Department of Applied Mathematics
Faculty of EEMCS

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands

Phone: +31-53-4893400
Fax: +31-53-4893114

Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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Coordination mechanisms for inventory control in three-echelon serial and distribution systems

W.H.M. Zijm and J.B. Timmer

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Henk Zijm and Judith Timmer *

Abstract

This paper is concerned with the coordination of inventory control in three-echelon serial and distribution systems under decentralized control. All installations in these supply chains track echelon inventories. Under decentralized control the installations will decide upon base stock levels that minimize their own inventory costs. In general these levels do not coincide with the optimal base stock levels in the global optimum of the chain under centralized control. Hence, the total cost under decentralized control is larger than under centralized control.

To remove this cost inefficiency, two simple coordination mechanisms are presented: one for serial systems and one for distribution systems. Both mechanisms are initiated by the most downstream installation(s). The upstream installation increases its base stock level while the downstream installation compensates the upstream one for the increase of costs and provides it with a part of its gain from coordination. It is shown that both coordination mechanisms result in the global optimum of the chain being the unique Nash equilibrium of the corresponding strategic game. Furthermore, all installations agree upon the use of these mechanisms because they result in lower costs per installation. The practical implementation of these mechanisms is discussed.

Key Words: supply chain, coordination mechanism, Nash equilibrium, strategic game, inventory control, multi-echelon system.

AMS Subject Classification: 90B50, 91A10, 91A35, 90B05.

1 Introduction

This paper is concerned with the coordination of inventory control in three-echelon serial and distribution systems under decentralized control. All installations in these supply chains track echelon inventories. Under decentralized control an installation will decide upon a base stock level that minimizes its inventory cost by taking into account only its own costs. It

*Stochastic Operations Research group, Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands. Email: w.h.m.zijm@ewi.utwente.nl, j.b.timmer@ewi.utwente.nl

neglects the external effects of its decision on others. Hence, the system consists of selfish installations and it is therefore not surprising that in general their decisions are not optimal from the perspective of the supply chain as a whole. Thus, the total cost of the system under decentralized control is larger than in the supply chain optimum.

To improve upon this cost inefficiency asks for some coordination between the installations. While maintaining decentralized decision-making, the goal of a coordination mechanism is to change the structure of the installations' costs such that the individual decisions of the installations are optimal for the system as a whole, that is, they coincide with the optimal decisions under centralized control. Furthermore, the coordination mechanism should make each installation better off, that is, it should result in lower cost. Applying such a mechanism results in the total cost of the system being as low as possible. All firms should agree upon the use of a certain coordination mechanism; this can be achieved by negotiations between the installations.

The basic systems under consideration, three-echelon serial and distribution systems, are widely studied in the literature. The concept of echelon stock was introduced in Clark and Scarf [5]. They showed the value of echelon stock for integrated control of serial systems compared to local stock, as well as the optimality of inventory control by means of echelon base stock levels. Langenhoff and Zijm [10] study multi-echelon production/distribution systems under centralized control. In case of a distribution system, the depot is allowed to keep stock. They derive optimal base stock policies for serial and distribution systems. A distribution system with a stockless depot (supplier) is considered in Eppen and Schrage [6]. For a review on multi-stage serial systems we refer to Van Houtum, Inderfurth and Zijm [9].

In case of both serial and distribution systems, the literature recognizes that decentralized control leads to larger total costs. Several papers study coordination mechanisms, mostly for distribution systems. A two-echelon serial supply chain is investigated by Cachon and Zipkin [3]. In that paper each installation may incur a consumer backorder penalty cost because the upper stage (supplier) is assumed to dislike backorders of his product at the retailer. Two noncooperative games are considered, based on whether local or echelon inventory is tracked. These games nearly always have a unique Nash equilibrium which differs from the global optimum. Under specific conditions the global optimum can be achieved if local inventory is tracked as a (possibly non-unique) Nash equilibrium by using a linear transfer payment.

Güllü, Van Houtum, Alişan and Erkip [8] analyze a decentralized supply chain consisting of a supplier and two independent retailers. When the supplier receives his orders, the retailers get the opportunity to redistribute their initial orders among themselves. This is done to improve the expected cost based upon the information that has become available in the meantime. It is shown that under mild conditions there exists a unique Nash equilibrium for the retailers base stock levels. This need not coincide with the global optimal levels.

A distribution system with one supplier and N retailers is studied in Cachon [2]. All firms pay inventory and backorder cost; the backorder cost for the supplier reflects the supplier's interest in the availability of his product at the retailers, as in [3]. A retailer continuously monitors his inventory and uses an (r, q) ordering policy for replenishment: whenever his inventory position drops to r he places an order for q units. Demand for the product is Poisson distributed. The supplier serves the retailers on a first-come-first-serve basis. It is shown that the competitive solution need not coincide with the global optimum. Three cooperation strategies are discussed, of which two lead to the global optimum being a Nash equilibrium.

The model of [3] is extended by Wang, Guo and Efstathiou [15] to a distribution system with 1 supplier and n different retailers, each with its own lead time and holding cost. After studying the system under decentralized control, coordination mechanisms are examined. In case the firms track echelon inventories, a contract is presented in which the retailers pay a nonlinear tariff (a nonlinear function of the base stock level) to the supplier. This contract ensures that the system optimal solution is the unique Nash equilibrium of the game. Nevertheless, nonlinear tariffs are not easy to implement.

Other models of coordination of inventories in serial or distribution systems include [1, 4, 7, 12, 13, 14].

In the literature, most coordination mechanisms are such that the retailer is induced to adopt the globally optimal base stock level. Opposed to this, using arguments from game theory, in our model the retailer (the installation downstream) will take the initiative to induce the supplier (the installation upstream) to change his decision. We present simple coordination mechanisms under the (natural) assumption that installations only communicate with their direct neighbors upstream and downstream in the supply chain. Further, all order decisions are based upon echelon inventory. The coordination is such that the upstream installation increases its base stock level while the downstream installation compensates the upstream one for the increase of costs. Besides, the downstream installation transfers a part of its cost savings less the compensation paid to the upstream installation, its gain from the coordination. Both the mechanisms for serial and distribution systems result in the installations choosing the global optimal base stock levels. This choice of base stock levels is the unique Nash equilibrium of the corresponding strategic game played by the installations. Further, all agree upon the use of these mechanisms because they result in lower costs per installation.

The outline of this paper is as follows. In the next Section we briefly recall results of serial systems under centralized control. In Section 3 serial systems under decentralized control are studied. A coordination mechanism for these systems and its practical implementation are presented in Section 4. Distribution systems under decentralized control are analysed in Section 5, and a coordination mechanism for these systems, as well as its implementation, is presented in Section 6. Section 7 concludes.

2 Three-echelon serial systems under centralized control

We start by studying inventory control for a single good in a three-echelon serial system. Such a system is a supply chain consisting of three installations in a series. The installations are numbered from 1, the most downstream installation, to 3. An echelon is a set of installations, starting from a certain installation and including all installations downstream. Echelons are numbered according to their most upstream installation. Hence, in the serial system under consideration, echelon i includes the installations i down to 1.

Demand for the good occurs only at the most downstream installation, installation 1. The distribution function of the l-period cumulative demand u_l is denoted by F_l. If $l = 1$ we write F instead of F_1. Let μ denote the expected demand per period.

The installations determine the quantity of the orders for replenishment of their stock on the basis of their echelon inventory position. The echelon stock of an installation consists of all stock at that installation plus all stock in transit to or on hand at any installation downstream minus eventual backlogs at installation 1. The echelon inventory position denotes the echelon stock plus materials that are already ordered but not yet delivered at the most upstream installation in the echelon.

All installations place their orders for replenishment of stock at the end of a period. A material shortage at installation 2 or 3 is possible, leading to incomplete fulfilment of the orders of installation 1 or 2. Any excess demand is backlogged. The ordered goods are delivered after a fixed lead time. Namely, it takes l_3 periods to transfer materials from the outside supplier, who can always deliver, to installation 3, and l_i periods are needed to transfer materials from installation $i+1$ to installation i, $i = 1, 2$.

The holding costs for installation i are $h_i + \ldots + h_3$ for goods at installation i and, if $i > 1$, for goods in transfer to installation $i - 1$. Installation 1 pays a penalty cost p for unfilled demand. All these costs are linear in time and quantity and occur at the end of a period.

In [10] a natural definition for the echelon cost functions of all installations is developed. We briefly repeat this. Let x_i denote the echelon stock associated with installation i at the beginning of a period before demand occurs. Assign the following one-period holding and penalty cost to installation 1:

$$L_1(x_1) = h_1 \int_0^\infty (x_1 - u)\mathrm{d}F(u) + (p + h_1 + h_2 + h_3) \int_{x_1}^\infty (u - x_1)\mathrm{d}F(u).$$

Assign

$$L_j(x_j) = h_j \int_0^\infty (x_j - u)\mathrm{d}F(u)$$

to installation j, $j = 2, 3$.

Let $D_N(y_1, \ldots, y_N)$ denote the average total cost of an N-echelon serial system if at the beginning of every period the echelon inventory position of echelon i is increased by installation

i to y_i, where $y_1 \leq y_2 \leq \ldots \leq y_N$. As shown by [10] the cost D_N is composed of N terms

$$D_N(y_1, \ldots, y_N) = C_1(y_1) + C_2(y_1, y_2) + \ldots + C_N(y_1, \ldots, y_N)$$

where

$$C_1(y_1) = \int_0^\infty L_1(y_1 - u_{l_1}) \mathrm{d}F_{l_1}(u_{l_1})$$

and

$$\begin{aligned} C_j(y_1, \ldots, y_j) &= \int_0^\infty L_j(y_j - u_{l_j}) \mathrm{d}F_{l_j}(u_{l_j}) \\ &+ \int_{y_j - y_{j-1}}^\infty [C_{j-1}(y_1, \ldots, y_{j-2}, y_j - u_{l_j}) - C_{j-1}(y_1, \ldots, y_{j-2}, y_{j-1})] \mathrm{d}F_{l_j}(u_{l_j}) \end{aligned}$$

for $j = 2, \ldots, N$. In the expression for C_j the second integral represents a penalty cost for installation j if it cannot fulfil the order of installation $j-1$, that is, if $y_j - u_{l_j} < y_{j-1}$.

