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Continuous Feedback Fluid Queues

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Abstract

We investigate a fluid buffer which is modulated by a stochastic background process, while the momentary behavior of the background process depends on the current buffer level in a continuous way. Loosely speaking the feedback is such that the background process behaves 'as a Markov process' with generator $Q(y)$ at times when the buffer level is y, where the entries of $Q(y)$ are continuous functions of y. Moreover, the flow rates for the buffer may also depend continuously on the current buffer level. Such models are interesting in the context of closed-loop telecommunication networks, in which sources interact with network buffers, but may also be deployed in the study of certain production systems.

After defining the feedback behavior precisely, we deduce the Kolmogorov forward equations for the joint background/buffer-process of interest under some regularity assumptions. Interestingly, these equations can also be viewed as the continuity equations for probability mass, and may also be found in this way. For the system with two states for the background process we find an explicit solution for the stationary distribution of the process, which we illustrate by a number of examples. We finally present a numerical method that enables us to find this distribution for the multiple-state cases.

Keywords: Fluid queue, feedback, forward equations, continuity equations, uniformization.

AMS Subject Classifications (2000) — 60K25

1 Introduction

In the area of modern telecommunication systems fluid queues are often used as burst scale models for multiplexers, see e.g. [22, Ch. 17], [25]. In such models the content process $\{C(t), t \geq 0\}$ of a fluid buffer changes at a rate determined by some autonomous modulating stochastic process $\{X(t),\ t \geq 0\}$. The net fluid input rate $r(t)$ at time t is then given by r_i at times when $X(t) = i$. Of particular importance are Markov-modulated fluid models in which the background process $\{X(t)\}$ is a Markov process, see e.g. [4, 15].

More general models, known as feedback fluid queues, were introduced in [3] and [24]. Here, the behavior of the fluid buffer content is determined by the background process as before, but in turn the behavior of the background process now depends on the current buffer level. Loosely speaking, the background process behaves as a continuous-time Markov chain, but its 'generator matrix' $Q(y)$ now depends on the current buffer level y. Due to this feedback, the background process is actually no longer an autonomous Markov process as in traditional Markov-modulated fluid models; however, the *joint* process $\{X(t), C(t), t \geq 0\}$ still is Markov.

Feedback fluid queues can be useful for studying certain production systems and for modeling modern telecommunication networks in which the network and the sources interact. For instance, the interaction between one or two TCP sources, (i.e., traffic sources that use the Transmission Control Protocol, as currently deployed in the Internet)

and some buffer in the network was analyzed in [10]. In [17, 18], feedback fluid queues were used to study feedback schemes in access networks. In the latter reference [17], the most general feedback fluid models studied so far are described. They have the general feature that the matrix $Q(y)$ is a piecewise constant function of the buffer level y. Moreover, the fluid rates may also depend on y as piecewise constant functions $r_i(y)$.

In this paper we analyze an even more general feedback model in which both $Q(y)$ and the fluid rates $r_i(y)$ depend *continuously* on y. This also means that the background process experiences an influence from the buffer, continuously over time (i.e. in each arbitrarily small interval of time), so that the title of our paper has a twofold interpretation. We restrict our analysis to the case in which the buffer is finite.

Apart from the relation with Markov-modulated and feedback fluid queues, our model also has a relation with extensions of the classical storage process, with state-dependent output. In the classical storage model, the input is a compound Poisson process, and the release rate (i.e. the rate at which the buffer is depleted) is constant, see [21]. Early extensions of this storage process are considered in [6, 12], and references therein, in which the release rate is state-dependent; in fact it is a strictly positive piecewise continuous function of the current buffer content. Another paper worth mentioning in this context is [5], where not only the release rate, but also the arrival rate of customers is state-dependent.

In all of the above models with state-dependent release rate, the input process is allowed to have jumps. Also in the context of fluid queues, where the input process is gradual, some models were studied where the input and/or output rates are state-dependent. The authors of [8] allow the net input rates, i.e., the difference between the input and the output rates, to be piecewise constant functions of the buffer content. In [14] the case is solved in which these rates are piecewise continuous functions. In fact these two models are special cases of the feedback models in [17] and the current paper respectively, since here also the fluid rates depend on the buffer content, while unlike [8, 14] also the behavior of the background process itself is influenced by the buffer content.

We did not yet explain in detail what we mean by saying that the background process behaves 'as a Markov chain, with a level-dependent generator matrix $Q(y)$'. A precise description of our model is given in Section 2, where also some technical assumptions are mentioned. In Section 3 we derive the Kolmogorov forward equations for the joint Markov process $\{X(t), C(t), t \geq 0\}$. We do this by carefully following the infinitesimal approach, also employed in standard Markov-modulated fluid models, but we also pay attention to the relation with the continuity equation from physics. Section 4 relates the partial differential equations of Section 3 to a system of ordinary differential equations, by taking Laplace transforms. In Section 5 we show by means of a uniformization approach that a stationary distribution exists, and we determine the differential equations that it must satisfy, including appropriate boundary conditions. In Section 6 we restrict ourselves to the model where $\{X(t)\}$ only has two states. In this case it is possible to find the stationary distribution in closed form. In Section 7 we present some examples for the two-state case, investigating the importance of some of the assumptions mentioned earlier, and we have a short look at a model with an infinite buffer. Finally, in Section 8 we present a numerical approach to find the stationary distribution when $\{X(t)\}$ has more than two states.

2 Model and Preliminaries

In this section we explain the fluid model of interest more precisely. We also state some assumptions on the functions involved and introduce some notation.

2.1 Model

Consider a fluid system consisting of a finite buffer of size B and one or more sources that transmit fluid into it. The background process, or source process, is denoted by $\{X(t), t \geq 0\}$ and has state space $\mathscr{X} = \{1, \ldots, N\}$ for some finite N. The buffer content process $\{C(t), t \geq 0\}$ takes values in the set $[0, B]$. When $X(t) = i$ and $C(t) = y$, fluid is transmitted into the buffer such that the instantaneous net input rate is given by the function $r_i(y)$. The behavior of the source process is, loosely speaking, as follows. At times when $C(t) = y$, the process $\{X(t)\}$ behaves instantaneously as a continuous-time Markov chain with generator $Q(y)$. More precisely, we mean the following:

Definition 2.1. *The behavior of the source process is such that for all $y \in [0, B]$, $i \neq j$, and for all $t \geq 0$:*

1. $\mathbb{P}\{X(t+h) = i | X(t) = i, C(t) = y\} = 1 + Q_{ii}(y)h + o(h)$,

2. $\mathbb{P}\{X(t+h) = j | X(t) = i, C(t) = y\} = Q_{ij}(y)h + o(h)$,

3. $\mathbb{P}\{X \text{ makes more than one transition in } [t, t+h] | X(t) = j, C(t) = y\} = o(h)$.

Here the function $Q_{ij}(y), j \neq i$, is said to be the transition rate at which the source process jumps from state i to j when $C(t) = y$, and $Q_{ii}(y) = -\sum_{j \neq i} Q_{ij}(y)$.

Clearly, $\{X(t)\}$ and $\{C(t)\}$ are not Markov processes, as the future of each process depends on its own past through the state of the other process. In particular, the transition rates of $\{X(t)\}$ depend on the momentary content level, which, in turn, depends on the past of the process $\{X(t)\}$. However, the joint process $\{X(t), C(t)\}$ *is* a Markov process, with state space given by $\mathscr{X} \times [0, B]$. The joint process is characterized by the matrix $Q(y)$ and the diagonal matrix with net input rates $R(y) = \text{diag}(r_1(y), \ldots, r_N(y))$.

The main goal of the paper is to study the transient and stationary behavior of the process $\{X(t), C(t)\}$. We therefore define the distribution functions

$$F_i(y, t) = \mathbb{P}\{X(t) = i, C(t) \leq y\}, \quad i \in \mathscr{X},\ y \in [0, B],$$

and

$$F_i(y) = \lim_{t \to \infty} F_i(y, t), \quad i \in \mathscr{X},\ y \in [0, B].$$

Remark 2.1. The model is such that the joint process $\{X(t), C(t), t \geq 0\}$ can be viewed as a piecewise-deterministic Markov process, as described in [7] (as is indeed also the case for more traditional fluid queueing processes). In [7], the process makes a jump immediately when it reaches a boundary, whereas in our model it stays there for an exponentially distributed time, but in principle this difference is not essential, as it can be lifted by adding some additional states for the boundaries. In this paper we will not follow this approach, trying to describe the (extended) generator of the joint process with its precise domain. Rather, we derive the Kolmogorov forward equations immediately, as is often done in the literature on fluid queues.

2.2 Assumptions

In the sequel we need some assumptions on $Q_{ij}(y)$ and $r_i(y)$, which we collect here for ease of reference. Let $C_k(X)$ denote the space of k-times continuously differentiable functions on the set X. We assume the following.

1. $Q_{ij}(y) \in C[0, B]$ for all $i, j \in \mathscr{X}$. Consequently, $Q_{ij}(y) < \infty$.

2. $r_i(y) \in C_1(0, B)$ and finite on $[0, B]$.

3. $\min_i \inf_{y \in (0,B)} |r_i(y)| > 0$, i.e., the functions r_i are strictly bounded away from 0 on $(0, B)$.

4. There exists at least one i and one j such that $r_i(y) < 0 < r_j(y)$, for some (and hence all) $y \in (0, B)$.

The continuity assumptions in 1 and 2 may be weakened; we refer to Remark 3.1 in Section 3 to see how models where $Q(y)$ and/or $R(y)$ have discontinuities can be analyzed. Due to the assumptions on r_i we can unambiguously define two disjoint subsets of $\mathscr{X}$: the set of *up-states* $\mathscr{X}_+ = \{i \in \mathscr{X} | r_i(y) > 0\}$, and the set of *down-states* $\mathscr{X}_- = \{i \in \mathscr{X} | r_i(y) < 0\}$, when $y \in (0, B)$. Let $N_- = |\mathscr{X}_-|$ and $N_+ = |\mathscr{X}_+|$. Clearly, by Assumption 3 we have that $\mathscr{X} = \mathscr{X}_+ \cup \mathscr{X}_-$. Assumption 4 ensures that both subsets are non-empty, thereby avoiding trivial models.

