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Constrained Isentropic Models of Tropospheric Dynamics*

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Summary

A two-layer isentropic model consisting of a tropospheric and a stratospheric layer is simplified using perturbation analysis while preserving the Hamiltonian structure. The first approximation applies when the thickness of the stratospheric layer is much larger than the tropospheric layer, such that the Froude number of the stratospheric layer is a small number. Using leading-order perturbation theory in the Hamiltonian formulation yields a conservative one-and-a-half isentropic layer model. Furthermore, when the Rossby number in this active lower layer is small, Hamiltonian theory either directly leads to (Salmon's) L1-dynamics using a geostrophic constraint, following a more concise derivation than shown before, or yields quasigeostrophic dynamics. The extension to multilayer isentropic balanced models for use in idealized climate forecasting is discussed.

Keywords: isentropic one-and-half layer models for flows in the troposphere and stratosphere   rigid-lid constraint   nearly geostrophic constraints   Hamiltonian formulation   perturbation theory

AMS SUBJECT CLASSIFICATION: 76M99, 76U05

* With an Appendix "Slaved Hamiltonian Dynamics" by O. Bokhove and T.G. Shepherd

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1. Introduction

We may want to explore the use of global or hemispheric, Hamiltonian, nearly geostrophic balanced models for idealized climate studies. Hamiltonian balanced models are hypothesized to be advantageous since their conservation properties are assumed to stabilize long-time integrations. Multilayer isentropic models are ideal parent models or candidates to start deriving such balanced models. Consequently, in these derivations we require the Hamiltonian formulation of such isentropic parent models. This paper constructs the Hamiltonian derivation of a one-and-a-half layer model and associated balanced models via a slaved Hamiltonian approach. (Models with a so-called "half layer" have a passive top layer in which the pressure is assumed constant at a fixed vertical level.) To simplify matters, we consider the dynamics solely on an f-plane instead of the more complex global geometry. In the end, we will conclude that there appears no further obstruction to apply this slaved Hamiltonian approach on the sphere in a multilayer setting, except for the inaccuracy of an imposed balance condition at the equator.

The atmosphere at mid-latitudes can roughly be divided in a troposphere with a layer thickness of about 15 km from the Earth's surface to the tropopause, and a stratosphere reaching from 15 km to about 60 km. In a simplified view on atmospheric dynamics, the troposphere and stratosphere can be presented as two isentropic layers with different but constant values of the entropy. Using hydrostatic balance and considering a horizontal velocity field that is uniform in the vertical in each layer, the dynamics is constrained to remain horizontal in each layer with a (weak) coupling between the layers. Such simplified models can be analyzed in much more detail than the full three-dimensional equations of motion, and play and have played a major role in gaining understanding of the atmosphere's dynamics (*e.g.* Starr, 1945).

We eventually intend to use $n + 1/2$-layer isentropic (balanced) models of the atmosphere for idealized weather and climate simulations with $n > 1$ a whole number. In these models, active layers are located in the troposphere and possibly in the stratosphere, while the upper, passive layer resides in the stratosphere. Consider a model with one tropospheric layer. Using z as the vertical coordinate normal to the Earth's surface, the tropospheric layer reaches from the static topographic surface at $z = z_2(x, y)$ with surface pressure $p_2(x, y, t)$ to the first dynamic interface at $z = z_1(x, y, t)$ with pressure $p_1(x, y, t)$. The stratospheric layer reaches from the interface at $z = z_1$ to the top, dynamic interface at $z = z_0(x, y, t)$ with constant pressure p_0, see also the defining sketch in Fig. 1. This latter layer will become passive and is subsequently counted as a half layer, because the top interface is constrained to be constant, that is, $z_0 = Z_0$ with Z_0 constant.

The equations of motion for the resulting one-and-half layer model are

$$\begin{aligned} \frac{\partial \sigma_\alpha}{\partial t} + \boldsymbol{\nabla}\cdot(\sigma_\alpha\, \mathbf{v}_\alpha) &= 0 \\ \frac{\partial \mathbf{v}_\alpha}{\partial t} + (\mathbf{v}_\alpha\cdot\boldsymbol{\nabla})\mathbf{v}_\alpha + f\,\hat{\mathbf{z}} \times \mathbf{v}_\alpha &= -\boldsymbol{\nabla} M_\alpha \end{aligned} \tag{1}$$

for $\alpha = 2$, with time t and horizontal gradient $\boldsymbol{\nabla}$, the pseudo-density $\sigma_2 = (p_2 - p_1)/g$ in the second layer defined as the pressure difference over the layer divided by the gravitational acceleration g, the horizontal velocity $\mathbf{v}_2 = \mathbf{v}_2(x, y, t)$

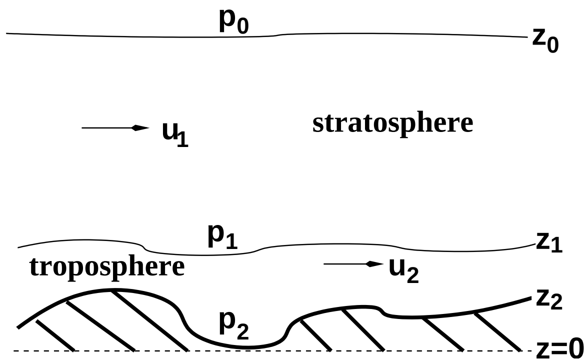

Figure 1. Sketch of the atmosphere with a tropospheric and stratospheric isentropic layer and the associated variables.

in that layer, a Coriolis parameter f, and the Montgomery potential M_2. The latter is related to p_2 and hence σ_2 as follows

$$M_2 = c_p\, \theta_2\, (p_2/p_r)^\kappa + g\, z_2, \tag{2}$$

where $\kappa = R/c_p$, the gas constant $R = c_p - c_v$, and $c_{p,v}$ is the specific heat at constant pressure and volume, respectively, p_r is a reference pressure and θ_2 is the constant potential temperature directly proportional to the natural logarithm of the constant entropy in the second layer. Note that the model is not yet closed, since we have to specify the relationship between p_1 and p_2 in the definition of pseudo-density σ_2. The equations of motion of an active, first layer are (1) for $\alpha = 1$, where the Montgomery potential

$$M_1 = c_p\, \theta_2\, (p_2/p_r)^\kappa + c_p\, (\theta_1 - \theta_2)\, (p_1/p_r)^\kappa + g\, (z_2 - Z_0) - c_p\, \theta_1\, (p_0/p_r)^\kappa \tag{3}$$

with θ_1 the constant entropy in layer one. One may show that $M_1 = g\,(z_0 - Z_0)$, see §2. So, in the one-and-half layer model we constrain z_0 to Z_0, whence $M_1 = 0$, which provides the desired definition of p_1 in terms of p_2. We note that such a one-and-half layer model has the advantage above a one-layer model that the pressure p_1 is active and not constrained to be constant, as is p_0. Furthermore, the values of the surface pressure p_2 are more realistic. This allows for a more accurate representation of the atmosphere within the context of layer models.

The one-and-half layer model appears to be inconsistent, since the constraint $M_1 = 0$ is not preserved in time by the original two continuity equations. Nevertheless, the closed one-and-half layer model (1) with $\alpha = 2$ and Montgomery potential M_2 results after taking $M_1 = 0$ and $\mathbf{v}_1 = 0$ in the momentum equation of the first layer and ignoring continuity in that layer.

With (idealized) climate modeling through balanced models as motivation and an apparent inconsistency in the parent layer models, we arrive at the following key questions: (i) Can we construct a Hamiltonian formulation of the one-and-half layer model? (ii) Can the apparent inconsistency in the derivation of this model be removed? (iii) Can we subsequently derive nearly geostrophic balanced models? Multilayer extensions of these (balanced) half-layer models with a passive upper layer can be readily derived, but the one-and-half layer model used here is arithmetically simpler, and suffices to explain a systematic, Hamiltonian approach.

To answer the first question, (i), we can simply use the Poisson bracket of the shallow water equations, and search ad hoc for the potential energy that yields the desired Montgomery potential M_2. Perhaps not surprisingly, the original potential energy of the two-layer model subject to the constraint $M_1 = g\,(z_0 - Z_0) = 0$ does give the desired potential energy of the one-and-half layer model. The second question, (ii), can be answered adequately by applying asymptotic analysis to the above two-layer system and by extending a finite-dimensional slaved Hamiltonian approach, detailed in Appendix A, to an infinite-dimensional case. The answer to the third question follows subsequently from the work of Vanneste and Bokhove (2002). Yet, we show that the slaved Hamiltonian approach provides a more concise derivation of balanced models based on velocity constraints. In addition, with this approach we also construct the well-known Hamiltonian formulation of the quasigeostrophic equations.

