
Department of Applied Mathematics
Faculty of EEMCS



University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands

Phone: +31-53-4893400

Fax: +31-53-4893114

Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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**A combinatorial approximation algorithm
for CDMA downlink rate allocation**

R.J. BOUCHERIE, A.F. BUMB,
A.I. ENDRAYANTO AND G.J. WOEGINGER

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A combinatorial approximation algorithm for CDMA downlink rate allocation*

R.J. Boucherie, A.F. Bumb, A.I. Endrayanto, G.J. Woeginger

Abstract

This paper presents a combinatorial algorithm for downlink rate allocation in Code Division Multiple Access (CDMA) mobile networks. By discretizing the coverage area into small segments, the transmit power requirements are characterized via a matrix representation that separates user and system characteristics. We obtain a closed-form analytical expression for the so-called Perron-Frobenius eigenvalue of that matrix, which provides a quick assessment of the feasibility of the power assignment for a given downlink rate allocation. Based on the Perron-Frobenius eigenvalue, we reduce the downlink rate allocation problem to a set of multiple-choice knapsack problems. The solution of these problems provides an approximation of the optimal downlink rate allocation and cell borders for which the system throughput, expressed in terms of utility functions of the users, is maximized.

Keywords: CDMA, feasibility transmit power, downlink rate allocation, multiple-choice knapsack, approximation scheme.

1 Introduction

One of the most important features of future wireless communication systems is their support of different user data rates. As a major complicating factor, due to their scarcity, the radio resources have to be used very efficiently. In Code Division Multiple Access (CDMA) systems, transmissions of different terminals are separated using (pseudo) orthogonal codes. The impact of multiple simultaneous calls is an increase in the interference level, that limits the capacity of the system. The assignment of transmission powers to calls is an important problem for network operation, since the interference caused by a call is directly related to the power. In the CDMA downlink, the transmission power is related to the downlink rates. Hence, for an efficient system utilization, it is necessary to adopt a rate allocation scheme in the transmission powers assignment.

The downlink rate assignment problem has been extensively studied in the literature [3, 6, 12, 14, 16]. In [6], Duan et al. present a procedure for finding the power and rate allocations that minimizes the total transmit power in one cell. In [12], Javidi analyzes several rate assignments in the context of the trade-off between fairness and over-all throughput. The rates are supposed to be continuous and the algorithms proposed for the rate allocation are based on solving the Lagrangean dual. Another approach for joint optimal rates and powers allocation, based on Perron-Frobenius theory, is proposed by Berggren ([3]) and by O'Neill et al. ([14]). Berggren ([3]) describes a distributed algorithm for

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assigning base station transmitter (BTSs) powers such that the common rate of the users is maximized, while in [14] multiple rates are considered. Again, both algorithms assume continuous rates. In [7], Endrayanto et al. present a model for characterizing downlink and uplink power assignment feasibility, for a single data rate.

In this paper we propose a rate and power allocation scheme for obtaining a close to optimum throughput for the downlink in a Universal Mobile Telecommunication System (UMTS) located on a highway. In accordance with the UMTS standard, the rates are chosen from a discrete set. Our goal is to assign rates to users, such that the utility of the system is maximized. The utility functions describing the satisfaction of the users have a very general form and do not have to satisfy any convexity requirement. For modeling the network, we use the model proposed in [7], which enables a characterization of downlink power feasibility via the Perron-Frobenius (PF) eigenvalue of a suitably chosen matrix. Moreover, an explicit analytical expression for the Perron Frobenius eigenvalue can be obtained. This explicit analytical expression of the PF eigenvalue reduces the rate optimization problem to a series of multiple choice knapsack problems, that can be solved efficiently by standard combinatorial optimization techniques. The algorithm we design is actually a fully polynomial time approximation scheme (FPTAS) for the rate optimization problem. The main advantages of this approach are that, by considering discrete rates, we avoid the rounding errors due to continuity assumptions and that, given an error bound ϵ , we can find a solution of value at least $(1 - \epsilon)$ times the optimum in polynomial time in the size of the input data and $\frac{1}{\epsilon}$. Moreover, the algorithm can be applied for a very large family of utility functions. Furthermore, our results indicate that the optimization problems for different cells are loosely coupled by a single interference parameter. If this parameter were known, the optimization problems for each cell could be independently solved.

The remainder of this paper is organized as follows. In Section 2 we present the model. In Section 3 we characterize the existence of a downlink power allocation for a given rate allocation via the Perron Frobenius theory. In Section 4 we formulate the rate optimization problem and present a FPTAS for finding a near optimal solution. We conclude our work and present ideas for further research in Section 5.

2 Model

This paper focuses on the modeling of downlink rate allocation in a CDMA system consisting of Base Transmitter Stations (BTSs) along a highway. Specifically, we focus on a two cells model, where only the area between the two base stations is taken into account.

For modeling a cell, we consider the discretized cell model proposed in [7]. This model permits, as we will see below, to characterize analytically the transmit power feasibility for a given rate allocation and users distribution. The discretized cell model can be described as follows. Let X and Y be the two base stations, situated at distance D from each other on a highway. The highway is divided into L small segments, from which segments $\{1, \dots, I\}$ are assigned to BTS X and segments $\{I + 1, \dots, L\}$ to BTS Y. We assume that in each segment, the subscribers are located in the middle of the segment and that they have the same data rate and power. Denote by n_i the number of users in segment $i, i \in \{1, \dots, L\}$.