Lemma 2.1 *The average cost of an N-echelon serial system, $N = 1, 2, 3$, can be written as*

$$D_1(y_1) = h_1(y_1 - (l_1 + 1)\mu) + (p + h_1 + h_2 + h_3) \int_{y_1}^\infty (u_{l_1+1} - y_1) dF_{l_1+1}(u_{l_1+1}),$$

$$D_2(y_1, y_2) = D_1(y_1) + h_2(y_2 - (l_2 + 1)\mu) + \int_{y_2 - y_1}^\infty [D_1(y_2 - u_{l_2}) - D_1(y_1)] dF_{l_2}(u_{l_2}),$$

and

$$\begin{aligned} D_3(y_1, y_2, y_3) &= D_2(y_1, y_2) + h_3(y_3 - (l_3 + 1)\mu) \\ &\quad + \int_{y_3 - y_2}^\infty [D_2(y_1, y_3 - u_{l_3}) - D_2(y_1, y_2)] dF_{l_3}(u_{l_3}). \end{aligned}$$

All proofs can be found in Section 8. This Lemma shows that the cost D_i of echelon $i > 1$ consist of the cost of echelon $i-1$ plus additional holding cost and a kind of penalty in case of a shortage, $y_i - u_{l_i} < y_{i-1}$. In [10] the minimum of $D_3(y_1, y_2, y_3)$ is found, which is repeated in the Lemma below.

Lemma 2.2 ([10]) *The cost function $D_3(y_1, y_2, y_3)$ is minimized in $(y_1, y_2, y_3) = (S_1, S_2, S_3)$ where S_1 minimizes $D_1(y_1)$, S_2 minimizes $D_2(S_1, y_2)$ and S_3 minimizes $D_3(S_1, S_2, y_3)$.*

To implement this optimal solution, centralized control is needed. But in practice, there is no centralized but decentralized control. The installations act on their own and independently decide about their base stock levels y_i. Thereafter these choices are implemented over an infinite horizon. Such a situation can be modeled as a noncooperative strategic game (see [11]) in which each installation, or player, has to make one decision. These decisions are made independently and simultaneously. Further, a strategic game consists of a set of players and for each player a strategy set and a cost function. Here, the three installations are the players.

The strategy set Y_i of player i is defined as the set of all possible echelon base stock levels. A triple of strategies (y_1, y_2, y_3) is also referred to as a strategy profile. The cost function H_i of player i is defined on all strategy profiles and will be specified further on. Each player will choose the base stock level that minimizes its cost.

Special interest goes to equilibria, or stable outcomes, of the game. A strategy profile $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ of base stock levels is called a Nash equilibrium of the game if unilateral deviations cannot reduce cost: $H_1(\bar{y}_1, \bar{y}_2, \bar{y}_3) \leq H_1(y_1, \bar{y}_2, \bar{y}_3)$, $H_2(\bar{y}_1, \bar{y}_2, \bar{y}_3) \leq H_2(\bar{y}_1, y_2, \bar{y}_3)$ and $H_3(\bar{y}_1, \bar{y}_2, \bar{y}_3) \leq H_3(\bar{y}_1, \bar{y}_2, y_3)$ for all $y_i \in Y_i$, $i = 1, 2, 3$. Let

$$r_1(y_2, y_3) = \{y_1 \in Y_1 | H_1(y_1, y_2, y_3) \leq H_1(y_1', y_2, y_3) \text{ for all } y_1' \in Y_1\}$$

be the set of best base stock levels for installation 1 given the levels y_2 and y_3 of the other installations. The function r_1 is called a best reply function of installation 1. The best reply function $r_2(y_1, y_3)$ of installation 2 and $r_3(y_1, y_2)$ of installation 3 are defined similarly. Now if a strategy profile $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ satisfies

$$\bar{y}_1 \in r_1(\bar{y}_2, \bar{y}_3), \quad \bar{y}_2 \in r_2(\bar{y}_1, \bar{y}_3) \quad \text{and} \quad \bar{y}_3 \in r_3(\bar{y}_1, \bar{y}_2)$$

then it is a Nash equilibrium. This definition of a Nash equilibrium will be used in the sequel.

3 Three-echelon serial systems under decentralized control

In this Section we assume that the three installations act on their own, that is, we are dealing with a three-echelon serial system under decentralized control. If y_i is the base stock level of installation i then the real order-up-to level as experienced by installation 3 is y_3, the desired order-up-to-level, because his supplier can always deliver. If $y_2 > y_3 - u_{l_3}$ then installation 2 is confronted with a shortage at installation 3 and receives only $y_3 - u_{l_3}$ instead of y_2. Therefore he experiences the order-up-to level $w_2 = \min(y_3 - u_{l_3}, y_2)$ and similarly installation 1 experiences $w_1 = \min(w_2 - u_{l_2}, y_1)$. These real order-up-to levels are useful in simplifying the expression for the average cost of the system.

Lemma 3.1 *The average cost of a three-echelon serial system is equal to*

$$D_3(y_1, y_2, y_3) = h_3(y_3 - (l_3 + 1)\mu) + \mathbb{E}D_2(y_1, \underline{w}_2)$$

where $\mathbb{E}$ *denotes taking the expectation, and where*

$$D_2(y_1, w_2) = h_2(w_2 - (l_2 + 1)\mu) + \mathbb{E}D_1(\underline{w}_1)$$

and

$$D_1(w_1) = h_1(w_1 - (l_1 + 1)\mu) + (p + h_1 + h_2 + h_3) \int_{w_1}^{\infty} (u_{l_1+1} - w_1) dF_{l_1+1}(u_{l_1+1}).$$

The above expression for the average cost $D_3(y_1, y_2, y_3)$ of the system was obtained from the assignment of certain one-period costs to the echelons. These costs are not the real one-period costs of the echelons. The true one-period cost for installation 1 consist of holding cost $h_1 + h_2 + h_3$ if there are goods in stock and penalty cost p otherwise,

$$\tilde{L}_1(x_1) = (h_1 + h_2 + h_3) \int_0^{x_1} (x_1 - u)\mathrm{d}F(u) + p \int_{x_1}^{\infty} (u - x_1)\mathrm{d}F(u).$$

This implies the expected true average cost

$$\begin{aligned} \tilde{D}_1(y_1) &= \int_0^{\infty} \tilde{L}_1(y_1 - u_{l_1})\mathrm{d}F_{l_1}(u_{l_1}) \\ &= D_1(y_1) + (h_2 + h_3)(y_1 - (l_1 + 1)\mu) \end{aligned} \tag{3.1}$$

for the one-echelon system with base stock level y_1. Since the real order-up-to level for installation 1 in a three-echelon system is not y_1 but w_1, the expected cost for this installation, $\tilde{H}_1(y_1, y_2, y_3)$, is $\tilde{H}_1(y_1, y_2, y_3) = \mathbb{E}\tilde{D}_1(\underline{w}_1)$. The true expected cost $\tilde{H}_2$ for installation 2 consists of inventory cost for goods in transit to 1 and inventory cost, if any, for goods that remain after fulfilling the order of echelon 1,

$$\tilde{H}_2(y_1, y_2, y_3) = (h_2 + h_3)l_1\mu + (h_2 + h_3)\mathbb{E} \int_0^{\infty} \max(\underline{w}_2 - u_{l_2} - y_1, 0)\mathrm{d}F_{l_2}(u_{l_2}).$$

The true cost $\tilde{H}_3$ for installation 3 are defined similarly as for 2. Let $\hat{w}_i = \mathbb{E}\underline{w}_i$ denote the expected order-up-to level of installation i. Rewriting these cost functions results in the following expressions.

Lemma 3.2 *The expected true costs for the three installations are equal to*

$$\tilde{H}_1(y_1, y_2, y_3) = \mathbb{E}D_1(\underline{w}_1) + (h_2 + h_3)(\hat{w}_1 - (l_1 + 1)\mu),$$

$$\tilde{H}_2(y_1, y_2, y_3) = (h_2 + h_3)l_1\mu + (h_2 + h_3)(\hat{w}_2 - l_2\mu - \hat{w}_1),$$

and

$$\tilde{H}_3(y_1, y_2, y_3) = h_3 l_2\mu + h_3(y_3 - l_3\mu - \hat{w}_2).$$

These cost functions distribute the total cost D_3 of the system among the installations.

Lemma 3.3 $\sum_{i=1}^{3} \tilde{H}_i(y_1, y_2, y_3) = D_3(y_1, y_2, y_3)$

In the strategic game with cost functions $\tilde{H}_i$ each installation will choose a base stock level that minimizes its cost. Installation 3 minimizes its cost $\tilde{H}_3$ in $y_3 = y_2$. Similarly, installation 2 will set $y_2 = y_1$. Hence, neither of these installations will keep any extra stock. Knowing this, installation 1 will minimize its cost $\tilde{H}_1$ in $y_1 = \tilde{S}_1$ where $\tilde{S}_1 > S_1$. Therefore, the Nash equilibrium of the game under decentralized control equals $(y_1, y_2, y_3) = (\tilde{S}_1, \tilde{S}_1, \tilde{S}_1)$. This implies that installation 1 is confronted with large costs because it is very likely that there

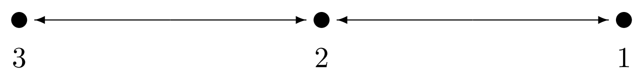

Figure 1: Serial communication between the three installations.

are material shortages at either installation 2 or 3. Because of this installation 1 would like the installations 2 and 3 to set $y_j > y_1$, $j = 2, 3$, such that the probability of a material shortage decreases and consequently the costs of 1 decrease. But $y_j > y_1$ increases the cost of installation j. Thus installation j is only willing to increase y_j to a level above y_1 if he is compensated for his extra costs. In the next Section we study a proposal by installation 1 for compensation of the installations 2 and 3. This proposal coordinates the serial chain.