To outline the current paper, we will arrive at a Hamiltonian formulation of the one-and-half layer model in a systematic manner in §2 by using a slaved Hamiltonian approach for continuous systems. Subsequently, in §3, we will also construct the Hamiltonian formulation of balanced models with general velocity constraints (cf. Vanneste and Bokhove, 2002) with a L1-balanced model (cf. Salmon, 1985) as particularization, and of the quasigeostrophic system for the one-and-half layer isentropic model. Although the latter derivations of nearly geostrophic balanced models are not novel, the presented derivation is Hamiltonian and more concise than the derivations in Salmon (1985, 1988), Allen and Holm (1996), Theiss (2000), Verkley (2001), McIntyre and Roulstone (2002), and Vanneste and Bokhove (2002). Consequently, the derivation of atmospheric balanced models with multiple, coupled isentropic layers will be easier.

2. Two-layer troposphere-stratosphere isentropic model

(a) *Further equations of motion, scaling and constraints*

We consider an extra-tropical atmospheric model consisting of the two, tropospheric and stratospheric, layers with constant but different values of the entropy (Starr, 1945). In each layer the pressure is hydrostatic

$$\frac{\partial p}{\partial z} = -\rho\, g \tag{4}$$

with density ρ. Hence, the equations of motion (1) for $\alpha = 1, 2$ can be derived from the three-dimensional Euler equations of motion with Montgomery potentials (2) and (3), and pseudo-densities

$$\sigma_1 = (p_1 - p_0)/g \qquad \text{and} \qquad \sigma_2 = (p_2 - p_1)/g. \tag{5}$$

Potential temperature θ is proportional to entropy and is defined relative to the Kelvin temperature T as follows

$$T/\theta = (p/p_r)^\kappa \tag{6}$$

with $p_r = 1000\, mb$ a reference pressure at sea level. Using hydrostatic equilibrium (4), (6), and the ideal gas law

$$p = \rho\, R\, T, \tag{7}$$

we find

$$\frac{1}{p}\frac{\partial p}{\partial z} = -\frac{1}{\kappa\, H}\,(p_r/p)^\kappa \qquad \text{and} \qquad \frac{\partial (T/\theta)}{\partial z} = -1/H, \tag{8}$$

where we have defined a scale height $H = c_p\,\theta/g$. Integration of the last equation in (8) over layers one and two, the upper and lower layer, respectively, and using (6) yields

$$\begin{aligned} \frac{T_1}{\theta_1} - \frac{T_2}{\theta_2} &= \left(\frac{p_1}{p_r}\right)^\kappa - \left(\frac{p_2}{p_r}\right)^\kappa = \frac{g\,(z_2 - z_1)}{c_p\,\theta_2} \\ \frac{T_0}{\theta_0} - \frac{T_1}{\theta_1} &= \left(\frac{p_0}{p_r}\right)^\kappa - \left(\frac{p_1}{p_r}\right)^\kappa = \frac{g\,(z_1 - z_0)}{c_p\,\theta_1} \end{aligned} \tag{9}$$

with $T_{0,1,2}$ the temperature at the surfaces at $z_{0,1,2}$, and $H_{1,2}$ the scale heights of the layers, respectively. Hence, from (3) we note that $M_1 = g\,(z_0 - Z_0)$. Given the differences in thickness of the lower, tropospheric and upper, stratospheric layer arising from the mean values $z_0 \approx 60\ km$, $z_1 \approx 15\ km$ and $z_2 \approx 0\ km$, we will consider the asymptotic limit

$$(z_1 - z_2)/(z_0 - z_1) \ll 1. \tag{10}$$

Taking $p_0 \ll p_1$, we find from (9) and (10) that

$$\theta_1\,p_1^\kappa \gg \theta_2\,(p_2^\kappa - p_1^\kappa). \tag{11}$$

Scaling the equations in the first layer with horizontal length and velocity scales L_1 and U_1, respectively, with corresponding scale height H_1, we obtain

$$\begin{aligned} \frac{\partial \sigma_1}{\partial t} + \boldsymbol{\nabla}\cdot(\sigma_1\,\mathbf{v}_1) &= 0 \\ \frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1\cdot\boldsymbol{\nabla})\mathbf{v}_1 + \frac{1}{R_1}\hat{\mathbf{z}} \times \mathbf{v}_1 &= -\frac{1}{F_1^2}\,\boldsymbol{\nabla} M_1 \end{aligned} \tag{12}$$

with Rossby number $R_1 = U_1/(f\,L_1)$ and Froude number $F_1^2 = U_1^2/(g\,H_1)$ on a f-plane, *i.e.* for a constant value of f. Assuming $R_1 = \mathcal{O}(1)$ and $F_1 \ll 1$, we find at leading order in F_1 in dimensional terms that $M_1 = g\,(z_0 - Z_0)$ is constant. We choose $M_1 = 0$ such that $z_0 = Z_0$. Hence, there are no undulations or surface gravity waves at the upper surface in the limit $F_1 \to 0$.

If we analyze the relative size of the pressure contributions in M_1, we note by using (11) and $p_0 \ll p_1$ that

$$M_1 \approx \bar{M}_1 = \bar{M}_1(p_1) = g\,(z_2 - Z_0) + c_p\,\theta_1\,(p_1/p_r)^\kappa \tag{13}$$

is a function of only the variable p_1 and hence by using (5) also only the variable σ_1. We define the basic state Σ_1 of σ_1 as the solution of $\bar{M}_1(\Sigma_1) = 0$. Consistency is met by insisting that the time derivative of $\bar{M}_1$ is zero. Hence, by using the continuity equation for σ_1 and the simplified constraint $\bar{M}_1 = 0$, we obtain $\boldsymbol{\nabla}\cdot(\Sigma_1\,\mathbf{v}_1) = 0$. Note that $\Sigma_1 = \Sigma_1(x, y)$ can be a function of x and y through the dependence on z_2. By formally writing the pseudo-density and the Montgomery potential as the sum of this basic state and a perturbation, $\sigma_1 = \Sigma_1 + F_1\,\sigma_1'$ and $M_1 = \bar{M}_1 + M_1'$, (12) becomes

$$\begin{aligned} \frac{\partial \sigma_1'}{\partial t} + \boldsymbol{\nabla}\cdot(\sigma_1'\,\mathbf{v}_1) + \frac{1}{F_1}\,\boldsymbol{\nabla}\cdot(\Sigma_1\,\mathbf{v}_1) &= 0 \\ \frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1\cdot\boldsymbol{\nabla})\mathbf{v}_1 + \frac{1}{R_1}\hat{\mathbf{z}} \times \mathbf{v}_1 &= -\frac{1}{F_1}\,\boldsymbol{\nabla} M_1'. \end{aligned} \tag{14}$$

So, a small Froude number analysis of the system yields at leading order in F_1 two constraints

$$\phi_1 = M_1 = 0 \qquad \text{and} \qquad D_1 = \boldsymbol{\nabla}\boldsymbol{\cdot}(\Sigma_1\,\mathbf{v}_1) = 0. \tag{15}$$

We will next consider the Hamiltonian dynamics on the manifold defined by these two constraints.

(b) Constrained Hamiltonian formulation

A dimensional Hamiltonian formulation of the two-layer system will be introduced to derive the formulation for the one-and-a-half layer system. It consists of the evolution

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \{\mathcal{F}, \mathcal{H}\} \tag{16}$$

with the shallow-layer Poisson bracket (e.g. Salmon, 1988) in both layers ($\alpha = 1, 2$)

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{\alpha=1}^{2} \iint \left\{ q_\alpha\,\hat{\mathbf{z}} \cdot \frac{\delta\mathcal{F}}{\delta\mathbf{v}_\alpha} \times \frac{\delta\mathcal{G}}{\delta\mathbf{v}_\alpha} - \frac{\delta\mathcal{F}}{\delta\sigma_\alpha}\boldsymbol{\nabla}\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{v}_\alpha} + \frac{\delta\mathcal{G}}{\delta\sigma_\alpha}\boldsymbol{\nabla}\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{v}_\alpha} \right\} \mathrm{d}x\,\mathrm{d}y \tag{17}$$

for arbitrary functions $\mathcal{F}$ and $\mathcal{G}$, a Hamiltonian

$$\mathcal{H} = \frac{1}{2} \iint \Bigg\{ \sum_{\alpha=1}^{2} \left(\frac{1}{2}\,\sigma_\alpha\,|\mathbf{v}_\alpha|^2 + g\,\sigma_\alpha\,z_2 \right) + \frac{c_p\,\theta_2}{g\,p_r^\kappa\,(\kappa+1)}\,(p_2^{\kappa+1} - p_1^{\kappa+1}) + \\ \frac{c_p\,\theta_1}{g\,p_r^\kappa\,(\kappa+1)}\,(p_1^{\kappa+1} - p_0^{\kappa+1}) - \sigma_1\left(g\,Z_0 + c_p\,\theta_1\,(p_0/p_r)^\kappa\right) \Bigg\}\,\mathrm{d}x\,\mathrm{d}y, \tag{18}$$

and the potential vorticity in each layer α

$$q_\alpha = \frac{f + \hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\times\mathbf{v}_\alpha}{\sigma_\alpha}. \tag{19}$$

The Hamiltonian follows from the one of the three-dimensional Euler equations by neglecting the vertical velocity relative to the horizontal velocities, by using hydrostatic balance and the ideal gas law, and integration in the vertical over each isentropic layer. In the latter integration, the horizontal velocity is assumed to be independent of the depth in each layer. Using the following functional derivatives of the Hamiltonian

$$\frac{\delta\mathcal{H}}{\delta\mathbf{v}_\alpha} = \sigma_\alpha\,\mathbf{v}_\alpha \qquad \text{and} \qquad \frac{\delta\mathcal{H}}{\delta\sigma_\alpha} = |\mathbf{v}_\alpha|^2/2 + M_\alpha, \tag{20}$$

it can be verified that (16)–(18) yield the equations of motion (1) in both layers.