We model the path loss propagation between a transmitter X and a receiver in segment i by a deterministic path loss propagation model of the following form

$$P_i^{rec} = P_i l_{i,X},$$

where $l_{i,X}$ depends only on the distance d_i between the middle of segment i and BTS X, P_i^{rec} is the received power in the i -th segment and P_i is the transmission power towards the i -th segment. If $l_{i,X} = d_i^{-\gamma}$, where $\gamma \geq 0$ is independent on the distance, we obtain the Okumura-Hata model, which performs reasonably in flat service areas (see [1, 10]).

A common measure of the quality of the transmission, is the *energy per bit to interference ratio*, $\left(\frac{E_b}{I_0}\right)$, that, for a user i , say, is defined as (see. e.g. [11])

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W \text{ useful signal power received by user } i}{R_i \text{ interference + thermal noise}},$$

where W is the system chip rate and R_i is the data rate in segment i . Under the described path loss model, with users in the same segment having the same power and the same rate and a constant noise N_0 , the energy per bit to interference ratio in the segments assigned to BTS X, respectively to BTS Y, becomes

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{R_i} \frac{P_i l_{i,X}}{\alpha l_{i,X} \left(\sum_{j=1}^I n_j P_j - P_i\right) + l_{i,Y} \sum_{j=I+1}^L n_j P_j + N_0}, \text{ for } i \in \{1, \dots, I\}, \quad (1)$$

respectively

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{R_i} \frac{P_i l_{i,Y}}{\alpha l_{i,Y} \left(\sum_{j=I+1}^L n_j P_j - P_i\right) + l_{i,X} \sum_{j=1}^I n_j P_j + N_0}, \text{ for } i \in \{I+1, \dots, L\}, \quad (2)$$

where α is the non-orthogonality factor. In order to ensure a certain quality of service, the energy per bit to interference ratio in each segment i has to be above a prespecified value ϵ_D^* . In the presence of perfect power control, we can actually assume that in each segment i , $\left(\frac{E_b}{I_0}\right)_i = \epsilon_D^*$.

We measure the satisfaction of a user in segment i , $i \in \{1, \dots, L\}$ by means of a positive utility function $u_i(R_i)$. For a presentation of the utility functions commonly used in the literature see [17].

Our goal is to allocate rates from a discrete and finite set $R = \{R_1, \dots, R_K\}$ to the users such that the total utility, i.e., the sum of the utilities of all users, is maximized under the condition that the prescribed quality of service is met for all users and that a feasible power assignment exists.

3 Downlink transmit power feasibility

In this section we derive a condition for the existence of a feasible power allocation when the rates allocated to users are known. For this, we will make use of the Perron Frobenius theory (see [15]), by analogy with the characterization of power feasibility for the uplink in [2, 8, 9].

For a rate allocation $r = (r_1, \dots, r_L)$, we say that a feasible power assignment exists if there exists a vector $p \in R^L$ verifying the following system

$$\begin{cases} \left(\frac{E_b}{I_0}\right)_i(r, p) = \epsilon_i, \text{ for each user in segment } i \\ p_i \geq 0 \text{ for each } i \in \{1, \dots, L\} \end{cases} \quad (3)$$

Before characterizing the feasibility of system (3) we introduce some notations. Let $\mathbf{N} = (n_1, \dots, n_L)$, $V(r_i) = \frac{\epsilon_D^* r_i}{W + \alpha \epsilon_D^* r_i}$, $\mathbf{L}_X = (l_1, \dots, l_I)$ and $\mathbf{L}_Y = (l_{I+1}, \dots, l_L)$, where

$$l_i = \begin{cases} \frac{l_{i,Y}}{l_{i,X}}, & \text{for } i \in \{1, \dots, I\} \\ \frac{l_{i,X}}{l_{i,Y}}, & \text{for } i \in \{I+1, \dots, L\}. \end{cases}$$

Based on (1) and (2), system (3) can be rewritten as:

$$\begin{cases} p_i = \alpha V(r_i) \sum_{j=1}^I p_j n_j + V(r_i) l_i \sum_{j=I+1}^L p_j n_j + V(r_i) l_{i,X}^{-1} N_0, & \text{for } i \in \{1, \dots, I\} \\ p_i = V(r_i) l_i \sum_{j=1}^I p_j n_j + \alpha V(r_i) \sum_{j=I+1}^L p_j n_j + V(r_i) l_{i,Y}^{-1} N_0, & \text{for } i \in \{I+1, \dots, L\}, \\ p \geq 0 \end{cases} \quad (4)$$

Note that system (4) has L equations, besides the positivity constraint of the power vector. Next we show that the feasibility of (4) is equivalent to the feasibility of a system with 2 equations (each of them characterizing one cell) and a positivity constraint.

Lemma 1 *System (4) is feasible if and only if the following system is feasible:*

$$\begin{cases} \left(1 - \alpha \sum_{j=1}^I V(r_j) n_j\right) x - \sum_{j=1}^I V(r_j) n_j l_j y = \sum_{j=1}^I V(r_j) n_j l_{j,X}^{-1} N_0 \\ - \sum_{j=I+1}^L V(r_j) n_j l_j x + \left(1 - \sum_{j=I+1}^L V(r_j) n_j\right) y = \sum_{j=I+1}^L V(r_j) n_j l_{j,Y}^{-1} N_0 \\ x \geq 0, y \geq 0 \end{cases} \quad (5)$$

Proof Let p be a positive solution of (4). In system (4) multiply each equation with the number of users in the corresponding segment and then add the first I equations and then the other $L - I$. It follows that $(x, y) = \left(\sum_{i=1}^I n_i p_i, \sum_{i=I+1}^L n_i p_i\right)$ verifies (5).