Because the boundaries 0 and B act as impenetrable barriers for the content process we have to assume

5. $r_i(0) = 0$ if $i \in \mathscr{X}_-$ and $r_j(B) = 0$ if $j \in \mathscr{X}_+$.

6. $r_i(0) = r_i(0+) > 0$ if $i \in \mathscr{X}_+$ and $r_j(B) = r_j(B-) < 0$ if $j \in \mathscr{X}_-$.

Here, and in the sequel, we use shorthands like $r_i(0+) = \lim_{y \downarrow 0} r_i(y)$ and $r_i(B-) = \lim_{y \uparrow B} r_i(y)$.

Our next concern is the irreducibility of the process. Because it seems difficult to formally characterize when the process is irreducible, we will only give a sufficient condition. Notice that by Assumptions 2 and 3 the integrals

$$\int_x^y \frac{du}{r_j(u)}, \quad j \in \mathscr{X}_+ \qquad \text{and} \qquad \int_y^x \frac{du}{-r_j(u)}, \quad j \in \mathscr{X}_-$$

are finite for all $x, y \in [0, B]$. These integrals represent the time it takes for the joint process to move from (j, x) to (j, y) without making jumps in between. The boundedness of $Q(y)$ then ensures that in both cases there is a positive probability that, starting from (j, x), the process will indeed move to (j, y) without jumps. As a consequence, the following condition is sufficient (but not necessary) for irreducibility.

7a. For all $j \in \mathscr{X}_-$ ($j \in \mathscr{X}_+$) there is some $i \in \mathscr{X}_+$ ($i \in \mathscr{X}_-$) such that $Q_{ij}(B) > 0$ ($Q_{ij}(0) > 0$);

b. The matrix $\tilde{Q}$ whose entries are defined as $\tilde{Q}_{ij} = 1\{i \in \mathscr{X}_+\}Q_{ij}(B) + 1\{i \in \mathscr{X}_-\}Q_{ij}(0)$ is irreducible,

where $1\{i \in \mathscr{X}_+\} = 1$ when $i \in \mathscr{X}_+$ and 0 otherwise. To illustrate the line of reasoning, suppose that we would like to show that it is possible to reach state (j, y) from (i, x) with $i, j \in \mathscr{X}_-$. Then by Assumption 7a, a state (k, B) exists with $k \in \mathscr{X}_+$ from which the process can jump to (j, B) (followed by a drift from (j, B) to (j, y)), while assumption 7b ensures that it is possible to reach state (k, B) from $(i, 0)$ (and hence from (i, x)). Similar arguments for the cases where i and/or j are not in $\mathscr{X}_-$ then establish that 7a and 7b indeed imply that any state (j, y) can be reached with positive probability from any starting state. For models that do no satisfy Assumption 7, we note that the results in this paper remain valid as long as the process under study is irreducible or has a single recurrent class. In many instances this is not difficult to verify.

Representing $F_i(y, t)$ in the form $F_i(y, t) = A_i(y, t) + D_i(y, t)$, where $A_i(y, t)$ is an absolutely continuous function of y for all t and $D_i(y, t)$ is a jump function of y, it is clear that when $D_i(y, 0) = 0,\ y \in [0, B]$, we also have $D_i(y, t) = 0,\ y \in (0, B)$ for all $t \geq 0$. Hence the densities $f_i(y, t) = \partial_y F_i(y, t)$ in that case exist for $y \in (0, B)$. In Section 3 and the Appendix we will actually need the following stronger assumption.

8. $F_i(y, t) \in C_2((0, B) \times [0, \infty)), \quad i \in \mathscr{X}$,

because it implies that $\partial_t f_i(y, t)$ and $\partial_y f_i(y, t)$ are continuous for $t \geq 0$ and that $\partial_y \partial_t F_i = \partial_t \partial_y F_i$. Note that Assumption 8 is of a different nature then the previous ones; in fact it is unclear what precise conditions on $R(y)$ and $Q(y)$ make sure that it is satisfied.

Finally, we emphasize the notation for the atoms of $F_i(y, t)$ at $y = 0$ and $y = B$, given by

$$D_i(0, t) = F_i(0, t) \qquad \text{and} \qquad D_i(B, t) = F_i(B, t) - F_i(B-, t).$$

3 Kolmogorov Differential Equations

In this section we derive the Kolmogorov forward equations for the joint process $\{X(t), C(t)\}$ by two different methods. The first is, in some sense, the standard method in which the probabilities $F_i(y, t + h)$ are expressed in terms of $F_i(y, t)$ for small h, see e.g. [4, 26]. In other words, here we fix an event and express its probability in terms of the distribution functions involved, both at time t and $t + h$, and finally equate these. The other method, which we describe in Section 3.2 is based on an interpretation of the forward equation in physical terms resulting in a continuity equation, see e.g. [13]. Here we fix a subset of the state space, rather than an event, and consider the in- and outflow of probability mass for this set. We believe that the discussion of the latter method is of interest because of its natural interpretation, and because it is a quick method that leads us directly to the correct equations.

3.1 Derivation of the Forward Equations

First we summarize the derivation of the forward equation for the situation in which Q, but not necessarily R, is constant as a function of the buffer content, see also [14]. Then we mention a modification of this forward equation to the feedback case (in which $Q(y)$ is a non-constant function of y) that seems obvious at first sight, but is in fact *not* valid. By a more careful analysis we then obtain the desired result.

3.1.1 Constant Q

The usual approach to derive the forward equations when Q is a constant matrix is to express the distribution function $F_i(y, t + h)$ in terms of $F_i(y, t)$ for sufficiently small h and $y \in (0, B)$, i.e.,

$$F_i(y, t + h) = (1 + hQ_{ii})F_i(y - r_i(y)h, t) + \sum_{j \neq i} hQ_{ji}F_j(y, t) + o(h). \tag{1}$$

By Assumption 8 we may write $F_i(y - r_i(y)h, t) = F_i(y,t) - hr_i(y)\partial_y F_i(y,t) + o(h)$. Rearranging terms, dividing by h, and letting h approach to zero, yields

$$\frac{\partial}{\partial t}F_i(y,t) = \lim_{h\to 0}\frac{F_i(y,t+h) - F_i(y,t)}{h} = -r_i(y)\frac{\partial}{\partial y}F_i(y,t) + \sum_j F_j(y,t)Q_{ji}. \tag{2}$$

This is precisely the result obtained in [14]. Notice that (2) has the same form as the corresponding result for traditional Markov-modulated fluid queues, where both R and Q are constant matrices, only with r_i replaced by $r_i(y)$. To complete the analysis we should also provide boundary conditions. However, as they are not relevant for the sequel, we will not dwell on this, but rather turn to the case of interest.

3.1.2 Variable Q

Focusing on the interval $(0, B)$, it may be tempting to assume that in case the coefficients of Q are non-constant functions of y, the partial differential equation (2) can be adapted simply by replacing Q_{ji} by $Q_{ji}(y)$, just as (2) is found from the equations for the standard Markov-modulated fluid model by replacing the constants r_i by $r_i(y)$. This is *not* the case, however, since the $Q_{ji}(y)$ are transition rate functions for the source process, *provided* $C(t) = y$, while the event $\{X(t) = j, C(t) \le y\}$ — the probability of which is given by $F_j(y,t)$ — does not at all imply that $C(t) = y$. This problem can circumvented by considering densities rather than distribution functions. Thus, we write $d_y F_j(y,t) \equiv \mathbb{P}\{X(t) = j, C(t) \in dy\} = f_j(y,t)dy$, the last equality being valid if $y \in (0, B)$, and find the following extension of (1) for $y \in (0, B)$:

$$F_i(y, t+h) = \int_0^{y - r_i(y)h} (1 + hQ_{ii}(x))d_x F_i(x,t) + h\sum_{j\neq i}\int_0^y Q_{ji}(x)d_x F_j(x,t) + o(h). \tag{3}$$

Again using $F_i(y - r_i(y)h, t) = F_i(y,t) - hr_i(y)\partial_y F_i(y,t) + o(h)$ and the fact that $\int_{y-r_i(y)h}^{y} hQ_{ii}(x)d_x F_i(x,t) = o(h)$, we find

$$F_i(y, t+h) = F_i(y,t) - hr_i(y)\frac{\partial}{\partial y}F_i(y,t) + h\sum_j\int_0^y Q_{ji}(x)d_x F_j(x,t) + o(h). \tag{4}$$

By subtracting $F_i(y,t)$ from both sides, dividing by h and taking the limit $h \to 0$, we find

$$\frac{\partial}{\partial t}F_i(y,t) = -r_i(y)\frac{\partial}{\partial y}F_i(y,t) + \int_0^y \sum_j Q_{ji}(x)d_x F_j(x,t). \tag{5}$$

This is the correct generalization of (2) at the interior of $[0, B]$. Notice that we were somewhat careless in the derivation above; in particular we did not prove assertion (3). To take care of this omission, a precise derivation of (4) is presented in the Appendix, with proper attention to $o(h)$ details.