The formulation (16)–(18) is Hamiltonian as it satisfies the following properties. The bracket $\{\mathcal{F}, \mathcal{G}\}$ is antisymmetric $\{\mathcal{F}, \mathcal{G}\} = -\{\mathcal{G}, \mathcal{F}\}$, and satisfies Jacobi's identity

$$\{\mathcal{F}, \{\mathcal{G}, \mathcal{K}\}\} + \{\mathcal{G}, \{\mathcal{K}, \mathcal{F}\}\} + \{\mathcal{K}, \{\mathcal{F}, \mathcal{G}\}\} = 0. \tag{21}$$

In the verification of these properties boundary conditions are required such as periodic boundaries, quiescence at infinity where σ_α is constant and $\mathbf{v}_\alpha = 0$, no-normal flow through walls, such that $\mathbf{v}_\alpha \cdot \hat{\mathbf{n}} = 0$ with $\hat{\mathbf{n}}$ the outward-pointing normal to the wall, or combinations of these boundary conditions. Furthermore, in these verifications the functional derivatives have to be restricted to satisfy corresponding boundary conditions.

Further in this section, we use for simplicity and without further notice periodic boundary conditions or quiescence at infinity. Given the constraints $\phi_1 = M_1 = 0$ and $D_1 = \nabla\cdot(\Sigma_1\, \mathbf{v}_1) = 0$, we can transform the Poisson bracket (17) in terms of the three variables $(\mathbf{v}_\alpha, \sigma_\alpha)$ to $(\phi_1, D_1, \omega_1, \mathbf{v}_2, \sigma_2)$ with $\omega_1 = \hat{\mathbf{z}}\cdot\nabla\times\mathbf{v}_1$ the vorticity in the top layer. We express the functional derivatives with respect to the former variables to functional derivatives in terms of the latter variables as follows

$$\begin{aligned}\frac{\delta\mathcal{F}}{\delta\mathbf{v}_1} &= \left(\nabla\frac{\delta\mathcal{F}}{\delta\omega_1}\right)\times\hat{\mathbf{z}} - \Sigma_1\,\nabla\frac{\delta\mathcal{F}}{\delta D_1}, \qquad \frac{\delta\mathcal{F}}{\delta\sigma_1} = G(p_1,p_2)\,\frac{\delta\mathcal{F}}{\delta\phi_1}\\ \frac{\delta\mathcal{F}}{\delta\sigma_2} &= \left.\frac{\delta\mathcal{F}}{\delta\sigma_2}\right|_{\phi_1} + \frac{c_p\,\theta_2\,\kappa\,g}{p_r}\left(\frac{p_2}{p_r}\right)^{\kappa-1}\frac{\delta\mathcal{F}}{\delta\phi_1}\end{aligned} \tag{22}$$

with

$$G(p_1,p_2) = \frac{c_p\,\theta_2\,\kappa\,g}{p_r}\left(\frac{p_2}{p_r}\right)^{\kappa-1} + \frac{c_p\,(\theta_1-\theta_2)\,\kappa\,g}{p_r}\left(\frac{p_1}{p_r}\right)^{\kappa-1}. \tag{23}$$

The transformed bracket then becomes

$$\begin{aligned}\{\mathcal{F},\mathcal{G}\} = \iint\Big\{ & q_1\,J\!\left(\frac{\delta\mathcal{F}}{\delta\omega_1},\frac{\delta\mathcal{G}}{\delta\omega_1}\right) + \Sigma_1^2\,q_1\,J\!\left(\frac{\delta\mathcal{F}}{\delta D_1},\frac{\delta\mathcal{G}}{\delta D_1}\right) +\\ & \Sigma_1\,q_1\left[\left(\nabla\frac{\delta\mathcal{G}}{\delta\omega_1}\right)\cdot\nabla\frac{\delta\mathcal{F}}{\delta D_1} - \left(\nabla\frac{\delta\mathcal{F}}{\delta\omega_1}\right)\cdot\nabla\frac{\delta\mathcal{G}}{\delta D_1}\right] +\\ & \frac{1}{F_1}\,G(p_1,p_2)\left[\frac{\delta\mathcal{G}}{\delta\phi_1}\,\nabla\cdot\left(\Sigma_1\nabla\frac{\delta\mathcal{F}}{\delta D_1}\right) - \frac{\delta\mathcal{F}}{\delta\phi_1}\,\nabla\cdot\left(\Sigma_1\nabla\frac{\delta\mathcal{G}}{\delta D_1}\right)\right] +\\ & q_2\,\hat{\mathbf{z}}\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{v}_2}\times\frac{\delta\mathcal{G}}{\delta\mathbf{v}_2} - \left(\frac{\delta\mathcal{F}}{\delta\sigma_2} + \frac{c_p\,\kappa\,\theta_2}{g}\,(p_2/p_r)^{\kappa-1}\frac{\delta\mathcal{F}}{\delta\phi_1}\right)\nabla\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{v}_2} +\\ & \left(\frac{\delta\mathcal{G}}{\delta\sigma_2} + \frac{c_p\,\kappa\,\theta_2\,g}{p_r}\,(p_2/p_r)^{\kappa-1}\frac{\delta\mathcal{G}}{\delta\phi_1}\right)\nabla\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{v}_2}\Big\}\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{24}$$

with $J(a,b) := (\partial_x a)(\partial_y b) - (\partial_x b)(\partial_y a)$ the Jacobian. Note that we have introduced the inverse of the Froude number F_1 in this dimensional bracket to indicate in front of which terms it would appear in the scaled bracket associated with both (14) and the equations in layer two.

Consistency requires that ϕ_1 and D_1 remain zero in time. Hence, we obtain

$$\begin{aligned} 0 &= \frac{\partial\phi_1(x,y,t)}{\partial t} = \{\phi_1(x,y,t),\mathcal{H}\}\\ &= -\left(\frac{1}{F_1}\,G(p_1,p_2)\,\nabla\cdot\left(\Sigma_1\nabla\frac{\delta\mathcal{H}}{\delta D_1}\right) + \frac{c_p\,\kappa\,\theta_2\,g}{p_r}\,(p_2/p_r)^{\kappa-1}\,\nabla\cdot\frac{\delta\mathcal{H}}{\delta\mathbf{v}_2}\right)\\ 0 &= \frac{\partial D_1(x,y,t)}{\partial t} = \{D_1(x,y,t),\mathcal{H}\}\\ &= -\nabla\cdot\left(\Sigma_1\,q_1\,\nabla\frac{\delta\mathcal{H}}{\delta\omega_1}\right) + J\!\left(\frac{\delta\mathcal{H}}{\delta\omega_1},\Sigma_1^2\,q_1\right) +\\ &\qquad \frac{1}{F_1}\,\nabla\cdot\left(\Sigma_1\nabla\left(G(p_1,p_2)\,\frac{\delta\mathcal{H}}{\delta\phi_1}\right)\right).\end{aligned} \tag{25}$$

Hence, to leading order in F_1 a solution is $\delta\mathcal{H}/\delta\phi_1 = \delta\mathcal{H}/\delta D_1 = 0$.