4 Coordination mechanism for serial systems

In practice it does not seem likely that the installations 1 and 3 communicate directly since installation 2 is in between. Instead, the installations only talk with their neighbors in the serial system. Figure 1 shows this situation, where the arrows indicate the communication possibilities. Assume that the installations communicate in this way.

Under decentralized control the installations in a three-echelon serial system will choose base stock levels $(y_1, y_2, y_3) = (\tilde{S}_1, \tilde{S}_1, \tilde{S}_1)$, as argued in the previous Section. In that situation installation 1 has a large base stock level and is therefore confronted with large costs. To be able to lower his base stock level, and consequently his cost, this installation will negotiate with installation 2 with the goal of achieving lower costs by means of coordination of actions.

4.1 Coordination between the installations 1 and 2

The following coordination mechanism is proposed. Installation 1 asks installation 2 to keep some stock, that is, to set y_2 such that $y_2 > y_1$. This increases the cost of installation 2 because now he also has to pay for the inventory cost of his extra stock. Installation 1 offers the following compensation. First, installation 2 is fully compensated for his increase in inventory cost. This implies that installation 2 has no additional cost compared to the situation before negotiation, but also no additional gain. Second, to persuade installation 2 to accept this offer, he also receives a part of the so-called surplus of 1, which is its cost savings less the compensation paid to installation 2. This is the gain of installation 2 from this negotiation.

If installation 2 sets $y_2 > y_1$ then his cost increases by

$$\tilde{H}_2(y_1, y_2, y_3) - \tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1)$$

whereas the cost of 1 decreases by

$$\tilde{H}_1(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - \tilde{H}_1(y_1,y_2,y_3).$$

Installation 1 fully compensates installation 2 for his cost increase. After this, the cost savings of 1 are reduced to

$$\begin{aligned}
&\tilde{H}_1(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - \tilde{H}_1(y_1,y_2,y_3) - (\tilde{H}_2(y_1,y_2,y_3) - \tilde{H}_2(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1)) \\
&= \quad D_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - \tilde{H}_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - (D_3(y_1,y_2,y_3) - \tilde{H}_3(y_1,y_2,y_3)),
\end{aligned}$$

where the equality follows from Lemma 3.3. Call this value the surplus of installation 1, or *surplus-1* in short. Installation 1 will give installation 2 a fraction α, $0 < \alpha < 1$, of this surplus. This provides an incentive for installation 2 to accept the offer by installation 1 because the costs of installation 2 are now lower than before the negotiation. Notice that the bounds for α are strict since installation 1 likes to keep a part of his surplus for himself while installation 2 wants to receive something extra next to being compensated for his cost increase.

Let H'_i denote the cost of installation i, $i = 1,2$, after the compensation. Then installation 1 pays his initial cost $\tilde{H}_1$, compensates installation 2 for his cost increase and gives him a fraction of surplus-1, resulting in

$$\begin{aligned}
&H'_1(y_1,y_2,y_3) \\
&= \quad \tilde{H}_1(y_1,y_2,y_3) + (\tilde{H}_2(y_1,y_2,y_3) - \tilde{H}_2(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1)) \\
&\quad\quad + \alpha(D_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - \tilde{H}_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - (D_3(y_1,y_2,y_3) - \tilde{H}_3(y_1,y_2,y_3))) \\
&= \quad (1-\alpha)(D_3(y_1,y_2,y_3) - \tilde{H}_3(y_1,y_2,y_3)) \\
&\quad\quad + \alpha(D_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - \tilde{H}_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1)) - \tilde{H}_2(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1).
\end{aligned}$$

Installation 2 pays his own cost $\tilde{H}_2$ and receives a compensation of installation 1 as well as a fraction of surplus-1. His cost change to

$$\begin{aligned}
&H'_2(y_1,y_2,y_3) \\
&= \quad \tilde{H}_2(y_1,y_2,y_3) - (\tilde{H}_2(y_1,y_2,y_3) - \tilde{H}_2(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1)) \\
&\quad\quad - \alpha(D_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - \tilde{H}_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - (D_3(y_1,y_2,y_3) - \tilde{H}_3(y_1,y_2,y_3))) \\
&= \quad \alpha(D_3(y_1,y_2,y_3) - \tilde{H}_3(y_1,y_2,y_3)) \\
&\quad\quad - \alpha(D_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1) - \tilde{H}_3(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1)) + \tilde{H}_2(\tilde{S}_1,\tilde{S}_1,\tilde{S}_1).
\end{aligned}$$

Notice that the new cost functions are a rearrangement of the former ones,

$$H'_1(y_1,y_2,y_3) + H'_2(y_1,y_2,y_3) = \tilde{H}_1(y_1,y_2,y_3) + \tilde{H}_2(y_1,y_2,y_3). \tag{4.1}$$

During these negotiations the base stock level of installation 3 remains unchanged, namely $y_3 = y_2$. This equality says that installation 3 keeps no stock and only has to pay the inventory cost of goods in transit to installation 2. This cost $\tilde{H}_3(y_1, y_2, y_2) = h_3 l_2 \mu$ is independent from y_1 and y_2. Therefore, minimizing both H_1' and H_2' under $y_3 = y_2$ is equivalent to minimizing $D_3(y_1, y_2, y_2)$. This minimization has a remarkable outcome.

Theorem 4.1 *The cost function $D_3(y_1, y_2, y_2)$ is minimized in $y_1 = S_1$, the optimal base stock level of installation 1 under* centralized *control, and $y_2 = \tilde{S}_2 > S_2$.*

Hence, installation 1 minimizes his new cost function by setting his base stock level equal to his optimal level under centralized control. Installation 2 picks a base stock level larger than his optimal level under centralized control. Both parties will agree on this outcome only if it results in decreased individual costs.

Theorem 4.2 *In the minimum $(y_1, y_2, y_3) = (S_1, \tilde{S}_2, \tilde{S}_2)$, surplus-1 equals $D_3(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) - D_3(S_1, \tilde{S}_2, \tilde{S}_2) > 0$. The installations 1 and 2 are better off under coordination because $H_i'(S_1, \tilde{S}_2, \tilde{S}_2) < \tilde{H}_i(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1)$, $i = 1, 2$.*

Thus, the coordination results in lower costs for the installations 1 and 2. Both are better off and agree to use this coordination mechanism.

4.2 Coordination between the installations 2 and 3

The negotiation between the installations 1 and 2 ends in the base stock levels $(S_1, \tilde{S}_2, \tilde{S}_2)$. The levels of the installations 2 and 3 are equal, because 3 still does not keep stock and consequently, 2 is confronted with large costs. He would like installation 3 to keep some stock, that is $y_3 > y_2$, so as to decrease his own costs. In exchange, installation 2 offers him a compensation for his increase in cost plus a part of the so-called surplus-2. All along, the base stock level of installation 1 remains S_1, the result of the previous coordination.

Although installation 2 negotiates with installation 3, the negotiations do not only affect installation 2 but also installation 1. Any resulting cost savings for installation 2 result in lower compensations to be paid by installation 1. In other words, the negotiations lead to cost savings for echelon 2. Knowing this, installation 1 will have no problems with installation 2 negotiating with installation 3 on behalf of echelon 2. And that is exactly what will happen.

A part of the compensation is surplus-2, which is the remainder of the cost savings of echelon 2 after compensating installation 3 for his increased costs:

$$\begin{aligned} \text{Surplus-2} &= H_1'(S_1, \tilde{S}_2, \tilde{S}_2) + H_2'(S_1, \tilde{S}_2, \tilde{S}_2) - (H_1'(S_1, y_2, y_3) + H_2'(S_1, y_2, y_3)) \\ &\quad - (\tilde{H}_3(S_1, y_2, y_3) - \tilde{H}_3(S_1, \tilde{S}_2, \tilde{S}_2)) \\ &= D_3(S_1, \tilde{S}_2, \tilde{S}_2) - D_3(S_1, y_2, y_3). \end{aligned}$$

Installation 3 receives a fraction β, $0 < \beta < 1$, of this surplus. This should persuade him to accept the deal with installation 2 because it results in costs that are lower than before the negotiation.

Let H_i denote the cost of installation i after the compensation. Then the installations 1 and 2 pay their own cost, compensate 3 for his cost increase and give him a part β of surplus-2:

$$\begin{aligned}&(H_1+H_2)(S_1,y_2,y_3)\\ &\quad= H_1'(S_1,y_2,y_3)+H_2'(S_1,y_2,y_3)+(\tilde{H}_3(S_1,y_2,y_3)-\tilde{H}_3(S_1,\tilde{S}_2,\tilde{S}_2))\\ &\qquad+\beta(D_3(S_1,\tilde{S}_2,\tilde{S}_2)-D_3(S_1,y_2,y_3))\\ &\quad= (1-\beta)D_3(S_1,y_2,y_3)-\tilde{H}_3(S_1,\tilde{S}_2,\tilde{S}_2)+\beta D_3(S_1,\tilde{S}_2,\tilde{S}_2).\end{aligned}$$

The compensation changes the cost of installation 3 to

$$\begin{aligned}&H_3(S_1,y_2,y_3)\\ &\quad= \tilde{H}_3(S_1,y_2,y_3)-(\tilde{H}_3(S_1,y_2,y_3)-\tilde{H}_3(S_1,\tilde{S}_2,\tilde{S}_2))-\beta(D_3(S_1,\tilde{S}_2,\tilde{S}_2)-D_3(S_1,y_2,y_3))\\ &\quad= \beta D_3(S_1,y_2,y_3)+\tilde{H}_3(S_1,\tilde{S}_2,\tilde{S}_2)-\beta D_3(S_1,\tilde{S}_2,\tilde{S}_2).\end{aligned}$$

One sees from these expressions that minimizing both cost functions boils down to minimizing $D_3(S_1,y_2,y_3)$.