We now provide the forward equations at the boundaries $y = 0$ and $y = B$. The equation for $y = 0$ follows easily by letting $y \downarrow 0$ in (5). Taking this limit yields

$$\frac{\partial}{\partial t}D_i(0,t) = -f_i(0+,t)r_i(0+) + \sum_j D_j(0,t)Q_{ji}(0). \tag{6}$$

To obtain the equation at $y = B$ we first write down the forward equations for the process $\{X(t)\}$:

$$\frac{\partial}{\partial t}F_i(B,t) = \int_0^B \sum_j Q_{ji}(x)d_x F_j(x,t), \tag{7}$$

which can be obtained by similar methods as those used in the Appendix to derive (5). Next, we take the limit $y \uparrow B$ in (5),

$$\frac{\partial}{\partial t}F_i(B-,t) = -f_i(B-,t)r_i(B-) + \int_0^{B-} \sum_j Q_{ji}(x)d_x F_j(x,t).$$

and subtract this from (7) to find

$$\frac{\partial}{\partial t}D_i(B,t) = f_i(B-,t)r_i(B-) + \sum_j D_j(B,t)Q_{ji}(B). \tag{8}$$

Another way to find this result is to use (6) and a symmetry argument. Finally, with respect to the boundary conditions it is clear, on physical grounds, that for $t > 0$ we must have

$$\mathbb{P}\{X(t) \in \mathscr{X}_-, C(t) = B\} = \mathbb{P}\{X(t) \in \mathscr{X}_+, C(t) = 0\} = 0.$$

With these boundary conditions, equations (5), (6) and (8) fully specify the stochastic behaviour of the process (together with initial conditions). Before stating our main result however, we prefer to turn the integro-differential equation (5) for $y \in (0, B)$ into a partial differential equation for densities. Thus, we differentiate with respect to y, using Assumptions 2 and 8 from Section 2.2 (so far we only needed $r_i(y) \in C(0,B)$ and $F_i(y,t) \in C_1((0,B) \times [0,\infty))$). The result is straightforward:

$$\frac{\partial}{\partial t}f_i(y,t) = -\frac{\partial}{\partial y}(f_i(y,t)r_i(y)) + \sum_j f_j(y,t)Q_{ji}(y). \tag{9}$$

Notice that this equation is exactly the same as the corresponding equation from Section 3.1.1 (which can be found by differentiating (2) with respect to y), only with Q_{ji} replaced by $Q_{ji}(y)$. Thus, although in equation (2) for distribution functions we could not simply replace Q_{ji} with $Q_{ji}(y)$, we apparently can in the equations for densities. This may not come as a surprise, in view of the discussion leading to equation (3).

The following theorem summarizes the results in vector form, where the row vector $f(y,t)$ (respectively $D(y,t)$) has components $f_i(y,t)$ (respectively $D_i(y,t)$), $i = 1, \ldots, N$.

Theorem 3.1. *Under the assumptions of Section 2.2, the Kolmogorov forward equations for the joint process* $\{X(t), C(t)\}$ *are, in row vector form,*

$$\frac{\partial}{\partial t}f(y,t) = -\frac{\partial}{\partial y}(f(y,t)R(y)) + f(y,t)Q(y) \tag{10a}$$

$$\frac{d}{dt}D(0,t) = -f(0+,t)R(0+) + D(0,t)Q(0) \tag{10b}$$

$$\frac{d}{dt}D(B,t) = f(B-,t)R(B-) + D(B,t)Q(B). \tag{10c}$$

The boundary conditions to be satisfied for $t > 0$ *are*

$$D_i(0,t) = D_j(B,t) \equiv 0, \quad \text{if } i \in \mathscr{X}_+, j \in \mathscr{X}_-. \tag{11}$$

Remark 3.1. Our analysis extends easily to the case in which $Q(y)$ and $R(y)$ depend piecewise continuously on the buffer content y by combining the ideas presented here with those of [17]. Considering thresholds $0 = B_0 < B_1 < \ldots < B_N = B$ in the buffer as in [17], we obtain a system of differential equations as in (10a) for each interval (B_k, B_{k+1}), while the equations (10b) and (10c) will be supplemented with similar equations for $D(B_k, t)$.

3.2 Different Interpretation of the Forward Equations

In this section we relate the forward equations for $f_i(y)$, i.e., (9), to *continuity equations*. These are well-known relations in e.g. the hydrodynamics and diffusion literature, expressing a conservation law in differential form, see e.g. [9, II.13.1] or for a probabilistic setting [13, Section II.3]. In the physics context the continuity equation in one dimension is given by

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}(\rho(x,t)v(x)), \tag{12}$$

where $\rho(x,t)$ is the density function of some conserved quantity, e.g. mass or electric charge, and $v(x)$ is the velocity at which this quantity is moving. Note that the partial derivative with respect to position operates on the *product* ρv, just as in (9). We now rederive (9) from this point of view.

The dynamics of *any* conserved quantity is governed by a conservation law (admittedly, this is tautological). In general such a law states, in words,

$$\text{'Rate of Change'} = \text{'Influx'} - \text{'Outflux'} + \text{'Source terms'} - \text{'Sink terms'}.$$

Note that this principle is slightly more general than the equation for the density ρ of (12) as it also incorporates contributions from sources and sinks. We will now interpret this equation in terms of the probability mass assigned to the interval $(a, y]$ such that $0 < a \leq y < B$. As such, we treat probability mass as the conserved quantity of interest and discuss each term of this equation separately.

Consider first the left hand side. The rate of change of the probability that $\{X(t), C(t)\} \in \{i\} \times (a, y]$ is given by

$$\frac{d}{dt}(F_i(y,t) - F_i(a,t)),$$

or, in integral form,

$$\frac{d}{dt}\int_a^y f_i(x,t)dx = \int_a^y \frac{\partial}{\partial t} f_i(x,t)dx,$$

where the second equality holds as, by assumption, $\partial_t f_i$ is continuous in y and t.

Now turn to the right hand side of the conservation law and consider an $i \in \mathscr{X}_+$, so that $r_i(y) > 0$ for $y \in (0, B)$. Then the rate at which probability mass flows out of the interval $(a, y]$ at the boundary y, i.e., the *probability outflux* at y, is $f_i(y,t)r_i(y)$. The *influx* at a is seen to be $f_i(a,t)r_i(a)$. Thus the *net* outflux from the interval $(a, y]$ is

$$f_i(y,t)r_i(y) - f_i(a,t)r_i(a).$$

When $i \in \mathscr{X}_-$ this expression also gives the net outflux, because now the outflux is given by $-f_i(a,t)r_i(a)$, while $-f_i(y,t)r_i(y)$ is the influx. Further, we interpret the first term in the expression

$$\int_a^y \sum_{j \neq i} f_j(x,t)Q_{ji}(x)dx - \int_a^y f_i(x,t)|Q_{ii}(x)|dx$$

as a source term of probability mass, and the second as a sink term.

So now, by the 'conservation of probability mass', the above combines into

$$\int_a^y \frac{\partial}{\partial t} f_i(x,t)dx = -f_i(y,t)r_i(y) + f_i(a,t)r_i(a) + \int_a^y \sum_{j \neq i} f_j(x,t)Q_{ji}(x)dx - \int_a^y f_i(x,t)|Q_{ii}(x)|dx.$$

As $(a, y]$ is an arbitrary interval, and $\partial_t f_i, \partial_y f_i$ and $\partial_y r_i$ are by assumption continuous we obtain the one-dimensional continuity equation (9) by differentiating with respect to y. Similar reasoning at the boundaries 0 and B immediately gives (6) and (8).

4 Transient Behavior

Assuming for the moment that all assumptions in Section 2.2 are satisfied (including Assumption 8), we take Laplace transforms with respect to time so that the partial differential equations (10a) are transformed into ordinary differential equations, and the ordinary differential equations (10b) and (10c) are turned into algebraic equations. This is of interest as there exist efficient numerical procedures to solve such systems of (ordinary) differential and algebraic equations for fixed $s > 0$, where s is the Laplace transform variable. Moreover, once such solutions are known for various well-chosen values of s, the transient solution $f_i(y,t)$ can be obtained by standard numerical Laplace inversion, see e.g. [1].

Thus, let us define for fixed $s > 0$ the Laplace transforms with respect to time, $\tilde{f}_i(y)$, $\tilde{D}_i(0)$ and $\tilde{D}_i(B)$, where $\tilde{f}_i(y) = \int_0^\infty e^{-st} f_i(y,t)dt$, etc.; note that we suppress the dependence on s in the transforms. Taking the transform of (10) is straightforward, so that we obtain in vector form,

$$-f(y,0) + s\tilde{f}(y) = -\frac{d}{dy}\left(\tilde{f}(y)R(y)\right) + \tilde{f}(y)Q(y) \tag{13a}$$

$$-D(0,0) + s\tilde{D}(0) = -\tilde{f}(0+)R(0+) + \tilde{D}(0)Q(0) \tag{13b}$$

$$-D(B,0) + s\tilde{D}(B) = \quad \tilde{f}(B-)R(B-) + \tilde{D}(B)Q(B). \tag{13c}$$

For fixed s, the solution of this system involves $3N$ constants, namely the $2N$ constants in $\tilde{D}(0)$ and $\tilde{D}(B)$, and N coefficients corresponding to the fundamental set of solutions of (13a). To check whether this system is well-defined we reason as follows. The transform of (11) yields N_+ conditions at $y = 0$ and N_- at $y = B$. Let I be the identity, and note that $R(y)$ and $sI - Q(y)$ are invertible for all y (due to Assumption 3 and the fact that the eigenvalues of $sI - Q$ have positive real part). Then it is clear that (13b) imposes N_+ constraints on the coefficients vector and relates $\tilde{D}_i(0), i \in \mathscr{X}_-$, to the remaining N_- degrees of freedom of $\tilde{f}(y)$. These N_- degrees of freedom are, in geometric terms, 'propagated' by (13a) from the level $y = 0$ to $y = B$. There the fact that $D_i(B) = 0, i \in \mathscr{X}_-$, provides the missing conditions so that the N_- degrees of freedom of the coefficients vector are removed.