The dynamics on the constrained manifold is governed by the slow variables $\{\omega_1, \mathbf{v}_2, \sigma_2\}$, since the dynamics of the fast variables $\{D_1, \sigma_1\}$ associated with the gravity waves in layer one have been removed.‡ Restricting the transformed bracket (24) to the constrained manifold and keeping all leading-order terms in F_1, the following constrained bracket emerges

$$\{\mathcal{F},\mathcal{G}\}_c = \iint \Big\{ \frac{(f+\omega_1)}{\Sigma_1} J\Big(\frac{\delta\mathcal{F}}{\delta\omega_1}, \frac{\delta\mathcal{G}}{\delta\omega_1}\Big) + q_2\, \hat{\mathbf{z}} \cdot \frac{\delta\mathcal{F}}{\delta\mathbf{v}_2} \times \frac{\delta\mathcal{G}}{\delta\mathbf{v}_\alpha} - \frac{\delta\mathcal{F}}{\delta\sigma_2}\boldsymbol{\nabla}\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{v}_2} + \frac{\delta\mathcal{G}}{\delta\sigma_2}\boldsymbol{\nabla}\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{v}_2}\Big\}\,\mathrm{d}x\,\mathrm{d}y. \tag{26}$$

Note that we have approximated $q_1 = (f+\omega_1)/(\Sigma_1 + F_1\,\sigma_1')$ to its leading-order term in F_1. Consider Jacobi's identity (21) for the bracket (24) of three functionals $\mathcal{F}, \mathcal{G}, \mathcal{K}$ of the slow variables. At leading order in F_1, Jacobi's identity for the bracket (24) of three functionals $\mathcal{F}, \mathcal{G}, \mathcal{K}$ depending only on the slow variables $\{\omega_1, \sigma_2, \mathbf{v}_2\}$ coincides with Jacobi's identity for the bracket (26). The reason is that the terms involving only slow variables in bracket (24) do not depend on the fast variables σ_1 and D_1 at leading order in F_1. Hence, as the original bracket satisfies Jacobi's identity at leading order, also (26) must satisfy that identity.

Finally, the dynamics on the constrained manifold is given by (16) with (26) and constrained Hamiltonian

$$\mathcal{H}_c = \frac{1}{2}\iint \Big\{ \frac{1}{2}\Sigma_1\,|\boldsymbol{\nabla}\Psi_1|^2 + g\,\sigma_1\,z_2 + \frac{1}{2}\sigma_2\,|\mathbf{v}_2|^2 + g\,\sigma_2\,z_2 + \frac{c_p\,\theta_2}{g\,p_r^\kappa\,(\kappa+1)}\,(p_2^{\kappa+1} - p_1^{\kappa+1}) + \frac{c_p\,\theta_1}{g\,p_r^\kappa\,(\kappa+1)}\,(p_1^{\kappa+1} - p_0^{\kappa+1}) - \sigma_1\Big(g\,Z_0 + c_p\,\theta_1\,(p_0/p_r)^\kappa\Big)\Big\}\,\mathrm{d}x\,\mathrm{d}y \tag{27}$$

with a transport streamfunction Ψ_1 in layer one related to the vorticity by $\omega_1 = \boldsymbol{\nabla}\cdot(\Sigma_1\boldsymbol{\nabla}\Psi_1)$, $\sigma_2 = (p_2 - p_1)/g$, and the constraint relating p_1 to p_2, that is,

$$M_1 = c_p\,\theta_2\,(p_2/p_r)^\kappa + c_p\,(\theta_1 - \theta_2)\,(p_1/p_r)^\kappa + g\,(z_2 - Z_0) - c_p\,\theta_1\,(p_0/p_r)^\kappa = 0. \tag{28}$$

The functional derivative of the potential and internal energy in (27) subject to constraint (28) is

$$\begin{aligned}\frac{\delta\mathcal{H}_{ci}}{\delta\sigma_2} &= \Big(z_2 + \frac{c_p\,\theta_2}{g}\,(p_2/p_r)^\kappa\Big)\,\delta p_2 + \Big(\frac{c_p\,(\theta_1-\theta_2)}{g}\,(p_1/p_r)^\kappa - Z_0 - \frac{c_p\,\theta_1}{g}\,(p_0/p_r)^\kappa\Big)\,\delta p_1\\ &= \Big(z_2 + c_p\,\theta_2\,(p_2/p_r)^\kappa\Big)\,\delta p_2 - \Big(\frac{c_p\,(\theta_1-\theta_2)}{g}\,(p_1/p_r)^\kappa - Z_0 - \frac{c_p\,\theta_1}{g}\,(p_0/p_r)^\kappa\Big)\,\frac{\partial p_1}{\partial p_2}\,\delta p_2\\ &= \Big(z_2 + \frac{c_p\,\theta_2}{g}\,(p_2/p_r)^\kappa\Big)\,(1 - \frac{\partial p_1}{\partial p_2})\,\delta p_2 = M_2\,\delta\sigma_2\end{aligned} \tag{29}$$

‡ The introduction of fast and slow variables based on the distinction between high-frequency and low-frequency waves in the linearized system can be found in Van Kampen (1985).

using the definition $g\,\sigma_2 = p_2 - p_1$. The equations of motion (1) for $\alpha = 2$ thus stay the same, but the dynamics in layer one is governed by a barotropic vorticity equation.

Recapitulating, we note that we have been able to construct the Hamiltonian formulation of the one-and-half layer model. *A posteriori*, we conclude that it is consistent to set $\mathbf{v}_1 = 0$, since in the small Froude number limit $D_1 = 0$, and we can initialize $\omega_1 = 0$ in layer one. The vorticity equation in layer one expresses the evolution in a one-layer rigid-lid model, as the thickness of layer two is negligible compared to the thickness of layer one and $z_0 = Z_0$. This derivation of the Hamiltonian formulation of a (weighted) incompressible flow with $\boldsymbol{\nabla} \cdot (\Sigma\, \mathbf{v}_1) = 0$ in the top layer, is similar to Salmon's (1988) constrained Hamiltonian derivation of incompressible flow from the shallow-water equations. The vorticity dynamics is decoupled from the dynamics in layer two, which is the dynamics of the original one-and-half layer model. When the thickness of each layer is comparable and we can not use (10), the vorticity dynamics in the top layer and the dynamics in the lower layer become intrinsically coupled. In the latter case, the unconstrained two-layer model has a more straightforward formulation, even though the dynamics of the constrained model may be conceptually simpler.

3. Hamiltonian formulation of nearly geostrophic balanced models

The Hamiltonian formulation of the one-and-half layer balanced model with $\omega_1 = 0$ will be used as the starting point to derive nearly geostrophic Hamiltonian approximate or balanced models. Geostrophic balance is a balance between the Coriolis force and the gradient of the Montgomery potential

$$f\,\mathbf{v} = -\boldsymbol{\nabla} M, \tag{30}$$

where we have dropped here and hereafter the layer subscript when no confusion arises. Geostrophy results as the leading-order balance in an expansion of the variables in terms of a small Rossby number $R = U/(f\,L)$, where U and L are velocity and length scales in the lower layer. We therefore start our approach with the bracket

$$\{\mathcal{F}, \mathcal{G}\}_c = \iint \left\{ q\,\hat{\mathbf{z}} \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \times \frac{\delta \mathcal{G}}{\delta \mathbf{v}} - \frac{\delta \mathcal{F}}{\delta \sigma} \boldsymbol{\nabla} \cdot \frac{\delta \mathcal{G}}{\delta \mathbf{v}} + \frac{\delta \mathcal{G}}{\delta \sigma} \boldsymbol{\nabla} \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \right\} \mathrm{d}x\, \mathrm{d}y. \tag{31}$$

(*a*) *Velocity constraints and L1 dynamics*

Defining $\mathbf{v} = (u_1, u_2)$, we consider general velocity constraints

$$\tilde{u}_i = u_i - u_i^C = 0 \tag{32}$$

for each component of the constraint horizontal velocity $\mathbf{v}^C$. The lowercase italic indices such as i, j run from 1 to 2. An example is, of course, the couple of geostrophic constraints

$$\tilde{u}_i = u_i + \epsilon_{ij} \partial_j M = 0 \tag{33}$$

with ϵ_{ij} the permutation symbol with $\epsilon_{ii} = 0$ and $\epsilon_{12} = -\epsilon_{21} = 1$. The variation of the constraint velocity is

$$\delta u_i^C = \mathsf{D}^i \delta\sigma \tag{34}$$

with the Fréchet derivative D^i (cf. Vanneste and Bokhove, 2002). We also require the adjoint of the Fréchet derivative, $\hat{\mathsf{D}}^i$, defined such that

$$\iint F\,\mathsf{D}^i G\,\mathrm{d}x\,\mathrm{d}y = \iint G\,\hat{\mathsf{D}}^i F\,\mathrm{d}x\,\mathrm{d}y \tag{35}$$

for arbitrary functions F and G. For the geostrophic constraint (33), we find

$$\mathsf{D}^i(\cdot) = -\frac{\epsilon_{ij}}{f}\,\partial_j\Big(\frac{\partial M_2}{\partial\sigma}(\cdot)\Big) \qquad \text{and} \qquad \hat{\mathsf{D}}^i(\cdot) = \frac{\partial M_2}{\partial\sigma}\frac{\epsilon_{ij}}{f}\,\partial_l(\cdot). \tag{36}$$