Theorem 4.3 *The cost function $D_3(S_1,y_2,y_3)$ is minimized in $y_2=S_2$ and $y_3=S_3$. Therefore, the base stock levels become $(S_1,y_2,y_3)=(S_1,S_2,S_3)$, the* global *optimum. In this minimum, surplus-2 equals $D_3(S_1,\tilde{S}_2,\tilde{S}_2)-D_3(S_1,S_2,S_3)>0$. The installations 1 and 2 and installation 3 are better off than before coordination because $(H_1+H_2)(S_1,S_2,S_3)<(H_1'+H_2')(S_1,\tilde{S}_2,\tilde{S}_2)$ and $H_3(S_1,S_2,S_3)<\tilde{H}_3(S_1,\tilde{S}_2,\tilde{S}_2)$.*

The coordination mechanism proposed in the negotiations result in each installation choosing its optimal base stock level as under centralized control. Hence, this mechanism ensures that the optimal individual decisions of the selfish installations are also optimal for the entire serial system.

The cost $(H_1+H_2)(S_1,S_2,S_3)$ of echelon 2 has to be divided among the installations 1 and 2. Recall that this cost consists of the individual costs $H_i'(S_1,S_2,S_3)$ of installation $i=1,2$ and the compensation paid to installation 3, $(\tilde{H}_3(S_1,S_2,S_3)-\tilde{H}_3(S_1,\tilde{S}_2,\tilde{S}_2))+\beta(D_3(S_1,\tilde{S}_2,\tilde{S}_2)-D_3(S_1,S_2,S_3))$. Naturally, each installation pays its own individual cost. Further, since installation 2 negotiates directly with installation 3 due to serial communication, he should pay the compensation to 3.

This cost division may seem unfavorable for installation 2 and resulting in large costs because he has to pay something extra besides his own cost, but that need not be true. The change in base stock levels from $(S_1,\tilde{S}_2,\tilde{S}_2)$ to (S_1,S_2,S_3) results in an increase of surplus-1 of $(H_1'+H_2')(S_1,\tilde{S}_2,\tilde{S}_2)-(H_1'+H_2')(S_1,S_2,S_3)$. Due to this increase, installation 2 receives an extra amount of $\alpha((H_1'+H_2')(S_1,\tilde{S}_2,\tilde{S}_2)-(H_1'+H_2')(S_1,S_2,S_3))$ from installation 1. This covers the compensation installation 2 has to pay to installation 3 if α satisfies

$$\alpha>\frac{\tilde{H}_3(S_1,S_2,S_3)-\tilde{H}_3(S_1,\tilde{S}_2,\tilde{S}_2)+\beta(D_3(S_1,\tilde{S}_2,\tilde{S}_2)-D_3(S_1,S_2,S_3))}{(H_1'+H_2')(S_1,\tilde{S}_2,\tilde{S}_2)-(H_1'+H_2')(S_1,S_2,S_3)}=:\underline{\alpha}.$$

The fraction α should be larger than some lower bound $\underline{\alpha}$, the compensation paid to installation 3 divided by the increase of surplus-1. Notice that $\underline{\alpha} < 1$.

Lemma 4.4 *The cost division decreases the costs of installation 1, $H_1'(S_1, S_2, S_3) < H_1'(S_1, \tilde{S}_2, \tilde{S}_2)$. Installation 2 saves cost, $H_2'(S_1, S_2, S_3) +$ compensation to 3 $< H_2'(S_1, \tilde{S}_2, \tilde{S}_2)$, if $\alpha > \underline{\alpha}$.*

We conclude that both installations save cost if α is large enough. The overall result of the negotiations is summarized in the Theorem below.

Theorem 4.5 *Consider the strategic game played by the installations, where the cost functions of the installations 1, 2 and 3 are H_1', H_2' and H_3 due to the coordination mechanism. If installation 2 pays the compensation to installation 3 and if $\alpha > \underline{\alpha}$ then the strategy profile $(y_1, y_2, y_3) = (S_1, S_2, S_3)$ is the unique Nash equilibrium in this game.*

This result shows that under the right incentives the selfish installations take decisions that are also optimal for the entire serial system. Besides all installations accept the incentives because each of them saves costs.

4.3 Implementation in practice

In this subsection we show how the coordination mechanism can be implemented in practice in a three-echelon serial system. As shown in Section 3, initially the players/installations play a strategic game in which the true cost per period of player i is $\tilde{H}_i(y_1, y_2, y_3)$. Individual optimization of the costs implies $(y_1, y_2, y_3) = (\tilde{S}_1, \tilde{S}_1, \tilde{S}_1)$, the outcome of the system under decentralized control. In this optimum, installation 1 faces large costs because the installations 2 and 3 keep minimal stocks, $y_3 = y_2$ and $y_2 = y_1$.

Installation 1 starts its coordination with installation 2 by asking him to set $y_2 > y_1$ instead of $y_2 = y_1$ under individual optimization. Suppose installation 2 does so. He announces the use of base stock level y_2 whereas installation 1 announces the use of y_1. Let v_{t_1,t_2} denote the realized cumulative demand over the periods $t_1, \ldots, t_2$.

At the beginning of period $t + l_3$ installation 2 decides to return his echelon inventory position to y_2. Due to a possible shortage at his supplier, installation 3, his actual echelon inventory position is $\min(y_3 - v_{t,t+l_3-1}, y_2)$. At the beginning of period $t + l_3 + l_2$ his echelon stock becomes $\min(y_3 - v_{t,t+l_3-1}, y_2) - v_{t+l_3,t+l_3+l_2-1}$. Now installation 1 wants to raise his echelon inventory position to y_1. This request by installation 1 can only be fulfilled by installation 2 if the amount requested is smaller than his stock, that is, $y_1 \leq \min(y_3 - v_{t,t+l_3-1}, y_2) - v_{t+l_3,t+l_3+l_2-1}$.

These changes in inventory position influence the costs of both players as follows. First, due to his larger base stock level y_2 installation 2 may be faced with unsold goods for which he has to pay additional inventory costs at the end of period $t + l_3 + l_2 + l_1$, namely

$$(h_2 + h_3)\max(\min(y_3 - v_{t,t+l_3-1}, y_2) - v_{t+l_3,t+l_3+l_2-1} - y_1, 0).$$

This extra cost will be refunded by installation 1. Its expected value is $\tilde{H}_2(y_1, y_2, y_3) - \tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1)$ and leads to the modified costs $\tilde{H}_1(y_1, y_2, y_3) + (\tilde{H}_2(y_1, y_2, y_3) - \tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1))$ for installation 1 and $\tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1)$ for installation 2. Notice that installation 2 is indifferent between being refunded and staying in the initial situation.

The second part of the compensation paid by installation 1 to installation 2 is a fraction of the actual surplus of installation 1. This actual surplus is the difference between the actual cost savings of installation 1 and the cost increase compensated to installation 2.

Initially, installation 1 uses base stock level $y_1 = \tilde{S}_1$. He can calculate this level because it minimizes his cost function, $\tilde{S}_1 = \arg\min_{y_1} \tilde{H}_1(y_1, y_1, y_1)$. At the end of period $t + l_3 + l_2 + l_1$ the actual inventory position of installation 1 is

$$IP_1 = \min(\min(y_3 - v_{t,t+l_3-1}, y_2) - v_{t+l_3,t+l_3+l_2-1}, y_1) - v_{t+l_3+l_2,t+l_3+l_2+l_1}$$

while it would have been $\tilde{S}_1 - v_{t,t+l_3+l_2+l_1}$ in the initial situation. Both these inventory positions can be measured by keeping track of demand in the past $l_3 + l_2 + l_1 + 1$ periods. The actual inventory cost of installation 1 at the end of period $t + l_3 + l_2 + l_1$ is

$$(h_1 + h_2 + h_3)\max(IP_1, 0) + p\max(-IP_1, 0)$$

while they are

$$(h_1 + h_2 + h_3)\max(\tilde{S}_1 - v_{t,t+l_3+l_2+l_1+1}, 0) + p\max(v_{t,t+l_3+l_2+l_1+1} - \tilde{S}_1, 0)$$

in the initial situation.

