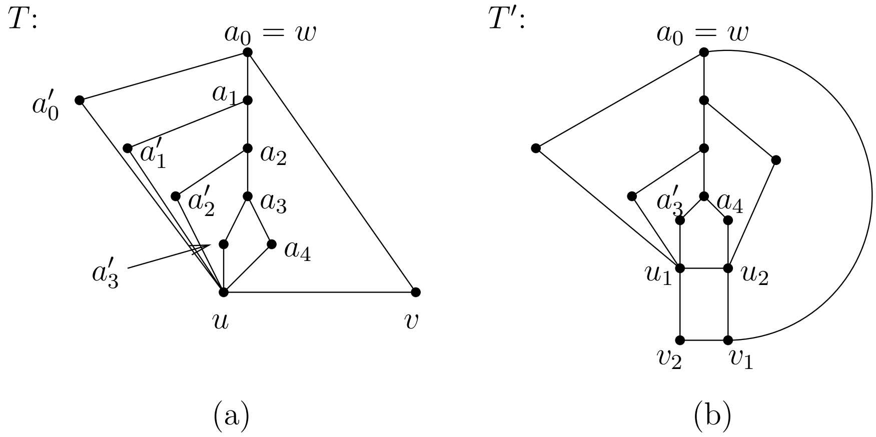

Figure 10: Examples of T and a corresponding T'

joint 2-connection 4-cycles exist. So in this decomposition, a B operation is always applied to an edge introduced by the previous B operation.

These observations narrow down the possibilities for T considerably: T is obtained from the triangle on u, v and w by first applying a B operation on uw (at least one B operation is used since T is split) and then, for an arbitrary number of steps, applying a B operation on one of the two edges that is part of the single 2-connection 4-cycle in edge component $F(uw)$ and that is incident with u. So, because of the symmetry, T is completely characterized by the number of B operations that are applied. See Figure 10(a). We use the vertex labeling as shown in this figure. w is given the label a_0. B operations are applied on edge ua_i, introducing vertices a'_i and a_{i+1}. So ua_n is an edge if and only if exactly n B operations are applied. Let n be the number of B operations that are applied. In T, $N(u) = \{v, a_n, a'_{n-1}, a'_{n-2}, .., a'_0\}$. W.l.o.g., in T' vertex a'_{n-1} has u_1 as neighbor and vertex a_n has u_2 as neighbor. See Figure 10(b) for an example of a graph T' that corresponds to T.

The first matching-cut we consider in T' is $M_1 = [S_1, \overline{S}_1]$ with $S_1 = \{a_n, a_{n-1}, a'_{n-1}\}$ (See Figure 11(a)). The second matching-cut we consider in T' is $M_2 = [S_2, \overline{S}_2]$ with $S_2 = \{v_2, u_1\} \cup \{a'_i : u_1 a'_i \in E(T')\}$ (See Figure 11(b)). Observe that $M_1 \cap M_2 = \emptyset$. Furthermore, $M_1 \cup M_2$ is a set of isolated edges plus the edge set of a path P of length 5 or 6 (depending on whether a'_{n-2} is adjacent to u_1 or u_2 in T') (see Figure 11(c)). Since x and y are incident with both M_1 and M_2, x and y are internal vertices of P. Also, if P' is the subpath of P from x to y, P' contains an odd number of edges from M_1 and an odd number of edges from M_2. Since the edges alternate and P has length at most 6 and x and y must be internal vertices of P, P' has exactly two edges. Now if $x = u_1$ and $y = a_{n-1}$, then G has a 2-connection 4-cycle. Therefore, only two possibilities for x and y remain: either $x = u_2$ and $y = a'_{n-1}$ or $x = a'_{n-1}$ and $y = a_{n-2}$ (or $y = v$ if $n = 1$). Consider these two cases:

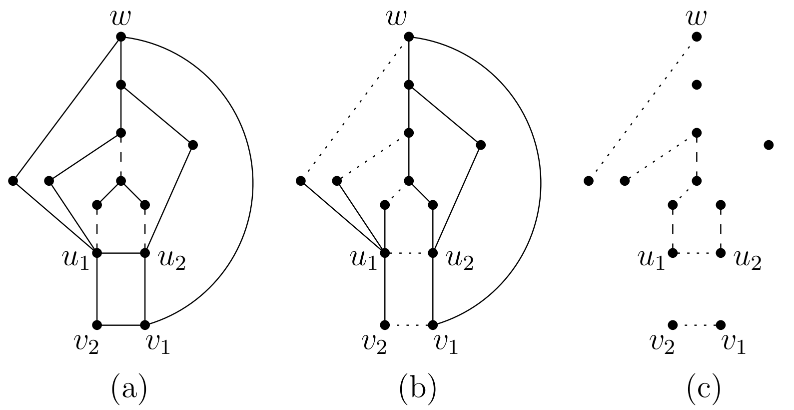

Figure 11: Two matching-cuts in T'