For each velocity constraint, we need to check whether boundary contributions need to appear in defining these Fréchet derivatives. Boundary terms do not need to appear for the geostrophic constraint, when we define them with some care, as in

$$\begin{aligned}
\delta u_i^C(\mathbf{x}') &= \iint \delta(\mathbf{x}'-\mathbf{x})\,\mathsf{D}^i\delta\sigma(\mathbf{x})\,\mathrm{d}x\,\mathrm{d}y\\
&= -\iint \frac{\epsilon_{ij}}{f}\partial'_j\delta(\mathbf{x}'-\mathbf{x})\,\frac{\partial M_2(\mathbf{x})}{\partial\sigma(\mathbf{x})}\,\delta\sigma(\mathbf{x})\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint \frac{\epsilon_{ij}}{f}\Big(\partial_j\delta(\mathbf{x}'-\mathbf{x})\Big)\,\frac{\partial M_2(\mathbf{x})}{\partial\sigma(\mathbf{x})}\,\delta\sigma(\mathbf{x})\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint \Big(\hat{\mathsf{D}}^i\delta(\mathbf{x}'-\mathbf{x})\Big)\,\delta\sigma(\mathbf{x})\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{37}$$

where we used the delta function $\delta(\mathbf{x}'-\mathbf{x})$.

Consider the transformation of functional derivatives from the set of variables $(\mathbf{v},\sigma)$ to the set $(\tilde{\mathbf{v}},\sigma)$

$$\frac{\delta\mathcal{F}}{\delta\sigma} = \frac{\delta\mathcal{F}}{\delta\sigma}\Big|_C - \hat{D}^i\frac{\delta\mathcal{F}}{\delta\tilde{u}_i} \qquad \text{and} \qquad \frac{\delta\mathcal{F}}{\delta u_i} = \frac{\delta\mathcal{F}}{\delta\tilde{u}_i}, \tag{38}$$

where $(\cdot)|_C$ denotes that we consider functional derivative of σ with $\tilde{\mathbf{v}}$ held fixed. This will become a constrained derivative, if we realize that by constraining u_i to u_i^C we obtain

$$\begin{aligned}
\delta\mathcal{F} &= \iint \Big(\frac{\delta\mathcal{F}}{\delta\sigma}\,\delta\sigma + \frac{\delta\mathcal{F}}{\delta u_i}\,\delta u_i^C\Big)\;\mathrm{d}x\,\mathrm{d}y\\
&= \iint \Big(\frac{\delta\mathcal{F}}{\delta\sigma} + \hat{D}^i\frac{\delta\mathcal{F}}{\delta u_i}\Big)\,\delta\sigma\;\mathrm{d}x\,\mathrm{d}y = \iint \frac{\delta\mathcal{F}}{\delta\sigma}\Big|_C\,\delta\sigma\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{39}$$

In particular, additional boundary conditions may be required. For example, when we consider the geostrophic constraint (33) the variations become

$$\begin{aligned}
\delta\mathcal{F} = \iint_D \Big[\frac{\delta\mathcal{F}}{\delta\sigma}\,\delta\sigma + \frac{\epsilon_{ij}}{f}\,\frac{\partial M_2}{\partial\sigma_2}\,\partial_i\Big(\frac{\delta\mathcal{F}}{\delta\tilde{u}_i}\Big)\,\delta\sigma + \frac{\delta\mathcal{F}}{\delta\tilde{u}_i}\,\delta u_i\Big]\,\mathrm{d}x\,\mathrm{d}y +\\
\int_{\partial D}\frac{\delta\mathcal{F}}{\delta\tilde{\mathbf{v}}}\cdot\hat{t}\,\delta\sigma\,\mathrm{d}l,
\end{aligned} \tag{40}$$

where $\mathrm{d}l$ is a cord along and $\hat{t}$ a unit vector tangent to the boundary. Hence, we set the tangential component of the (functional derivative of) ageostrophic velocity $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{v}^C$ to zero (cf. Salmon, 1985) as additional boundary condition.

Substitution of (38) into the bracket (31) yields

$$\{\mathcal{F},\mathcal{G}\}_c = \iint \Big\{ q\,\hat{\mathbf{z}}\cdot\frac{\delta\mathcal{F}}{\delta\tilde{\mathbf{v}}}\times\frac{\delta\mathcal{G}}{\delta\tilde{\mathbf{v}}} - \frac{\delta\mathcal{F}}{\delta\sigma}\Big|_C \boldsymbol{\nabla}\cdot\frac{\delta\mathcal{G}}{\delta\tilde{\mathbf{v}}} + \frac{\delta\mathcal{G}}{\delta\sigma}\Big|_C \boldsymbol{\nabla}\cdot\frac{\delta\mathcal{F}}{\delta\tilde{\mathbf{v}}} \\ + \Big(\hat{D}^i\frac{\delta\mathcal{F}}{\delta\tilde{u}_i}\Big)\boldsymbol{\nabla}\cdot\frac{\delta\mathcal{G}}{\delta\tilde{\mathbf{v}}} - \Big(\hat{D}^i\frac{\delta\mathcal{G}}{\delta\tilde{u}_i}\Big)\boldsymbol{\nabla}\cdot\frac{\delta\mathcal{F}}{\delta\tilde{\mathbf{v}}}\Big\}\,\mathrm{d}x\,\mathrm{d}y. \tag{41}$$

Consistency requires that

$$0 = \frac{\partial\tilde{u}_i}{\partial t} = q\,\epsilon_{ij}\frac{\delta\mathcal{H}}{\delta\tilde{u}_j} - \partial_i\frac{\delta\mathcal{H}}{\delta\sigma}\Big|_C + D^i\partial_j\frac{\delta\mathcal{H}}{\delta\tilde{u}_j} + \partial_i\Big(\hat{D}^j\frac{\delta\mathcal{H}}{\delta\tilde{u}_j}\Big) \tag{42}$$

in which

$$q = (f + \epsilon_{ij}\,\partial_i u^C_j)/\sigma \tag{43}$$

is the constrained potential vorticity. Hence,

$$\mathcal{L}^{ij}\frac{\delta\mathcal{H}}{\delta\tilde{u}} = \partial_i\frac{\delta\mathcal{H}}{\delta\sigma}\Big|_C, \tag{44}$$

where we introduced the linear operator

$$\mathcal{L}^{ij} = D^i\partial_j\frac{\delta\mathcal{H}}{\delta\tilde{u}_j} + \partial_i\Big(\hat{D}^j\frac{\delta\mathcal{H}}{\delta\tilde{u}_j}\Big) + q\,\epsilon_{ij}\frac{\delta\mathcal{H}}{\delta\tilde{u}_j} \tag{45}$$

(cf. Vanneste and Bokhove, 2002). Finally, on the constrained manifold defined by the velocity constraints, we obtain the Dirac bracket (cf. Vanneste and Bokhove, 2002) by substituting (44) into (41)

$$\{\mathcal{F},\mathcal{G}\}_C = \iint \partial_i\frac{\delta\mathcal{F}}{\delta\sigma}\Big|_C \mathcal{L}^{-ij}\,\partial_j\frac{\delta\mathcal{G}}{\delta\sigma}\Big|_C \mathrm{d}x\,\mathrm{d}y \tag{46}$$

after an integration by parts and after formally assuming that $\mathcal{L}$ is invertible. (In these manipulations, we have assumed that boundary terms disappear. For periodic boundary conditions, for example, this is true.)

In contrast with previous derivations (Allen and Holm, 1996; Vanneste and Bokhove, 2002), we have not introduced any Lagrangian variables or Lagrange multipliers. In Appendix A, it is shown for general finite-dimensional systems that the slaved Hamiltonian approach yields the Dirac bracket (Dirac, 1964). Dirac (1958) proved that this finite-dimensional Dirac bracket satisfies Jacobi's identity, while the antisymmetric property of this finite-dimensional Dirac bracket follows more directly. Since $\mathcal{L}^{ij}$ is antisymmetric under suitable boundary conditions and under the assumption that it is invertible, we note that the Dirac bracket (46) is antisymmetric.