So far, we have worked with (Laplace transforms of) the density functions $f_i(y,t)$, complemented with the atoms $D_i(0,t)$ and $D_i(B,t)$. We could also, as in [14], derive equations for the Laplace transforms of the distribution functions, i.e. for $\tilde{F}_i(y) = \int_0^\infty e^{-st} F_i(y,t)dt$. It is not difficult to find from (5) that this yields for $0 \le y < B$,

$$-F(y,0) + s\tilde{F}(y) = -\frac{d\tilde{F}(y)}{dy} R(y) + \int_0^y d_x \tilde{F}(x) \cdot Q(x), \tag{14}$$

while for $y = B$ the first term on the right-hand side vanishes. The notation in the integral is similar as in Section 3.1.2, only generalized to matrix notation, so that e.g. the lower limit of the integral yields the term $\tilde{D}(0)Q(0)$. By differentiation and taking some appropriate limits we can obtain the more appealing equations (13) from (14). Notice however that for (5) and hence (14) to hold we only need to assume that the initial distribution has no atoms, i.e. $F_i(y,0) \in C_1([0,B])$, instead of the more stringent Assumption 8 in Section 2.2. Moreover, we can find any initial distribution as a weak limit of a sequence of distributions in $C_1([0,B])$, so that (14) must in fact hold for any initial distribution, possibly including atoms. This broader applicability can also be carried over to (13), since initial atoms at the boundaries are explicitly taken into account, while atoms in the open interval (0,B) can be included by means of Dirac delta functions. Thus we arrive at the following theorem.

Theorem 4.1. *Under Assumptions 1–7 in Section 2.2 and for any initial distribution, the Laplace transforms* $\tilde{f}_i(y)$, $\tilde{D}_i(0)$ *and* $\tilde{D}_i(B)$ *satisfy* (13)*.*

Let us now specialize to the case with a two-state source, taking $\mathscr{X} = \{1,2\}$, with

$$Q(y) = \begin{pmatrix} -\lambda(y) & \lambda(y) \\ \mu(y) & -\mu(y) \end{pmatrix}.$$

Considering just the differential equation (13a), we find

$$-f_1(y,0) + s\tilde{f}_1(y) + (r_1(y)\tilde{f}_1(y))' = -\lambda(y)\tilde{f}_1(y) + \mu(y)\tilde{f}_2(y) \tag{15a}$$

$$-f_2(y,0) + s\tilde{f}_2(y) + (r_2(y)\tilde{f}_2(y))' = \quad \lambda(y)\tilde{f}_1(y) - \mu(y)\tilde{f}_2(y). \tag{15b}$$

Now express $\tilde{f}_2$ by means of (15a) in terms of $\tilde{f}_1$ and $\tilde{f}_1'$, differentiate this with respect to y so that $\tilde{f}_2'$ can be written in terms of $\tilde{f}_1$, $\tilde{f}_1'$ and $\tilde{f}_1''$, and put these relations in (15b). The result is, after some algebra, the following inhomogeneous second order differential equation for $\tilde{f}_1$:

$$\begin{aligned} \tilde{f}_1'' + \left(\frac{s + r_2' + \mu}{r_2} - \frac{\mu'}{\mu} + \frac{s + 2r_1' + \lambda}{r_1} \right) \tilde{f}_1' + \left(\frac{s + r_1' + \lambda}{r_1} \left(\frac{s + r_2' + \mu}{r_2} - \frac{\mu'}{\mu} \right) + \frac{\lambda' + r_1''}{r_1} - \frac{\lambda\mu}{r_1 r_2} \right) \tilde{f}_1 \\ = \left(\frac{s + r_2' + \mu}{r_2} - \frac{\mu'}{\mu} \right) \frac{f_1(y,0)}{r_1} + \frac{f_1'(y,0)}{r_1} + \frac{\mu f_2(y,0)}{r_1 r_2}, \end{aligned} \tag{16}$$

where we suppress the dependence of λ, μ, r_1 and r_2 on y for conciseness.

We now consider two particular choices for the functions r_1, r_2, λ and μ. If we take the fluid rate functions of the form $r_i(y) = \alpha_i y + \beta_i$, and take λ and μ to be independent of y, the corresponding homogeneous equation can be rewritten as the differential equation for the hypergeometric function, see e.g. [16, p. 349–351]. Indeed, precisely this is done in Kella and Stadje [14], where the transient behavior of this system is found explicitly.

By another choice, where the functions λ and μ are non constant, namely $\lambda(y) = \mu(y) = (1+y)^{-1}$ and $r_2 = -r_1 = 1$, it is possible to solve the homogeneous equation corresponding to (16) in terms of a Kummer U-function.

The homogeneous equation in this case becomes

$$\tilde{f}_1''(y) + \frac{1}{1+y}\tilde{f}_1'(y) - \left(s^2 + \frac{3s}{1+y}\right)\tilde{f}_1(y) = 0.$$

Using the transformation $\tilde{f}_1(y) = w(2s(1+y))/\sqrt{1+y}$ we find

$$w''(y) + \left(-1/4 - \frac{3/2}{y} + \frac{1/4}{y^2}\right) w(y) = 0.$$

The two independent solutions of the latter are the Whittaker functions with parameters $-3/2$ and 0, see [2, chapter 13]. Using the Kummer M- and U-functions, these can be expressed as

$$\begin{aligned} w_1(y) &= \sqrt{y}\, e^{-y/2}\, M(2,1,y) \;=\; \sqrt{y}\, e^{y/2}\,(1+y), \\ w_2(y) &= \sqrt{y}\, e^{-y/2}\, U(2,1,y). \end{aligned}$$

It follows that

$$\tilde{f}_1(y) = ae^{sy}(1+2s(1+y)) + be^{-sy}U(2,1,2s(1+y)), \tag{17}$$

for some a and b that only depend on s. Since by (15a) we have that $\tilde{f}_2(y) = (1+s(1+y))\tilde{f}_1(y) - (1+z)\tilde{f}_1'(y)$, and from the theory of special functions we know that

$$\begin{aligned} U(2,1,y) &= (1+y)U(1,1,y) - 1, \\ U'(2,1,y) &= (2+y)U(1,1,y) - 1 - 1/y, \end{aligned}$$

we find for $\tilde{f}_2(y)$,

$$\tilde{f}_2(y) = ae^{sy} + be^{-sy}U(1,1,2s(1+y)). \tag{18}$$

Taking as initial conditions e.g. $D_2(0,0) = 1$ and $D_1(0,0) = D_1(B,0) = D_2(B,0) = 0$ with $f_1(y,0) = f_2(y,0) \equiv 0$ on $(0,B)$, we easily find that in this case (13b) and (13c) become

$$\begin{aligned} (1+s)\tilde{D}_1(0) &= \tilde{f}_1(0), \\ -D_2(0,0) + (1+s)\tilde{D}_2(0) &= -\tilde{f}_2(0), \\ D_2(B) &= (1+B)\tilde{f}_1(B), \\ (1+s(1+B))D_2(B) &= (1+B)\tilde{f}_2(B). \end{aligned}$$

Substituting equations (17) and (18), we thus find four equations for $D_1(0)$, $D_2(B)$, a and b. Solving these functions of s yields the Laplace transforms of the transient distribution functions and point masses.

5 Stationary Behavior

First we will show that a stationary distribution actually exists. The approach we follow is of interest in its own right, and may be described best as the uniformization of the joint process $\{X(t), C(t)\}$. It is similar to the approach in [14], where the background process is uniformized. In our context, however, the process $\{X(t)\}$ is not a Markov process, but it turns out that we can still follow the same line of reasoning.

We commence by choosing a finite constant λ such that $\lambda \geq \sup_{(i,y)\in\mathscr{X}\times[0,B]} |Q_{ii}(y)|$, which is possible by Assumption 1. Then we define for $y \in [0,B]$,

$$p_{ij}(y) = \begin{cases} Q_{ij}(y)/\lambda & \text{if } i \neq j, \\ 1 + Q_{ii}(y)/\lambda & \text{if } i = j. \end{cases}$$

Although we cannot consider the background process as a discrete time Markov chain embedded at (i.e. just before) the points of increase of an independent Poisson process with intensity λ, we *can* do so for the joint process. The transition kernel of the resulting joint discrete-time process has the following form. When $j \in \mathscr{X}_+$,

$$\begin{aligned} P_{i,x}(j,(y,B]) &= \begin{cases} p_{ij}(x), & \text{if } 0 \le y \le x, \\ p_{ij}(x)\exp\left(-\lambda\int_x^y \frac{du}{r_j(u)}\right), & \text{if } x \le y < B, \end{cases} \\ P_{i,x}(j,\{B\}) &= p_{ij}(x)\exp\left(-\lambda\int_x^B \frac{du}{r_j(u)}\right), \end{aligned} \tag{19a}$$

and when $j \in \mathscr{X}_-$ (recall $r_j < 0$ when $j \in \mathscr{X}_-$),

$$\begin{aligned} P_{i,x}(j,[0,y)) &= \begin{cases} p_{ij}(x), & \text{if } x \le y < B, \\ p_{ij}(x)\exp\left(-\lambda\int_x^y \frac{du}{r_j(u)}\right), & \text{if } 0 < y \le x, \end{cases} \\ P_{i,x}(j,\{0\}) &= p_{ij}(x)\exp\left(-\lambda\int_x^0 \frac{du}{r_j(u)}\right). \end{aligned} \tag{19b}$$

The proof of the following proposition shows that the uniformized process has the same jump-behavior as the source process specified in Definition 2.1, and may actually serve to better understand this definition.