Let (x, y) be a solution of (5). Define:

$$p_i = \begin{cases} V(r_i) l_i y + \alpha V(r_i) x + V(r_i) l_{i,X}^{-1} N_0, & \text{for } i \in \{1, \dots, I\} \\ V(r_i) l_i x + \alpha V(r_i) y + V(r_i) l_{i,Y}^{-1} N_0 & \text{for } i \in \{I+1, \dots, L\} \end{cases} \quad (6)$$

By simple substitution in (4) it can be shown that p is a solution of (4). ■

Lemma 1 reduces the amount of calculations involved in characterizing the power feasibility, since it is straightforward to verify that a system with 2 equations in 2 positive variables is feasible.

System (5) can be rewritten in the following form:

$$(\mathbf{I} - \mathbf{T}) \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{c} \quad (7)$$

where

$$\mathbf{T} = \begin{pmatrix} \alpha \sum_{i=1}^I V(r_i) n_i & \sum_{i=1}^I V(r_i) n_i l_i \\ \sum_{i=I+1}^L V(r_i) n_i l_i & \alpha \sum_{i=I+1}^L V(r_i) n_i \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} \sum_{i=1}^I V(r_i) N_0 n_i l_{i,X}^{-1} \\ \sum_{i=I+1}^L V(r_i) N_0 n_i l_{i,Y}^{-1} \end{pmatrix}.$$

Since matrix \mathbf{T} is a non-negative matrix, according to the Perron-Frobenius theorem (see [15]), the feasibility of (7) is determined by the Perron-Frobenius (PF) eigenvalue $\lambda(\mathbf{T})$ of the matrix \mathbf{T} i.e.,

$$p \geq \mathbf{0} \text{ exist and } p = (\mathbf{I} - \mathbf{T})^{-1} \mathbf{c} \iff \lambda(\mathbf{T}) < 1. \quad (8)$$

The explicit expression of the PF eigenvalue of \mathbf{T} can be calculated easily

$$\lambda(\mathbf{T}) = \frac{1}{2} \left(\sum_{i=1}^I \alpha V(r_i) n_i + \sum_{i=I+1}^L \alpha V(r_i) n_i \right) + \frac{1}{2} \sqrt{\alpha^2 \left(\sum_{i=1}^I V(r_i) n_i - \sum_{i=I+1}^L V_i n_i \right)^2 + 4 \left(\sum_{i=1}^I V(r_i) n_i l_i \right) \left(\sum_{i=I+1}^L V_i n_i l_i \right)}.$$

Further note that the condition $\lambda(T) < 1$ is equivalent with the following system:

$$\begin{cases} \sum_{i=1}^I \alpha V(r_i) n_i + \sum_{i=I+1}^L \alpha V(r_i) n_i \leq 2 \\ (1 - \sum_{i=1}^I \alpha V(r_i) n_i) (1 - \sum_{i=I+1}^L \alpha V(r_i) n_i) > \left(\sum_{i=1}^I V(r_i) n_i l_i \right) \left(\sum_{i=I+1}^L V_i n_i l_i \right). \end{cases} \quad (9)$$

Since $\sum_{i=1}^I \alpha V(r_i) n_i$ and $\sum_{i=I+1}^L \alpha V(r_i) n_i$ cannot be both larger than 1 without violating the first inequality of (9), system (9) is equivalent with

$$\begin{cases} \sum_{i=1}^I \alpha V(r_i) n_i < 1 \\ \sum_{i=I+1}^L \alpha V(r_i) n_i < 1 \\ (1 - \sum_{i=1}^I \alpha V(r_i) n_i) (1 - \sum_{i=I+1}^L \alpha V(r_i) n_i) > \left(\sum_{i=1}^I V(r_i) n_i l_i \right) \left(\sum_{i=I+1}^L V_i n_i l_i \right). \end{cases}$$

Hence, we have proved the following theorem.

Theorem 2 For a given rate allocation r , a feasible power allocation exists, i.e., system (4) is feasible, if and only if

$$\begin{cases} \sum_{i=1}^I \alpha V(r_i) n_i < 1 \\ \sum_{i=I+1}^L \alpha V(r_i) n_i < 1 \\ (1 - \sum_{i=1}^I \alpha V(r_i) n_i) (1 - \sum_{i=I+1}^L \alpha V(r_i) n_i) > \left(\sum_{i=1}^I V(r_i) n_i l_i \right) \left(\sum_{i=I+1}^L V(r_i) n_i l_i \right). \end{cases}$$

Theorem 2 provides a clear motivation for discretizing the cells into segments, since it facilitates obtaining an analytical model for characterizing the transmit power feasibility for a certain rate allocation and a certain user distribution. Moreover, we observe that the first two conditions we obtained characterize the two cells separately and the third contains products of factors depending only of one cell. In the next section we will show how these nice properties lead to a fast algorithm for finding a close to optimal rate allocation.

4 The rate optimization problem

Let $R = \{R_1, R_2, \dots, R_K\}$ be the set of admissible rates, where $R_1 < R_2 < \dots < R_K$. The decision of dropping the users of a segment is equivalent with assigning zero rate to the respective segment, case in which $R_1 = 0$.