Balanced equations of motion (cf. Vanneste and Bokhove, 2002) appear when we use (16), (46) and the constrained Hamiltonian

$$\mathcal{H}_c = \frac{1}{2}\iint\Big\{\frac{1}{2}\,\sigma\,|\mathbf{v}^C|^2 + g\,\sigma\,z_2 + g\,\sigma_1\,z_2 + \\ \frac{c_p\,\theta_2}{g\,p_r^\kappa\,(\kappa+1)}\,(p_2^{\kappa+1} - p_1^{\kappa+1}) + \frac{c_p\,\theta_1}{g\,p_r^\kappa\,(\kappa+1)}\,(p_1^{\kappa+1} - p_0^{\kappa+1}) - \\ \sigma_1\Big(g\,Z_0 + c_p\,\theta_1\,(p_0/p_r)^\kappa\Big)\Big\}\,\mathrm{d}x\,\mathrm{d}y \tag{47}$$

with $M_1 = 0$ [(28)]. In particular, we find L1 dynamics for a one-and-half layer isentropic model after we substitute the Fréchet derivatives (36) for the geostrophic constraint. The only difference with the result for one layer in Vanneste and Bokhove (2002) is the Hamiltonian, which includes additional internal-energy terms in the one-and-half layer model.

(*b*) *Quasigeostrophic dynamics*

In this section, the domain has solid walls and may have multiple connections. Consider the linearized potential vorticity as slow variable

$$\Omega = \omega - \frac{f}{\Sigma_2}\,\sigma' \tag{48}$$

with $\sigma = \Sigma_2 + R\,\sigma'$. Although we have not scaled the equations, the Rossby number R has been placed in its position to remind us what the order is of various terms. Typically, $R = U/(f\;L) \approx 0.1$ for large-scale atmospheric dynamics with $f = 10^{-4}\;s^{-1}$, $U = 10\;m/s$ and $L = 10^6\;m$. In terms of σ' geostrophic balance is rewritten as

$$u_i = -\epsilon_{ij}\partial_j\psi \tag{49}$$

with the streamfunction $\psi = M'(\Sigma_2)\sigma'/f$ and $M' = dM/d\sigma$. The fast variables are the divergence D and geostrophic imbalance Υ defined as

$$D = \boldsymbol{\nabla}\cdot\mathbf{v} \qquad \text{and} \qquad \Upsilon = f\;\omega - \nabla^2\psi. \tag{50}$$

The distinction between slow and fast variables (Van Kampen, 1985) follows by considering the equations after scaling: in the slow equation for Ω no linear terms of $\mathcal{O}(1/R)$ appear, while these do appear in the fast equations for D and Υ. The equations for the slow and fast variables linearized around a state of rest on the slow time scale t read

$$\begin{aligned} \frac{\partial\Omega}{\partial t} &= 0 \\ \frac{\partial D}{\partial t} + \frac{1}{R}\,\Upsilon &= 0 \\ \frac{\partial\Upsilon}{\partial t} + \frac{1}{R}\left(f^2 - M'(\Sigma_2)\,\Sigma_2\nabla^2\right)D &= 0. \end{aligned} \tag{51}$$

The fast equations thus support high-frequency gravity waves with a frequency of $\mathcal{O}(1/R)$. The additional nonlinear terms in the slow and fast equations are $\mathcal{O}(1)$. Hence, at leading order in R, we find the constraints

$$D = 0 \qquad \text{and} \qquad \Upsilon = 0. \tag{52}$$

This motivates the transformation of functional derivatives from the trio of variables $(\mathbf{v}, \sigma')$ to $(\Omega, D, \Upsilon, \tilde{\Gamma})$, that is,

$$\delta\mathcal{F} = \iint_D \Big(\frac{\delta\mathcal{F}}{\delta\Omega}\,\delta\Omega + \frac{\delta\mathcal{F}}{\delta D}\,\delta D + \frac{\delta\mathcal{F}}{\delta\Upsilon}\,\delta\Upsilon \Big)\ \mathrm{d}x\,\mathrm{d}y + \int_{\partial D} \frac{\delta\mathcal{F}}{\delta\tilde{\Gamma}}\,\delta\tilde{\Gamma}\,\mathrm{d}l,$$
$$= \iint_{D_h} \Big[\boldsymbol{\nabla}\Big(\frac{\delta\mathcal{F}}{\delta\Omega} + f\,\frac{\delta\mathcal{F}}{\delta\Upsilon}\Big) \times \hat{z}\cdot\delta\mathbf{v} - \Big(\boldsymbol{\nabla}\frac{\delta\mathcal{F}}{\delta D}\Big)\cdot\delta\mathbf{v} -$$
$$\Big(\frac{f}{\Sigma_2}\,\frac{\delta\mathcal{F}}{\delta\Omega} + \nabla^2\frac{\delta\mathcal{F}}{\delta\Upsilon}\Big)\,\delta\sigma'\Big]\ \mathrm{d}x\,\mathrm{d}y + \tag{53}$$
$$\int_{\partial D_h} \Big[\Big(\frac{\delta\mathcal{F}}{\delta\Omega} + f\,\frac{\delta\mathcal{F}}{\delta\Upsilon}\Big)\,\hat{t}\cdot\delta\mathbf{v} +$$
$$\frac{\delta\mathcal{F}}{\delta D}\,\hat{n}\cdot\delta\mathbf{v} - \frac{\delta\mathcal{F}}{\delta\Upsilon}\,\boldsymbol{\nabla}\delta\psi\cdot\hat{n} + \boldsymbol{\nabla}\frac{\delta\mathcal{F}}{\delta\Upsilon}\cdot\hat{n}\,\delta\psi\Big]\ \mathrm{d}l,$$

whence we identify $\delta\mathcal{F}/\delta\tilde{\Gamma} = \delta\mathcal{F}/\delta\Omega + f\,\delta\mathcal{F}/\delta\Upsilon$, $\tilde{\Gamma} = \hat{t}\,\mathbf{v}$ and use $\hat{n}\cdot\delta\mathbf{v} = 0$. We further assume that $\delta\psi$ and $\delta\mathcal{F}/\delta\Upsilon$ are zero at the boundary, which is reasonable as ψ is the streamfunction and $\Upsilon = 0$ at leading order in the Rossby number limit $R \to 0$, when $D = \Upsilon = 0$. The bracket (31) expressed in terms of these new variables becomes

$$\{\mathcal{F},\mathcal{G}\}_c = \iint_{D_h} \Big\{ q\,\Big[J\Big(\frac{\delta\mathcal{F}}{\delta\Omega}, \frac{\delta\mathcal{G}}{\delta\Omega}\Big) + J\Big(\frac{\delta\mathcal{F}}{\delta D}, \frac{\delta\mathcal{G}}{\delta D}\Big) +$$
$$\boldsymbol{\nabla}\frac{\delta\mathcal{F}}{\delta D}\cdot\boldsymbol{\nabla}\Big(\frac{\delta\mathcal{G}}{\delta\Omega} + f\,\frac{\delta\mathcal{G}}{\delta\Upsilon}\Big) - \boldsymbol{\nabla}\frac{\delta\mathcal{G}}{\delta D}\cdot\boldsymbol{\nabla}\Big(\frac{\delta\mathcal{F}}{\delta\Omega} + f\,\frac{\delta\mathcal{F}}{\delta\Upsilon}\Big)\Big] -$$
$$\frac{1}{R}\Big(\nabla^2\frac{\delta\mathcal{G}}{\delta D}\Big)\Big(\nabla^2 - \frac{f^2}{\Sigma_2}\Big)\frac{\delta\mathcal{F}}{\delta\Upsilon} + \frac{1}{R}\Big(\nabla^2\frac{\delta\mathcal{F}}{\delta D}\Big)\Big(\nabla^2 - \frac{f^2}{\Sigma_2}\Big)\frac{\delta\mathcal{G}}{\delta\Upsilon}\Big\}\ \mathrm{d}x\,\mathrm{d}y +$$
$$\int_{\partial D_h} \frac{f}{\Sigma_2}\,\hat{n}\cdot\Big\{\frac{\delta\mathcal{G}}{\delta\tilde{\Gamma}}\,\boldsymbol{\nabla}\frac{\delta\mathcal{F}}{\delta D} - \frac{\delta\mathcal{F}}{\delta\tilde{\Gamma}}\,\boldsymbol{\nabla}\frac{\delta\mathcal{G}}{\delta D}\Big\}\ \mathrm{d}l. \tag{54}$$

Note that we introduced $1/R$-terms to indicate the order of the terms in the bracket. On the constrained manifold, we find that

$$\Omega = \nabla^2\psi - \frac{f}{\Sigma_2}\,\sigma'. \tag{55}$$

At leading order in R, the Hamiltonian minus the potential energy of the rest state reads

$$\mathcal{H}_{QG} = \frac{1}{2}\iint \big\{\Sigma_2\,|\boldsymbol{\nabla}\psi|^2 + M_2'(\Sigma_2)\,\sigma'^2\big\}\ \mathrm{d}x\,\mathrm{d}y \tag{56}$$

with $\psi = M_2'(\Sigma_2)\,\sigma'/f$. Variation of (56) yields

$$\frac{\delta\mathcal{H}_{QG}}{\delta\Omega} = -\Sigma_2\,\psi \qquad \text{and} \qquad \frac{\delta\mathcal{H}_{QG}}{\delta\tilde{\Gamma}} = \Sigma_2\,\psi. \tag{57}$$