Proposition 5.1. *The jump behavior of the uniformized process above is in agreement with Definition 2.1.*

Proof. By conditioning on the number of jumps by the uniformizing Poisson process we find that

$$\begin{aligned} &P\{X(t+h) = i | X(t) = i, C(t) = x\} \\ &= \sum_k \exp(-\lambda h)\frac{(\lambda h)^k}{k!}\, P\{X(t+h) = i | X(t) = i, C(t) = x,\ k \text{ jumps in } [t,t+h]\} \\ &= (1 - \lambda h + o(h))\,(1 + p_{ii}(x+\epsilon)\lambda h + o(h)) \;=\; 1 + Q_{ii}(x+\epsilon)h + o(h) \\ &= 1 + Q_{ii}(x)h + o(h), \end{aligned}$$

since $|\epsilon| < h \sup_{(i,y)\in\mathscr{X}\times[0,B]} |r_i(y)|$, and similarly

$$\begin{aligned} &P\{X(t+h) = j | X(t) = i, C(t) = x\} \\ &= \sum_k \exp(-\lambda h)\frac{(\lambda h)^k}{k!}\, P\{X(t+h) = j | X(t) = i, C(t) = x,\ k \text{ jumps in } [t,t+h]\} \\ &= (1 - \lambda h + o(h))\,(0 + p_{ij}(x+\epsilon)\lambda h + o(h)) \;=\; Q_{ij}(x+\epsilon)h + o(h) \\ &= Q_{ij}(x)h + o(h). \end{aligned}$$

Finally, the process also satisfies the third part of Definition 2.1, since the number of jumps by $\{X(t)\}$ in $[t,t+h]$, is bounded from above by the number of jumps of the uniformizing Poisson process in $[t,t+h]$, which is $o(h)$. □

We next prove the existence of a stationary distribution for the discrete-time process by applying a theorem of Meyn and Tweedie [19].

Lemma 5.2. *The discrete Markov chain governed by the Markov transition kernel P defined by (19a) and (19b) is (strong) Feller. Therefore, by Theorem 12.0.1 of [19], there exists at least one invariant, i.e., stationary, distribution.*

Proof. The transition kernel P acts on (measurable) functions on $\mathscr{X} \times [0,B]$ as an operator defined by

$$Th_i(x) = \sum_j \int_0^B P_{i,x}(j,dy)h_j(y).$$

According to Theorem 6.1.1 of Meyn and Tweedie we have to show that for T to be strong Feller, it maps all bounded measurable functions h (on $\mathscr{X} \times [0,B]$) to continuous functions. If we interpret the functions $h_{\,i}$ as the coordinate

functions of a bounded function h, it is clear that it suffices to show that $Th_i(x)$ is continuous for any bounded measurable h_i. From (19),

$$\begin{aligned}
Th_i(x) &= \sum_j \int_0^B P_{i,x}(j,dy)h_j(y) \\
&= \sum_{j\in\mathscr{X}_-} p_{ij}(x)\exp\left(-\lambda\int_x^0 \frac{du}{r_j(u)}\right)h_j(0) \\
&\quad - \sum_{j\in\mathscr{X}_-} p_{ij}(x)\int_0^x \exp\left(-\lambda\int_x^y \frac{du}{r_j(u)}\right)\frac{\lambda}{r_j(y)}h_j(y)dy \\
&\quad - \sum_{j\in\mathscr{X}_+} p_{ij}(x)\int_x^B \exp\left(-\lambda\int_x^y \frac{du}{r_j(u)}\right)\frac{\lambda}{r_j(y)}h_j(y)dy \\
&\quad + \sum_{j\in\mathscr{X}_+} p_{ij}(x)\exp\left(-\lambda\int_x^B \frac{du}{r_j(u)}\right)h_j(B).
\end{aligned}$$

Due to the continuity of p_{ij}, r_j and the boundedness of h_j this expression is continuous in x. Finally, regarding the tightness condition of Theorem 12.0.1 in [19], we remark that our state space is already compact. □

Finally, using Pasta it is clear that the stationary distribution of the continuous-time process $\{X(t), C(t)\}$ exists and is the same as the stationary distribution of the discrete-time process. From (10) it is now clear that the following theorem holds.

Theorem 5.3. *Under Assumptions 1–7 in Section 2.2, a stationary distribution for the process $\{X(t), C(t)\}$ exists. It satisfies the following system of (ordinary) differential and algebraic equations,*

$$\frac{d}{dy}(f(y)R(y)) = f(y)Q(y) \tag{20a}$$

$$f(0+)R(0+) = D(0)Q(0) \tag{20b}$$

$$-f(B-)R(B-) = D(B)Q(B), \tag{20c}$$

with boundary conditions

$$D_i(0) = D_j(B) = 0, \quad \text{if } i \in \mathscr{X}_+, j \in \mathscr{X}_-, \tag{20d}$$

and normalization condition

$$\sum_j F_j(B) = \sum_j D_j(0) + \sum_j \int_0^B f_j(x)dx + \sum_j D_j(B) = 1. \tag{20e}$$

Let us check that the number of equations suffices to make the system complete. The number of unknowns is, formally, $3N$: the N coefficients appearing in the general solution of (20a), and $2N$ constants in the form of $D_i(0)$ and $D_i(B)$. The required number of conditions should then also equal $3N$. It is evident that (20d) and (20e) together contain $N+1$ conditions. However, the number of conditions provided by (20b) and (20c) is not immediately obvious. Note that in the presence of (20d) we may replace both $Q(0)$ and $Q(B)$ in these equations by the matrix $\tilde{Q}$, which we introduced in Assumption 7b. Since we assumed there that this matrix is irreducible, its rank is $N-1$. So, formally speaking, it might seem that (20b) and (20c) provide only $2N-2$ conditions, and that one condition is lacking. Interestingly, this sought-after condition lies hidden in (20a), (20b) and (20c) and is stated in the following lemma.

Lemma 5.4. *Any solution of (20) satisfies the condition*

$$\sum_i f_i(y)r_i(y) = 0, \quad y \in (0,B).$$

To see this, note that the row sums of $Q(y)$ equal 0 for all $y \in [0, B]$. Hence, taking this sum in (20a) and then integrating yields that $\sum_i f_i(y) r_i(y) = C$ for all $y \in (0, B)$ and some constant C. From (20b) and (20c) it immediately follows that $C = 0$. We end this section by mentioning that the statement in Lemma 5.4 also has a probabilistic interpretation, in the form of a level crossing argument as is also employed in e.g. [5]. The argument is based on the fact that, in stationarity, the number of times that the buffer level moves up through some level y, must in the long run balance the number of times it moves down through the same level y. It is not difficult to see that this reasoning also leads to $\sum_i f_i(y) r_i(y) = 0$.

6 Explicit Stationary Solution for the Two-State Model

The goal of this section is to find a closed-form expression for the solution of (20) when $\mathscr{X} = \{1, 2\}$, which is done in two steps. The first step is concerned with finding a fundamental solution that satisfies Lemma 5.4. In the second step we find the constants involved. For the sake of clarity we summarize the results of each step in a lemma, and state the final result in a theorem.

Let us start by writing down (20a) in full detail for the present case. Here

$$R(y) = \text{diag}(r_1(y), r_2(y)),$$

where, without loss of generality, $r_1(y) < 0 < r_2(y), y \in (0, B)$, and

$$Q(y) = \begin{pmatrix} -\lambda(y) & \lambda(y) \\ \mu(y) & -\mu(y) \end{pmatrix}.$$

Hence, the system of differential equations (20a) becomes

$$f_1'(y) r_1(y) + f_1(y) r_1'(y) = -\lambda(y) f_1(y) + \mu(y) f_2(y) \tag{21a}$$
$$f_2'(y) r_2(y) + f_2(y) r_2'(y) = \lambda(y) f_1(y) - \mu(y) f_2(y). \tag{21b}$$

Lemma 6.1. *Any positive solution of the system (21) that satisfies Lemma 5.4 is given by*

$$f(y) = a e^{-g(y)} \left(-\frac{1}{r_1(y)}, \frac{1}{r_2(y)} \right), \tag{22}$$

where a is a positive constant, and

$$g(y) = \int_0^y \left(\frac{\lambda(x)}{r_1(x)} + \frac{\mu(x)}{r_2(x)} \right) dx. \tag{23}$$

Proof. When the meaning is clear we suppress in the proofs the functional dependence of $r_1(y), f_1(y)$, etc., on y, i.e., we write $r_1 = r_1(y), f_1 = f_1(y)$, etc.

For ease of notation, we prefer to analyze

$$g_i = f_i r_i,$$

which is equivalent to analyzing f_i as $|r_i(y)| \geq \epsilon$ for some $\epsilon > 0$ and for all $y \in (0, B)$. By substitution in (21) we obtain the equivalent problem

$$g_1' = -\frac{\lambda}{r_1} g_1 + \frac{\mu}{r_2} g_2 \tag{24a}$$
$$g_2' = \frac{\lambda}{r_1} g_1 - \frac{\mu}{r_2} g_2. \tag{24b}$$

By Lemma 5.4 we have that $g_1 = -g_2$. Therefore (24a) becomes

$$g_1' = -\left(\frac{\lambda}{r_1} + \frac{\mu}{r_2} \right) g_1.$$

Its solution is seen to be of the form

$$g_1(y) = -a e^{-g(y)}, \tag{25}$$

where $g(y)$ is given by (23) and a is some constant. Finally, writing $f_1 = g_1/r_1$ and $f_2 = -g_1/r_2$ we find the fundamental solution (22), which is positive when $a > 0$. □

Lemma 6.2. *The flux relations (20b) and (20c) together with the boundary conditions (20d) and the normalization (20e) imply that*

$$\begin{aligned} a &= \lambda(0)D_1(0) \\ D_1(0) &= \left[1 + \lambda(0)\int_0^B e^{-g(x)}\left(\frac{-1}{r_1(x)} + \frac{1}{r_2(x)}\right)dx + e^{-g(B)}\frac{\lambda(0)}{\mu(B)}\right]^{-1} \\ D_2(B) &= e^{-g(B)}\frac{\lambda(0)}{\mu(B)}D_1(0). \end{aligned} \tag{26}$$