The problem of allocating rates from the set R to users such that the total utility of the users is maximized, under the condition of ensuring the required Quality of Service and a feasible power assignment, can be formulated as follows:

$$(P) \quad \begin{aligned} \max \quad & \sum_{i=1}^L u_i(r_i) \\ \text{s.t.} \quad & \left(\frac{E_b}{I_0} \right)_i(r, p) = \epsilon_D^*, \text{ for each user in segment } i \\ & r_i \in \{R_1, \dots, R_K\}, \text{ for each } i \in \{1, \dots, L\} \\ & p_i \geq 0 \text{ for each } i \in \{1, \dots, L\} \end{aligned}$$

where r_i , respectively p_i represent the rate, respectively the power allocated to segment i and ϵ_D^* is the threshold for the the energy per bit to interference ratio.

We are interested in designing an algorithm for assigning rates to segments in such a way that a throughput of at least $(1 - \epsilon)$ times the optimum is obtained, in a time polynomial in the size of an instance and $\frac{1}{\epsilon}$. Such an algorithm would be a fully polynomial approximation scheme (FPTAS) for problem (P). We distinguish three main steps in the design of the algorithm:

- First we show that finding an optimal solution of (P) can be reduced to solving a set of optimization problems $\{P_1(t), P_2(t) | t \in [t_{min}, t_{max}]\}$, where $P_1(t)$ characterize the first cell, $P_2(t)$ characterize the second cell and the interval $[t_{min}, t_{max}]$ is an interval depending on the system and the user distribution.
- Then we show that $P_1(t)$, respectively $P_2(t)$ are multiple choice knapsack problems, for which efficient algorithms are known.

- Finally, we will prove that, for finding a solution of value at least $(1 - \epsilon)$ times the optimum, for an $\epsilon > 0$, we only have to solve $P_1(t)$ and $P_2(t)$ for $O(\frac{1}{\epsilon})$ t 's in $[t_{min}, t_{max}]$. Since to solve $P_1(t)$, respectively $P_2(t)$ we can apply known FPTAS (see e.g. [4]) for the multiple choice knapsack problem, the algorithm we propose is a FPTAS for (P) .

We proceed with the first step of the analysis. Theorem 2 implies that the optimization problem (10) is equivalent with the following problem:

$$\begin{aligned}
(P') \quad & \max \sum_{i=1}^L u_i(r_i) \\
& \sum_{i=1}^I \alpha V(r_i) n_i < 1 \\
& \sum_{i=I+1}^L \alpha V(r_i) n_i < 1 \\
& (1 - \sum_{i=1}^I \alpha V(r_i) n_i) (1 - \sum_{i=I+1}^L \alpha V(r_i) n_i) > \left(\sum_{i=1}^I V(r_i) n_i l_i \right) \left(\sum_{i=I+1}^L V_i(r_i) n_i l_i \right) \\
& r_i \in \{R_1, \dots, R_K\}, \text{ for each } i \in \{1, \dots, L\}
\end{aligned}$$

Note that if the rate assignment in one of the cells is known, the problem of assigning rates to the segments of the other cell reduces to a multiple choice knapsack problem. The multiple choice knapsack problem is a NP-hard problem, for which a FPTAS based on dynamical programming is proposed in [4]. In a multiple choice knapsack problem the following data are given: the sizes and the profits of a set of objects, which are divided into disjoint classes, and the volume of a knapsack. The goal is to choose the set of objects with maximum profit among the sets of objects that fit into the knapsack and contain one object from each class. If, for example, the rates in the cell assigned to BTS Y were known, then, based on (P') , the problem of allocating rates to the segments in the cell assigned to BTS X becomes:

$$\begin{aligned}
& \max \sum_{i=1}^L u_i(r_i) \\
& \sum_{i=1}^I V(r_i) n_i \left(\alpha + l_i \frac{\sum_{i=I+1}^L \alpha V(r_i) n_i l_i}{1 - \sum_{i=I+1}^L \alpha V(r_i) n_i} \right) < 1 \\
& r_i \in \{R_1, \dots, R_K\}, \text{ for each } i \in \{1, \dots, I\}
\end{aligned}$$

This is a multiple choice knapsack problem with the following data: the objects are the pairs $\{(i, s), i \in \{1, \dots, I\}, s \in \{1, \dots, K\}\}$, a class consists of the objects corresponding to the same segment, the profit of an object (i, s) is $u_i(R_s)$ and its size is $V(R_s) n_i (\alpha +$

$l_i \frac{\sum_{i=I+1}^L \alpha V(r_i) n_i l_i}{1 - \sum_{i=I+1}^L \alpha V(r_i) n_i})$. The volume of the knapsack is 1.

Hence, if we knew the rate allocation in one of the cells, we could find a rate allocation for the segments in the other cell by applying an algorithm for the multiple choice knapsack problem. Since this also holds for the case where all the segments in one cell receive zero rate, in the following we may assume that in cell X there is at least one segment which receives non-zero rate.

Under these assumptions, problem (P') can be rewritten as:

$$(P') \quad \begin{aligned} & \max \sum_{i=1}^L u_i(r_i) \\ & \sum_{i=1}^I \alpha V(r_i) n_i < 1 \end{aligned} \quad (10)$$

$$\begin{aligned} & \sum_{i=I+1}^L \alpha V(r_i) n_i < 1 \\ & \frac{1 - \sum_{i=1}^I \alpha V(r_i) n_i}{\sum_{i=1}^I V(r_i) n_i l_i} > \frac{\sum_{i=I+1}^L V_i(r_i) n_i l_i}{1 - \sum_{i=I+1}^L \alpha V(r_i) n_i} \end{aligned} \quad (11)$$

$$\sum_{i=1}^I r_i > 0 \quad (12)$$

$$r_i \in \{R_1, \dots, R_K\}, \text{ for each } i \in \{1, \dots, L\}$$

Constraint (12) ensures that at least one segment in cell X will receive non zero rate. Remark that the variables and parameters characterizing the two cells are well separated in (P') . This suggests a decomposition of (P') into a set of problems corresponding to the first cell and one corresponding to the second cell. Denote by

$$t_{min} = \min_{r \in R^L} \frac{\sum_{i=I+1}^L V_i(r_i) n_i l_i}{1 - \sum_{i=I+1}^L \alpha V(r_i) n_i} \text{ and } t_{max} = \max_{r \in R^L, r \neq 0} \frac{1 - \sum_{i=1}^I \alpha V(r_i) n_i}{\sum_{i=1}^I V(r_i) n_i l_i}.$$