We therefore restrict the functional derivatives such that $\delta\mathcal{F}/\delta\tilde{\Gamma}$ is constant along the boundary. To obtain the bracket on the constraint manifold $D = \Upsilon = 0$, we only require the leading-order expressions $\delta\mathcal{H}/\delta D$ and $\delta\mathcal{F}/\delta\Upsilon$, since $q = f/(R\,\Sigma_2) + \Omega/\Sigma_2 + \mathcal{O}(R)$. We introduce the circulation $\Gamma_i = \int_{\partial D_{hi}} \tilde{\Gamma}\,\mathrm{d}l$ on each connected piece $\partial D_{h\mathsf{i}}$ of the domain wall with $\mathsf{i} = 1, \ldots, N-1$ for N connections (on one connected piece we can set $\psi = 0$). When $\delta\mathcal{F}/\delta\tilde{\Gamma}$ is constant along the boundary, we find, for example

$$\frac{\delta\mathcal{F}}{\delta\Gamma_{\mathsf{i}}} \int_{\partial D_{h\mathsf{i}}} \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \frac{\delta\mathcal{G}}{\delta D} \mathrm{d}l \tag{58}$$

on each connected piece of the domain. Hence, the boundary terms vanish if we assume $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}(\delta\mathcal{G}/\delta D) = 0$ as additional boundary condition. To underpin that assumption, we consider

$$0 = \hat{\mathbf{n}} \cdot \mathbf{v} = \hat{\mathbf{n}} \cdot (\hat{\mathbf{z}} \times \boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\chi) \tag{59}$$

at the boundary, where χ is the velocity potential. Since $\hat{\mathbf{t}} \cdot \boldsymbol{\nabla}\psi = 0$ at the boundary it follows that $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\chi \cdot = 0$. Moreover, $\delta\mathcal{H}/\delta D = \chi + \mathcal{O}(R)$, and the extra boundary condition on the functional derivative seems therefore reasonable. In terms of the perturbation pseudo-density σ' and the circulation Γ_i on each connected piece, save one, we obtain the bracket for quasigeostrophic dynamics

$$\{\mathcal{F}, \mathcal{G}\}_{QG} = \iint_{D_h} \frac{\Omega}{\Sigma_2} J\Big(\frac{\delta\mathcal{F}}{\delta\Omega}, \frac{\delta\mathcal{G}}{\delta\Omega}\Big) \, \mathrm{d}x \, \mathrm{d}y. \tag{60}$$

Using the Hamiltonian (56) and the bracket (60), we find the familiar quasi-geostrophic vorticity and circulation equations

$$\frac{\partial\Omega}{\partial t} + J(\psi, \Omega) = 0 \qquad \text{and} \qquad \frac{d\Gamma_{\mathsf{i}}}{dt} = 0 \qquad \text{with} \qquad \mathsf{i} = 1, \ldots, N-1. \tag{61}$$

This inclusion of the circulation contributions in a *derivation* of the Hamiltonian formulation of quasigeostrophic dynamics from the parent, shallow-layer dynamics appears to be novel.

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Appendix A

Slaved Hamiltonian Dynamics
—By O. Bokhove and T.G. Shepherd—

We consider generalized, finite-dimensional Hamiltonian systems subject to an even number of constraints $\tilde{u} = u - U(s) = 0$ between the variables u and variables s, where u and s are both vectors. u is said to be slaved to s via the constraints. The matrix of the generalized Poisson bracket of the constraints is assumed to be invertible. Under this assumption, we show that the slaved Hamiltonian dynamics on the constrained manifold $\tilde{u} = 0$ follows directly from the equations of motion for s and $\tilde{u}$, subject to the consistency condition that the time evolution of the constraints is zero, *i.e.* $d\tilde{u}/dt = 0$. The resulting generalized Poisson bracket on the constrained manifold is the Dirac bracket.

In many Hamiltonian systems, the variables can be divided into slow variables s and fast variables f. The ratio of slow to fast timescales then defines a small parameter ε. In these systems, the matrix of the bracket of constraints is guaranteed to be invertible at leading order in ε. Finally, it can be shown that the slaved Hamiltonian approach can be simplified to a leading-order Hamiltonian slow dynamics on the approximated constrained manifold $f = 0$.

B. Hamiltonian constrained dynamics

(a) *Preliminaries*

Consider a (generalized) Hamiltonian structure, which is written as

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \{z, H\} \tag{B.1}$$

for a Hamiltonian $H(z)$ and a (generalized) Poisson bracket $\{\cdot, \cdot\}$ that will both be defined below. The Poisson bracket $\{F, G\}$ of two functions $F(z)$ and $G(z)$ is a derivation, $\{F, GK\} = \{F, G\}K + G\{F, K\}$, which is antisymmetric, $\{F, G\} = -\{G, F\}$, and which obeys Jacobi's identity,

$$\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0 \tag{B.2}$$

(e.g., Olver, 1986). More specifically, we write

$$\{F, G\} = (\partial F/\partial z^i)\{z^i, z^j\}(\partial G/\partial z^j). \tag{B.3}$$

For the asymptotics we adopt the following notation: Any function written in the form $F(z;\varepsilon)$ is understood to be of $\mathcal{O}(1)$, meaning that $\lim_{\varepsilon\to 0} F(z;\varepsilon)$ is finite. By $\mathcal{O}(\varepsilon^n)$ we mean $\varepsilon^n F(z;\varepsilon)$ for some $F(z;\varepsilon)$.

We let $z = (s, u)$ with $s \in \mathbf{R}^p$ and $u \in \mathbf{R}^q$. Moreover, q is an even number. We denote $\partial_s := \partial/\partial s$ and $\partial_u := \partial/\partial u$. Individual components will be denoted by superscripts: z^i, s^i, u^i, ∂_s^i and ∂_u^i. Moreover, $(\partial_u g)^{ij} := \partial g^j/\partial u^i$, etcetera. Repeated indices are understood to be summed over the relevant ranges.

(b) *Constraints and consistency*

We consider constraints

$$\tilde{u} = u - U(s) = 0. \tag{B.4}$$

Otherwise stated, u is slaved to s. We suppose that these constraints arise after suitable analysis of the physics involved in combination with an asymptotic analysis such that the constraints are at least consistent to leading order in a suitably chosen small parameter ϵ. More particularly, we assume that the matrix of the generalized Poisson bracket of constraints, $\{\tilde{u}^i, \tilde{u}^j\}$, is invertible. This assumption will be motivated in §C using an asymptotic approach.

The variations of a function $F(z)$ in terms of variables $z = (s, u)$ are related to the ones with variables $z = (s, \tilde{u})$ as follows, *i.e.*, before we apply the constraints,

$$\delta F = \partial_s F\big|_C\, \delta s + \partial_{\tilde{u}} F\, \delta\tilde{u} = \Big(\partial_s F\big|_C - \partial_{\tilde{u}} F\, \partial_s U\Big)\, \delta s + \partial_{\tilde{u}} F\, \delta u, \tag{B.5}$$

where $(\cdot)|_C$ denotes that we consider derivatives of s with $\tilde{u}$ held fixed. Hence

$$\partial_s F = \partial_s F\big|_C - \partial_{\tilde{u}} F\, \partial_s U \qquad \text{and} \qquad \partial_u F = \partial_{\tilde{u}} F. \tag{B.6}$$

Consistency of the constraints requires that

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} = 0. \tag{B.7}$$

Combining the transformed equations with (B.7) yields

$$\begin{aligned}
\frac{\mathrm{d}s^i}{\mathrm{d}t} &= \{s^i, s^j\}\, \partial_s^j H\big|_C + \Big(\{s^i, u^j\} - \{s^i, s^l\}\, \partial_s^l U^j\Big)\, \partial_{\tilde{u}}^j H \\
&= \{s^i, s^j\}\, \partial_s^j H\big|_C + \{s^i, \tilde{u}^j\}\, \partial_{\tilde{u}}^j H \\
0 = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} &= \Big(\{u^i, s^j\} - \partial_s^k U^i\, \{s^k, s^j\}\Big)\, \partial_s^j H\big|_C + \\
&\Big(\{u^i, u^j\} - \partial_s^k U^i\, \{s^k, u^j\} - \{u^i, s^k\}\, \partial_s^k U^j + \partial_s^k U^i \{s^k, s^l\} \partial_s^l U^j\Big)\, \partial_{\tilde{u}}^j H \\
&= \{\tilde{u}^i, s^j\}\, \partial_s^j H\big|_C + \{\tilde{u}^i, \tilde{u}^j\}\, \partial_{\tilde{u}}^j H.
\end{aligned} \tag{B.8}$$