The transfer paid by installation 1 to installation 2 in this period is a fraction α of the actual cost savings of installation 1 minus the compensation for installation 2

$$\begin{aligned}
&(h_1 + h_2 + h_3)\max(\tilde{S}_1 - v_{t,t+l_3+l_2+l_1+1}, 0) + p\max(v_{t,t+l_3+l_2+l_1+1} - \tilde{S}_1, 0) \\
&- ((h_1 + h_2 + h_3)\max(IP_1, 0) + p\max(-IP_1, 0)) \\
&- (h_2 + h_3)\max(\min(y_3 - v_{t,t+l_3-1}, y_2) - v_{t+l_3,t+l_3+l_2-1} - y_1, 0).
\end{aligned}$$

Installation 3's policy remains $y_3 = y_2$ and therefore the expected transfer $T(y_1, y_2, y_2)$ is

$$\begin{aligned}
T(y_1, y_2, y_2) &= \alpha\left[\tilde{H}_1(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) - \tilde{H}_1(y_1, y_2, y_2) - (\tilde{H}_2(y_1, y_2, y_2) - \tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1))\right] \\
&= \alpha\left[D_3(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) - D_3(y_1, y_2, y_2)\right]
\end{aligned}$$

where the last equality follows from Lemma 3.3 and $\tilde{H}_3(y_1, y_2, y_2) = \tilde{H}_3(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) = h_3 l_2 \mu$. The payment changes the costs of the players to

$$\tilde{H}_1(y_1, y_2, y_3) + (\tilde{H}_2(y_1, y_2, y_3) - \tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1)) + T(y_1, y_2, y_2) = H_1'(y_1, y_2, y_2)$$

for player 1 and

$$\tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) - T(y_1, y_2, y_2) = H_2'(y_1, y_2, y_2)$$

for player 2. The cost functions H'_i are the costs resulting from the coordination between the installations 1 and 2 as described in Section 4.1. Minimizing the cost of player i boils down to minimizing the total cost $D_3(y_1, y_2, y_2)$ of the supply chain under $y_3 = y_2$. The optimal choice of base stock levels is $(y_1, y_2, y_2) = (S_1, \tilde{S}_2, \tilde{S}_2)$ (see Theorem 4.1). Hence, the costs resulting from this coordination are $H'_i(S_1, \tilde{S}_2, \tilde{S}_2)$ for installation i. To summarize: coordination between the installations 1 and 2 can be implemented in practice by considering actual costs and keeping track of demands in the past $l_3 + l_2 + l_1 + 1$ periods.

Among similar lines the coordination between the installations 2 and 3, as described in Section 4.2, can be implemented.

5 Three-echelon distribution systems under decentralized control

A three-echelon distribution system consists of one supplier delivering goods to two local retailers. These retailers are denoted by indices 1 and 2, and the supplier by index 3. The leadtime for delivery of goods to the supplier is l_2 while the leadtime for all retailers is l_1. The distribution function of the random demand $u^{(n)}$ at retailer n is denoted by $F^{(n)}$. The cumulative demand per period is denoted by u and its expectation is μ.

The true one-period cost for retailer n consists of holding and penalty costs. If x_n is the echelon stock of retailer n at the beginning of a period then the true cost at the end of that period are

$$\tilde{L}_n(x_n) = (h_1 + h_2) \int_0^{x_n} (x_n - u^{(n)}) \mathrm{d}F^{(n)}(u^{(n)}) + p \int_{x_n}^{\infty} (u^{(n)} - x_n) \mathrm{d}F^{(n)}(u^{(n)}).$$

Using $C_n(y_n) = \mathbb{E}\tilde{L}_n(y_n - u_{l_1}^{(n)}) - h_2(y_n - (l_1 + 1)\mu^{(n)})$ as in [10], the expected average cost for retailer n can be written as

$$\tilde{D}_n(y_n) = \int_0^{\infty} \tilde{L}_n(y_n - u_{l_1}^{(n)}) \mathrm{d}F_{l_1}^{(n)}(u_{l_1}^{(n)}) = C_n(y_n) + h_2(y_n - (l_1 + 1)\mu^{(n)})$$

(compare this to (3.1)).

The retailers place their orders for replenishment of stock at the supplier. Under the balance assumption in [10], the supplier can distribute his echelon stock $y_3 - u_{l_2}$ among the retailers such that both retailers have an equal probability of stock-out, a so-called equal fractile position [6]. Denote by $z_n[y_3 - u_{l_2}]$ the amount allocated to retailer n according to this distribution.

The real order-up-to level w_n for retailer n depends on the base stock levels y_n, $n = 1, 2$, and the echelon stock $y_3 - u_{l_2}$ of the supplier. If the supplier's stock is large enough then the requests of the retailers will be fulfilled. Otherwise, we assume that the supplier distributes his stock among the retailers according to the allocation functions z_n. Thus,

$$w_n = \begin{cases} y_n, & y_1 + y_2 \leq y_3 - u_{l_2} \\ z_n[y_3 - u_{l_2}], & y_1 + y_2 > y_3 - u_{l_2} \end{cases} \tag{5.1}$$

and $w_1 + w_2 = \min(y_1 + y_2, y_3 - u_{l_2})$. Notice that in (5.1) the allocation function z_n is used if the supplier has a shortage, $y_1 + y_2 > y_3 - u_{l_2}$. In the next Section, where a coordination mechanism for distribution systems is discussed, the allocation function z_n is also used if the supplier has no shortage, $y_3 - u_{l_2} \geq y_1 + y_2$, to determine the compensation of each retailer to the supplier.

The expected true average cost $\tilde{H}_n$ per period for retailer n is $\tilde{H}_n(y_1, y_2, y_3) = \mathbb{E}\tilde{D}_n(w_n)$. The expected cost $\tilde{H}_3$ per period for the supplier consist of the inventory cost of goods in transit to the retailers and on stock:

$$\tilde{H}_3(y_1, y_2, y_3) = h_2 l_1 \mu + h_2 \int_0^\infty \max(y_3 - u_{l_2} - (y_1 + y_2), 0) \mathrm{d}F_{l_2}(u_{l_2})$$

The cost functions $\tilde{H}_i$ divide the total cost $D^{(3)}(y_1, y_2, y_3)$ of the distribution system among the installations, $\sum_{i=1}^{3} \tilde{H}_i(y_1, y_2, y_3) = D^{(3)}(y_1, y_2, y_3)$. This total cost $D^{(3)}(y_1, y_2, y_3)$ is minimized in the optimum under centralized control (the so-called global optimum) $(y_1, y_2, y_3) = (S_1, S_2, S_3)$, as shown in [10]. For retailer n the value S_n also minimizes $C_n(y_n)$.

Under decentralized control the supplier will keep its echelon base stock level y_3 as low as possible, namely $y_3 = y_1 + y_2$. This way, its costs are only $\tilde{H}_3(y_1, y_2, y_1 + y_2) = h_2 l_1 \mu$. This low base stock level implies $w_n = z_n[y_1 + y_2 - u_{l_2}] < y_n$; the retailers always receive less than they ordered. Their costs are

$$\tilde{H}_n(y_1, y_2, y_1 + y_2) = \mathbb{E}C_n(z_n[y_1 + y_2 - u_{l_2}]) + h_2(\mathbb{E}z_n[y_1 + y_2 - u_{l_2}] - (l_1 + 1)\mu^{(n)})$$

Minimizing this cost results in $y_n = \tilde{S}_n \neq S_n$. The outcome of the game under decentralized control is $(y_1, y_2, y_3) = (\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2)$, which is unequal to the global optimum.

6 Coordination mechanism for distribution systems

Both retailers are not happy with the outcome under decentralized control, in which the supplier keeps a minimal base stock level $y_3 = y_1 + y_2$. The following coordination mechanism is proposed. Both retailers ask the supplier to increase his base stock level such that $y_3 > y_1 + y_2$. This implies a cost increase for the supplier of size

$$\tilde{H}_3(y_1, y_2, y_3) - \tilde{H}_3(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) = h_2 \int_0^\infty (y_3 - u_{l_2} - (w_1 + w_2)) \mathrm{d}F_{l_2}(u_{l_2}).$$

This extra cost will be compensated by the retailers. Retailer n will pay the part

$$h_2 \int_0^\infty (z_n[y_3 - u_{l_2}] - w_n) \mathrm{d}F_{l_2}(u_{l_2}) = h_2 \mathbb{E}(z_n[y_3 - u_{l_2}] - w_n).$$

This is the expected holding cost of the extra amount received by retailer n if the supplier would always distribute the quantity $y_3 - u_{l_2}$ according to the allocation function z_n instead of supplying w_n. Due to this compensation the supplier is indifferent between cooperating with the retailers and working on his own because both result in equal costs.

The surplus of retailer n, his cost savings minus the compensation to the supplier, equals

$$\tilde{H}_n(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) - \tilde{H}_n(y_1, y_2, y_3) - h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n).$$

To provide an incentive for the supplier to cooperate with the retailers, he receives a fraction γ, $0 < \gamma < 1$, of the surplus of the retailers. Retailer n now faces his own cost, the compensation to the supplier for the cost increase and the payment of a fraction γ of his surplus,

$$\begin{aligned} & H_n(y_1, y_2, y_3) \\ = \quad & \tilde{H}_n(y_1, y_2, y_3) + h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n) \\ & + \gamma(\tilde{H}_n(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) - \tilde{H}_n(y_1, y_2, y_3) - h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n)) \\ = \quad & (1-\gamma)(\tilde{H}_n(y_1, y_2, y_3) + h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n)) + \gamma\tilde{H}_n(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2). \end{aligned}$$

These costs are minimized if the retailer sets his base stock level equal to his optimal level under centralized control.

Lemma 6.1 *Retailer n minimizes his cost $H_n(y_1, y_2, y_3)$ in $y_n = S_n$.*

The cost of the supplier after being compensated by the retailers is

$$\begin{aligned} & H_3(y_1, y_2, y_3) \\ = \quad & \tilde{H}_3(y_1, y_2, y_3) - \sum_{n=1}^{2} h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n) \\ & - \sum_{n=1}^{2} \gamma(\tilde{H}_n(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) - \tilde{H}_n(y_1, y_2, y_3) - h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n)) \\ = \quad & \gamma D^{(3)}(y_1, y_2, y_3) + \tilde{H}_3(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) - \gamma D^{(3)}(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2). \end{aligned}$$

Obviously, if the retailers set $y_n = S_n$ then $y_3 = S_3$ minimizes the cost of the supplier. This immediately implies the following result, which is presented without proof.

Theorem 6.2 *Consider the strategic game played by the supplier and the retailers, where firm i has cost function H_i due to the coordination mechanism. The strategy profile $(y_1, y_2, y_3) = (S_1, S_2, S_3)$ is the unique Nash equilibrium in this game.*

The coordination mechanism results in each firm choosing its global optimal base stock level. The incentive to use this mechanism is also present, as shown in the Theorem hereafter.