From (10)-(12) follows that (P') is feasible if and only if $\alpha V(R_1) \min_{i \in \{I+1, \dots, L\}} n_i l_i < 1$ and $t_{min} \leq t_{max}$. In what follows, we suppose that these two conditions are always satisfied.

For each $t \in [t_{min}, t_{max}]$ consider the following problems:

$$\begin{aligned}
P_1(t) \quad & \max \sum_{i=1}^I u_i(r_i) \\
& \frac{1 - \sum_{i=1}^I \alpha V(r_i) n_i}{\sum_{i=1}^I V(r_i) n_i l_i} > t \\
& \sum_{i=1}^I r_i > 0 \\
& r_i \in \{R_1, \dots, R_K\}, \text{ for each } i \in \{1, \dots, I\}
\end{aligned}$$

and

$$\begin{aligned}
P_2(t) \quad & \max \sum_{i=I+1}^L u_i(r_i) \\
& t > \frac{\sum_{i=I+1}^L V_i(r_i) n_i l_i}{1 - \sum_{i=I+1}^L \alpha V(r_i) n_i} \\
& r_i \in \{R_1, \dots, R_K\}, \text{ for each } i \in \{I+1, \dots, L\}.
\end{aligned}$$

Let OPT denote the optimal value of the optimization problem (P') and $OPT_1(t)$, respectively $OPT_2(t)$, be the optimal values of $P_1(t)$, respectively $P_2(t)$. In the following lemma we prove that we can find OPT by solving $P_1(t)$ and $P_2(t)$ for all $t \in [t_{min}, t_{max}]$.

Lemma 3 $OPT = \max_{t \in [t_{min}, t_{max}]} OPT_1(t) + OPT_2(t)$

Proof Consider a $t \in [t_{min}, t_{max}]$. Let $(\bar{r}_1, \dots, \bar{r}_I)$, respectively $(\tilde{r}_{I+1}, \dots, \tilde{r}_L)$, be optimal solutions of $P_1(t)$, respectively $P_2(t)$. Clearly, $(\bar{r}_1, \dots, \bar{r}_I, \tilde{r}_{I+1}, \dots, \tilde{r}_L)$ is a feasible solution of (P'), and therefore $OPT_1(t) + OPT_2(t) \leq OPT$. We proved that $\max_{t \in [t_{min}, t_{max}]} OPT_1(t) + OPT_2(t) \leq OPT$.

In order to prove the reverse inequality, consider an optimal solution r^* of (P). Let $t = \frac{1 - \alpha \sum_{i=1}^I V(r_i^*) n_i}{\sum_{i=1}^I V(r_i^*) n_i l_i}$. Since (r_1^*, \dots, r_I^*) is feasible for $P_1(t)$ and $(r_{I+1}^*, \dots, r_L^*)$ is feasible for $P_2(t)$, $OPT \leq OPT_1(t) + OPT_2(t)$. ■

Lemma 3 implies that an optimal rate allocation can be found by solving independently the set of optimization problems $\{P_1(t) | t \in [t_{min}, t_{max}]\}$ and $\{P_2(t) | t \in [t_{min}, t_{max}]\}$ where each set characterizes only one cell, the cells interacting only through the parameter t .

Next we show that $P_1(t)$ and $P_2(t)$ are multiple choice knapsack problems, which can be efficiently solved. For this, we rewrite $P_1(t)$ and $P_2(t)$ in the following form:

$$\begin{aligned}
P_1(t) \quad & \max \sum_{i=1}^I u_i(r_i) \\
& \sum_{i=1}^I V(r_i)n_i(\alpha + l_i t) < 1 \\
& \sum_{i=1}^I r_i > 0 \\
& r_i \in \{R_1, \dots, R_K\}, \text{ for each } i \in \{1, \dots, I\}
\end{aligned}$$

and

$$\begin{aligned}
P_2(t) \quad & \max \sum_{i=I+1}^L u_i(r_i) \\
& \sum_{i=I+1}^L V(r_i)n_i(\alpha t + l_i) < t \\
& r_i \in \{R_1, \dots, R_K\}, \text{ for each } i \in \{I+1, \dots, L\}.
\end{aligned}$$

The input to the multiple choice knapsack problems $P_1(t)$, respectively $P_2(t)$ is: the objects are the pairs $\{(i, s), i \in \{1, \dots, I\}, s \in \{1, \dots, K\}\}$, respectively $\{(i, s), i \in \{I+1, \dots, L\}, s \in \{1, \dots, K\}\}$; a class consists of the objects corresponding to the same segment; the profit of an object (i, s) is $u_i(R_s)$ and its size is $V(R_s)n_i(\alpha + l_i t)$ for $i \in \{1, \dots, I\}$, respectively $V(R_s)n_i(\alpha t + l_i)$ for $i \in \{I+1, \dots, L\}$. The volumes of the knapsacks are 1, respectively t . In $P_1(t)$ an extra condition is imposed, namely that the zero rate cannot be allocated to all users in cell X.