We rewrite the last equation in (B.8) as

$$L^{ij}\, \partial_{\tilde{u}}^j H = -\{\tilde{u}^i, s^j\}\, \partial_s^j H\big|_C \tag{B.9}$$

with the skew-symmetric operator

$$L^{ij} = \{\tilde{u}^i, \tilde{u}^j\}. \tag{B.10}$$

Let L^{ij} be invertible (cf. the assumption stated above). Note that L thus has an even number of rows and columns. After using (B.10) to reorder (B.8), the dynamics on the constrained manifold becomes

$$\frac{\mathrm{d}s^i}{\mathrm{d}t} = \Big(\{s^i, s^j\} - \{s^i, \tilde{u}^k\}\, (L^{-1})^{kl}\, \{\tilde{u}^l, s^j\}\Big)\, \partial_s^j H \tag{B.11}$$

with the associated Dirac bracket (Dirac, 1964)

$$\{F, G\}_C = \partial_s F\, \Big(\{s, s\} - \{s, \tilde{u}\}\, L^{-1}\, \{\tilde{u}, s\}\Big)\, \partial_s G, \tag{B.12}$$

where we have dropped the reference to the constrained derivatives and where we have substituted $\tilde{u} = 0$ in the Hamiltonian and the resulting Dirac bracket. Note that the bracket (B.12) is antisymmetric. Dirac (1958) proved that the bracket (B.12) satisfies Jacobi's identity for a canonical Poisson bracket, but this proof can be extended given any generalized Poisson bracket $\{\cdot, \cdot\}$.

C. Leading-order Hamiltonian slow dynamics

In many applications one encounters problems with two timescales in which the dependent variables $z = (s, u)$ can be divided into slow variables s and fast variables f after a suitable scaling and a transformation of variables (Van Kampen, 1985). The ratio of slow to fast timescales then defines a small parameter ε.

Hence, the formulation

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = \{z^i, H\} = \{z^i, z^j\}(\partial H / \partial z^j) \tag{C.13}$$

can be written in terms of variables s and f. The bracket $\{z^i, z^j\}$ is by hypothesis given by

$$\begin{aligned}
\{s^i, s^j\} &= J^{ij} &&= J_0^{ij}(s) + \varepsilon J_1^{ij}(s, f; \varepsilon) \\
\{f^i, s^j\} &= K^{ij} &&= K_0^{ij}(s, f) + \varepsilon K_1^{ij}(s, f; \varepsilon) \\
\{f^i, f^j\} &= -\frac{1}{\varepsilon}T^{ij} + Y^{ij} &&= -\frac{1}{\varepsilon}T^{ij} + Y_0^{ij}(s, f; \varepsilon).
\end{aligned} \tag{C.14}$$

We note that T is a constant invertible skew-symmetric matrix, and that antisymmetry of the Poisson bracket dictates that $\{s, f\} = -K^{\mathrm{T}}$, where the superscript $^{\mathrm{T}}$ denotes matrix transpose. Here and in the rest of this paper, J_0, T, Y_0, and also A and R_0 introduced below are understood to denote fixed *functions* of their arguments. That J_0 cannot depend on f can be seen by considering Jacobi's identity $\{s, \{s, f\}\} + \cdots = 0$ at $\mathcal{O}(1/\varepsilon)$.

We take $H(z; \varepsilon)$ of the form

$$H(s, f; \varepsilon) = \tfrac{1}{2} f^{\mathrm{T}} A\, f + R_0(s) + \varepsilon R_1(s, f; \varepsilon), \tag{C.15}$$

where A is a constant symmetric matrix. Hence, (C.13) and the Poisson bracket (C.14) imply equations of motion of the generic form

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}t} &= J\, \partial_s H - K^{\mathrm{T}}\, \partial_f H \\
\frac{\mathrm{d}f}{\mathrm{d}t} &= K\, \partial_s H - \frac{1}{\varepsilon}\Gamma A^{-1}\, \partial_f H + Y\, \partial_f H
\end{aligned} \tag{C.16}$$

with $T = \Gamma A^{-1}$ and Γ a constant invertible skew-hermitian matrix (or a linear operator with purely imaginary eigenvalues). The latter property implies that f undergoes rapid energy-conserving oscillations in the limit $\varepsilon \to 0$. Hence the fast motions are waves, not damped motion; this property distinguishes the present approach from center manifold theory (Carr, 1981).

Following Van Kampen (1985), we can define the constraints

$$\tilde{u} = f - U(s). \tag{C.17}$$

Otherwise stated, f is slaved to s. In general, $U(s)$ is determined by expanding the fast variable f into a power series in ϵ or by iterating f, in both cases with s as the independent variable. Clearly at leading order or as the leading iteration, we find $f = U_0(s) = 0$ from (C.15) and (C.16). Transforming to the variables $\{s, \tilde{u}\}$, we find the skew-symmetric operator L^{ij} defined in (B.10) to be

$$\begin{aligned}
L^{ij} &= \{f^i, f^j\} - \partial_s^k U^i\, \{s^k, f^j\} - \{f^i, s^k\}\, \partial_s^k U^j + \partial_s^k U^i \{s^k, s^l\} \partial_s^l U^j \\
&= -\frac{1}{\varepsilon}T^{ij} + Y_0^{ij}(s, f; \varepsilon) + \partial_s^k U^i \left(K_0^{jk} + \varepsilon K_1^{jk}\right) - \\
&\qquad \left(K_0^{ik} + \varepsilon K_1^{ik}\right) \partial_s^k U^j + \partial_s^k U^i \left(J_0^{kl} + \varepsilon J_1^{kl}\right) \partial_s^l U^j.
\end{aligned} \tag{C.18}$$

Hence, $L^{-1} \to 0$ in the limit $\varepsilon \to 0$, since

$$L^{-ij} = -\varepsilon\, T^{-ij} + \mathcal{O}(\varepsilon^2). \tag{C.19}$$

At leading order, we thus find $\partial H / \partial \tilde{u} = 0$ by combining (B.9) and (C.19). The Hamiltonian slow dynamics on the constrained manifold $f = 0$, at leading order, is therefore

$$\frac{\mathrm{d}s}{\mathrm{d}t} = J_0\, \partial_s H_0 \tag{C.20}$$

with the leading-order Hamiltonian $H_0(s) = R_0(s)$, cf. (C.15). The associated bracket is

$$\{F, G\}_0 = \partial_s F\, J_0(s) \partial_s G. \tag{C.21}$$

Jacobi's identity $\{s, \{s, s\}\} + \cdots = 0$ evaluated at $\mathcal{O}(1)$, only involves $J_0(s)$. Hence, the leading-order (Dirac) bracket (C.21) satisfies Jacobi's identity.

D. Concluding remarks

We have considered finite-dimensional Hamiltonian systems with an even number of constraints $\tilde{u} = u - U(s)$ which arise from a slaving approach. We assume that the ratio of fast and slow timescales defines a small parameter ε. At leading and higher order, an asymptotic or iterative approach then defines a sequence of constraints of the form $\tilde{u} = 0$. Furthermore, we assume that the matrix of the generalized Poisson bracket of constraints is invertible, which is arguably the case when $\varepsilon \to 0$. The consistency requirement $d\tilde{u}/dt = 0$ combined with the dynamics for s, then yields a slaved Hamiltonian dynamics on the constrained manifold. The associated bracket is the Dirac bracket arising in Dirac's constrained Hamiltonian approach. Our derivation is more concise than Dirac's derivation (Dirac, 1958, 1964), because our slaving constraints are special. In Dirac's approach, the dynamics is modified by introducing forces that constrain the dynamics to the constrained manifold. The constraints are enforced with the use of Lagrange multipliers. These additional forces are small if the constraints are already a good approximation to the original system. In particular, when the constraints are exactly obeyed by the original system, these constraining forces are zero. Our slaved Hamiltonian approach also appears to correspond with a direct substitution of the constraints in a relevant variational principle. However, such a variational principle underlying the Hamiltonian dynamics may not always be (readily) available.

The presented slaved Hamiltonian approach in this appendix can be extended to infinite-dimensional systems on a case by case basis, for example in the field of geophysical fluid dynamics. In the latter field, it can simplify the derivations of the Hamiltonian reduced or balanced dynamics in Salmon (1985, 1988), Allen and Holm (1996), Theiss (2000), McIntyre and Roulstone (2002), and of the Dirac bracket in Bokhove and Vanneste (2002). Alternatively, it can provide other derivations of approximate Hamiltonian fluid systems, as the derivation of the one-and-a-half isentropic layer model in the main text illustrates.

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