Theorem 6.3 *All firms have a lower cost in the Nash equilibrium than in the initial situation, $H_i(S_1, S_2, S_3) < \tilde{H}_i(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2)$.*

Therefore, also for distribution systems there exists a coordination mechanism that aligns the incentives of the installations with those of the supply chain and results in cost savings for each installation.

6.1 Implementation in practice

In this subsection we show how the coordination mechanism can be implemented in practice in a three-echelon distribution system with a supplier and two retailer. This implementation is analogous to the one for serial systems in Section 4.3. As shown in Section 5, initially the players play a strategic game in which the true cost per period of player i is $\tilde{H}_i(y_1, y_2, y_3)$. Assuming the players know the allocation functions z_n each of them can optimize its individual cost, resulting in $(y_1, y_2, y_3) = (\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2)$, the outcome of the system under decentralized control. This outcome implies costs $\tilde{H}_i(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2)$ for player i. In this optimum, the retailers face large costs because the supplier keeps a minimal stock, $y_3 = y_1 + y_2$.

The retailers start their coordination with the supplier by asking him to set $y_3 > y_1 + y_2$ instead of $y_3 = y_1 + y_2$ under individual optimization. Suppose the supplier does so. He announces the use of base stock level y_3 whereas the retailers announce the use of y_1 and y_2. Let $v^{(n)}_{t_1,t_2}$ denote the realized cumulative demand over the periods $t_1, \dots, t_2$ at retailer n and let v_{t_1,t_2} be the total demand at both retailers.

At the beginning of period t the supplier decides to return his echelon inventory position to y_3. Then at the beginning of period $t + l_2$ the echelon stock of the supplier becomes $y_3 - v_{t,t+l_2-1}$. Now retailer n wants to raise his echelon inventory position to y_n. The requests for replenishments by the retailers can only be fulfilled by the supplier if the total amount requested is smaller than his echelon stock, that is, $y_1 + y_2 \le y_3 - v_{t,t+l_2-1}$. Otherwise, the supplier will allocate his stock by means of the rationing functions z_n. Hence, at time $t + l_2$ the inventory position of retailer n is raised to

$$\begin{cases} y_n, & y_3 - v_{t,t+l_2-1} \ge y_1 + y_2, \\ z_n[y_3 - v_{t,t+l_2-1}], & y_3 - v_{t,t+l_2-1} < y_1 + y_2. \end{cases}$$

These changes in inventory position influence the costs of the players as follows. First, due to his larger base stock level y_3 the supplier may be faced with unsold goods for which he has to pay (additional) inventory costs at the end of period $t + l_2 + l_1$, namely

$$h_2 \max(y_3 - v_{t,t+l_2-1} - (y_1 + y_2), 0).$$

This extra cost will be refunded by the retailers. Retailer n refunds

$$h_2 \max(z_n[y_3 - v_{t,t+l_2-1}] - y_n, 0).$$

to the supplier. The expected value of this refund is $h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n)$ and leads to the modified costs

$$\tilde{H}_n(y_1, y_2, y_3) + h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n)$$

for retailer n and

$$\tilde{H}_3(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2)$$

for the supplier, who is indifferent between being refunded and staying in the initial situation.

The second part of the compensation paid by the retailers to the supplier is a fraction of the actual surplus of the retailers. This actual surplus is the difference between the actual cost savings of the retailer and the cost refunded to the supplier.

Initially, retailer n uses base stock level $y_n = \tilde{S}_n$. The retailer can calculate this level because it minimizes his cost function, $\tilde{S}_n = \arg\min_{y_n} \tilde{H}_n(y_1, y_2, y_1 + y_2)$. At the end of period $t + l_2 + l_1$ the actual inventory position of retailer n is

$$IP_n = \begin{cases} y_n - v^{(n)}_{t+l_2,t+l_2+l_1}, & y_3 - v_{t,t+l_2-1} \geq y_1 + y_2 \\ z_n[y_3 - v_{t,t+l_2-1}] - v^{(n)}_{t+l_2,t+l_2+l_1}, & y_3 - v_{t,t+l_2-1} < y_1 + y_2. \end{cases}$$

instead of $\tilde{S}_n - v^{(n)}_{t,t+l_2+l_1}$ in the initial situation. Both these inventory positions can be measured by keeping track of demand in the past $l_2 + l_1 + 1$ periods. The actual inventory cost of the retailer at the end of period $t + l_2 + l_1$ is

$$(h_1 + h_2)\max(IP_n, 0) + p\max(-IP_n, 0)$$

while they are

$$(h_1 + h_2)\max(\tilde{S}_n - v^{(n)}_{t,t+l_2+l_1}, 0) + p\max(v^{(n)}_{t,t+l_2+l_1} - \tilde{S}_n, 0)$$

in the initial situation. The transfer paid by retailer n to the supplier in this period is a fraction γ of the actual cost savings of the retailer minus the cost refunded to the supplier

$$\begin{aligned} &(h_1 + h_2)\max(\tilde{S}_n - v^{(n)}_{t,t+l_2+l_1}, 0) + p\max(v^{(n)}_{t,t+l_2+l_1} - \tilde{S}_n, 0) \\ &- ((h_1 + h_2)\max(IP_n, 0) + p\max(-IP_n, 0)) \\ &- h_2\max(z_n[y_3 - v_{t,t+l_2-1}] - y_n, 0). \end{aligned}$$

Now the expected transfer $T_n(y_1, y_2, y_3)$ is

$$T_n(y_1, y_2, y_3) = \gamma\left[\tilde{H}_n(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) - \tilde{H}_n(y_1, y_2, y_3) - h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n)\right]$$

This transfer changes the costs of the players to

$$\begin{aligned} &\tilde{H}_n(y_1, y_2, y_3) + h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n) + T_n(y_1, y_2, y_3) \\ &\quad = (1-\gamma)(\tilde{H}_n(y_1, y_2, y_3) + h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n)) + \gamma\tilde{H}_n(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) \end{aligned}$$

for retailer n and

$$\begin{aligned} &\tilde{H}_3(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) + T_1(y_1, y_2, y_3) + T_2(y_1, y_2, y_3) \\ &\quad = \gamma D^{(3)}(y_1, y_2, y_3) + \tilde{H}_3(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) + \gamma D^{(3)}(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) \end{aligned}$$

for the supplier. These costs are identical to those resulting from the coordination. The optimal base stock levels are $y_n = S_n$ and $y_3 = S_3$ (see Theorem 6.2). Hence, the minimal costs resulting from this coordination are

$$(1-\gamma)(\tilde{H}_n(S_1, S_2, S_3) + h_2\mathbb{E}(z_n[S_3 - u_{l_2}] - w_n)) + \gamma\tilde{H}_n(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2)$$

for retailer n and

$$\gamma D^{(3)}(S_1, S_2, S_3) + \tilde{H}_3(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2) + \gamma D^{(3)}(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1 + \tilde{S}_2)$$

for the supplier. To summarize: coordination of installations in a distribution system can be implemented in practice by considering actual costs and keeping track of demands in the past $l_2 + l_1 + 1$ periods at both retailers.

7 Conclusions

In this paper coordination mechanisms for three-echelon serial and distribution systems under decentralized control are studied. This decentralized control implies that the echelon base stock levels set by the installations need not be optimal from the perspective of the supply chain as a whole. The selfish installations act in their own self interest, which conflicts with global interests. For serial and distribution systems, coordination mechanisms are presented that work under very mild conditions. Both mechanisms are based upon the idea that installations should be fully compensated for cost increases due to larger base stock levels and should also receive something extra that should persuade the installations to join the cooperation. The mechanisms alter the costs in such a way that the global optimum is the unique Nash equilibrium of the corresponding strategic game. The practical implementation of these mechanisms is discussed.

All the results in this paper can easily be extended to N-echelon serial and distribution systems, where $N > 3$. Directions for future research include the extension of these results to distribution systems with unequal lead time and holding costs for the retailers, as well as asymmetric information availability in serial and distribution systems.

8 Proofs

Proof of Lemma 2.1.
According to the definition of D_N we derive subsequently

$$\begin{aligned} D_1(y_1) &= \int_0^\infty L_1(y_1 - u_{l_1})\mathrm{d}F_{l_1}(u_{l_1}) \\ &= h_1(y_1 - (l_1+1)\mu) + (p + h_1 + h_2 + h_3)\int_{y_1}^\infty (u_{l_1+1} - y_1)\mathrm{d}F_{l_1+1}(u_{l_1+1}) \end{aligned}$$

for a one-echelon system,

$$\begin{aligned} D_2(y_1, y_2) &= D_1(y_1) + \int_0^\infty L_2(y_2 - u_{l_2})\mathrm{d}F_{l_2}(u_{l_2}) \\ &\quad + \int_{y_2 - y_1}^\infty [D_1(y_2 - u_{l_2}) - D_1(y_1)]\mathrm{d}F_{l_2}(u_{l_2}) \\ &= D_1(y_1) + h_2(y_2 - (l_2+1)\mu) + \int_{y_2 - y_1}^\infty [D_1(y_2 - u_{l_2}) - D_1(y_1)]\mathrm{d}F_{l_2}(u_{l_2}) \end{aligned}$$

for a two-echelon serial system. Finally, notice that for $j \geq 2$

$$\begin{aligned}
& C_{j-1}(y_1, \ldots, y_{j-2}, y_j - u_{l_j}) - C_{j-1}(y_1, \ldots, y_{j-2}, y_{j-1}) \\
&= D_{j-1}(y_1, \ldots, y_{j-2}, y_j - u_{l_j}) - D_{j-1}(y_1, \ldots, y_{j-2}, y_{j-1}).
\end{aligned}$$