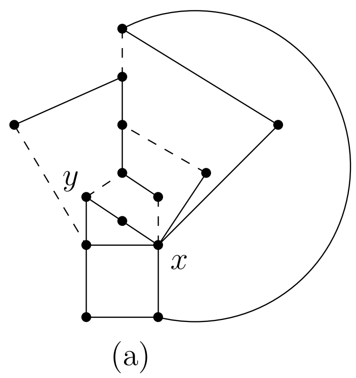

Figure 12: Two examples of G if $x = u'$ and $y = a'_{n-1}$

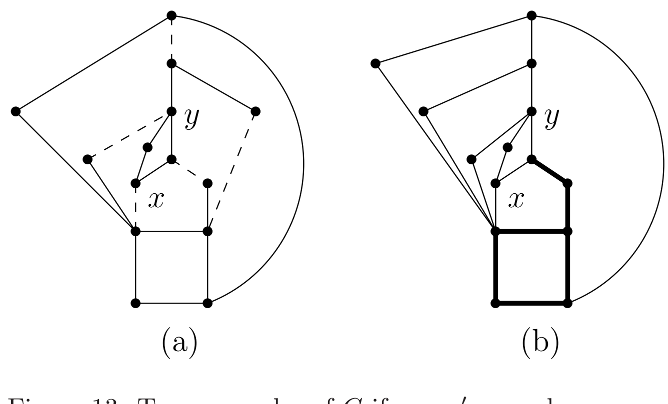

Figure 13: Two examples of G if $x = a'_{n-1}$ and $y = a_{n-2}$

- Suppose $x = u_2$ and $y = a'_{n-1}$. If at least one vertex a'_i is adjacent to u_1 in T' ($i \neq n-1$), then $[S, \overline{S}]$ with $S = \{a'_i, a_i, a_{i+1}, \ldots, a_n\}$ is a matching-cut in G (see Figure 12(a)). Therefore, in T' all vertices a'_i are adjacent to u_2 (except for a'_{n-1}). See Figure 12(b). So $d_G(u_1) = 3$. In this case, the forbidden structure from Claim 35 is present in G, which is shown by the bold edges in Figure 12(b), a contradiction.
- Suppose $x = a'_{n-1}$ and $y = a_{n-2}$ (or $y = v$ if $n = 1$). If at least one vertex a'_i is adjacent to u_2 in T', then $[S, \overline{S}]$ with $S = \{a'_i, a_i, a_{i+1}, .., a_{n-1}, a'_{n-1}, z\}$ is a matching-cut in G (see Figure 13(a)). Therefore, every vertex a'_i is adjacent to u_1 in T'. Now $d_G(u_2) = 3$, and the forbidden structure from Claim 35 is again present, as indicated by the bold edges in Figure 13(b), a contradiction.

So in every case a contradiction is obtained, which shows that G cannot contain a C_4. □

9 A minimum counterexample cannot contain a C_3

In this section, we prove the following lemma:

Lemma 49 *If G is a minimum counterexample, G does not contain a C_3.*

The proof is by contradiction. Suppose C is a triangle in G on vertices v_1, v_2 and v_3. We may assume w.l.o.g. that $d(v_1) \geq 3$ and $d(v_2) \geq 3$, otherwise a 1-connection triangle is present, which leads to a contradiction with Claim 38.

Applying operation C3 on C such that the resulting vertex is vertex v gives a new graph G'. This graph is again extremal immune (Lemma 6), and by our assumption, must be an ABC graph.

Consider a decomposition of G' with A-components $T_1, \ldots, T_k$, and if the order of G' is even, C-component P. The edge sets of these components induce the components $T'_1, \ldots, T'_k$ resp. P' in G. Observe that the edges of these components together with the edges $\{v_1v_2, v_2v_3, v_1v_3\}$ of C give a partition of the edges of G.