Proof. Combining (20b) with the boundary condition $D_2(0) = 0$ yields

$$(f_1(0+)r_1(0+), f_2(0+)r_2(0+)) = (-\lambda(0)D_1(0), \lambda(0)D_1(0)).$$

With (22) this becomes

$$a(-1, 1) = (-\lambda(0)D_1(0), \lambda(0)D_1(0)),$$

implying that

$$a = \lambda(0)D_1(0).$$

At the boundary $y = B$ we find from (20c) and $D_1(B) = 0$ that

$$-f_1(B-)r_1(B-) = ae^{-g(B)} = \mu(B)D_2(B).$$

Hence,

$$D_2(B) = e^{-g(B)}\frac{\lambda(0)}{\mu(B)}D_1(0).$$

The last step is to find $D_1(0)$. This follows from (20e), i.e.,

$$D_1(0) + \int_0^B (f_1(y) + f_2(y))dy + D_2(B) = 1.$$

□

Theorem 6.3. *The unique solution of (20a) satisfying (20b), (20c), (20d), and (20e) is given by*

$$f(y) = \lambda(0)D_1(0)e^{-g(y)}\left(-\frac{1}{r_1(y)}, \frac{1}{r_2(y)}\right), \tag{27}$$

where $D_1(0)$ and $D_2(B)$ are given by (26)*, and $g(y)$ is defined by* (23)*. In terms of the distribution function we have*

$$F(y) = \left(D_1(0)\,,\, D_2(B)1_{\{y=B\}}\right) + \lambda(0)D_1(0)\int_0^y e^{-g(x)}\left(\frac{-1}{r_1(x)}, \frac{1}{r_2(x)}\right)dx, \qquad 0 \le y \le B, \tag{28}$$

where $1_{\{y=B\}} = 1$ when $y = B$ and 0 else.

Remark 6.1. The systems (15) and (21) have a similar structure. However, the seemingly innocuous presence of $s\tilde{f}_1(y)$ and $s\tilde{f}_1(y)$ complicates the solvability of (15). To see this, note that the reasoning leading to Lemma 5.4 yields for equation (15) of the transient case

$$s\tilde{f}_1(y) - f_1(y, 0) + (r_1(y)\tilde{f}_1(y))' + s\tilde{f}_2(y) - f_2(y, 0) + (r_2(y)\tilde{f}_2(y))' = 0.$$

Based on this, it is clear that now we cannot conclude $(r_1(y)\tilde{f}_1(y))' + (r_2(y)\tilde{f}_2(y))' = 0$, and thus, we cannot establish Lemma 5.4 for the transient case.

Remark 6.2. Finding an explicit solution when the source process has more than two states seems to be exceedingly difficult. To see this, suppose that $N = 3$. Lemma 5.4 enables us to reduce the number of differential equations in (20a) by one, leaving a two dimensional homogeneous linear system of differential equations with variable coefficients. For such systems once one fundamental solution is known, the second (linearly independent) solution can be found by a mere integration, see e.g. [16, p. 367]. However, there is no standard theory on how to find the first solution.

Actually, for the two-dimensional system (21) we used the method indicated in [16] to compute the second solution, based on the first solution given in (22). This second solution turns out not to satisfy Lemma 5.4, as expected.

Remark 6.3. When λ and μ do not depend on y, say $\lambda(y) \equiv \lambda$ and $\mu(y) \equiv \mu$ as in [14], it is not difficult to see that

$$\lambda \int_0^y e^{-g(x)} \frac{-1}{r_1(x)} dx - \mu \int_0^y e^{-g(x)} \frac{1}{r_2(x)} dx = e^{-g(y)} - 1,$$

so that by taking $y = B$ in the above, equation (26) yields

$$D_1(0) = \frac{\mu}{\lambda + \mu} \left[e^{-g(B)} + \int_0^B e^{-g(x)} \frac{\mu}{r_2(x)} dx \right]^{-1}.$$

After some algebra we can actually write (28) in the form

$$F_1(y) = \frac{\mu}{\lambda + \mu} \frac{e^{-g(y)} + H(y)}{e^{-g(B)} + H(B)}, \qquad F_2(y) = \frac{\lambda}{\lambda + \mu} \frac{1_{\{y=B\}} e^{-g(B)} + H(y)}{e^{-g(B)} + H(B)}, \tag{29}$$

where

$$H(y) = \int_0^B e^{-g(x)} \frac{\mu}{r_2(x)} dx.$$

This result indeed coincides with the one in [14] (notice that our downstate 0 is in their setting called state 1 and vice-versa, and that their function $\exp(g(y))$ for $y_0 = 0$ is our $\exp(-g(y))/r_2(y)$). Notice also that it is not necessary to make the particular choice for the functions r_1 and r_2 as done in [14].

7 Examples

To illustrate our results, we present three examples, each of which has $\mathscr{X} = \{1, 2\}$. The first example, presented in Section 7.1, discusses the case in which the rate at which the source turns on depends on the current buffer content. Section 7.2 shows that not all assumptions we stated for $R(y)$ and $Q(y)$ in Section 2.2 are required. However, as the example shows, such cases should be handled with care. Finally, the third example in Section 7.3, makes clear that the solution obtained for $B < \infty$ can sometimes be extended by taking the limit $B \to \infty$.

7.1 Discouraged Two-State Source

Suppose that the rate at which the source process changes states from state 1 to 2 depends on the content level: the higher the level, the less willing, or more 'discouraged', the source is to make a transition from state 1 to 2. One simple model for this behavior is to take

$$\lambda(y) = \lambda_0 \left(1 - \frac{y}{B}\right), \tag{30}$$

with $\lambda_0 > 0$. We assume $\mu(y) \equiv \mu$, and $r_1(y) \equiv -r_2(y) \equiv -r$, where μ and r are some positive constants. Using Theorem 6.3 we find that

$$f(y) = \frac{\lambda_0 D_1(0)}{r} e^{-g(y)} \left(1\,,\,1\right),$$

where $g(y)$ can be found from (23) to be

$$\begin{aligned} g(y) &= \frac{1}{r} \int_0^y \left(\mu - \lambda_0 \left(1 - \frac{x}{B}\right)\right) dx \\ &= \frac{\mu - \lambda_0}{r} y + \frac{\lambda_0}{2rB} y^2. \end{aligned}$$

Hence we obtain for $F(y)$, $0 \le y \le B$,

$$F(y) = \left(D_1(0)\,,\, D_2(B) 1_{\{y=B\}}\right) + \frac{\lambda_0 D_1(0)}{r} \int_0^y \exp\left(-\frac{\mu - \lambda_0}{r} x - \frac{\lambda_0}{2rB} x^2\right) dx \left(1\,,\,1\right),$$

where $D_2(B) = D_1(0)e^{-g(B)}\lambda_0/\mu$ and $D_1(0)$ follows from normalization. Notice that $F(y)$ can also be expressed as

$$F(y) = \Big(D_1(0)\,,\, D_2(B)1_{\{y=B\}}\Big) + \frac{\lambda_0 D_1(0)}{r}\sqrt{2\pi}\,\sigma\,\exp\left(\mu^2/2\sigma^2\right)\left(\Phi_{\mu,\sigma^2}(y) - \Phi_{\mu,\sigma^2}(0)\right)\,\Big(1\,,\,1\Big),$$

where Φ_{μ,σ^2} is the distribution function of a normal random variable with mean $\mu = B(1-\mu/\lambda_0)$ and variance $\sigma^2 = rB/\lambda_0$.

7.2 Two Recurrent Classes

The condition that $r_i(y)$ should be bounded away from 0 is not always necessary to find a solution for the system (20). However, some care is required, as is demonstrated by this example.

Consider the two-state model of Section 6. Set for $\alpha \in (0,1)$,

$$\begin{aligned} \lambda(y) &= \lambda & \mu(y) &= \mu \\ r_1(y) &= -r_1 y^\alpha & r_2(y) &= r_2 y^\alpha. \end{aligned}$$

and choose positive constants λ, μ, r_1, r_2 such that

$$\gamma = \frac{\mu}{r_2} - \frac{\lambda}{r_1} \neq 0.$$

Now (23) is seen to be

$$g(y) = \int_0^y \left(\frac{\lambda(x)}{r_1(x)} + \frac{\mu(x)}{r_2(x)}\right) dx = \frac{\gamma}{1-\alpha} y^{1-\alpha}.$$

It is possible to evaluate the integral

$$\begin{aligned} \int_0^y e^{-g(x)} \left(\frac{1}{r_2(x)} - \frac{1}{r_1(x)}\right) dx &= \frac{r_1+r_2}{r_1 r_2}\int_0^y \frac{1}{x^\alpha}\exp\left(-\frac{\gamma}{1-\alpha}x^{1-\alpha}\right) dx \\ &= \frac{r_1+r_2}{\mu r_1 - \lambda r_2}\left[1 - e^{-g(y)}\right]. \end{aligned}$$

Therefore, it seems to follow straightforwardly from (26) that

$$D_1^{-1}(0) = 1 + \lambda\frac{r_1+r_2}{\mu r_1 - \lambda r_2}\left[1 - e^{-g(B)}\right] + \frac{\lambda}{\mu}e^{-g(B)},$$

from which the coefficient a and the constant $D_2(B)$ follow (see Lemma 6.2). While this appears to be an entirely correct solution, this is not the case. The reason is that (20b) implies there is *no* exchange of probability mass between the subspaces $\{1,2\}\times\{0\}$ and $\{1,2\}\times(0,B]$ of the state space of the process $\{X(t), C(t)\}$, i.e., the state space consists of two disjoint recurrent classes. In fact, since $r_2(0) = 0$, it is no longer required that $D_2(0) = 0$.