Since $P_1(t)$ and $P_2(t)$ are multiple choice knapsack problems, close to optimal solutions can be found by applying for example the FPTAS described in [4]. For an $\epsilon > 0$ and $t \in [t_{\min}, t_{\max}]$, let $K_1(t, \epsilon)$ and $K_2(t, \epsilon)$, be the value of the solution given by a FPTAS for $P_1(t)$, respectively $P_2(t)$. Hence,

$$K_1(t, \epsilon) \geq (1 - \epsilon)OPT_1(t)$$

and

$$K_2(t, \epsilon) \geq (1 - \epsilon)OPT_2(t).$$

Let t^* be the value for which $OPT_1(t^*) + OPT_2(t^*) = OPT$.

In next lemma we will prove that a feasible solution of (P') of value at least $(1 - \epsilon)OPT$ can be found using only the values $K_1(t, \epsilon)$ and $K_2(t, \epsilon)$, for $t \in [t_{\min}, t_{\max}]$.

Lemma 4 *For each $\epsilon > 0$, the following relation holds*

$$\max_{t \in [t_{\min}, t_{\max}]} \{K_1(t, \epsilon) + K_2(t, \epsilon)\} \geq (1 - \epsilon)OPT.$$

Proof From Lemma 3 follows

$$\begin{aligned} \max_{t \in [t_{min}, t_{max}]} \{K_1(t, \epsilon) + K_2(t, \epsilon)\} &\geq K_1(t^*, \epsilon) + K_2(t^*, \epsilon) \\ &\geq (1 - \epsilon)OPT_1(t^*) + (1 - \epsilon)OPT_2(t^*) \\ &\geq (1 - \epsilon)OPT, \end{aligned}$$

where for the second inequality we have used that $K_1(t^*, \epsilon)$, respectively $K_2(t^*, \epsilon)$ are values returned by a FPTAS for $P_1(t^*)$, respectively $P_2(t^*)$. ■

However, if $\epsilon \geq \frac{1}{2}$, in order to find a solution of value $(1 - \epsilon)OPT$ it is not necessary to calculate $\max_{t \in [t_{min}, t_{max}]} \{K_1(t, \epsilon) + K_2(t, \epsilon)\}$. Let $r = \{r_1, \dots, r_I\}$ and $r' = \{r_{I+1}, \dots, r_L\}$ be two rate allocations that give a total utility for cell 1, respectively cell 2, of value at least $\frac{1}{2}OPT_1(t_{min})$, respectively $\frac{1}{2}OPT_2(t_{max})$. Since $OPT_1(t)$ is a decreasing function and $OPT_2(t)$ is an increasing function, it follows that the rate allocation $r'' = (r_1, \dots, r_I, r_{I+1}, \dots, r_L)$ gives a total utility of value at least $\frac{1}{2}OPT$. The rate allocations r and r' with the above mentioned properties can be found by applying standard methods (see [4]).

In the sequel, we suppose that $\epsilon \leq \frac{1}{2}$.

The only bottleneck in finding a solution of (P') of value at least $(1 - \epsilon)OPT$ is that we have to calculate $K_1(t, \epsilon)$ and $K_2(t, \epsilon)$ for all $t \in [t_{min}, t_{max}]$. However, as we will see below, we can still obtain a solution close to optimum by analysing only a polynomial number of values of t .

For $\epsilon > 0$, let t_{app} be the value of t for which

$$K_1(t_{app}, \epsilon) + K_2(t_{app}, \epsilon) = \max_{t \in [t_{min}, t_{max}]} \{K_1(t, \epsilon) + K_2(t, \epsilon)\}.$$

Note that $OPT_1(t)$, respectively $OPT_2(t)$ are step functions and have at most 2^{KI} , respectively 2^{KJ} jump points, the number of the possible rate assignments in each cell. Therefore, for finding t_{app} , it would suffice to check only the jump points of the two functions.

Next lemma's further reduce the set of t 's that must be considered for obtaining a solution of value at least $(1 - \epsilon)OPT$.

Lemma 5 *For each $\epsilon < \frac{1}{2}$, the following holds*

$$t_{app} \in [t_{min}, t_{max}] \setminus \{t \mid K_1(t_{app}, \epsilon) < \epsilon K_1(t_{min}, \epsilon) \text{ and } K_2(t_{app}, \epsilon) < \epsilon K_2(t_{max}, \epsilon)\}$$

Proof Suppose that $K_1(t_{app}, \epsilon) < \epsilon K_1(t_{min}, \epsilon)$ and $K_2(t_{app}, \epsilon) < \epsilon K_2(t_{max}, \epsilon)$. Hence,

$$K_1(t_{app}, \epsilon) + K_2(t_{app}, \epsilon) < \epsilon (K_1(t_{min}, \epsilon) + K_2(t_{max}, \epsilon)),$$

which, since $\epsilon < \frac{1}{2}$, leads to a contradiction with

$$K_1(t_{app}, \epsilon) + K_2(t_{app}, \epsilon) \geq \frac{1}{2} (K_1(t_{min}, \epsilon) + K_2(t_{min}, \epsilon) + K_1(t_{max}, \epsilon) + K_2(t_{max}, \epsilon)).$$