Using this identity, we obtain

$$\begin{aligned}
& D_3(y_1, y_2, y_3) \\
&= D_2(y_1, y_2) + \int_0^\infty L_3(y_3 - u_{l_3}) \mathrm{d}F_{l_3}(u_{l_3}) \\
&\quad + \int_{y_3 - y_2}^\infty [D_2(y_1, y_3 - u_{l_3}) - D_2(y_1, y_2)] \mathrm{d}F_{l_3}(u_{l_3}) \\
&= D_2(y_1, y_2) + h_3(y_3 - (l_3 + 1)\mu) + \int_{y_3 - y_2}^\infty [D_2(y_1, y_3 - u_{l_3}) - D_2(y_1, y_2)] \mathrm{d}F_{l_3}(u_{l_3})
\end{aligned}$$

for a three-echelon serial system. □

Proof of Lemma 3.1.
Start with $D_3(y_1, y_2, y_3)$ as formulated in Lemma 2.1:

$$\begin{aligned}
& D_3(y_1, y_2, y_3) \\
&= D_2(y_1, y_2) + h_3(y_3 - (l_3 + 1)\mu) + \int_{y_3 - y_2}^\infty [D_2(y_1, y_3 - u_{l_3}) - D_2(y_1, y_2)] \mathrm{d}F_{l_3}(u_{l_3}) \\
&= h_3(y_3 - (l_3 + 1)\mu) + \int_0^{y_3 - y_2} D_2(y_1, y_2) \mathrm{d}F_{l_3}(u_{l_3}) + \int_{y_3 - y_2}^\infty D_2(y_1, y_3 - u_{l_3}) \mathrm{d}F_{l_3}(u_{l_3}) \\
&= h_3(y_3 - (l_3 + 1)\mu) + \int_0^\infty D_2(y_1, \min(y_3 - u_{l_3}, y_2)) \mathrm{d}F_{l_3}(u_{l_3}) \\
&= h_3(y_3 - (l_3 + 1)\mu) + \mathbb{E} D_2(y_1, \underline{w}_2).
\end{aligned}$$

Among similar lines we obtain

$$\begin{aligned}
D_2(y_1, w_2) &= D_1(y_1) + h_2(w_2 - (l_2 + 1)\mu) + \int_{w_2 - y_1}^\infty [D_1(w_2 - u_{l_2}) - D_1(y_1)] \mathrm{d}F_{l_2}(u_{l_2}) \\
&= h_2(w_2 - (l_2 + 1)\mu) + \mathbb{E} D_1(\underline{w}_1),
\end{aligned}$$

and $D_1(w_1)$ follows directly from Lemma 2.1. □

Proof of Lemma 3.2.
The true cost for installation 1 is

$$\begin{aligned}
\tilde{H}_1(y_1, y_2, y_3) &= \mathbb{E} \tilde{D}_1(\underline{w}_1) \\
&= \mathbb{E} \left(D_1(\underline{w}_1) + (h_2 + h_3)(\underline{w}_1 - (l_1 + 1)\mu) \right) \\
&= \mathbb{E} D_1(\underline{w}_1) + (h_2 + h_3)(\hat{w}_1 - (l_1 + 1)\mu)
\end{aligned}$$

where the second equality is due to (3.1) and $\hat{w}_1 = \mathbb{E}\underline{w}_1$. The true expected cost for installation 2 is

$$\begin{aligned}
&\tilde{H}_2(y_1, y_2, y_3) \\
&= (h_2 + h_3)l_1\mu + (h_2 + h_3)\mathbb{E}\int_0^\infty \max(\underline{w}_2 - u_{l_2} - y_1, 0)\mathrm{d}F_{l_2}(u_{l_2}) \\
&= (h_2 + h_3)l_1\mu + (h_2 + h_3)\mathbb{E}\int_0^\infty (\underline{w}_2 - u_{l_2} - y_1 - \min(\underline{w}_2 - u_{l_2} - y_1, 0))\mathrm{d}F_{l_2}(u_{l_2}) \\
&= (h_2 + h_3)l_1\mu + (h_2 + h_3)\mathbb{E}\left(\underline{w}_2 - l_2\mu - y_1 - \int_0^\infty (\underline{w}_1 - y_1)\mathrm{d}F_{l_2}(u_{l_2})\right) \\
&= (h_2 + h_3)l_1\mu + (h_2 + h_3)(\hat{w}_2 - l_2\mu - \hat{w}_1),
\end{aligned}$$

where the third equality follows from $\min(\underline{w}_2 - u_{l_2} - y_1, 0) = \underline{w}_1 - y_1$. Similarly one obtains $\tilde{H}_3(y_1, y_2, y_3) = h_3 l_2\mu + h_3(y_3 - l_3\mu - \hat{w}_2)$, the true expected cost for installation 3. □

Proof of Lemma 3.3.
Adding $\tilde{H}_1$ and $\tilde{H}_2$ gives

$$\begin{aligned}
&(\tilde{H}_1 + \tilde{H}_2)(y_1, y_2, y_3) \\
&= \mathbb{E}D_1(\underline{w}_1) + (h_2 + h_3)(\hat{w}_1 - (l_1 + 1)\mu) + (h_2 + h_3)l_1\mu + (h_2 + h_3)(\hat{w}_2 - l_2\mu - \hat{w}_1) \\
&= \mathbb{E}D_1(\underline{w}_1) + (h_2 + h_3)(\hat{w}_2 - (l_2 + 1)\mu) \\
&= \mathbb{E}D_2(y_1, \underline{w}_2) + h_3(\hat{w}_2 - (l_2 + 1)\mu). \qquad (8.1)
\end{aligned}$$

The first equality is due to Lemma 3.2 and the last one follows from Lemma 3.1. Adding $\tilde{H}_3$ results in

$$\begin{aligned}
&(\tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3)(y_1, y_2, y_3) \\
&= \mathbb{E}D_2(y_1, \underline{w}_2) + h_3(\hat{w}_2 - (l_2 + 1)\mu) + h_3 l_2\mu + h_3(y_3 - l_3\mu - \hat{w}_2) \\
&= \mathbb{E}D_2(y_1, \underline{w}_2) + h_3(y_3 - (l_3 + 1)\mu) \\
&= D_3(y_1, y_2, y_3)
\end{aligned}$$

Again, Lemma 3.2 and Lemma 3.1 are used. □

Proof of Theorem 4.1.
Under $y_3 = y_2$ the real order-up-to-level w_2 for installation 2 becomes $w_2' = \min(y_2 - u_{l_3}, y_2) = y_2 - u_{l_3}$. This level has an expected value of $\hat{w}_2' = y_2 - l_3\mu$. Substituting this in (8.1) results in

$$\begin{aligned}
(\tilde{H}_1 + \tilde{H}_2)(y_1, y_2, y_2) &= \mathbb{E}D_2(y_1, y_2 - u_{l_3}) + h_3(y_2 - (l_3 + l_2 + 1)\mu) \\
&= (h_2 + h_3)(y_2 - (l_3 + l_2 + 1)\mu) + \mathbb{E}D_1(\underline{w}_1)
\end{aligned}$$

where the second equality follows from Lemma 3.1. From the same Lemma, one can see that this expression for $(\tilde{H}_1 + \tilde{H}_2)(y_1, y_2, y_2)$ is equal to the total cost $D_2(y_1, y_2)$ of a two-echelon serial system in case installation 2 has holding cost $h_2 + h_3$ and lead time $l_2 + l_3$ (instead of h_2 and l_2 respectively). Therefore, by Lemma 2.2, the cost $(\tilde{H}_1 + \tilde{H}_2)(y_1, y_2, y_2)$ is minimized in $y_1 = S_1$ and $y_2 = \tilde{S}_2 > S_2$. The larger lead time leads to a higher base stock level for installation 2. □

Proof of Theorem 4.2.
In the optimum $(S_1, \tilde{S}_2, \tilde{S}_2)$, surplus-1 is positive because

$$D_3(S_1, \tilde{S}_2, \tilde{S}_2) = \min_{y_1, y_2} D_3(y_1, y_2, y_2) < \min_{y_1} D_3(y_1, y_1, y_1) = D_3(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1).$$

Furthermore,

$$\begin{aligned} & H_1'(S_1, \tilde{S}_2, \tilde{S}_2) - \tilde{H}_1(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) \\ = \; & (1-\alpha) D_3(S_1, \tilde{S}_2, \tilde{S}_2) + \alpha D_3(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) - h_3 l_2 \mu - \tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) - \tilde{H}_1(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) \\ = \; & (1-\alpha) D_3(S_1, \tilde{S}_2, \tilde{S}_2) + \alpha D_3(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) - (\tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3)(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) \\ = \; & (1-\alpha) D_3(S_1, \tilde{S}_2, \tilde{S}_2) - (1-\alpha) D_3(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) \\ < \; & 0 \end{aligned}$$

After compensating installation 2, installation 1 is better off than before the coordination (due to the positive surplus). For installation 2 we derive

$$H_2'(S_1, \tilde{S}_2, \tilde{S}_2) - \tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) = \alpha D_3(S_1, \tilde{S}_2, \tilde{S}_2) - \alpha D_3(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) < 0.$$

The compensation leads to lower costs for installation 2. We conclude that both installations gain from the coordination. □