G can be constructed from G' with two edge expansions and an edge addition. So in this section we can use the terminology of Section 2.1 and consider whether components of G' are split or not by these operations: an edge induced component $G'[M]$ is split if it is not isomorphic to $G[M]$.

If the order of G' is even, one C operation is used in every decomposition of G'. In the decomposition we consider, x and y will denote the vertices in G' on which the C operation is applied, and z will denote the vertex introduced by the C operation. So the C-component P consists of edge components $F(xz)$ and $F(yz)$.

With G, G', v etc. defined as above, we first state a number of claims before Lemma 49 can be proved.

Claim 50 *G' has a decomposition with at least one A-component.*

Proof: The proof is very similar to the proof of Claim 40. □

Claim 51 *In G', if an A-component T_i is split then the order of G' is even and T_i contains both x and y and $x \neq y$.*

Proof: Let T_i be an A-component that is split, so $v \in V(T_i)$. First we construct a matching-cut M in T_i'.

W.l.o.g. T_i' can be obtained from T_i by a non-trivial edge expansion of v into v_1v_2 and deletion of v_1v_2, followed by an edge expansion of v_2 into v_2v_3, and deleting v_2v_3. Since a decomposition of T_i exists such that v is a triangle vertex (Corollary 22), Lemma 32 shows that an edge cut M exists in T_i that becomes a matching after the first edge expansion, and therefore is a matching-cut in T_i'. Observe that M is a matching-cut in G if and only if it is an edge cut in G' (the C3 operation can be reversed by adding a loop and applying two edge expansions. These operations do not destroy edge cuts). If G' is odd, then T_i is a block of G' (Lemma 16), so M is also an edge cut in G' (Observation 3). So G' is even, and T_i is not a block of G' and therefore $x \neq y$ in G'.

Suppose that in G', x is not incident with edges from M. Consider any matching-cut M' in $F(xz)$ that separates x from z. Since edges from M and M' share no end vertices (Lemma 16), either M or $M \cup M'$ is a matching-cut in G (Corollary 17). A similar matching-cut can be constructed if y is not incident with edges from M. We conclude that x and y are both incident with edges from M and therefore part of T_i. □

Claim 52 *G' is 2-connected.*

Proof: If G' is not 2-connected, then it contains a 1-connection A-component T_i (Corollary 18). If T_i is split, then it contains x and y, and $x \neq y$ (Claim 51). In this case x and y are two different connection vertices of T_i, a contradiction. So T_i is not split. In that case, the connection vertex of T_i in G' also corresponds to a cut vertex in G, a contradiction (Claim 38). □

Claim 53 *The order of G' is even.*

Proof: Suppose the order of G' is odd. Then by Lemma 16 and Claim 52, G' has only one A-component. By Claim 51, it cannot be split. It follows that v_1 is a cut vertex in G, a contradiction (Claim 38). □

Claim 54 *There is a decomposition of G' such that v is part of an A-component.*

Proof: Suppose $v \in V(P) \backslash \{x, y\}$. Since G' is 2-connected (Claim 52), x and y are the only connection vertices of P (Lemma 16), and v is only incident with edges from P. So if P is not split, v_1 is a cut vertex in G, a contradiction (Claim 38). We conclude that P is split. The rest of the proof is illustrated

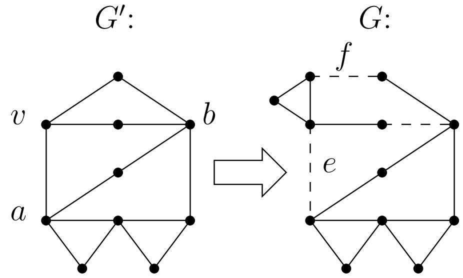

Figure 14: P is split

in Figure 14. Consider a decomposition of P, and consider the operation that introduces v. If $v = z$ this is the C operation, otherwise a B operation. In both cases, after this operation, v is incident with exactly two edges av and bv. In P we consider the edge components $F(av)$ and $F(bv)$, which only have connection vertices v and a resp. b, and the P_3-component Q consisting of both of these. Since P is split and v is only incident with edges from Q, we can find edges $e \in F(av)$ and $f \in F(bv)$ both incident with v, that are not adjacent in G. Let M_1 be a matching-cut in $F(av)$ that contains e and separates a and v, and let M_2 be a matching-cut in $F(bv)$ that contains f and separates b and v (Claim 28). Together these form an edge cut $M_1 \cup M_2$ for Q that does not separate a and b. Since a and b are the only connection vertices of Q in G', this is also an edge cut in G'. So by choice of the edges e and f, this is a matching-cut in G. □