■

Consider the sets $A_l(\epsilon)$ and $\overline{A}_l(\epsilon)$, for $l \in \{0, 1, \dots, \lfloor \frac{1}{\epsilon} \ln \frac{1}{\epsilon} \rfloor + 1\}$ defined as

$$A_0(\epsilon) = \{t | K_1(t_{min}, \epsilon) < K_1(t, \epsilon)\}$$

$$\overline{A}_0(\epsilon) = \{t | K_2(t_{max}, \epsilon) < K_2(t, \epsilon)\}$$

$$A_l(\epsilon) = \{t | (1 - \epsilon)^l K_1(t_{min}, \epsilon) < K_1(t, \epsilon) < (1 - \epsilon)^{l-1} K_1(t_{min}, \epsilon)\}, \text{ for } l \geq 1$$

$$\overline{A}_l(\epsilon) = \{t | (1 - \epsilon)^l K_2(t_{max}, \epsilon) < K_2(t, \epsilon) < (1 - \epsilon)^{l-1} K_2(t_{max}, \epsilon)\}, \text{ for } l \geq 1$$

Remark 6 From the fact that $(1 - \epsilon)^{\frac{1}{\epsilon} \ln \frac{1}{\epsilon}} < \epsilon$, and from Lemma 5 follows that $t_{app} \in \bigcup_{l=0}^{\lfloor \frac{1}{\epsilon} \ln \frac{1}{\epsilon} \rfloor + 1} (A_l(\epsilon) \cup \overline{A}_l(\epsilon))$

Further we will prove that by choosing only one element from each set A_l , respectively \overline{A}_l , we will not deviate significantly from the optimum. This will reduce the number of t 's to consider to at most $\lfloor \frac{2}{\epsilon} \ln \frac{1}{\epsilon} \rfloor + 2$.

Lemma 7 a) If $t_{app} \in A_l(\epsilon)$, then for each $t \in A_l(\epsilon)$, $(1 - \epsilon)K_1(t_{app}, \epsilon) \leq K_1(t, \epsilon)$.
b) If $t_{app} \in \overline{A}_l(\epsilon)$, then for each $t \in \overline{A}_l(\epsilon)$, $(1 - \epsilon)K_2(t_{app}, \epsilon) \leq K_2(t, \epsilon)$

Proof a) For $l = 0$,

$$K_1(t_{min}, \epsilon) \geq (1 - \epsilon)OPT_1(t_{min}) \geq (1 - \epsilon)OPT_1(t_{app}) \geq (1 - \epsilon)K_1(t_{app}, \epsilon),$$

where for the second inequality we used the monotonicity of OPT_1 . For $l \in \{1, \dots, \lfloor \frac{1}{\epsilon} \ln \frac{1}{\epsilon} \rfloor + 1\}$ the proof follows immediately from the definition of A_l . ■

Let $J_1(\epsilon)$ be the set containing the maximal element from each nonempty set $A_l(\epsilon)$ and $J_2(\epsilon)$ the set containing the minimal element from each nonempty set $\overline{A}_l(\epsilon)$.

The following lemma shows that in order to find a feasible solution of (P) of value at least $(1 - \epsilon)OPT$ it is enough to calculate $K_1(t, \epsilon')$ and $K_2(t, \epsilon')$ only for $t \in J_1(\epsilon') \cup J_2(\epsilon')$, for a well chosen ϵ' .

Lemma 8 For $\epsilon' = 1 - \sqrt[3]{1 - \epsilon}$ the following relation holds

$$\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t, \epsilon') + K_2(t, \epsilon')\} \geq (1 - \epsilon)OPT.$$

Proof We have seen in Remark 6 that $t_{app} \in \bigcup_{l=0}^{\lfloor \frac{1}{\epsilon} \ln \frac{1}{\epsilon} \rfloor + 1} (A_l(\epsilon') \cup \overline{A}_l(\epsilon'))$. Suppose that $t_{app} \in A_k(\epsilon') \cap \overline{A}_l(\epsilon')$. Let $t_k = J_1(\epsilon') \cap A_k(\epsilon')$ and $\overline{t}_l = J_2(\epsilon') \cap \overline{A}_l(\epsilon')$.

From Lemma 7 follows that

$$K_1(t_k, \epsilon') \geq (1 - \epsilon')K_1(t_{app}, \epsilon') \tag{13}$$

and

$$K_2(\overline{t}_l, \epsilon') \geq (1 - \epsilon')K_2(t_{app}, \epsilon'). \tag{14}$$

Suppose that $t_k \geq \bar{t}_l$. Since $OPT_2(t)$ is an increasing function, the following relations hold:

$$\begin{aligned} K_2(t_k, \epsilon') &\geq (1 - \epsilon')OPT_2(t_k) \geq (1 - \epsilon')OPT_2(\bar{t}_l) \\ &\geq (1 - \epsilon')K_2(\bar{t}_l, \epsilon') \end{aligned} \quad (15)$$

Combining (13), (14), (15) and Lemma 4, we obtain

$$\begin{aligned} K_1(t_k, \epsilon') + K_2(t_k, \epsilon') &\geq (1 - \epsilon')(K_1(t_k, \epsilon') + K_2(\bar{t}_l, \epsilon')) \\ &\geq (1 - \epsilon')^2(K_1(t_{app}, \epsilon') + K_2(t_{app}, \epsilon')) \\ &\geq (1 - \epsilon')^3OPT, \end{aligned}$$

where the first inequality follows from (13), the second from (14) and (15), and the third from Lemma 4. Substituting $\epsilon' = 1 - \sqrt[3]{1 - \epsilon}$ in the last relation, we get

$$\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t, \epsilon') + K_2(t, \epsilon')\} \geq (1 - \epsilon)OPT.$$