The correct solution depends on the initial conditions. When the buffer is empty at $t = 0$, the solution $D(0) = (D_1(0), D_2(0))$ is the stationary distribution of $Q(0)$, while $a = 0$, and $D(B) = 0$. When the initial buffer content is positive, it is clear that $D(0) = 0$. The solution in this case is found by following the procedure of Section 6 without making use of (20b) and the boundary condition $D_2(0) = 0$. This results in solving for a and $D_2(B)$ in the relations

$$\begin{aligned} a^{-1} &= \int_0^B e^{-g(x)}\left(\frac{1}{r_2(x)} - \frac{1}{r_1(x)}\right) dx + \frac{e^{-g(B)}}{\mu(B)} \\ D_2(B) &= a\frac{e^{-g(B)}}{\mu(B)}. \end{aligned}$$

7.3 Infinite-buffer Systems

In the entire analysis so far we assumed that $B < \infty$. The reason is that it seems difficult to find conditions for non-trivial $Q(y)$ and $R(y)$ such that a limiting distribution for the process $\{X(t), C(t), t \geq 0\}$ exists. In fact, if we let the vector $\pi(y)$ be the corresponding stationary distribution for $Q(y)$, i.e. the vector that satisfies $\pi(y)Q(y) = 0$ and $\sum_i \pi_i(y) = 1$, it may *seem* intuitively clear that a condition for stability is the following: for all y larger than some value (y_0, say) we must have $\sum \pi_i(y) r_i(y) < \infty$. After a moments thought it is clear that this is not a necessary condition, since we can adapt the matrix function Q in some stable model such that above each level y_0 there are some (very small) regions of the buffer space where the condition does not hold, while the adapted model is still stable. That the proposed condition is also not sufficient, however, is less obvious. A counterexample is given below for a simple two-state model. We first compute the solution in case $B < \infty$, and then consider the limit $B \to \infty$. This procedure seems reasonable when it turns out that normalization is possible.

To define our model, set for $\mathscr{X} = \{1, 2\}$

$$\lambda(y) = \frac{\lambda_0}{1+y} \qquad \mu(y) = \frac{\mu_0}{1+y}$$
$$r_1(y) = -1 \qquad r_2(y) = 1,$$

where λ_0 and μ_0 are positive constants. Now (23) yields

$$g(y) = \int_0^y \left(\frac{-\lambda_0}{1+x} + \frac{\mu_0}{1+x} \right) dx \; = \; (\mu_0 - \lambda_0) \log(1+y),$$

so that (27) becomes

$$f(y) = D_1(0)\lambda_0 \left(\frac{1}{1+y} \right)^{\mu_0 - \lambda_0} \Big(1 \, , \, 1 \Big).$$

After some algebra we find

$$D_1^{-1}(0) = \frac{\mu_0 + \lambda_0 - 1}{\mu_0 - \lambda_0 - 1} - \frac{\lambda_0}{\mu_0} \frac{\mu_0 + \lambda_0 + 1}{\mu_0 - \lambda_0 - 1} \left(\frac{1}{1+B} \right)^{\mu_0 - \lambda_0 - 1}.$$

Clearly, when $\mu_0 > \lambda_0 + 1$, $D_1(0)$ has a finite limit when $B \to \infty$. In that case,

$$F(y) = \left(\frac{\mu_0 - 1}{\mu_0 + \lambda_0 - 1}, \frac{\lambda_0}{\mu_0 + \lambda_0 - 1} \right) - \frac{\lambda_0}{\mu_0 + \lambda_0 - 1} \left(\frac{1}{1+y} \right)^{\mu_0 - \lambda_0 - 1} \Big(1 \, , \, 1 \Big).$$

For this particular model our previously mentioned condition $\sum \pi_i(y) r_i(y) < \infty$ leads to $\mu_0 > \lambda_0$, while we just established that the correct condition must be $\mu_0 > \lambda_0 + 1$. One other aspect worth mentioning for this example is that when $\lambda_0 < \mu_0 < \lambda_0 + 1$ the normalization breaks down due to the fact that the integrals of the densities f_1 and f_2 become infinite for $B \to \infty$, and not because the value $D_2(B)$ does not approach zero (which in fact it does). Hence, the condition that $D_2(B) \to 0$ as $B \to \infty$ is a necessary, but not a sufficient condition for proper normalization, and hence for the stability of the infinite-buffer model.

The fact that we did not find any elegant stability conditions is the main reason why we did not include any theoretical results for the infinite buffer case, although the limiting procedure described above works well in many cases. As a matter of fact, we can also consider the limit $B \to \infty$ for the examples in Sections 7.1 and 7.2. In the first case we expect that the impact of the factor $1 - y/B$ in (30) disappears. Indeed this is the case, resulting in the familiar expression for a fluid model without feedback,

$$F(y) = \left(\frac{\mu}{\lambda_0 + \mu}, \frac{\lambda_0}{\lambda_0 + \mu} \right) - \frac{\lambda_0}{\lambda_0 + \mu} e^{\frac{\lambda_0 - \mu}{r} y} \Big(1 \, , \, 1 \Big).$$

For the second example it is clear that $\gamma > 0$ ensures that $g(B) \to \infty$ as $B \to \infty$. Consequently the normalization works well since $D_1(0) \to (\mu r_1 - \lambda r_2)/(\mu r_1 + \mu r_2) > 0$.

8 Numerical Method

For the case when $N > 2$, exact results cannot be expected (see Remark 6.2), and we wish to resort to numerical methods to obtain the stationary distribution. Because our problem (20) is not an initial boundary value problem, but a two-point boundary value problem, this is not entirely trivial. We discuss first a simple method to solve such problems, and then demonstrate its use by solving an extension of Example 7.1.

8.1 The Method

We start by rewriting the differential equation (20a) as $g'(y) = g(y)R^{-1}(y)Q(y)$, where $g(y) = f(y)R(y)$. With the fundamental matrix G, i.e., the matrix that satisfies

$$G'(y) = G(y)R^{-1}(y)Q(y), \tag{31}$$

we write the solution as $g(y) = aG(y)$, where a is a row vector to be determined based on the boundary conditions. In the following we will choose the matrix $G(y)$ such that $G(0) = I$, so that in fact $g(y) = g(0)G(y)$. Thus, the first step is to numerically solve (31) with $G(0) = I$, which can be done by standard methods.

The next step is to use the expression $g(B) = g(0)G(B)$ in the flux equations (20b) and (20c). In the present setting these become

$$\begin{aligned} D(0)Q(0) = \quad & f(0+)R(0+) \; = \quad g(0+), \\ D(B)Q(B) = -f(B-)R(B-) &= -g(B-) = -g(0)G(B), \end{aligned}$$

so that substituting the first into the second yields $D(B)Q(B) = -D(0)Q(0)G(B)$, or

$$D(0)Q(0) + D(B)Q(B)G^{-1}(B) = 0. \tag{32}$$

To express the boundary conditions (20d) efficiently in matrix form we need projection operators I_- and I_+. The first operator is the projection on $\mathscr{X}_-$, i.e.,

$$(I_-)_{ij} = \begin{cases} 1 & \text{if } i = j \in \mathscr{X}_-, \\ 0 & \text{else.} \end{cases}$$

The second operator $I_+ = I - I_-$ evidently projects on $\mathscr{X}_+$. We are now ready to define the row vectors

$$\begin{aligned} D_-(0) &= D(0)I_- & \qquad D_+(0) &= D(0)I_+ \\ D_-(B) &= D(B)I_- & \qquad D_+(B) &= D(B)I_+, \end{aligned}$$

so that we can include the boundary conditions $D_+(0) = 0$ and $D_-(B) = 0$ in (32) to obtain

$$D_-(0)Q(0) + D_+(B)Q(B)G^{-1}(B) = 0.$$

To solve $D_-(0)$ and $D_+(B)$ from the above equation, we rewrite it by using the property of projection operators, $I_- = I_-^2$ and $I_+ = I_+^2$. Defining the ancillary vector $v = D_-(0) + D_+(B)$, for which we also have

$$D_-(0) = vI_- \qquad\qquad D_+(B) = vI_+, \tag{33}$$

the problem is to solve v from

$$v[I_-Q(0) + I_+Q(B)G^{-1}(B)] = 0.$$

This can be done, up to normalization, in a numerically robust manner by means of Singular Value Decomposition, see for instance [11]. Once v is known it is immediate, by (33), to also find $D_-(0)$ and $D_+(B)$, up to normalization.

The last steps are to integrate the differential equation $g'(y) = g(y)R^{-1}(y)Q(y)$ with initial condition $g(0) = D(0)Q(0)$, to compute $f(y) = g(y)R^{-1}(y)$ and to normalize according to (20e).

Remark 8.1. Formally, the matrix $G(B)$ is invertible as it is a fundamental set of solutions, see e.g. [20] for a proof. However, *numerically* the state of affairs may be less agreeable as the problem (31) is principally ill-conditioned. When N is large, the two-point boundary value problem can be solved by a more efficient method, see e.g. [23]. Still, for relatively small buffer sizes and a small number of source states the above method can be successful.

8.2 Three Discouraged Sources

We applied the method above to compare a model with three independent identical discouraged sources—compare the source model of Example 7.1—to a model without feedback, namely the three-source AMS model [4], with a finite buffer. The Q-matrices of interest in these models are denoted by $Q(y)$ and Q_{AMS}, respectively, and are given by

$$Q(y) = \begin{pmatrix} -3\lambda(y) & 3\lambda(y) & 0 & 0 \\ \mu & -2\lambda(y)-\mu & 2\lambda(y) & 0 \\ 0 & 2\mu & -\lambda(y)-2\mu & \lambda(y) \\ 0 & 0 & 3\mu & -3\mu \end{pmatrix}$$

$$Q_{\text{AMS}} = \begin{pmatrix} -3\lambda_0 & 3\lambda_0 & 0 & 0 \\ \mu & -2\lambda_0-\mu & 2\lambda_0 & 0 \\ 0 & 2\mu & -\lambda_0-2\mu & \lambda_0 \\ 0 & 0 & 3\mu & -3\mu \end{pmatrix},$$

where $\lambda(y) = \lambda_0(1 - y/B)$, as in Example 7.1. The fluid rate matrices are equal for both models, given by

$$R = \begin{pmatrix} -l & & & 0 \\ & r-l & & \\ & & 2r-l & \\ 0 & & & 3r-l \end{pmatrix}.$$

To produce Figure 1 we set $\lambda_0 = 0.5$, $\mu = 1$, $r = 1$, $l = 0.5$ and $B = 2$. We conclude from the panels of the figure that the atoms at $y = B$ ($y = 0$) are larger (smaller) in the setting without feedback, as could be expected.