Proof of Theorem 4.3.
According to Lemma 2.2 $D_3(S_1, y_2, y_3)$ is minimized in $(S_1, y_2, y_3) = (S_1, S_2, S_3)$. In this minimum, surplus-2 is equal to $D_3(S_1, \tilde{S}_2, \tilde{S}_2) - D_3(S_1, S_2, S_3)$. This surplus is positive because

$$D_3(S_1, S_2, S_3) = \min_{y_1, y_2, y_3} D_3(y_1, y_2, y_3) < \min_{y_1, y_2} D_3(y_1, y_2, y_2) = D_3(S_1, \tilde{S}_2, \tilde{S}_2)$$

where the first equality is due to Lemma 2.2. Using this we obtain

$$\begin{aligned} & (H_1 + H_2)(S_1, S_2, S_3) \\ = \; & (1-\beta) D_3(S_1, S_2, S_3) - \tilde{H}_3(S_1, \tilde{S}_2, \tilde{S}_2) + \beta D_3(S_1, \tilde{S}_2, \tilde{S}_2) \\ = \; & (1-\beta) D_3(S_1, S_2, S_3) - (1-\beta) D_3(S_1, \tilde{S}_2, \tilde{S}_2) + D_3(S_1, \tilde{S}_2, \tilde{S}_2) - \tilde{H}_3(S_1, \tilde{S}_2, \tilde{S}_2) \\ < \; & D_3(S_1, \tilde{S}_2, \tilde{S}_2) - \tilde{H}_3(S_1, \tilde{S}_2, \tilde{S}_2) \\ = \; & (H_1' + H_2')(S_1, \tilde{S}_2, \tilde{S}_2) \end{aligned}$$

for the installations 1 and 2 and

$$\begin{aligned} H_3(S_1, S_2, S_3) \; &= \; \beta D_3(S_1, S_2, S_3) + \tilde{H}_3(S_1, \tilde{S}_2, \tilde{S}_2) - \beta D_3(S_1, \tilde{S}_2, \tilde{S}_2) \\ &< \; \tilde{H}_3(S_1, \tilde{S}_2, \tilde{S}_2) \end{aligned}$$

for installation 3. Both inequalities follow from surplus-2 being positive in the minimum (S_1, S_2, S_3). $\square$

Proof of Lemma 4.4.
The cost distribution results in a cost saving for installation 1 because

$$\begin{aligned} & H'_1(S_1, S_2, S_3) \\ = \;& (1-\alpha)(D_3 - \tilde{H}_3)(S_1, S_2, S_3) + \alpha(D_3 - \tilde{H}_3)(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) - \tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) \\ < \;& (1-\alpha)(D_3 - \tilde{H}_3)(S_1, \tilde{S}_2, \tilde{S}_2) + \alpha(D_3 - \tilde{H}_3)(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) - \tilde{H}_2(\tilde{S}_1, \tilde{S}_1, \tilde{S}_1) \\ = \;& H'_1(S_1, \tilde{S}_2, \tilde{S}_2), \end{aligned}$$

where $D_3(S_1, S_2, S_3) < D_3(S_1, \tilde{S}_2, \tilde{S}_2)$ and $\tilde{H}_3(S_1, S_2, S_3) > \tilde{H}_3(S_1, \tilde{S}_2, \tilde{S}_2)$ are used. Installation 2 saves cost if

$$H'_2(S_1, S_2, S_3) + \text{compensation to } 3 < H'_2(S_1, \tilde{S}_2, \tilde{S}_2),$$

or if the compensation to 3 is smaller than $H'_2(S_1, \tilde{S}_2, \tilde{S}_2) - H'_2(S_1, S_2, S_3)$. Concentrate on this latter difference:

$$\begin{aligned} & H'_2(S_1, \tilde{S}_2, \tilde{S}_2) - H'_2(S_1, S_2, S_3) \\ = \;& \alpha(D_3 - \tilde{H}_3)(S_1, \tilde{S}_2, \tilde{S}_2) - \alpha(D_3 - \tilde{H}_3)(S_1, S_2, S_3) \\ = \;& \alpha\left((H'_1 + H'_2)(S_1, \tilde{S}_2, \tilde{S}_2) - (H'_1 + H'_2)(S_1, S_2, S_3)\right), \end{aligned}$$

which is larger than the compensation paid to 3 if $\alpha > \underline{\alpha}$. Equation (4.1) is used in the final equality. $\square$

Proof of Theorem 4.5.
The coordination mechanism leads to the unique choice (S_1, S_2, S_3) of base stock levels. Furthermore, in this optimum all players have lower cost than on their own and the players 1 and 2 also have lower cost compared to the first round of negotiation if $\alpha > \underline{\alpha}$. We conclude that (S_1, S_2, S_3) is the unique Nash equilibrium of the corresponding strategic game. $\square$

Proof of Lemma 6.1.
Minimizing $H_n(y_1, y_2, y_3)$ with respect to y_n is equivalent to minimizing

$$\tilde{H}_n(y_1, y_2, y_3) + h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - w_n)$$

because the other terms in $H_n(y_1, y_2, y_3)$ are constants. If we recall the definition of $\tilde{H}_n(y_1, y_2, y_3)$, this expression can be rewritten to

$$\mathbb{E}C_n(w_n) + h_2\mathbb{E}(z_n[y_3 - u_{l_2}] - u^{(n)}_{l_1+1}).$$

The second term is independent of y_n. Therefore minimizing $H_n(y_1, y_2, y_3)$ boils down to minimizing $\mathbb{E}C_n(w_n)$ with respect to y_n, where

$$\mathbb{E}C_n(w_n) = C_n(y_n)F_{l_2}(y_3 - (y_1 + y_2)) + \int_{y_3-(y_1+y_2)}^{\infty} C_n(z_n[y_3 - u_{l_2}])\mathrm{d}F_{l_2}(u_{l_2}).$$

The first order condition for a minimum of $\mathbb{E}C_n(w_n)$ is

$$C'_n(y_n)F_{l_2}(y_3-(y_1+y_2))-(C_n(y_n)-C_n(z_n[y_1+y_2]))f_{l_2}(y_3-(y_1+y_2))=0.$$

Due to $z_n[y_1+y_2]=y_n$ this first order condition reduces to

$$C'_n(y_n)F_{l_2}(y_3-(y_1+y_2))=0.$$

This equality holds if $y_n=S_n$ because S_n minimizes C_n. The second order condition for a minimum is also satisfied. □

Proof of Theorem 6.3.
First, consider the retailers. Denote w_n by $w_n(y_1,y_2,y_3)$ to explicitly show the dependence on (y_1,y_2,y_3). Due to Lemma 6.1

$$\begin{aligned}
&H_n(S_1,S_2,S_3)\\
&\quad< \ H_n(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)\\
&\quad= \ (1-\gamma)(\tilde H_n(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)+h_2\mathbb{E}(z_n[\tilde S_1+\tilde S_2-u_{l_2}]-w_n(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)))\\
&\qquad\ +\gamma\tilde H_n(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)\\
&\quad= \ \tilde H_n(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)
\end{aligned}$$

The last equality results from $w_n(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)=z_n[\tilde S_1+\tilde S_2-u_{l_2}]$.

Second, consider the supplier.

$$\begin{aligned}
H_3(S_1,S_2,S_3) &= \gamma D^{(3)}(S_1,S_2,S_3)+\tilde H_2(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)-\gamma D^{(3)}(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)\\
&< \tilde H_3(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)
\end{aligned}$$

where the inequality follows from $D^{(3)}(S_1,S_2,S_3)<D^{(3)}(\tilde S_1,\tilde S_2,\tilde S_1+\tilde S_2)$. □

References

[1] S. Axsäter, W-F. Zhang (1999) A joint replenishment policy for multi-echelon inventory control. *International Journal of Production Economics*, Vol. 59, pp. 243–250.

[2] G.P. Cachon (2001) Stock wars: inventory competition in a two-echelon supply chain with multiple retailers. *Operations Research*, Vol. 49, pp. 658–674.

[3] G.P. Cachon, P.H. Zipkin (1999) Competitive and cooperative inventory policies in a two-stage supply chain. *Management Science*, Vol. 45, pp. 936–953.

[4] F. Chen, A. Federgruen, Y-S. Zheng (2001) Coordination mechanisms for a distribution system with one supplier and multiple retailers. *Management Science*, Vol. 47, pp. 693–708.

[5] A.J. Clark, H. Scarf (1960) Optimal policies for a multi-echelon inventory problem. *Management Science*, Vol. 6, pp. 475–490.

[6] G. Eppen, L. Schrage (1981) Centralized ordering policies in a multi-warehouse system with lead times and random demand. In: L.B. Schwarz (ed.), *Multi-level production/inventory control systems: theory and practice*, Studies in the Management Sciences, Vol. 16, North-Holland, Amsterdam, pp. 51–67.

[7] J. Gjerdrum, N. Shah, L.G. Papageorgiou (2002) Fair transfer price and inventory holding policies in two-enterprise supply chains. *European Journal of Operational Management*, Vol. 143, pp. 582–599.

[8] R. Güllü, G.J. van Houtum, Z. Alişan, N. Erkip (2003) Analysis of a decentralized supply chain under partial cooperation. *Beta working paper series* no. 90.

[9] G.J. van Houtum, K. Inderfurth, W.H.M. Zijm (1996) Materials coordination in stochastic multi-echelon systems. *European Journal of Operational Research*, Vol. 95, pp. 1–23.

[10] L.J.G. Langenhoff, W.H.M. Zijm (1990) An analytical theory of multi-echelon production/distribution systems. *Statistica Neerlandica*, Vol. 44, pp. 149–174.

[11] M.J. Osborne, A. Rubinstein (1994) *A course in game theory.* MIT Press, Cambridge.

[12] Ö. Özer (2003) Replenishment strategies for distribution systems under advance demand information. *Management Science*, Vol. 49, pp. 255–272.

[13] T.A. Taylor (2002) Supply chain coordination under channel rebates with sales effort effects. *Management Science*, Vol. 48, pp. 992–1007.

[14] S. Viswanathan, R. Piplani (2001) Coordinating supply chain inventories through common replenishment epochs. *European Journal of Operational Research*, Vol. 129, pp. 277–286.

[15] H. Wang, M. Guo, J. Efstathiou (2004) A game-theoretical cooperative mechanism design for a two-echelon decentralized supply chain. *European Journal of Operational Research*, Vol. 157, pp. 372–388.