A similar analysis can be done if $t_k \leq \bar{t}_l$, but based on the monotonicity of $OPT_1(t)$. ■

Hence, the number of points we are looking at in order to find a solution close to the optimum is reduced to $|J_1(\epsilon)| + |J_2(\epsilon)| = \frac{2}{\epsilon} \ln \frac{1}{\epsilon} + 2 = O(\frac{1}{\epsilon} \ln \frac{1}{\epsilon})$. Note that the points in $J_1(\epsilon) \cup J_2(\epsilon)$ can be found while running the FPTAS presented in [4] for obtaining $K_1(t_{min}, \epsilon')$, respectively for $K_2(t_{max}, \epsilon')$. This implies that the following procedure is a FPTAS for problem (P):

- Let $\epsilon' = 1 - \sqrt[3]{1 - \epsilon}$.
- Find the sets $J_1(\epsilon')$ and $J_2(\epsilon')$.
- For all $t \in J_1(\epsilon') \cup J_2(\epsilon')$, calculate $K_1(t, \epsilon')$ and $K_2(t, \epsilon')$, by using a FPTAS for the multiple choice knapsack problem.
- Find $t_{app} \in J_1(\epsilon') \cup J_2(\epsilon')$ for which $\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t, \epsilon') + K_2(t, \epsilon')\}$ is attained.
- Return the rate allocation obtained by solving $K_1(t_{app}, \epsilon')$ and $K_2(t_{app}, \epsilon')$.

If, for solving the multiple choice knapsack problems, one uses the FPTAS described in [4], which, for a given ϵ , runs in time $O(\frac{K^3 L}{\epsilon})$, then the running time of the algorithm presented above is $O(\frac{K^3 L}{\epsilon^2} \ln \frac{1}{\epsilon})$.

We conclude this section with several remarks on the algorithm.

Remark 1 The rate allocation provided in this paper should be seen as an almost optimal allocation (with respect to the utility functions) in an ideal setting. Most notably, it requires the base stations to have perfect and complete information on location and path loss of the mobile terminals. This information is clearly not available at the base station. Implementation of rate allocation in a UMTS system will most likely be based on heuristics that use an approximation of location and path loss. For example, from the required power the base station can approximate the location and path loss. In order to characterize the performance of such a heuristic and of a rate allocation, one can use as a benchmark the ideal solution proposed in this paper.

Remark 2 The aim of this paper is to demonstrate that the rate allocation problem reduces to solving coupled multiple choice knapsack problems. For solving such knapsack problems, various approaches are available in the literature. If one is not necessarily interested in obtaining a FPTAS for the rate allocation problem, one can use other approximation or exact algorithms described in the literature (see *e.g.* [5] for a fast branch and bound algorithm). Clearly, any algorithm for the multiple choice knapsack problem, should take into account the specific choice for the utility function. An extensive treatment of the influence of the utility function on the efficiency of the algorithms for solving the multiple choice knapsack problem is beyond the scope of this paper.

Remark 3 Note that the rate allocation algorithm proposed above can be easily adapted to the case where, for each segment, a different set of rates are required by users. The only change will be in the definition of the classes in the underlying multiple choice knapsack problems. More precisely, if, for a segment i , only the rates in the set $\{R_{k_1}, \dots, R_{k_2}\}$, with $k_1, k_2 \in \{1, \dots, K\}$ are required, the class of objects corresponding to segment i will become $\{(i, s), s \in \{k_1, \dots, k_2\}\}$.

Remark 4 The algorithm presented considers differentiated rate allocation in a two cell UMTS system, which goes beyond results described in the literature that usually consider single cell case (see [6] and [13]). For a UMTS network that covers a road, which is the main application intended in this paper, interference among cells will be most likely restricted to neighbouring cells. The main bottleneck in applying our results for general networks, taking into account interference among more than two cells, is the explicit formula for the Perron-Frobenius eigenvalue that is underlying our decomposition among cells. Developing heuristics for more general networks, based on our results, seems possible.

To this end, let us illustrate a possible heuristic for a three cell system. First consider cell 1 and 2. Once a rate allocation has been determined for cells 1 and 2, consider cell 2 and cell 3 and incorporate the interference from cell 1 as noise. Now consider cell 1 and cell 3 and the interference from cell 2 as noise, etc. This procedure may be followed until sufficient convergence is reached. It is among our aims for further research to develop a fixed point scheme for a multi cell UMTS system.

5 Summary and Further Research

This paper has provided a combinatorial algorithm for finding a downlink rate allocation in a CDMA network, that, for an $\epsilon > 0$, achieves a throughput of value at least $(1 - \epsilon)$ times the optimum. Based on the Perron-Frobenius eigenvalue of the power assignment matrix, we have reduced the downlink rate allocation problem to a set of multiple-choice knapsack problems, for which efficient algorithms are known. This approach proves to have several advantages. First, the discrete optimization approach has eliminated the rounding errors due to continuity assumptions of the downlink rates. Using our model, the exact rate that should be allocated to each user can be indicated. Second, the rate allocation approximation we proposed guarantees that the solution obtained is close to the optimum. Moreover, the algorithm works for very general utility functions. Furthermore, our results indicate that the optimal downlink rate allocation can be obtained in a distributed way: the allocation in each cell can be optimized independently, interference being incorporated in a single parameter t .

It is among our aims for further research to develop a downlink rate algorithm that takes into account mobility of users and limited transmit powers of cells. We may also focus on how the efficiency of the algorithm may be improved when specific utility functions are chosen.

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