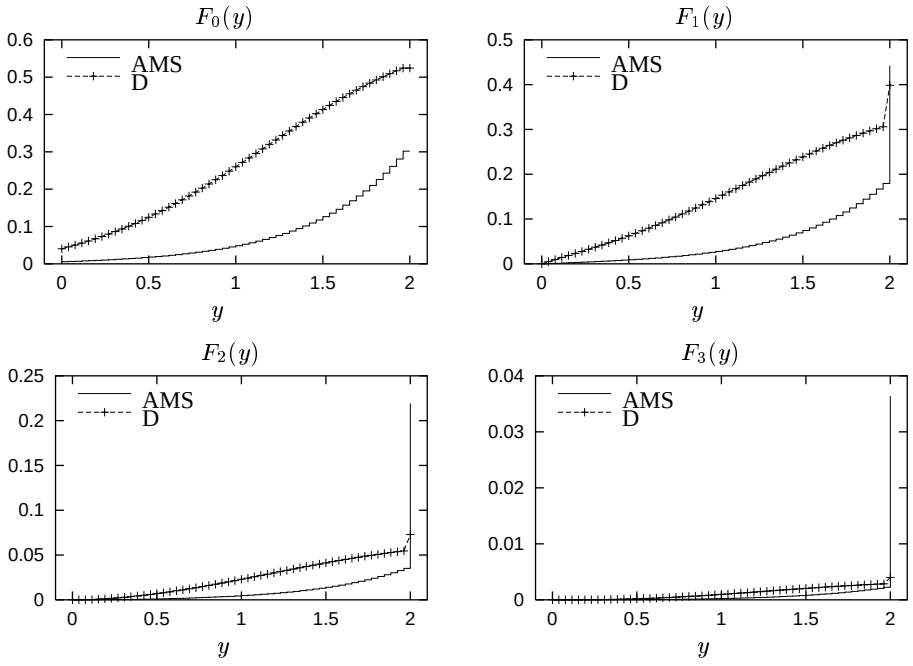

Figure 1: The distribution functions for the AMS model and the discouraged users model (D). Note the different scales on the vertical axes.

A Appendix

In Section 3.1 we expressed the dynamics of $F_i(y, t+h)$ for $h > 0$ sufficiently small and $y \in (0, B)$ in terms of the expansion

$$F_i(y, t+h) = \int_0^{y - r_i(y)h} (1 + hQ_{ii}(x)) d_x F_i(x, t) + h \sum_{j \neq i} \int_0^y Q_{ji}(x) d_x F_j(x, t) + o(h).$$

It is not immediately obvious that this expansion is indeed correct. For instance, the upper limit of integration y in the second term suggests that we assume that the content level remains constant during $[t, t+h]$, while this is certainly *not* the case. Compensating for this effect is difficult since the knowledge that $X(t) = j, X(t+h) = i$, and $C(t) = y$ is not sufficient to determine $C(t+h)$. To do that, we also need the epoch $\tau \in [t, t+h]$ at which the source makes its transition from state j to i, which is unknown. Similarly, we notice that the upper limits of both integrals should incorporate some $o(h)$ term. In the following we prove that, assuming that the assumptions of Section 2.2 are satisfied, the impact of such subtleties can be absorbed in the term $o(h)$. In fact we immediately prove (4), from which (5) then easily follows.

Lemma A.1. *Under the assumptions of Theorem 3.1, the expansion* (5) *in Section 3.1.2, i.e.,*

$$F_i(y, t+h) = F_i(y, t) - hr_i(y) \frac{\partial}{\partial y} F_i(y, t) + h \sum_j \int_0^y Q_{ji}(x) d_x F_j(x, t) + o(h)$$

is valid for any $i \in \mathscr{X}$.

Proof. The proof consists of four steps. First we state three differential equations that bound all possible paths of $\{C(s), s \in [t, t+h]\}$. Then we define a family of transition functions that allow us in the third step to express the functions $F_i(y, t+h)$ in terms of some integrals, each involving the functions $F_j(x, t)$. In the last step we rewrite these and thereby prove the lemma.

In the sequel consider $i \in \mathscr{X}$ fixed. First we wish to introduce three differential equations with the aim to relate events at time $t+h$, e.g., $\{C(t+h) \leq y\}$, to events at time t. To that end we define for $y \in (0, B)$,

$$\underline{r}(y) = \min(r_1(y), \ldots, r_N(y)), \qquad \overline{r}(y) = \max(r_1(y), \ldots, r_N(y)).$$

Clearly, $\underline{r}(y)$ and $\overline{r}(y)$ are continuous and finite functions on $(0, B)$ by the continuity and finiteness of the functions $r_i(y), 1 \leq i \leq N$. The three terminal value problems of interest are now given as follows,

$$\begin{aligned} \dot{y}(s) &= r_i(y(s)), & y(t+h) &= y, \\ \dot{\underline{y}}(s) &= \underline{r}(\underline{y}(s)), & \underline{y}(t+h) &= y, \\ \dot{\overline{y}}(s) &= \overline{r}(\overline{y}(s)), & \overline{y}(t+h) &= y, \end{aligned}$$

where $y \in (0, B)$ is some fixed terminal value. Notice that these are well defined on $s \in [t, t+h]$, provided that h is so small that $\underline{y}(t), \overline{y}(t) \in (0, B)$ if $y \in (0, B)$. The solutions to these terminal value problems are unique; consequently, trajectories with different terminal conditions do not cross. In the remainder we will be particularly interested in these solutions evaluated at t, for which we have

$$\overline{y}(t) \leq y(t) \leq \underline{y}(t)$$

Now, let J_0, J_1 and J_2 denote the events that the source process makes respectively $0, 1$, or more than 1 transitions in the interval $[t, t+h]$. Then we can define the following transition functions:

$$P_n(i, y, t+h; j, x, t) = \mathbb{P}\{X(t+h) = i, C(t+h) \leq y, J_n \mid X(t) = j, C(t) = x\}.$$

¿From our definition of $\overline{y}(t), y(t)$ and $\underline{y}(t)$ and the definition in Section 2.1 it can be seen that for h sufficiently small

we actually have

$$\begin{aligned}
P_0(i,y,t+h;j,x,t) &= \begin{cases} 1+hQ_{ii}(x)+o(h) & \text{if } j=i \text{ and } x \le y(t) \\ 0 & \text{else} \end{cases} \\
P_1(i,y,t+h;j,x,t) &= \begin{cases} hQ_{ji}(x)+o(h) & \text{if } j \neq i \text{ and } x \le \overline{y}(t) \\ 0 & \text{if } j \neq i \text{ and } x \ge \underline{y}(t), \text{ or if } j=i \end{cases} \\
P_1(i,y,t+h;j,x,t) &\le hQ_{ji}(x)+o(h) \qquad \text{if } j \neq i \text{ and } \overline{y}(t) \le x \le \underline{y}(t), \\
P_2(i,y,t+h;j,x,t) &= o(h).
\end{aligned}$$

Moreover, by conditioning we know that

$$F_i(y,t+h) = \mathbb{P}\{X(t+h)=i, C(t+h) \le y\} = I_0 + I_1 + I_2, \tag{34}$$

where

$$I_n = \sum_{j \in \mathscr{X}} \int_0^B P_n(i,y,t+h;j,x,t) d_x F_j(x,t).$$

We consider the individual integrals I_0, I_1, and I_2, consecutively.

With the expression above for $P_0(i,y,t+h;j,x,t)$, the integral I_0 becomes

$$I_0 = \int_0^{y(t)} (1+hQ_{ii}(x)) d_x F_i(x,t) + o(h),$$

where we used dominated convergence to establish that $\int_0^{y(t)} o(h) d_x F_i(x,t) = o(h)$. We rewrite this expression further by using Assumption 8 and the Taylor expansion of $y(t)$ around $t+h$, i.e.,

$$y(t) = y(t+h) - h\dot{y}(t+h) + o(h) = y - hr_i(y) + o(h).$$

The result becomes

$$I_0 = F_i(y,t) - hr_i(y)\frac{\partial}{\partial y}F_i(y,t) + \int_0^y hQ_{ii}(x) d_x F_i(x,t) + \int_y^{y(t)} hQ_{ii}(x) d_x F_i(x,t) + o(h). \tag{35}$$

Notice that the second integral may be omitted since it is $o(h)$.

For the second integral I_1 we can derive that

$$-\sum_{j \neq i} \int_{\overline{y}(t)}^{y} hQ_{ji}(x) d_x F_j(x,t) + o(h) \le I_1 - \sum_{j \neq i} \int_0^y hQ_{ji}(x) d_x F_j(x,t) \le \sum_{j \neq i} \int_y^{\underline{y}(t)} hQ_{ji}(x) d_x F_j(x,t) + o(h),$$

and since the left-hand side and right-hand side are both $o(h)$, we find

$$I_1 = \sum_{j \neq i} \int_0^y hQ_{ji}(x) d_x F_j(x,t) + o(h). \tag{36}$$

Finally, it is clear that $I_2 = o(h)$. Combining this with (34), (35) and (36) yields the desired result. □

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