Claim 55 *The C-component of G' consists only of edges xz and yz.*

Proof: The proof is the same as the proof of Claim 43: if the C-component P is split w.l.o.g. it is split at x (Claim 54), which leads to a matching-cut (Lemma 33). So P is not split and if at least one B operation is applied in the decomposition of P, it leads to a 2-connection 4-cycle in G, a contradiction (Claim 34). □

Claim 56 *At least one A-component of G' is split.*

Proof: Assume no A-components are split. Since G' is 2-connected, a decomposition of G' exists such that no A operation is applied after the C operation. So using Claim 55, $G' - z$ is an AB graph. v is part of an A-component T_i in $G' - z$ (Claim 54). By Claim 26, there is a decomposition of $G' - z$ that starts with the construction of T_i. By Corollary 22, there is a decomposition of T_i that starts with v. Instead of starting only with v, start with the triangle on v_1, v_2 and v_3. Continue with the rest of the decomposition. If an A-component T_j is introduced by an A operation on v, then T'_j is only incident with v_1, v_2 or v_3 in G, since T_j is not split. So apply an A operation on v_1, v_2 resp. v_3 instead.

This is similar for the C-component. This yields an ABC decomposition of G. □

Proof of Lemma 49: The above claims show that at least one A-component T is split (Claim 56), so T contains x and y (Claim 51), and therefore this is the only A-component (Claim 52, Lemma 16). The C-component consists only of edges xz and yz (Claim 55). By Lemma 39 we know that G cannot contain a C_4, so every C_4 in G' is split and therefore incident with v. In addition, it follows that T is not a triangle, since then the three edges of T together with one edge from $\{v_1v_2, v_1v_3, v_2v_3\}$ would form a C_4 in G.

Consider $T' = G - z$. Note that even though $T = T_1$, $T' \neq T_1'$ since the edges of cycle C are included in T'. We will describe a number of matching-cuts for T', and show that no matter how x and y are chosen, one of these matching-cuts is also a matching-cut in G. Throughout this proof we use the fact that if M is a matching-cut in T', it is easily extended to a matching-cut in G unless M separates x from y and x and y are both incident with edges from M.

For every $u \in V(T)$ that is adjacent to v, we know that a decomposition of G' exists where u and v are triangle vertices of T (Corollary 22). Let w be the third triangle vertex of T in this decomposition. $F'(vw)$ is the subgraph of T' induced by the edges of edge component $F(vw)$. $F(uw)$ is a single edge, otherwise T contains a C_4 that is not incident with v. Since T is not a triangle, $F(vw)$ is not a single edge. Then $F(vw)$ is split, otherwise $F'(vw)$ contains a C_4. Now we can find a matching-cut in T' that does not contain uv.

Recall that w.l.o.g. $F'(vw)$ can be obtained from $F(vw)$ by a non-trivial edge expansion of v into v_1v_2 and deleting v_1v_2, followed by an edge expansion of v_2 into v_2v_3 and deleting v_2v_3.

Claim 30 shows that an edge cut M for $F(vw)$ exists that separates w from v, such that w is not incident with edges from M, and M becomes a matching after the first edge expansion. So M is a matching-cut in $F'(vw)$ that is not incident with w. $M \cup \{uw\}$ is a matching-cut for T' that does not include uv (See Figure 15(b)). We conclude that for every neighbor u of v, we can find a matching-cut in T' that does not contain uv. As a corollary, we find that it is not possible that $x = v$ and y is a neighbor of v in G'.

Since T is not a triangle, in a decomposition of T, at least one B operation is used. Consider the last B operation. The 2-connection 4-cycle corresponding to this operation is split so it has connection vertices v and another vertex u. Let a_1 and a_2 be the other vertices of this 2-connection 4-cycle. If we consider the intermediate graph in the decomposition from which T is obtained, we know it has a decomposition with triangle vertices u and v (Corollary 22), so T also has a decomposition with triangle vertices u, v and another vertex w. W.l.o.g. we assume that a_1 is adjacent to v_1 in T', and a_2 is adjacent to v_2 in T'. Now we can find a matching-cut in G: $F(uw)$ is a single edge again. In T', edge set $M = \{uw, a_1v_1, a_2v_2\}$ is a matching-cut (see Figure 15(c)). If the distance from x to y in T' is two, then G contains a C_4. So since M is not part of a matching-cut in G, the distance from x to y is one. Above we showed that it is not possible that $x = v$ and y is a neighbor of v in G', so the only remaining

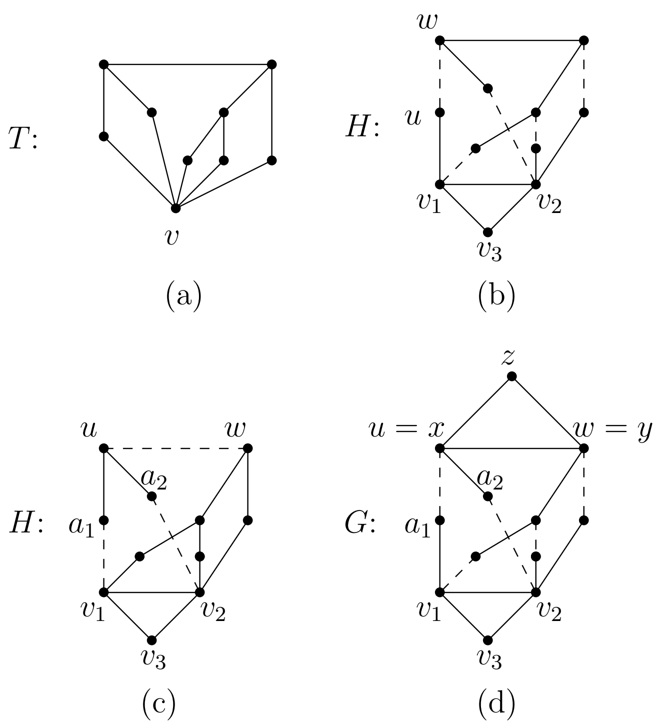

Figure 15: Two matching-cuts in T' and one in G

possibility is $x = u$ and $y = w$.

Finally consider the following matching-cut: take any matching-cut M for $F(vw)$ that separates v from w. If M is incident with v_1 in T', $M \cup \{a_2v_2, a_1u\}$ is a matching-cut in G, otherwise $M \cup \{a_1v_1, a_2u\}$ is a matching-cut (see Figure 15(d)).

Now in every case we can find a matching-cut in G, a contradiction. □

10 The proof of the conjecture

In this section, we prove Conjecture 2:

Theorem 57 *If graph $G = (V, E)$ is extremal immune, then G is ABC.*

Proof: Suppose this is not true, so there exist counterexamples, which are extremal immune graphs that are not ABC. Let G be a graph with minimum size among these counterexamples. Claim 37 shows that G is simple, so no C2 operation can be applied to it. Lemma 39 shows that G cannot contain a 4-cycle, so the C4 operation cannot be applied to it. Lemma 49 shows that G cannot contain a triangle, so no C3 operation can be applied.

Suppose a P2 operation can be applied to G. Then in G there are neighbors u and v, with $d(u) = 3$ and $d(v) = 2$. x and y are the other neighbors of u, and z is the other neighbor of v. If $z = x$ or $z = y$ then G contains a C_3, a contradiction. If x and z are neighbors or y and z are neighbors, then G contains a C_4, also a contradiction. So after the P2 operation is applied, graph G' is obtained that contains edge xy and vertex z, and z is not equal to or adjacent to x or y. G' is an ABC graph since it is again extremal immune (Lemma 6). Clearly, G' cannot be a K_1, a C_2 or a C_3. For every ABC graph G' other than these three graphs and every edge $e \in E(G')$, we can show that there is a 4-cycle or triangle that does not contain e:

If $G' \neq C_2$, we may assume there is at least one A-component (Claim 24). If G' contains an A-component of order at least 5, then the statement follows from Claim 21. Otherwise, if G' has at least two A-components, then two disjoint triangles are easily found. So G' must consist of a triangle and a C-component. If on this C-component at least one B operation is applied, then we have a C_4 and a C_3 which are disjoint. So G' is a diamond (a K_4 minus one edge). For this graph the statement is also true.

So in G' there is a 4-cycle or triangle that does not contain xy. This corresponds to a 4-cycle or triangle in G, a contradiction.

We conclude that no C2, C3, C4 or P2 operation can be applied to G, which is a contradiction with Lemma 7. Therefore a minimum counterexample does not exist. □

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